AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 5(1): 385–398.
DOI:10.3934/math.2020026
Received: 30 July 2019
Accepted: 17 November 2019
Published: 22 November 2019

Research article

Existence results for fractional order boundary value problem with nonlocal non-separated type multi-point integral boundary conditions

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Abstract: In this article, we discuss the existence of solutions of a fractional boundary value problem of order $m \in (1, 2]$, with nonlocal non-separated type integral multipoint boundary conditions. Shaefer type and Krasnoselskii's fixed point theorems are used to prove existence results for the given problem. To establish the uniqueness of solutions Banach contraction principle is used. The criteria for Hyers-Ulam stability of the given boundary value problem is also discussed. Some examples are included for the illustration of our results.

Keywords: nonlocal; non-separated; fractional BVP; existence; unique solution; Hyers-Ulam stability
Mathematics Subject Classification: 05A15, 15A18

1. Introduction

Fractional calculus is a natural generalization of ordinary calculus when the order of the derivative is non-integer. Many fractional operators are defined like, Riemman-Liouville, Hadamard, Caputo and Grunwald-Letnikov [12, 20, 21]. The choice of the operator depends upon the considered system. Due to the variations in the physical systems, many researchers are defining a number of fractional operators, for details, we refer the readers to the article [24].

Many of the complex physical problems may be better understood in the framework of fractional differential equations. One can find applications of fractional calculus in diverse fields like biology, chemistry, physics, fluid mechanics, economics and social sciences, etc. [1, 16, 17, 21, 26].

For theoretical details of fractional differential equations see [16, 18] and the references cited therein. The existence results for non-integer order differential equations are discussed in many articles, for example, see [10, 11, 13, 14]. Stability analysis of fractional differential equations is an important aspect of the qualitative theory of fractional differential equations form numerical and optimization point of view. Ulam in [25], raised the question "Under what conditions there exist an additive mapping near to an approximate additive mapping?". This question initiated the study of stability of differential equations and the answer of Ulam's question was given by Hyers in [15]. Later on Rassias [22], developed a technique for Hyers-Ulam stability of linear and nonlinear mappings. The theory of fractional equations involving different kinds of boundary conditions has always been remained a field of interest in physical sciences. Non-local and integral boundary conditions are widely used where classical boundary conditions fail to examine many physical properties of the models. Many researchers have already been involved in the existence theory of boundary value problems involving non-local and integral boundary conditions, for example [2–4, 6–9, 19, 23]. Recently Alsaedi et al. [5], discussed the existence theory of the following second order boundary value problem with non-local non-separated type integral multi-point boundary conditions on an arbitrary domain.

$$\begin{cases} u''(t) = f(t, u(t)), \; a < t < T, \; a, \; T \in \mathbb{R}, \\ \alpha_1 u(a) + \alpha_2 u(T) = \alpha_3 \int_a^\zeta u(\tau)d\tau + \sum\limits_{\iota=1}^{m} \gamma_\iota u(v_\iota), \\ \beta_1 \, u'(a) + \beta_2 \, u'(T) = \beta_3 \int_a^\zeta \, u'(s)ds + \sum\limits_{\iota=1}^{m} \rho_\iota \, u'(v_\iota), \end{cases} \tag{1.1}$$

where $a \leq \zeta \leq v_i \leq T$. Motivated by [5], we prove the existence results for solution of a fractional boundary value problem involving Caputo fractional derivative of order $1 < m \leq 2$ and nonlocal non-separated type integral multipoint boundary conditions involving Caputo fractional derivative of order $0 < p \leq 1$ on an arbitrary domain. In fact we consider the following fractional differential equation

$$^cD^m u(t) = F(t, u(t)) \text{ where } 1 < m \leq 2, \tag{1.2}$$

with boundary conditions

$$\begin{aligned} \gamma_1 u(a) + \gamma_2 u(T) &= \gamma_3 \int_a^\zeta u(\tau)d\tau + \gamma_4 \sum_{\iota=1}^{e} \vartheta_\iota u(v_\iota) \qquad (1.3) \\ \varrho_1 \, {}^cD^p u(a) + \varrho_2 \, {}^cD^p u(T) &= \varrho_3 \int_a^\zeta {}^cD^p u(\tau)d\tau + \varrho_4 \sum_{\iota=1}^{e} \rho_\iota \, {}^cD^p u(v_\iota), \; 0 < p \leq 1. \end{aligned}$$

Clearly for $m = 2$, $p = 1$ and $\gamma_4 = \varrho_4 = 1$, the above problems (1.2)–(1.3) reduces to (1.1).

