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Research article

Blow-up criterion for the 3D nematic liquid crystal flows via one velocity and vorticity components and molecular orientations

Qiang Li and Baoquan Yuan*

School of Mathematics and Information Science, Henan Polytechnic University, Henan, 454000, China.

* **Correspondence:** Email: bqyuan@hpu.edu.cn.

Abstract: In this paper, we are devoted to investigating the blow-up criteria for the three dimensional nematic liquid crystal flows. More precisely, we proved that the smooth solution (u, d) can be extended beyond T, provided that $\int_0^T (\|\omega_3\|_{L^p}^{\frac{2p}{2p-3}} + \|u_3\|_{L^q}^{\frac{2q}{q-3}} + \|\nabla d\|_{\dot{B}^0_{\infty,\infty}}^2)\mathrm{d}t < \infty, \frac{3}{2} < p \leq \infty, 3 < q \leq \infty$.

Keywords: nematic liquid crystal flow; blow-up criteria; smooth solution
Mathematics Subject Classification: 35B65, 35Q35, 76A15

1. Introduction

In this paper, we are interested in the following hydrodynamic system modeling the flow of the nematic liquid crystal materials in 3-dimensions:

$$\begin{cases} \partial_t u + u \cdot \nabla u - \mu \Delta u + \nabla p = -\lambda \nabla \cdot (\nabla d \odot \nabla d), \\ \partial_t d + u \cdot \nabla d = \gamma (\Delta d + |\nabla d|^2 d), \\ \nabla \cdot u = 0, |d| = 1, \\ u(x,0) = u_0(x), d(x,0) = d_0(x), \end{cases} \tag{1.1}$$

where u is the velocity field, d is the macroscopic average of molecular orientation field and p represents the scalar pressure. And μ is the kinematic viscosity, λ is the competition between the kinetic and potential energies, and γ is the microscopic elastic relation time for the molecular orientation field. The notation $\nabla d \odot \nabla d$ represents the 3×3 matrix, of which the $(i, j)th$ component can be denoted by $\partial_i d_k \partial_j d_k (i, j \leq 3)$.

The model of the hydrodynamic theory for liquid crystals was established by Ericksen and Leslie [8, 12, 13], and the system (1.1) was first introduced by Lin [14] as a simplified version to the

Ericksen-Leslie system describing the flow of nematic liquid crystals. Later, Lin and Liu had done many significant works such as [15, 16].

When the orientation field d equals a constant, the above equations become the incompressible Navier-Stokes equations. Many regularity results on the weak solutions to the three-dimensional Navier-Stokes equations have been well studied, for example see [3–7, 9, 17, 18, 21–23, 30, 32, 33], and references therein, where they have proved that the solution is a smooth one if the velocity, or vorticity, or the gradient of velocity, or their components are regular. In their famous work [2], J. Beale et al. proved that the smooth solution u blows up at a finite time $t = T^*$ for the 3D Euler equations, if $\int_0^{T^*} \|\omega\|_{L^\infty} \mathrm{d}t = \infty$, which also holds for the Navier-Stokes equations. In [31], Zhang has investigated a regularity criterion via one velocity and one vorticity component. On the other hand, when the velocity field $u = 0$, the system (1.1) becomes to the heat flow of harmonic maps onto a sphere. Wang proved in [24] that, if $0 < T^* < \infty$ is the maximal time for the unique smooth solution $d \in C^\infty(\mathbb{R}^n; (0, T^*])$, then $\|\nabla d\|_{L^n}$ blows up as time t tends to T^*. Motivated by these developments, the global smooth solution on the nematic liquid crystal model (1.1) are studied in a series papers [10, 19, 20, 26–29]. Huang and Wang [10] established a BKM type blow-up criterion for the system (1.1). That is, if T^* is the maximal time, $0 < T^* < \infty$, then

$$\int_0^{T^*} (\|\omega\|_{L^\infty} + \|\nabla d\|_{L^\infty}^2)\mathrm{d}t = \infty. \tag{1.2}$$

