Research article

Blow-up criterion for the 3D nematic liquid crystal flows via one velocity and vorticity components and molecular orientations

Qiang Li and Baoquan Yuan∗

School of Mathematics and Information Science, Henan Polytechnic University, Henan, 454000, China.

* Correspondence: Email: bqyuan@hpu.edu.cn.

Abstract: In this paper, we are devoted to investigating the blow-up criteria for the three dimensional nematic liquid crystal flows. More precisely, we proved that the smooth solution \((u, d)\) can be extended beyond \(T\), provided that
\[
\int_0^T \left( ||\omega_3||_{L^p}^{2p} + ||u_3||_{L^q}^{2q} + ||\nabla d||_{L^\infty}^{2} \right) dt < \infty, \quad \frac{3}{2} < p \leq \infty, \quad 3 < q \leq \infty.
\]

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1. Introduction

In this paper, we are interested in the following hydrodynamic system modeling the flow of the nematic liquid crystal materials in 3-dimensions:

\[
\begin{cases}
\partial_t u + u \cdot \nabla u - \mu \Delta u + \nabla p = -\lambda \nabla \cdot (\nabla d \otimes \nabla d), \\
\partial_t d + u \cdot \nabla d = \gamma (\Delta d + |\nabla d|^2 d), \\
\nabla \cdot u = 0, |d| = 1, \\
u(x, 0) = u_0(x), d(x, 0) = d_0(x),
\end{cases}
\]

\[(1.1)\]

where \(u\) is the velocity field, \(d\) is the macroscopic average of molecular orientation field and \(p\) represents the scalar pressure. And \(\mu\) is the kinematic viscosity, \(\lambda\) is the competition between the kinetic and potential energies, and \(\gamma\) is the microscopic elastic relation time for the molecular orientation field. The notation \(\nabla d \otimes \nabla d\) represents the \(3 \times 3\) matrix, of which the \((i, j)th\) component can be denoted by \(\partial_i d_k \partial_j d_k\) \((i, j \leq 3)\).

The model of the hydrodynamic theory for liquid crystals was established by Ericksen and Leslie [8, 12, 13], and the system (1.1) was first introduced by Lin [14] as a simplified version to the
Ericksen-Leslie system describing the flow of nematic liquid crystals. Later, Lin and Liu had done many significant works such as [15, 16].

When the orientation field $d$ equals a constant, the above equations become the incompressible Navier-Stokes equations. Many regularity results on the weak solutions to the three-dimensional Navier-Stokes equations have been well studied, for example see [3–7, 9, 17, 18, 21–23, 30, 32, 33], and references therein, where they have proved that the solution is a smooth one if the velocity, or vorticity, or the gradient of velocity, or their components are regular. In their famous work [2], J. Beale et al. proved that the smooth solution $u$ blows up at a finite time $t = T^*$ for the 3D Euler equations, if $\int_0^{T^*} \|\omega\|_{L^\infty} dt = \infty$, which also holds for the Navier-Stokes equations. In [31], Zhang has investigated a regularity criterion via one velocity and one vorticity component. Our main results are stated as follows:

**Theorem 1.1.** Assume the initial data $u_0 \in H^3(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$, and $d_0 \in H^4(\mathbb{R}^3, \mathbb{S}^2)$, $(u, d)$ is a smooth solution to the equations of (1.1) on $[0, T)$ for some $0 < T < \infty$. Then $(u, d)$ can be extended beyond $T$, provided that

$$\int_0^T (\|\omega\|_{L^{\infty}}^{2p} + \|\nabla d\|_{B^{2p,\infty}_{\mathbb{S}^2}}^2) dt < \infty, \quad 0 < p < 2,$$

then the solution $(u, d)$ can be extended smoothly beyond $T$.

Recently, Yuan and Wei [27] consider the blow-up criterion in terms of the vorticity in Besov space of negative index and the orientation field in the homogeneous Besov space. If

$$\int_0^T (\|\omega\|_{B^{r,\infty}_{\mathbb{S}^2}}^{2r} + \|\nabla d\|_{B^{2r,\infty}_{\mathbb{S}^2}}^2) dt < \infty, \quad 0 < r < 2,$$

then the solution $(u, d)$ can be extended smoothly beyond $T$.