The remaining part of the article is arranged as follows. Section 2 contains the existence results for the fractional boundary value problems (1.2)–(1.3) which are proved by applying Schaefer type and Krasnoselskii's fixed point theorems. In Section 3, we prove an existence and uniqueness result by using Banach contraction principle. Hyers-Ulam stability is discussed in Section 4.

2. Existence results via Schaefer type and Krasnoselskii's fixed point theorems

Let $H = C[a, T]$ be the Banach space of all continuous functions defined on closed interval $[a, T] \subseteq \mathbb{R}$ with norm

$$\|u\| = \sup_{t \in [a,T]} |u(t)| \tag{2.1}$$

for $u \in H$. Set

$$\alpha_1 = \gamma_1 + \gamma_2 - \gamma_3(\zeta - a) - \gamma_4 \sum_{\iota=1}^{e} \vartheta_\iota \tag{2.2}$$

and

$$\beta = \varrho_2 \frac{(T-a)^{1-p}}{\Gamma(2-p)} - \varrho_3 \frac{(\zeta-a)^{2-p}}{\Gamma(3-p)} - \varrho_4 \sum_{\iota=1}^{e} \rho_\iota \frac{(v_\iota - a)^{1-p}}{\Gamma(2-p)}. \tag{2.3}$$

The following lemma will be crucial for coming existence results.

Lemma 2.1. Let $\sigma \in H$ and α_1 and β are nonzero. Then the solution of the following linear problem (2.4) with boundary conditions (1.3)

$${}^{c}D^{m} u(t) = \sigma(t), \text{ for } a < t < T \tag{2.4}$$

has an integral representation given as

$$\begin{aligned} u(t) &= \int_a^t \frac{(t-r)^{m-1}}{\Gamma(m)} \sigma(r) dr + \frac{1}{K} \int_a^\zeta \left(\beta\gamma_3 \frac{(\zeta-r)^p}{\Gamma(m+1)} + \frac{\varrho_3}{\Gamma(m-p+1)} \eta(t) \right) (\zeta - r)^{m-p} \sigma(r) dr \\ & -\frac{1}{K} \int_a^T \left(\beta\gamma_2 \frac{(T-r)^p}{\Gamma(m)} + \frac{\varrho_2}{\Gamma(m-p)} \eta(t) \right) (T-r)^{m-p-1} \sigma(r) dr \\ & +\frac{1}{K} \sum_{\iota=1}^{e} \int_a^{v_\iota} \left(\beta\gamma_4 \vartheta_\iota \frac{(v_\iota - r)^p}{\Gamma(m)} + \frac{\varrho_4}{\Gamma(m-p)} \rho_\iota \eta(t) \right) (v_\iota - r)^{m-p-1} \sigma(r) dr \end{aligned} \tag{2.5}$$

where

$$\alpha_2 = \gamma_2 (T-a) - \gamma_3 \frac{(\zeta-a)^2}{2} - \gamma_4 \sum_{\iota=1}^{e} \vartheta_\iota (v_\iota - a), \tag{2.6}$$

$$\eta(t) = \alpha_1 (t-a) - \alpha_2 \tag{2.7}$$

and

$$K = \beta\alpha_1. \tag{2.8}$$

Proof. The general solution for linear problem

$${}^{c}D^{m} u(t) = \sigma(t), \ a < t < T \text{ and } 1 < m \leq 2$$

as given in [16], is

$$u(t) = \int_a^t \frac{(t-r)^{m-1}}{\Gamma(m)} \sigma(r) dr - d_1 - d_2 (t-a) \tag{2.9}$$