This result is improved by Zhao [29] via two velocity components and molecular orientations. More precisely, the smooth solution (u, d) of the system (1.1) blows up at time $t = T^* < \infty$, if and only if

$$\int_0^{T^*} (\|\nabla_h u^h\|_{\dot{B}^0_{p,\frac{2p}{3}}}^q + \|\nabla d\|_{\dot{B}^0_{\infty,\infty}}^2)\mathrm{d}t = \infty,\ \text{with}\ \frac{3}{p} + \frac{2}{q} = 2, \frac{3}{2} < p \leq \infty. \tag{1.3}$$

Recently, Yuan and Wei [27] consider the blow-up criterion in terms of the vorticity in Besov space of negative index and the orientation field in the homogeneous Besov space. If

$$\int_0^{T} (\|\omega\|_{\dot{B}^{-r}_{\infty,\infty}}^{\frac{2}{2-r}} + \|\nabla d\|_{\dot{B}^0_{\infty,\infty}}^2)\mathrm{d}t < \infty,\ 0 < r < 2, \tag{1.4}$$

then the solution (u, d) can be extended smoothly beyond T.

Inspired by [27] and [31], we are aimed to replace the gradient of velocity in (1.3) and the vorticity in (1.4) by one velocity and one vorticity component. Our main results are stated as follows:

Theorem 1.1. *Assume the initial data* $u_0 \in H^3(\mathbb{R}^3)$ *with* $\nabla \cdot u_0 = 0$, *and* $d_0 \in H^4(\mathbb{R}^3, \mathbb{S}^2)$, (u, d) *is a smooth solution to the equations of (1.1) on* $[0, T)$ *for some* $0 < T < \infty$. *Then* (u, d) *can be extended beyond T, provided that*

$$\int_0^{T} (\|\omega_3\|_{L^p}^{\frac{2p}{2p-3}} + \|u_3\|_{L^q}^{\frac{2q}{q-3}} + \|\nabla d\|_{\dot{B}^0_{\infty,\infty}}^2)\mathrm{d}t < \infty,\ \text{with}\ \frac{3}{2} < p \leq \infty,\ 3 < q \leq \infty. \tag{1.5}$$

Remark 1.2. *As we know, if the initial data* $u_0 \in H^s(\mathbb{R}^n)$ *with* $\nabla \cdot u_0 = 0$ *and* $d_0 \in H^{s+1}(\mathbb{R}^n, \mathbb{S}^2)$ *for* $s \geq n$, *then there exists a positive time T depending only on the initial value such that system (1.1) has a unique smooth solution* $(u, d) \in (\mathbb{R}^n \times [0, T))$ *satisfying (see for example [25])*

$$u \in C([0, T]; H^s(\mathbb{R}^n)) \cap C^1([0, T]; H^{s-2}(\mathbb{R}^n)),$$
$$d \in C([0, T]; H^{s+1}(\mathbb{R}^n)) \cap C^1([0, T]; H^{s-1}(\mathbb{R}^n)).$$

In the following part, we shall use simplified notations. we shall use the letter C to denote a generic constant which may be different from line to line, and write $\partial_t u = \frac{\partial u}{\partial t}, \partial_i = \frac{\partial}{\partial x_i}$. Since the concrete values of the constants μ, λ, γ play no role in our discussion, to simplify the presentation, we shall assume that $\mu = \lambda = \gamma = 1$ in this paper.

2. Preliminaries

In this section, we shall recall the interpolation inequality in [1] and the commutator estimate in [11], which will be used in the process of the proof of Theorem 1.1.

Lemma 2.1. *(Page 82 in [1]). Let* $1 < q < p < \infty$ *and* α *be a positive real number. Then there exists a constant C such that*

$$\|f\|_{L^p} \leq C\|f\|_{\dot{B}^{-\alpha}_{\infty,\infty}}^{1-\theta}\|f\|_{\dot{B}^{\beta}_{q,q}}^{\theta}, \text{with } \beta = \alpha(\frac{p}{q} - 1), \theta = \frac{q}{p}.$$

In particular, when $\beta = 1$, $q = 2$ *and* $p = 4$, *we have* $\alpha = 1$ *and*

$$\|f\|_{L^4} \leq C\|f\|_{\dot{B}^{-1}_{\infty,\infty}}^{\frac{1}{2}}\|f\|_{\dot{H}^1}^{\frac{1}{2}}.$$