**Remark 1.2.** As we know, if the initial data $u_0 \in H^s(\mathbb{R}^n)$ with $\nabla \cdot u_0 = 0$ and $d_0 \in H^{s+1}(\mathbb{R}^n, \mathbb{S}^2)$ for $s \geq n$, then there exists a positive time $T$ depending only on the initial value such that system (1.1) has a unique smooth solution $(u, d) \in (\mathbb{R}^n \times [0, T))$ satisfying (see for example [25])

$$u \in C([0, T]; H^s(\mathbb{R}^n)) \cap C^1([0, T]; H^{s-2}(\mathbb{R}^n)),$$

$$d \in C([0, T]; H^{s+1}(\mathbb{R}^n)) \cap C^1([0, T]; H^{s-1}(\mathbb{R}^n)).$$
In the following part, we shall use simplified notations. we shall use the letter C to denote a generic constant which may be different from line to line, and write \( \partial_t u = \frac{\partial u}{\partial t}, \partial_i = \frac{\partial}{\partial x_i} \). Since the concrete values of the constants \( \mu, \lambda, \gamma \) play no role in our discussion, to simplify the presentation, we shall assume that \( \mu = \lambda = \gamma = 1 \) in this paper.

2. Preliminaries

In this section, we shall recall the interpolation inequality in [1] and the commutator estimate in [11], which will be used in the process of the proof of Theorem 1.1.

Lemma 2.1. (Page 82 in [1]). Let \( 1 < q < p < \infty \) and \( \alpha \) be a positive real number. Then there exists a constant \( C \) such that

\[
\|f\|_{L^p} \leq C\|f\|^{1-\frac{\alpha}{q}}_{B^{1,\infty}_{q,q}} \|f\|^\frac{\alpha}{q}_{L^q}, \quad \text{with} \quad \beta = \alpha\left(\frac{p}{q} - 1\right), \quad \theta = \frac{q}{p},
\]

In particular, when \( \beta = 1, q = 2 \) and \( p = 4 \), we have \( \alpha = 1 \) and

\[
\|f\|_{L^1} \leq C\|f\|^{\frac{1}{2}}_{B^{1,\infty}_{2,2}} \|f\|^{\frac{1}{2}}_{L^2}.
\]

Lemma 2.2. (Commutator estimate [11]). Let \( s > 0, 1 < p < \infty \), and \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4} \) with \( p_2, \ p_3 \in (1, +\infty) \) and \( p_1, \ p_4 \in [1, +\infty) \). Then,

\[
\|\Lambda^s (fg)\|_{L^p} \leq C(\|f\|_{L^p_1} \|\Lambda^s f\|_{L^p_2} + \|\Lambda^s g\|_{L^p_3} \|f\|_{L^p_4}),
\]

\[
\|[\Lambda^s, f \cdot \nabla]g\|_{L^p} \leq C(\|\nabla f\|_{L^p_1} \|\Lambda^s g\|_{L^p_2} + \|\Lambda^s f\|_{L^p_3} \|\nabla g\|_{L^p_4}).
\]

3. Proof of Theorem 1.1

In this section, we will prove Theorem 1.1 by energy methods. Under the condition (1.5), it suffices to show that, there exists a constant \( C \) such that

\[
\int_0^T (\|\omega\|_{L^p} + \|\nabla d\|_{L^2}^2)dt < C,
\]

which is enough to guarantee the extension of smooth solution \((u, d)\) beyond the time \( T \), for details refer to [10].