for some real constants d_1 and d_2. Making use of (1.3) , (2.2) and (2.6) in (2.9) , we have

$$\begin{aligned} d_1\alpha_1 + d_2\alpha_2 &= \gamma_2 \int_a^T \frac{(T-r)^{m-1}}{\Gamma(m)} \sigma(r) dr - \gamma_3 \int_a^\zeta \frac{(\zeta-r)^m}{\Gamma(m+1)} \sigma(r) dr \\ & -\gamma_4 \sum_{\iota=1}^{e} \vartheta_\iota \int_a^{v_\iota} \frac{(v_\iota - r)^{m-1}}{\Gamma(m)} \sigma(r) dr. \end{aligned}$$

$$d_1 = \frac{1}{\alpha_1} \left(-d_2\alpha_2 + \gamma_2 \int_a^T \frac{(T-r)^{m-1}}{\Gamma(m)} \sigma(r) dr \right.$$

$$-\gamma_3\int_a^\zeta \frac{(\zeta-r)^m}{\Gamma(m+1)}\sigma(r)\,dr-\gamma_4\sum_{\iota=1}^{e}\vartheta_\iota\int_a^{v_\iota}\frac{(v_\iota-r)^{m-1}}{\Gamma(m)}\sigma(r)dr\Bigg). \tag{2.10}$$

From (2.9)

$$u(t)=\int_a^t \frac{(t-r)^{m-1}}{\Gamma(m)}\sigma(r)dr-d_1-d_2t$$

taking Caputo fractional derivative of order $p\in(0,1)$ on both sides and using its properties, we get

$${}^cD^pu(t)=\int_a^t \frac{(t-r)^{m-p-1}}{\Gamma(m-p)}\sigma(r)dr-d_2\frac{(t-a)^{1-p}}{\Gamma(2-p)}. \tag{2.11}$$

Now using the second boundary condition from (1.3) in (2.11) we get

$$\begin{aligned}
&\varrho_2\int_a^T \frac{(T-r)^{m-p-1}}{\Gamma(m-p)}\sigma(r)dr-\varrho_2d_2\frac{(T-a)^{1-p}}{\Gamma(2-p)}\\
=\;&\varrho_3\int_a^\zeta\int_a^\tau \frac{(\tau-r)^{m-p-1}}{\Gamma(m-p)}\sigma(r)drd\tau-\varrho_3d_2\frac{(\zeta-a)^{2-p}}{(2-p)\Gamma(2-p)}\\
&+\varrho_4\sum_{\iota=1}^{e}\rho_\iota\int_a^{v_\iota}\frac{(v_\iota-r)^{m-p-1}}{\Gamma(m-p)}\sigma(r)dr-\varrho_4\sum_{\iota=1}^{e}\rho_\iota d_2\frac{(v_\iota-a)^{1-p}}{\Gamma(2-p)},
\end{aligned}$$

simplifying, we obtain

$$\begin{aligned}
&d_2\left(\varrho_2\frac{(T-a)^{1-p}}{\Gamma(2-p)}-\varrho_3\frac{(\zeta-a)^{2-p}}{(2-p)\Gamma(2-p)}-\varrho_4\sum_{\iota=1}^{e}\rho_\iota\frac{(v_\iota-a)^{1-p}}{\Gamma(2-p)}\right)\\
=\;&\varrho_2\int_a^T \frac{(T-r)^{m-p-1}}{\Gamma(m-p)}\sigma(r)dr-\varrho_4\sum_{\iota=1}^{e}\rho_\iota\int_a^{v_\iota}\frac{(v_\iota-r)^{m-p-1}}{\Gamma(m-p)}\sigma(r)dr\\
&-\varrho_3\int_a^\zeta \frac{(\zeta-r)^{m-p}}{(m-p)\Gamma(m-p)}\sigma(r)\,dr.
\end{aligned}$$