Lemma 2.2. *(Commutator estimate [11]). Let* $s > 0$, $1 < p < \infty$, *and* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$ *with* p_2, $p_3 \in (1, +\infty)$ *and* p_1, $p_4 \in [1, +\infty]$. *Then,*

$$\|\Lambda^s(fg)\|_{L^p} \leq C(\|g\|_{L^{p_1}}\|\Lambda^s f\|_{L^{p_2}} + \|\Lambda^s g\|_{L^{p_3}}\|f\|_{L^{p_4}}),$$
$$\|[\Lambda^s, f\cdot\nabla]g\|_{L^p} \leq C(\|\nabla f\|_{L^{p_1}}\|\Lambda^s g\|_{L^{p_2}} + \|\Lambda^s f\|_{L^{p_3}}\|\nabla g\|_{L^{p_4}}).$$

3. Proof of Theorem 1.1

In this section, we will prove Theorem 1.1 by energy methods. Under the condition (1.5), it suffices to show that, there exists a constant C such that

$$\int_0^T (\|\omega\|_{L^\infty} + \|\nabla d\|_{L^\infty}^2)\mathrm{d}t < C, \tag{3.1}$$

which is enough to guarantee the extension of smooth solution (u, d) beyond the time T, for details refer to [10].

Firstly, taking the L^2 inner product with u and $-\Delta d$ to the equations $(1.1)_1$ and $(1.1)_2$ respectively, and adding them together, it follows that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|_{L^2}^2 + \|\nabla d\|_{L^2}^2) + \|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2 &= -\int_{\mathbb{R}^3}|\nabla d|^2 d\Delta d\mathrm{d}x\\ &= \int_{\mathbb{R}^3}|d\Delta d|^2\mathrm{d}x \leq \|\Delta d\|_{L^2}^2,\end{aligned} \tag{3.2}$$

where we have used the facts $|d| = 1$, $|\nabla d|^2 = -d\cdot\Delta d$, and the following equalities, due to $\nabla\cdot u = 0$,

$$\int_{\mathbb{R}^3}(u\cdot\nabla u)\cdot u\mathrm{d}x = 0, \int_{\mathbb{R}^3}\nabla p\cdot u\mathrm{d}x = 0,$$

$$\int_{\mathbb{R}^3}[(u\cdot\nabla d)\cdot\Delta d-\nabla\cdot(\nabla d\odot\nabla d)\cdot u]\mathrm{d}x = \int_{\mathbb{R}^3}(u_i\partial_i d\partial_j\partial_j d-\partial_i d\partial_j\partial_j du_i-\partial_i\partial_j d\partial_j du_i)\mathrm{d}x$$
$$= \int_{\mathbb{R}^3}-\partial_i(\frac{|\partial_j d|^2}{2})u_i\mathrm{d}x=0.$$

Integrating (3.2) in time, we get

$$\sup_{0<t<T}(\|u(t)\|_{L^2}^2+\|\nabla d(t)\|_{L^2}^2)+\int_0^T\|\nabla u(t)\|_{L^2}^2\mathrm{d}t\le\|u_0\|_{L^2}^2+\|\nabla d_0\|_{L^2}^2.$$

Next, we are devoted to obtaining the the H^1 estimate of u and ∇d. Applying Δ to the Eq. $(1.1)_2$, and taking the inner product with Δd, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta d\|_{L^2}^2+\|\nabla\Delta d\|_{L^2}^2=-\int_{\mathbb{R}^3}\Delta(u\cdot\nabla d)\cdot\Delta d\mathrm{d}x+\int_{\mathbb{R}^3}\Delta(|\nabla d|^2 d)\cdot\Delta d\mathrm{d}x. \tag{3.3}$$

Multiplying $(1.1)_1$ by $-\Delta u$, and integrating by parts, one has

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|_{L^2}^2+\|\Delta u\|_{L^2}^2=\int_{\mathbb{R}^3}(u\cdot\nabla)u\cdot\Delta u\mathrm{d}x+\int_{\mathbb{R}^3}\nabla\cdot(\nabla d\odot\nabla d)\cdot\Delta u\mathrm{d}x. \tag{3.4}$$