Firstly, taking the \( L^2 \) inner product with \( u \) and \(-\Delta d\) to the equations (1.1)_1 and (1.1)_2 respectively, and adding them together, it follows that

\[
\frac{1}{2} \frac{d}{dt} (\|u\|_{L^2}^2 + \|\nabla d\|_{L^2}^2) + \|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2 = -\int_{\mathbb{R}^3} |\nabla d|^2 d\Delta dx = \int_{\mathbb{R}^3} |d\Delta d|^2 dx \leq \|\Delta d\|_{L^2}^2,
\]

where we have used the facts \( |d| = 1, |\nabla d|^2 = -d \cdot \Delta d \), and the following equalities, due to \( \nabla \cdot u = 0 \),

\[
\int_{\mathbb{R}^3} (u \cdot \nabla u) \cdot u dx = 0, \quad \int_{\mathbb{R}^3} \nabla p \cdot u dx = 0,
\]
\[
\int_{\mathbb{R}^3} [(u \cdot \nabla d) \cdot \Delta d - \nabla \cdot (\nabla d \otimes \nabla d) \cdot u] \, dx = \int_{\mathbb{R}^3} (u_i \partial_i d \partial_j d - \partial_i d \partial_j d u_i - \partial_j d \partial_j d u_i) \, dx \\
= \int_{\mathbb{R}^3} -\partial_i \left( \frac{|\partial_i d|^2}{2} \right) u_i \, dx = 0.
\]

Integrating (3.2) in time, we get
\[
\sup_{0 < t < T} (\|u(t)\|^2_{L^2} + \|\nabla d(t)\|^2_{L^2}) + \int_0^T \|\nabla u(t)\|^2_{L^2} \, dt \leq \|u_0\|^2_{L^2} + \|\nabla d_0\|^2_{L^2}.
\]

Next, we are devoted to obtaining the the \(H^1\) estimate of \(u\) and \(\nabla d\). Applying \(\Delta\) to the Eq. (1.1)\(_2\), and taking the inner product with \(\Delta d\), we obtain
\[
\frac{1}{2} \frac{d}{dt} \|\Delta d\|^2_{L^2} + \|\nabla \Delta d\|^2_{L^2} = -\int_{\mathbb{R}^3} \Delta (u \cdot \nabla d) \cdot \Delta d \, dx + \int_{\mathbb{R}^3} \Delta (|\nabla d|^2 d) \cdot \Delta d \, dx. \tag{3.3}
\]

Multiplying (1.1) by \(-\Delta u\), and integrating by parts, one has
\[
\frac{1}{2} \frac{d}{dt} \|\nabla u\|^2_{L^2} + \|\Delta u\|^2_{L^2} = \int_{\mathbb{R}^3} (u \cdot \nabla) u \cdot \Delta u \, dx + \int_{\mathbb{R}^3} \nabla \cdot (\nabla d \otimes \nabla d) \cdot \Delta u \, dx. \tag{3.4}
\]

Summing up (3.3) and (3.4) , it could be derived that
\[
\frac{1}{2} \frac{d}{dt} (\|\nabla u\|^2_{L^2} + \|\Delta d\|^2_{L^2}) + \|\Delta u\|^2_{L^2} + \|\nabla \Delta d\|^2_{L^2} = \int_{\mathbb{R}^3} (u \cdot \nabla) u \cdot \Delta u \, dx + \int_{\mathbb{R}^3} \nabla \cdot (\nabla d \otimes \nabla d) \cdot \Delta u \, dx \\
- \int_{\mathbb{R}^3} \Delta (u \cdot \nabla d) \cdot \Delta d \, dx + \int_{\mathbb{R}^3} \Delta (|\nabla d|^2 d) \cdot \Delta d \, dx \\
:= I_1 + I_2 + I_3 + I_4. \tag{3.5}
\]

For the term \(I_1\) one may refer to [31], for the completeness, We here give the deduction as follows:
\[
I_1 = \int_{\mathbb{R}^3} (u \cdot \nabla) u \cdot \Delta u \, dx \\
= \int_{\mathbb{R}^3} \sum_{i,j=1}^3 u_j \partial_j u_i \partial_k u_i \, dx \\
= -\int_{\mathbb{R}^3} \sum_{i,j=1}^3 \partial_k u_j \partial_j u_i \partial_k u_i \, dx.
\]