Using (2.3) in above equation, we have

$$\begin{aligned}
d_2\;=\;&\frac{1}{\beta}\Bigg(\varrho_2\int_a^T \frac{(T-r)^{m-p-1}}{\Gamma(m-p)}\sigma(r)dr-\varrho_3\int_a^\zeta \frac{(\zeta-r)^{m-p}}{\Gamma(m-p+1)}\sigma(r)\,dr\\
&-\varrho_4\sum_{\iota=1}^{e}\rho_\iota\int_a^{v_\iota}\frac{(v_\iota-r)^{m-p-1}}{\Gamma(m-p)}\sigma(r)dr\Bigg),
\end{aligned} \tag{2.12}$$

substituting (2.12) in (2.10) gives

$$\begin{aligned}
d_1\;=\;&\frac{-\alpha_2}{\beta\alpha_1}\Bigg(\varrho_2\int_a^T \frac{(T-r)^{m-p-1}}{\Gamma(m-p)}\sigma(r)dr-\varrho_3\int_a^\zeta \frac{(\zeta-r)^{m-p}}{\Gamma(m-p+1)}\sigma(r)\,dr\\
&-\varrho_4\sum_{\iota=1}^{e}\rho_\iota\int_a^{v_\iota}\frac{(v_\iota-r)^{m-p-1}}{\Gamma(m-p)}\sigma(r)dr\Bigg)+\frac{1}{\alpha_1}\Bigg(\gamma_2\int_a^T \frac{(T-r)^{m-1}}{\Gamma(m)}\sigma(r)dr\\
&-\gamma_3\int_a^\zeta \frac{(\zeta-r)^m}{\Gamma(m+1)}\sigma(r)\,dr-\gamma_4\sum_{\iota=1}^{e}\vartheta_\iota\int_a^{v_\iota}\frac{(v_\iota-r)^{m-1}}{\Gamma(m)}\sigma(r)dr\Bigg).
\end{aligned} \tag{2.13}$$

Making use of (2.7) , (2.8) , (2.12) and (2.13) in (2.9) , we get the required solution

$$\begin{aligned} u(t) &= \int_a^t \frac{(t-r)^{m-1}}{\Gamma(m)}\sigma(r)dr \\ &+\frac{1}{K}\int_a^\zeta \left(\beta\gamma_3\frac{(\zeta-r)^p}{\Gamma(m+1)}+\frac{\varrho_3}{\Gamma(m-p+1)}\eta(t)\right)(\zeta-r)^{m-p}\,\sigma(r)dr \\ &-\frac{1}{K}\int_a^T \left(\beta\gamma_2\frac{(T-r)^p}{\Gamma(m)}+\frac{\varrho_2}{\Gamma(m-p)}\eta(t)\right)(T-r)^{m-p-1}\sigma(r)dr \\ &+\frac{1}{K}\sum_{\iota=1}^{e}\int_a^{v_\iota}\left(\beta\gamma_4\vartheta_\iota\frac{(v_\iota-r)^p}{\Gamma(m)}+\frac{\varrho_4\rho_\iota}{\Gamma(m-p)}\eta(t)\right)(v_\iota-r)^{m-p-1}\sigma(r)dr. \end{aligned}$$

□

The solution of boundary value problems (1.2)–(1.3) exists if and only if the following operator $Q : H \to H$ defined by

$$\begin{aligned} &Q\,(u(t)) = \int_a^t \frac{(t-r)^{m-1}}{\Gamma(m)}F\,(r,u\,(r))\,dr \\ &+\frac{1}{K}\int_a^\zeta \left(\beta\gamma_3\frac{(\zeta-r)^p}{\Gamma(m+1)}+\frac{\varrho_3}{\Gamma(m-p+1)}\eta(t)\right)(\zeta-r)^{m-p}F\,(r,u\,(r))\,dr \\ &-\frac{1}{K}\int_a^T \left(\beta\gamma_2\frac{(T-r)^p}{\Gamma(m)}+\frac{\varrho_2}{\Gamma(m-p)}\eta(t)\right)(T-r)^{m-p-1}F\,(r,u\,(r))\,dr \\ &+\frac{1}{K}\sum_{\iota=1}^{e}\int_a^{v_\iota}\left(\beta\gamma_4\vartheta_\iota\frac{(v_\iota-r)^p}{\Gamma(m)}+\frac{\varrho_4\rho_\iota}{\Gamma(m-p)}\eta(t)\right)(v_\iota-r)^{m-p-1}F\,(r,u\,(r))\,dr, \end{aligned} \tag{2.14}$$

has a fixed point.