Summing up (3.3) and (3.4) , it could be derived that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla u\|_{L^2}^2+\|\Delta d\|_{L^2}^2)+\|\Delta u\|_{L^2}^2+\|\nabla\Delta d\|_{L^2}^2\\
=&\int_{\mathbb{R}^3}(u\cdot\nabla)u\cdot\Delta u\mathrm{d}x+\int_{\mathbb{R}^3}\nabla\cdot(\nabla d\odot\nabla d)\cdot\Delta u\mathrm{d}x\\
-&\int_{\mathbb{R}^3}\Delta(u\cdot\nabla d)\cdot\Delta d\mathrm{d}x+\int_{\mathbb{R}^3}\Delta(|\nabla d|^2 d)\cdot\Delta d\mathrm{d}x\\
:=&I_1+I_2+I_3+I_4.
\end{aligned} \tag{3.5}$$

For the term I_1 one may refer to [31], for the completeness, We here give the deduction as follows:

$$\begin{aligned}
I_1 &= \int_{\mathbb{R}^3}(u\cdot\nabla)u\cdot\Delta u\mathrm{d}x\\
&= \int_{\mathbb{R}^3}\sum_{i,j=1}^3 u_j\partial_j u_i\partial_k\partial_k u_i\mathrm{d}x\\
&= -\int_{\mathbb{R}^3}\sum_{i,j=1}^3\partial_k u_j\partial_j u_i\partial_k u_i\mathrm{d}x.
\end{aligned}$$

We classify the the terms $\partial_k u_j\partial_j u_i\partial_k u_i$, $1\le i,j,k\le 3$ as
(1) If $k=j=3$, or $j=i=3$, or $k=i=3$, we then invoke the divergence free condition to replace $\partial_3 u_3$ by $-\partial_1 u_1-\partial_2 u_2$;
(2) Otherwise, at least two indices belong to $\{1,2\}$. Thus I_1 will be

$$I_1 = \sum_{i,j,k,l=1}^3\int_{\mathbb{R}^3}\alpha_{11ijkl}\partial_1 u_1\partial_i u_j\partial_k u_l+\sum_{i,j,k,l=1}^3\int_{\mathbb{R}^3}\alpha_{12ijkl}\partial_1 u_2\partial_i u_j\partial_k u_l$$

$$
\begin{aligned}
&+ \sum_{i,j,k,l=1}^{3} \int_{\mathbb{R}^3} \alpha_{21ijkl} \partial_2 u_1 \partial_i u_j \partial_k + \sum_{i,j,k,l=1}^{3} \int_{\mathbb{R}^3} \alpha_{22ijkl} \partial_2 u_2 \partial_i u_j \partial_k \\
&= I_{11} + I_{12} + I_{21} + I_{22},
\end{aligned}
$$

where α_{mnijkl}, $1 \leq m, n \leq 2$, $1 \leq i, j, k, l \leq 3$, are suitable integers. Next, we want to represent $\partial_m u_n, 1 \leq m, n \leq 2$ by u_3 and ω_3. Denoting by $\Delta_h = \partial_1\partial_1 + \partial_2\partial_2$ the horizontal Laplacian, we have

$$
\begin{aligned}
\Delta_h u_1 &= \partial_1\partial_1 u_1 + \partial_2\partial_2 u_1 \\
&= \partial_1(-\partial_2 u_2 - \partial_3 u_3) + \partial_2\partial_2 u_1 \\
&= -\partial_2(\partial_1 u_2 - \partial_2 u_1) - \partial_1\partial_3 u_3 \\
&= -\partial_2\omega_3 - \partial_1\partial_3 u_3,
\end{aligned}
$$

$$
\Delta_h u_2 = \partial_1\omega_3 - \partial_2\partial_3 u_3.
$$

Based on the computations above, we can use the two-dimension Riesz transformation $\mathfrak{R}_m = \frac{\partial_m}{\sqrt{-\Delta_h}}$ to denote the term $\partial_m u_n, 1 \leq m, n \leq 2$,

$$
\partial_m u_1 = \frac{\partial_2}{\sqrt{-\Delta_h}} \frac{\partial_m}{\sqrt{-\Delta_h}} \omega_3 + \frac{\partial_1}{\sqrt{-\Delta_h}} \frac{\partial_m}{\sqrt{-\Delta_h}} \partial_3 u_3 = \mathfrak{R}_2\mathfrak{R}_m\omega_3 + \mathfrak{R}_1\mathfrak{R}_m\partial_3 u_3, \tag{3.6}
$$

$$
\partial_m u_2 = \mathfrak{R}_1\mathfrak{R}_m\omega_3 + \mathfrak{R}_2\mathfrak{R}_m\partial_3 u_3. \tag{3.7}
$$