We classify the the terms \(\partial_k u_j \partial_j u_i \partial_k u_i, 1 \leq i, j, k \leq 3\) as
(1) If \(k = j = 3, \) or \(j = i = 3, \) or \(k = i = 3, \) we then invoke the divergence free condition to replace \(\partial_3 u_3\) by \(-\partial_1 u_1 - \partial_2 u_2;\)
(2) Otherwise, at least two indices belong to \{1, 2\}. Thus \(I_1\) will be
\[
I_1 = \sum_{i,j,k,l=1}^3 \alpha_{i,j,k,l} \partial_1 u_1 \partial_j u_j \partial_k u_k + \sum_{i,j,k,l=1}^3 \alpha_{i,j,k,l} \partial_2 u_2 \partial_j u_j \partial_k u_k
\]
Based on the computations above, we can use the two-dimension Riesz transformation \( \mathcal{R}_m = \frac{\partial_m}{\sqrt{-\Delta_h}} \) to denote the term \( \partial_m u_n, 1 \leq m, n \leq 2 \), and using Hölder and Gagliardo-Nirenberg inequalities yields

\[
\Delta_h u_1 = \partial_1 \partial_1 u_1 + \partial_2 \partial_2 u_1 \\
= \partial_1 (-\partial_2 u_2 - \partial_3 u_3) + \partial_2 \partial_2 u_1 \\
= -\partial_2 (\partial_1 u_2 - \partial_2 u_1) - \partial_1 \partial_3 u_3 \\
= -\partial_2 \omega_3 - \partial_1 \partial_3 u_3,
\]

\[
\Delta_h u_2 = \partial_1 \omega_3 - \partial_2 \partial_3 u_3.
\]

Based on the computations above, we can use the two-dimension Riesz transformation \( \mathcal{R}_m = \frac{\partial_m}{\sqrt{-\Delta_h}} \) to denote the term \( \partial_m u_n, 1 \leq m, n \leq 2 \),

\[
\partial_m u_1 = \frac{\partial_2}{\sqrt{-\Delta_h}} \frac{\partial_m}{\sqrt{-\Delta_h}} \omega_3 + \frac{\partial_1}{\sqrt{-\Delta_h}} \frac{\partial_m}{\sqrt{-\Delta_h}} \partial_3 u_3 = \mathcal{R}_2 \mathcal{R}_m \omega_3 + \mathcal{R}_1 \mathcal{R}_m \partial_3 u_3,
\]

(3.6)

\[
\partial_m u_2 = \mathcal{R}_1 \mathcal{R}_m \omega_3 + \mathcal{R}_2 \mathcal{R}_m \partial_3 u_3.
\]

(3.7)

By (3.6), the term \( I_{11} \) could be turned into

\[
I_{11} = \sum_{i,j,k,l=1}^{3} \int_{\mathbb{R}^3} \alpha_{11ijkl} \partial_i u_1 \partial_j u_2 \partial_k u_3 \partial_l u_4 \, dx
\]

\[
= \sum_{i,j,k,l=1}^{3} \int_{\mathbb{R}^3} \alpha_{11ijkl} (\mathcal{R}_2 \mathcal{R}_1 \omega_3 + \mathcal{R}_1 \mathcal{R}_1 \partial_3 u_3) \partial_i u_1 \partial_j u_2 \partial_k u_3 \partial_l u_4 \, dx
\]

\[
= \sum_{i,j,k,l=1}^{3} \int_{\mathbb{R}^3} \alpha_{11ijkl} \mathcal{R}_2 \mathcal{R}_1 \omega_3 \partial_i u_1 \partial_j u_2 \partial_k u_3 \partial_l u_4 \, dx
\]

\[
- \sum_{i,j,k,l=1}^{3} \int_{\mathbb{R}^3} \alpha_{11ijkl} \mathcal{R}_1 \mathcal{R}_1 \partial_3 u_3 (\partial_i u_1 \partial_j u_2 \partial_k u_3 + \partial_i \partial_j \partial_k u_4) \, dx.
\]