To reduce computational complexities we set

$$\begin{aligned} c &= \frac{(T-a)^m}{\Gamma(m+1)}+\frac{1}{|\alpha_1|}\left(\left|\gamma_3\right|\frac{(\zeta-a)^{m+1}}{\Gamma(m+2)}+\left|\gamma_2\right|\frac{(T-a)^m}{\Gamma(m+1)}+\sum_{\iota=1}^{e}\left|\gamma_4\right|\frac{\vartheta_\iota(v_\iota-a)^m}{\Gamma(m+1)}\right) \\ &+\frac{|\alpha_1(T-a)-\alpha_2|}{K}\left(\left|\varrho_3\right|\frac{(\zeta-a)^{m-p+1}}{\Gamma(m-p+2)}+\left|\varrho_2\right|\frac{(T-a)^{m-p}}{\Gamma(m-p+1)}+\sum_{\iota=1}^{e}\left|\varrho_4\right|\frac{\rho_\iota(v_\iota-a)^{m-p}}{\Gamma(m-p+1)}\right). \end{aligned} \tag{2.15}$$

$$\begin{aligned} d &= \left|\frac{1}{\alpha_1}\right|\left(\left|\gamma_3\right|\frac{(\zeta-a)^{m+1}}{\Gamma(m+2)}+\left|\gamma_2\right|\frac{(T-a)^m}{\Gamma(m+1)}+\left|\gamma_4\right|\sum_{\iota=1}^{e}\vartheta_\iota\frac{(v_\iota-a)^m}{\Gamma(m+1)}\right) \\ &+\left|\frac{\alpha_1(T-a)-\alpha_2}{K}\right|\left(\left|\varrho_3\right|\frac{(\zeta-a)^{m-p+1}}{\Gamma(m-p+2)}+\left|\varrho_2\right|\frac{(T-a)^{m-p}}{\Gamma(m-p+1)}+\left|\varrho_4\right|\sum_{\iota=1}^{e}\frac{\rho_\iota(v_\iota-a)^{m-p}}{\Gamma(m-p+1)}\right). \end{aligned} \tag{2.16}$$

Theorem 2.2 (**Schaefer**). [16] Let H be a Banach space and $Q : H \to H$ be a completely continuous mapping. Then either the equation $y = \lambda Qy$ has a solution for $\lambda = 1$, or the set $\{y \in H : y = \lambda Qy \text{ for some } \lambda \in (0,1)\}$ is unbounded.

The following is the existence results by applying above fixed point theorem.

Theorem 2.3. Let $F : [a,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. If there exists $L_1 > 0$ such that $|F(t,u(t))| \le L_1$ for all $t \in [a,T]$, $u \in \mathbb{R}$, then there exists a solution of the boundary value problems (1.2)–(1.3).

Proof. First it will be verified that the operator $Q : H \to H$ is completely continuous. Since F is continuous, this implies the continuity of Q. For a positive constant ϵ, define $B_\epsilon = \{u \in H : \|u\| \le \epsilon\}$ a bounded set in H. We prove that the operator Q maps bounded sets into bounded sets of H. For $u \in B_\epsilon$, $t \in [a,T]$, we consider