By (3.6), the term I_{11} could be turned into

$$
\begin{aligned}
I_{11} &= \sum_{i,j,k,l=1}^{3} \int_{\mathbb{R}^3} \alpha_{11ijkl} \partial_1 u_1 \partial_i u_j \partial_k u_l \mathrm{d}x \\
&= \sum_{i,j,k,l=1}^{3} \int_{\mathbb{R}^3} \alpha_{11ijkl} (\mathfrak{R}_2\mathfrak{R}_1\omega_3 + \mathfrak{R}_1\mathfrak{R}_1\partial_3 u_3) \partial_i u_j \partial_k u_l \mathrm{d}x \\
&= \sum_{i,j,k,l=1}^{3} \int_{\mathbb{R}^3} \alpha_{11ijkl} \mathfrak{R}_2\mathfrak{R}_1\omega_3 \partial_i u_j \partial_k u_l \mathrm{d}x \\
&- \sum_{i,j,k,l=1}^{3} \int_{\mathbb{R}^3} \alpha_{11ijkl} \mathfrak{R}_1\mathfrak{R}_1 u_3 (\partial_3\partial_i u_j \partial_k u_l + \partial_i u_j \partial_3\partial_k u_l) \mathrm{d}x.
\end{aligned}
$$

Because of the Riesz transformation being bounded in $L^p(\mathbb{R}^2)$ to $L^p(\mathbb{R}^2)$ for $1 < p < \infty$, and using Hölder and Gagliardo-Nirenberg inequalities yields

$$
\begin{aligned}
I_{11} &\leq C\|\omega_3\|_{L^p} \|\nabla u\|^2_{L^{\frac{2p}{p-1}}} + C\|u_3\|_{L^q} \|\nabla u\|_{L^{\frac{2q}{q-2}}} \|\nabla^2 u\|_{L^2} \\
&\leq C\|\omega_3\|_{L^p} \|\nabla u\|_{L^2}^{\frac{2p-3}{p}} \|\nabla^2 u\|_{L^2}^{\frac{3}{p}} + C\|u_3\|_{L^q} \|\nabla u\|_{L^2}^{\frac{q-3}{q}} \|\nabla^2 u\|_{L^2}^{\frac{q+3}{q}} \\
&\leq C(\|\omega_3\|_{L^p}^{\frac{2p}{2p-3}} + \|u_3\|_{L^q}^{\frac{2q}{q-3}}) \|\nabla u\|^2_{L^2} + \frac{1}{32}\|\Delta u\|^2_{L^2}.
\end{aligned}
$$

The estimates of terms I_{12}, I_{21}, I_{22} are similar to I_{11}, thus we can get

$$I_1 \leq C(\|\omega_3\|_{L^p}^{\frac{2p}{2p-3}} + \|u_3\|_{L^q}^{\frac{2q}{q-3}})\|\nabla u\|_{L^2}^2 + \frac{1}{8}\|\Delta u\|_{L^2}^2. \tag{3.8}$$

Next, we estimate the terms I_2, I_3, I_4.