Because of the Riesz transformation being bounded in \( L^p(\mathbb{R}^2) \) to \( L^p(\mathbb{R}^2) \) for \( 1 < p < \infty \), and using Hölder and Gagliardo-Nirenberg inequalities yields

\[
I_{11} \leq C ||\omega_3||_{L^p} ||\nabla u||_{L^2}^2 + C ||u_3||_{L^p} ||\nabla u||_{L^2}^2 ||\nabla^2 u||_{L^2}^2
\]

\[
\leq C ||\omega_3||_{L^p} ||\nabla u||_{L^2}^{\frac{2}{3}p} ||\nabla^2 u||_{L^2}^3 \leq C ||u_3||_{L^p} ||\nabla u||_{L^2}^{\frac{2}{3}} ||\nabla^2 u||_{L^2}^3
\]

\[
\leq C (||\omega_3||_{L^p}^{\frac{2}{3}p} + ||u_3||^{\frac{2}{3}}_{L^p}) ||\nabla u||_{L^2}^{\frac{2}{3}} + \frac{1}{32} ||\Delta u||_{L^2}^2.
\]
The estimates of terms $I_{12}, I_{21}, I_{22}$ are similar to $I_{11}$, thus we can get

$$I_1 \leq C(||\omega_3||_{L^{\frac{2p}{p-1}}}^2 + ||u_3||_{L^p}^2)||\nabla u||_{L^2}^2 + \frac{1}{8}||\Delta u||_{L^2}^2. \tag{3.8}$$

Next, we estimate the terms $I_2, I_3, I_4$.

$$I_2 + I_3 = \int_{\mathbb{R}^3} \partial_j(\partial_i d_k \partial_j d_k) \partial_i u_i dx - \int_{\mathbb{R}^3} \partial_i(\partial_i d_k \partial_j d_k) \partial_j d_k dx$$

$$= \int_{\mathbb{R}^3} \partial_i d_i \partial_j d_k \partial_i u_i dx + \int_{\mathbb{R}^3} \partial_i(\partial_i d_k \partial_j d_k) \partial_j d_i dx$$

$$= -\int_{\mathbb{R}^3} \partial_i d_i \partial_j d_j \partial_i u_i dx - \int_{\mathbb{R}^3} \partial_i d_i \partial_j d_j \partial_i u_i dx + \int_{\mathbb{R}^3} u_i \partial_i d_i \partial_j d_j dx$$

We deduce from the Lemma 2.1 that

$$I_2 + I_3 \leq ||\nabla u||_{L^2} ||\Delta d||_{L^2}$$

$$\leq C||\nabla u||_{L^2} ||\nabla d||_{B^{0,\infty}_{2,\infty}} ||\Delta d||_{L^2}$$

$$\leq C||\nabla d||_{B^{0,\infty}_{2,\infty}}^2 ||\nabla u||_{L^2}^2 + \frac{1}{4} ||\nabla d||_{L^2}^2, \tag{3.9}$$

For $I_4$, it is easy to check that

$$I_4 = \int_{\mathbb{R}^3} \Delta(\nabla d^2) \cdot \Delta d dx = -\int_{\mathbb{R}^3} \nabla(\nabla d^2) \nabla \Delta d dx$$

$$= -\int_{\mathbb{R}^3} (\nabla d^2 \nabla d + d \nabla d^2 d) \nabla \Delta d dx = \int_{\mathbb{R}^3} |\nabla d|^2 \nabla^2 d \Delta d - d \nabla d^2 d \nabla \Delta d dx$$

$$\leq C||\nabla d||_{L^2}^2 ||\Delta d||_{L^2}^2 + C||\nabla d||_{L^2} ||\nabla \Delta d||_{L^2}$$

$$\leq C||\nabla d||_{L^2}^2 ||\Delta d||_{L^2}^2 + \frac{1}{8} ||\nabla \Delta d||_{L^2}^2 \tag{3.10}$$

$$\leq C||\nabla d||_{B^{0,\infty}_{2,\infty}}^2 ||\Delta d||_{L^2}^2 + \frac{1}{8} ||\nabla \Delta d||_{L^2}^2$$