$$\begin{aligned}
\|(Qu)\| = \sup_{t\in[a,T]} \Bigg| & \int_a^t \frac{(t-r)^{m-1}}{\Gamma(m)} F(r,u(r))dr \\
& + \tfrac{1}{K}\int_a^\zeta \left(\beta\gamma_3 \tfrac{(\zeta-r)^p}{\Gamma(m+1)} + \tfrac{\varrho_3}{\Gamma(m-p+1)}\eta(t)\right)(\zeta-r)^{m-p} F(r,u(r))dr \\
& - \tfrac{1}{K}\int_a^T \left(\beta\gamma_2 \tfrac{(T-r)^p}{\Gamma(m)} + \tfrac{\varrho_2}{\Gamma(m-p)}\eta(t)\right)(T-r)^{m-p-1} F(r,u(r))dr \\
& + \tfrac{1}{K}\sum_{\iota=1}^{e}\int_a^{v_\iota} \left(\beta\gamma_4\vartheta_\iota \tfrac{(v_\iota-r)^p}{\Gamma(m)} + \tfrac{\varrho_4\rho_\iota}{\Gamma(m-p)}\eta(t)\right)(v_\iota-r)^{m-p-1} F(r,u(r))dr \Bigg| \\
\le |F(r,u(r))| \Bigg[& \tfrac{(t-a)^m}{\Gamma(m+1)} + \tfrac{1}{|\alpha_1|}\left(\left|\gamma_3\right| \tfrac{(\zeta-a)^{m+1}}{\Gamma(m+2)} + \left|\gamma_2\right| \tfrac{(T-a)^m}{\Gamma(m+1)} + \sum_{\iota=1}^{e}\left|\gamma_4\right| \tfrac{\vartheta_\iota(v_\iota-a)^m}{\Gamma(m+1)}\right) \\
& + \tfrac{|\alpha_1(t-a)-\alpha_2|}{k}\left(\left|\varrho_3\right| \tfrac{(\zeta-a)^{m-p+1}}{\Gamma(m-p+2)} + \left|\varrho_2\right| \tfrac{(T-a)^{m-p}}{\Gamma(m-p+1)} + \left|\varrho_4\right| \tfrac{\rho_\iota(v_\iota-a)^{m-p}}{\Gamma(m-p+1)}\right)\Bigg] \\
\le {} & L_1 c, \qquad (2.17)
\end{aligned}$$

where c is defined by (2.15).

Next we show that operator $Q : H \to H$ maps bounded set into equicontinuous set of H. For $a \le t_1 \le t_2 \le T$ and $u \in B_\epsilon$, we consider

$$\begin{aligned}
|(Qu)(t_2) - (Qu)(t_1)| \le & \left| \int_a^{t_2} \frac{(t_2-r)^{m-1}}{\Gamma(m)} F(r,u(r))dr \right| \\
& + \left| \frac{1}{K}\int_a^\zeta \left(\beta\gamma_3 \frac{(\zeta-r)^p}{\Gamma(m+1)} + \frac{\varrho_3}{\Gamma(m-p+1)}\eta(t_2)\right)(\zeta-r)^{m-p} F(r,u(r))dr \right| \\
& + \left| \frac{1}{K}\int_a^T \left(\beta\gamma_2 \frac{(T-r)^p}{\Gamma(m)} + \frac{\varrho_2}{\Gamma(m-p)}\eta(t_2)\right)(T-r)^{m-p-1} F(r,u(r))dr \right| \\
& + \left| \frac{1}{K}\sum_{\iota=1}^{e}\int_a^{v_\iota} \left(\beta\gamma_4\vartheta_\iota \frac{(v_\iota-r)^p}{\Gamma(m)} + \frac{\varrho_4\rho_\iota}{\Gamma(m-p)}\eta(t_2)\right)(v_\iota-r)^{m-p-1} F(r,u(r))dr \right| \\
& + \left| \int_a^{t_1} \frac{(t_1-r)^{m-1}}{\Gamma(m)} F(r,u(r))dr \right| \\
& + \left| \frac{1}{K}\int_a^\zeta \left(\beta\gamma_3 \frac{(\zeta-r)^p}{\Gamma(m+1)} + \frac{\varrho_3}{\Gamma(m-p+1)}\eta(t_1)\right)(\zeta-r)^{m-p} F(r,u(r))dr \right| \\
& + \left| \frac{1}{K}\int_a^T \left(\beta\gamma_2 \frac{(T-r)^p}{\Gamma(m)} + \frac{\varrho_2}{\Gamma(m-p)}\eta(t_1)\right)(T-r)^{m-p-1} F(r,u(r))dr \right|
\end{aligned}$$