$$\begin{aligned}
I_2 + I_3 &= \int_{\mathbb{R}^n} \partial_j(\partial_i d_k \partial_j d_k)\partial_{ll} u_i \mathrm{d}x - \int_{\mathbb{R}^3} \partial_{ll}(u_i \partial_i d_k)\partial_{jj} d_k \mathrm{d}x \\
&= \int_{\mathbb{R}^3} \partial_i d_k \partial_{jj} d_k \partial_{ll} u_i \mathrm{d}x + \int_{\mathbb{R}^3} \partial_l(u_i \partial_i d_k)\partial_l \partial_{jj} d_k \mathrm{d}x \\
&= -\int_{\mathbb{R}^3} \partial_i \partial_l d_k \partial_{jj} d_k \partial_l u_i \mathrm{d}x - \int_{\mathbb{R}^3} \partial_i d_k \partial_l \partial_{jj} d_k \partial_l u_i \mathrm{d}x \\
&+ \int_{\mathbb{R}^3} \partial_l u_i \partial_i d_k \partial_l \partial_{jj} d_k \mathrm{d}x + \int_{\mathbb{R}^3} u_i \partial_i \partial_l d_k \partial_l \partial_{jj} d_k \mathrm{d}x \\
&= -\int_{\mathbb{R}^3} \partial_i \partial_l d_k \partial_{jj} d_k \partial_l u_i \mathrm{d}x + \int_{\mathbb{R}^3} u_i \partial_i \partial_l d_k \partial_l \partial_{jj} d_k \mathrm{d}x \\
&= -\int_{\mathbb{R}^3} \partial_i \partial_l d_k \partial_{jj} d_k \partial_l u_i \mathrm{d}x - \int_{\mathbb{R}^3} \partial_l u_i \partial_i \partial_l d_k \partial_{jj} d_k \mathrm{d}x.
\end{aligned}$$

We deduce from the Lemma 2.1 that

$$\begin{aligned}
I_2 + I_3 &\leq \|\nabla u\|_{L^2}\|\Delta d\|_{L^4}^2 \\
&\leq C\|\nabla u\|_{L^2}\|\nabla d\|_{\dot{B}^0_{\infty,\infty}}\|\nabla\Delta d\|_{L^2} \\
&\leq C\|\nabla d\|_{\dot{B}^0_{\infty,\infty}}^2\|\nabla u\|_{L^2}^2 + \frac{1}{4}\|\nabla\Delta d\|_{L^2}^2,
\end{aligned} \tag{3.9}$$

For I_4, it is easy to check that

$$\begin{aligned}
I_4 &= \int_{\mathbb{R}^3} \Delta(|\nabla d|^2 d)\cdot\Delta d \mathrm{d}x = -\int_{\mathbb{R}^3} \nabla(|\nabla d|^2 d)\nabla\Delta d \mathrm{d}x \\
&= -\int_{\mathbb{R}^3} (|\nabla d|^2\nabla d + d\nabla d\nabla^2 d)\nabla\Delta d \mathrm{d}x = \int_{\mathbb{R}^3} |\nabla d|^2\nabla^2 d\Delta d - d\nabla d\nabla^2 d\nabla\Delta d \mathrm{d}x \\
&\leq C\|\nabla d\|_{L^4}^2\|\Delta d\|_{L^4}^2 + C\|\nabla d\|_{L^4}\|\Delta d\|_{L^4}\|\nabla\Delta d\|_{L^2} \\
&\leq C\|\nabla d\|_{L^4}^2\|\Delta d\|_{L^4}^2 + \frac{1}{8}\|\nabla\Delta d\|_{L^2}^2 \\
&\leq C\|\Delta d\|_{L^2}\|\nabla d\|_{\dot{B}^0_{\infty,\infty}}\|\nabla\Delta d\|_{L^2} + \frac{1}{8}\|\nabla\Delta d\|_{L^2}^2 \\
&\leq C\|\nabla d\|_{\dot{B}^0_{\infty,\infty}}^2\|\Delta d\|_{L^2}^2 + \frac{1}{4}\|\nabla\Delta d\|_{L^2}^2.
\end{aligned} \tag{3.10}$$

Inserting (3.8), (3.9) and (3.10) into (3.5) yields

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) &+ \|\Delta u\|_{L^2}^2 + \|\nabla\Delta d\|_{L^2}^2 \\
&\leq C(\|\omega_3\|_{L^p}^{\frac{2p}{2p-3}} + \|u_3\|_{L^q}^{\frac{2q}{q-3}} + \|\nabla d\|_{\dot{B}^0_{\infty,\infty}}^2)(\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2).
\end{aligned}$$

Applying the Gronwall inequality leads to

$$\begin{aligned}
\|\nabla u\|_{L^2}^2 &+ \|\Delta d\|_{L^2}^2+\int_0^T(\|\Delta u\|_{L^2}^2+\|\nabla\Delta d\|_{L^2}^2)\mathrm{d}t \\
&\leq \exp\int_0^T(\|\omega_3\|_{L^p}^{\frac{2p}{2p-3}}+\|u_3\|_{L^q}^{\frac{2q}{q-3}}+\|\nabla d\|_{\dot{B}^0_{\infty,\infty}}^2\mathrm{d}t)(\|\nabla u_0\|_{L^2}^2+\|\Delta d_0\|_{L^2}^2) \\
&< C
\end{aligned} \tag{3.11}$$

At last, under the H^1 estimates of ∇u and Δd, we will show

$$\int_0^T(\|\nabla\Delta u\|_{L^2}^2+\|\Delta^2 d\|_{L^2}^2)\mathrm{d}t<C, \tag{3.12}$$

where C is a constant.