Inserting (3.8), (3.9) and (3.10) into (3.5) yields

$$\frac{d}{dt}(||\nabla u||_{L^2}^2 + ||\Delta d||_{L^2}^2) + ||\Delta u||_{L^2}^2 + ||\nabla d||_{L^2}^2 \leq C(||\omega_3||_{L^{\frac{2p}{p-1}}}^2 + ||u_3||_{L^p}^2 ||\nabla u||_{L^2}^2 + ||\Delta d||_{L^2}^2).$$
Applying the Gronwall inequality leads to
\[ \|\nabla u\|_{L^2}^2 + ||\Delta d||_{L^2}^2 + \int_0^T (||\Delta u||_{L^2}^2 + ||\nabla \Delta d||_{L^2}^2) dt \leq \exp \left( \int_0^T (||\omega_j||_{L^2}^2 + ||\omega_3||_{L^2}^2 + ||\nabla d||_{B_{C_{1+}}^2}^2) dt \right) (||\nabla u_0||_{L^2}^2 + ||\Delta d_0||_{L^2}^2) \]
(3.11)

At last, under the \( H^1 \) estimates of \( \nabla u \) and \( \Delta d \), we will show
\[ \int_0^T (||\nabla u||_{L^2}^2 + ||\Delta^2 d||_{L^2}^2) dt < C, \]
(3.12)
where \( C \) is a constant.

Applying \( \Delta \) and \( \nabla \Delta \) to the Eqs. (1.1)_1 and (1.1)_2 respectively, and taking the \( L^2 \) inner product with \((\Delta u, \nabla \Delta d)\), we obtain that
\[
\frac{1}{2} \frac{d}{dt} (||\Delta u||_{L^2}^2 + ||\nabla \Delta d||_{L^2}^2) + ||\nabla \Delta u||_{L^2}^2 + ||\Delta^2 d||_{L^2}^2 \\
= - \int_{\mathbb{R}^3} \Delta (u \cdot \nabla u) \cdot \Delta u dx - \int_{\mathbb{R}^3} \Delta (\nabla d_j \cdot \Delta d_j) \cdot \Delta u dx \\
- \int_{\mathbb{R}^3} \nabla \Delta (u \cdot \nabla d) \cdot \nabla \Delta d dx - \int_{\mathbb{R}^3} \nabla \Delta (||\nabla d||_{L^2}^2) \cdot \nabla \Delta d dx \\
:= J_1 + J_2 + J_3 + J_4. \]
(3.13)

Using the inequality (3.11) and commutator estimate, \( J_1, J_2, J_3 \) can be estimated by

\[ J_1 = - \int_{\mathbb{R}^3} [\Delta, u \cdot \nabla] u \cdot \Delta u dx \]
\[ \leq ||[\Delta, u \cdot \nabla] u||_{L^2} \cdot ||\Delta u||_{L^2} \]
\[ \leq C(\|\nabla u\|_{L^2} \|\Delta u\|_{L^2} + \|\Delta u\|_{L^2} \|\nabla u\|_{L^2}) \|\Delta u\|_{L^2} \]
\[ \leq C||\nabla u||_{L^2}^2 ||\Delta u||_{L^2} \]
\[ \leq C||\nabla u||_{L^2} \]
\[ \leq C||\nabla u||_{L^2}^2 + \frac{1}{6} ||\nabla \Delta u||_{L^2}^2, \]
(3.14)