$$+\left|\frac{1}{K}\sum_{\iota=1}^{e}\int_a^{v_\iota}\left(\beta\gamma_4\vartheta_\iota\frac{(v_\iota-r)^p}{\Gamma(m)}+\frac{\varrho_4\rho_\iota}{\Gamma(m-p)}\eta(t_1)\right)(v_\iota-r)^{m-p-1}F(r,u(r))dr\right|$$

$$\begin{aligned}&\leq|F(r,u(r))|\Big[\tfrac{(t_2-t_1)^m}{\Gamma(m+1)}+\tfrac{(t_1-a)^m-(t_2-a)^m}{\Gamma(m+1)}+\tfrac{(t_2-t_1)^m}{\Gamma(m+1)}\\&+\left|\tfrac{\alpha_1(t_2-t_1)}{K}\right|\left(\left|\varrho_3\right|\tfrac{(\zeta-a)^{m-p+1}}{\Gamma(m-p+2)}+\left|\varrho_2\right|\tfrac{(T-a)^{m-p}}{\Gamma(m-p+1)}+\sum_{\iota=1}^{e}\left|\varrho_4\right|\tfrac{\rho_\iota(v_\iota-a)^{m-p}}{\Gamma(m-p+1)}\right)\Big]\\&\leq L_1\Big[\tfrac{(t_2-t_1)^m}{\Gamma(m+1)}+\tfrac{(t_1-a)^m-(t_2-a)^m}{\Gamma(m+1)}+\tfrac{(t_2-t_1)^m}{\Gamma(m+1)}\\&+\left|\tfrac{\alpha_1(t_2-t_1)}{K}\right|\left(\left|\varrho_3\right|\tfrac{(\zeta-a)^{m-p+1}}{\Gamma(m-p+2)}+\left|\varrho_2\right|\tfrac{(T-a)^{m-p}}{\Gamma(m-p+1)}+\sum_{\iota=1}^{e}\left|\varrho_4\right|\tfrac{\rho_\iota(v_\iota-a)^{m-p}}{\Gamma(m-p+1)}\right).\end{aligned}$$

As $t_1 \to t_2$, the above expression approaches to zero independent of $u \in B_\epsilon$. Hence, by the Arzelá Ascoli theorem, the operator $Q : H \to H$ is completely continuous. Finally, we show that the set $V = \{u \in H : u = \lambda Qu,\ 0 \leq \lambda \leq 1\}$ is bounded. For $u \in V$ and $t \in [a, T]$, and using inequalities (2.17) and (2.15), we have

$$\|u\| = \sup_{t\in[a,T]} |\lambda(Qu)t| \leq L_1 c.$$

Hence by Theorem 2.2, Q has a fixed point in H. □

Theorem 2.4 (**Krasnoselskii).** [16] Let Ω be a closed bounded, convex and nonempty subset of a Banach space X. Let g_1, g_2 be the operators such that

(i) $g_1y_1 + g_2y_2 \in \Omega$, whenever $y_1, y_2 \in \Omega$.

(ii) g_1 is compact and continuous;

(iii) g_2 is a contraction.

Then there exist $y_3 \in \Omega$ such that $y_3 = g_1y_3 + g_2y_3$.

We apply the above theorem to prove the following existence result.

Theorem 2.5. Let $F : [a, T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying:

(H_1) $|F(t,u) - F(t,v)| \leq L|u - v|$, for all $t \in [a, T]$, $L > 0$, and $u, v \in \mathbb{R}$,

(H_2) there exists a function $\mu \in C([a, T], \mathbb{R}^+)$ with $|F(t,u)| \leq \mu(t)$, $\forall\ (t,u) \in [a, T] \times \mathbb{R}$.