Applying Δ and $\nabla\Delta$ to the Eqs. $(1.1)_1$ and $(1.1)_2$ respectively, and taking the L^2 inner product with $(\Delta u,\nabla\Delta d)$, we obtain that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\Delta u\|_{L^2}^2+\|\nabla\Delta d\|_{L^2}^2)+\|\nabla\Delta u\|_{L^2}^2+\|\Delta^2 d\|_{L^2}^2 \\
=\ & -\int_{\mathbb{R}^3}\Delta(u\cdot\nabla u)\cdot\Delta u\mathrm{d}x-\int_{\mathbb{R}^3}\Delta(\nabla d_j\cdot\Delta d_j)\cdot\Delta u\mathrm{d}x \\
-\ & \int_{\mathbb{R}^3}\nabla\Delta(u\cdot\nabla d)\cdot\nabla\Delta d\mathrm{d}x-\int_{\mathbb{R}^3}\nabla\Delta(|\nabla d|^2 d)\cdot\nabla\Delta d\mathrm{d}x \\
:=\ & J_1+J_2+J_3+J_4.
\end{aligned} \tag{3.13}$$

Using the inequality (3.11) and commutator estimate, J_1, J_2, J_3 can be estimated by

$$\begin{aligned}
J_1 &= -\int_{\mathbb{R}^3}[\Delta,u\cdot\nabla]u\cdot\Delta u\mathrm{d}x \\
&\leq \|[\Delta,u\cdot\nabla]u\|_{L^{\frac{4}{3}}}\|\Delta u\|_{L^4} \\
&\leq C(\|\nabla u\|_{L^2}\|\Delta u\|_{L^4}+\|\Delta u\|_{L^4}\|\nabla u\|_{L^2})\|\Delta u\|_{L^4} \\
&\leq C\|\nabla u\|_{L^2}\|\Delta u\|_{L^4}^2 \\
&\leq C\|\Delta u\|_{L^2}^{\frac{1}{2}}\|\nabla\Delta u\|_{L^2}^{\frac{3}{2}} \\
&\leq C\|\Delta u\|_{L^2}^2+\frac{1}{6}\|\nabla\Delta u\|_{L^2}^2,
\end{aligned} \tag{3.14}$$

$$\begin{aligned}
J_2 &= \int_{\mathbb{R}^3}\nabla(\nabla d_j\cdot\Delta d_j)\nabla\Delta u\mathrm{d}x \\
&\leq \|\Delta(\nabla d_j\cdot\Delta d_j)\|_{L^2}\|\Delta u\|_{L^2} \\
&\leq (\|\nabla d\|_{L^4}\|\nabla\Delta d\|_{L^4}+\|\Delta d\|_{L^4}\|\Delta d\|_{L^4})\|\nabla\Delta u\|_{L^2} \\
&\leq C(\|\Delta d\|_{L^2}\|\nabla\Delta d\|_{L^4}+\|\Delta d\|_{L^4}\|\Delta d\|_{L^4})\|\nabla\Delta u\|_{L^2} \\
&\leq C\|\nabla\Delta d\|_{L^4}^2+C\|\Delta d\|_{L^4}^4+\frac{1}{6}\|\nabla\Delta u\|_{L^2}^2 \\
&\leq C\|\nabla\Delta d\|_{L^2}^{\frac{1}{2}}\|\Delta^2 d\|_{L^2}^{\frac{3}{2}}+C\|\Delta d\|_{L^2}^{\frac{5}{2}}\|\Delta^2 d\|_{L^2}^{\frac{3}{2}}+\frac{1}{6}\|\nabla\Delta u\|_{L^2}^2
\end{aligned} \tag{3.15}$$