\[ J_2 = \int_{\mathbb{R}^3} \nabla (\nabla d_j \cdot \Delta d_j) \nabla \Delta u dx \]
\[ \leq ||\Delta (\nabla d_j \cdot \Delta d_j)||_{L^2} \|\Delta u\|_{L^2} \]
\[ \leq (||\nabla d||_{L^2} \|\nabla \Delta d||_{L^2} + ||\Delta d||_{L^2} \|\Delta d||_{L^2}) \|\nabla \Delta u||_{L^2} \]
\[ \leq C(||\nabla d||_{L^2} \|\nabla \Delta d||_{L^2} + ||\Delta d||_{L^2} \|\Delta d||_{L^2}) \|\nabla \Delta u||_{L^2} \]
\[ \leq C||\nabla \Delta d||_{L^2}^2 + C||\Delta d||_{L^2}^4 + \frac{1}{6} ||\nabla \Delta u||_{L^2}^2 \]
(3.15)
\[ \leq C||\nabla \Delta d||_{L^2}^2 ||\Delta^2 d||_{L^2}^{\frac{3}{2}} + C||\Delta d||_{L^2}^5 ||\Delta^2 d||_{L^2}^{\frac{3}{2}} + \frac{1}{6} ||\nabla \Delta u||_{L^2}^2 \]
\[ J_3 = - \int_{\mathbb{R}^3} [\nabla \Delta, u \cdot \nabla] d \cdot \nabla \Delta d x \]
\[ \leq ||[\nabla \Delta, u \cdot \nabla] d||_{L^4}^2 ||\nabla \Delta d||_{L^4} \]
\[ \leq C(||\nabla u||_{L^2} ||\nabla \Delta d||_{L^4} + ||\nabla d||_{L^4} ||\nabla \Delta u||_{L^2}) ||\nabla \Delta d||_{L^4} \]
\[ \leq C(||\nabla u||_{L^2} ||\nabla \Delta d||_{L^4} + ||\Delta d||_{L^2} ||\nabla \Delta u||_{L^2}) ||\nabla \Delta d||_{L^4} \]
\[ \leq C||\nabla \Delta d||_{L^2}^2 + \frac{1}{6} ||\nabla \Delta d||_{L^2}^2 \]
\[ \leq C||\nabla \Delta d||_{L^2}^2 + \frac{1}{6} ||\nabla \Delta d||_{L^2}^2 + \frac{1}{6} ||\nabla \Delta u||_{L^2}^2. \tag{3.16} \]

To bound \( J_4 \), by the facts \(|d| = 1, \nabla d|^2 = -d \cdot \Delta d \), it follows that
\[ J_4 = - \int_{\mathbb{R}^3} \nabla \Delta ([\nabla d]^2 d) \cdot \nabla \Delta d x = \int_{\mathbb{R}^3} \Delta ([\nabla d]^2 d) \Delta^2 d \]
\[ = \int_{\mathbb{R}^3} [\Delta ([\nabla d]^2 d) + 2\nabla [\nabla d]^2 \nabla d + [\nabla d]^2 \Delta d] \Delta^2 d \]
\[ \leq C(||\Delta d \Delta d||_{L^2} + ||\nabla \Delta d \Delta d||_{L^2} + ||\nabla \nabla \Delta d \Delta d||_{L^2} + ||d \Delta \Delta d \Delta d||_{L^2}) ||\Delta^2 d||_{L^2} \]
\[ \leq C(||\Delta d||_{L^4}^2 + ||\nabla d||_{L^6} ||\nabla \Delta d||_{L^2}) ||\Delta^2 d||_{L^2} \]
\[ \leq C||\Delta d||_{L^2}^2 ||\Delta^2 d||_{L^2}^2 + C||\nabla \Delta d||_{L^2}^2 ||\Delta^2 d||_{L^2} + C||\nabla \Delta d||_{L^2}^2 ||\Delta^2 d||_{L^2} \]
\[ \leq C(||\nabla \Delta d||_{L^2}^2 + 1) + \frac{1}{6} ||\Delta^2 d||_{L^2}^2. \tag{3.17} \]

Putting the estimates (3.14)–(3.17) to (3.13), we get
\[ \frac{d}{dt} (||\Delta u||_{L^2}^2 + ||\nabla \Delta d||_{L^2}^2 + ||\nabla \Delta u||_{L^2}^2 + ||\Delta^2 d||_{L^2}^2 \leq C(||\Delta u||_{L^2}^2 + ||\nabla \Delta d||_{L^2}^2 + 1), \]
which gives us the desired result (3.12) by the Gronwall inequality. Finally, by using the Sobolev embedding \( H^2(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3) \), (3.12) leads to the BKM’s criterion (3.1) immediately, which completes the proof of Theorem 1.1.

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**Conflict of interest**

All authors declare no conflicts of interest in this paper.
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