$$\leq \quad C(\|\nabla\Delta d\|_{L^2}^2 + 1) + \frac{1}{6}\|\Delta^2 d\|_{L^2}^2 + \frac{1}{6}\|\nabla\Delta u\|_{L^2}^2,$$

$$\begin{aligned}
J_3 &= -\int_{\mathbb{R}^3} [\nabla\Delta, u\cdot\nabla]d\cdot\nabla\Delta d\,\mathrm{d}x \\
&\leq \|[\nabla\Delta, u\cdot\nabla]d\|_{L^{\frac{4}{3}}}\|\nabla\Delta d\|_{L^4} \\
&\leq C(\|\nabla u\|_{L^2}\|\nabla\Delta d\|_{L^4} + \|\nabla d\|_{L^4}\|\nabla\Delta u\|_{L^2})\|\nabla\Delta d\|_{L^4} \\
&\leq C(\|\nabla u\|_{L^2}\|\nabla\Delta d\|_{L^4} + \|\Delta d\|_{L^2}^{\frac{1}{2}}\|\nabla\Delta u\|_{L^2})\|\nabla\Delta d\|_{L^4} \\
&\leq C\|\nabla\Delta d\|_{L^4}^2 + \frac{1}{6}\|\nabla\Delta u\|_{L^2}^2 \qquad (3.16)\\
&\leq C\|\nabla\Delta d\|_{L^2}^{\frac{1}{2}}\|\Delta^2 d\|_{L^2}^{\frac{3}{2}} + \frac{1}{6}\|\nabla\Delta u\|_{L^2}^2 \\
&\leq C\|\nabla\Delta d\|_{L^2}^2 + \frac{1}{6}\|\Delta^2 d\|_{L^2}^2 + \frac{1}{6}\|\nabla\Delta u\|_{L^2}^2.
\end{aligned}$$

To bound J_4, by the facts $|d| = 1,\ |\nabla d|^2 = -d\cdot\Delta d$, it follows that

$$\begin{aligned}
J_4 &= -\int_{\mathbb{R}^3} \nabla\Delta(|\nabla d|^2 d)\cdot\nabla\Delta d\,\mathrm{d}x = \int_{\mathbb{R}^3} \Delta(|\nabla d|^2 d)\Delta^2 d \\
&= \int_{\mathbb{R}^3} [\Delta(|\nabla d|^2)d + 2\nabla|\nabla d|^2\nabla d + |\nabla d|^2\Delta d]\Delta^2 d \\
&\leq C(\|\Delta d\Delta d\|_{L^2} + \|\nabla d\nabla\Delta d\|_{L^2} + \|\nabla d\nabla d\Delta d\|_{L^2} + \|d\Delta d\Delta d\|_{L^2})\|\Delta^2 d\|_{L^2} \\
&\leq C(\|\Delta d\|_{L^4}^2 + \|\nabla d\|_{L^4}\|\nabla\Delta d\|_{L^4})\|\Delta^2 d\|_{L^2} \qquad (3.17)\\
&\leq C\|\Delta d\|_{L^2}^{\frac{5}{4}}\|\Delta^2 d\|_{L^2}^{\frac{3}{4}}\|\Delta^2 d\|_{L^2} + C\|\nabla\Delta d\|_{L^2}^{\frac{1}{4}}\|\Delta^2 d\|_{L^2}^{\frac{3}{4}}\|\Delta^2 d\|_{L^2} \\
&\leq C(\|\nabla\Delta d\|_{L^2}^2 + 1) + \frac{1}{6}\|\Delta^2 d\|_{L^2}^2.
\end{aligned}$$

Putting the estimates (3.14)–(3.17) to (3.13), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\Delta u\|_{L^2}^2 + \|\nabla\Delta d\|_{L^2}^2) + \|\nabla\Delta u\|_{L^2}^2 + \|\Delta^2 d\|_{L^2}^2 \leq C(\|\Delta u\|_{L^2}^2 + \|\nabla\Delta d\|_{L^2}^2 + 1),$$

which gives us the desired result (3.12) by the Gronwall inequality. Finally, by using the Sobolev embedding $H^2(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$, (3.12) leads to the BKM's criterion (3.1) immediately, which completes the proof of Theorem 1.1.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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