If d in (2.16) satisfies

$$d < \frac{1}{L}, \tag{2.18}$$

then there exists a solution of problems (1.2)–(1.3).

Proof. Consider a set $B_r = \{u \in H : \|u\| \leq r\}$ with $r \geq c\|u\|$, clearly B_r is a closed subset of H, where c is given in (2.15). We decompose the operator Q defined in (2.14) into sum of two operators Q_1 and Q_2 on B_r as follows:

$$(Q_1u)(t) = \int_a^t \frac{(t-r)^{m-1}}{\Gamma(m)}F(r,u(r))dr$$

and

$$\begin{aligned}(Q_2u)(t) &= \frac{1}{K}\int_a^{\zeta}\left(\beta\gamma_3\frac{(\zeta-r)^p}{\Gamma(m+1)}+\frac{\varrho_3}{\Gamma(m-p+1)}\eta(t)\right)(\zeta-r)^{m-p}F(r,u(r))dr\\&\quad-\frac{1}{K}\int_a^{T}\left(\beta\gamma_2\frac{(T-r)^p}{\Gamma(m)}+\frac{\varrho_2}{\Gamma(m-p)}\eta(t)\right)(T-r)^{m-p-1}F(r,u(r))dr\end{aligned}$$

$$+\frac{1}{K}\sum_{\iota=1}^{e}\int_a^{\nu_\iota}\left(\beta\gamma_4\vartheta_\iota\frac{(\nu_\iota-r)^p}{\Gamma(m)}+\frac{\varrho_4\rho_\iota}{\Gamma(m-p)}\eta(t)\right)(\nu_\iota-r)^{m-p-1}F(r,u(r))dr.$$

For $u, v \in B_r$, consider

$$\begin{aligned}
&\|Q_1u+Q_2v\|\\
&=\sup_{t\in[a,T]}\Big|\int_a^t\tfrac{(t-r)^{m-1}}{\Gamma(m)}F(r,u(r))dr\\
&\quad+\tfrac{1}{K}\int_a^\zeta\left(\beta\gamma_3\tfrac{(\zeta-r)^p}{\Gamma(m+1)}+\tfrac{\varrho_3}{\Gamma(m-p+1)}\eta(t)\right)(\zeta-r)^{m-p}F(r,v(r))dr\\
&\quad-\tfrac{1}{K}\int_a^T\left(\beta\gamma_2\tfrac{(T-r)^p}{\Gamma(m)}+\tfrac{\varrho_2}{\Gamma(m-p)}\eta(t)\right)(T-r)^{m-p-1}F(r,v(r))dr\\
&\quad+\tfrac{1}{K}\sum_{\iota=1}^{e}\int_a^{\nu_\iota}\left(\beta\gamma_4\vartheta_\iota\tfrac{(\nu_\iota-r)^p}{\Gamma(m)}+\tfrac{\varrho_4\rho_\iota}{\Gamma(m-p)}\eta(t)\right)(\nu_\iota-r)^{m-p-1}F(r,v(r))dr\Big|\\
&\le\|\mu\|\sup_{t\in[a,T]}\Big[\tfrac{(t-a)^m}{\Gamma(m+1)}+\left|\tfrac{\beta\gamma_3}{k}\right|\tfrac{(\zeta-a)^{m+1}}{\Gamma(m+2)}+\left|\tfrac{\beta\gamma_2}{k}\right|\tfrac{(T-a)^m}{\Gamma(m+1)}+\sum_{\iota=1}^{e}\left|\tfrac{\beta\gamma_4}{k}\right|\tfrac{\vartheta_\iota(\nu_\iota-a)^m}{\Gamma(m+1)}\\
&\quad+\left|\tfrac{\varrho_3\eta(t)}{k}\right|\tfrac{(\zeta-a)^{m-p+1}}{\Gamma(m-p+2)}+\left|\tfrac{\varrho_2\eta(t)}{k}\right|\tfrac{(T-a)^{m-p}}{\Gamma(m-p+1)}+\left|\tfrac{\varrho_4\eta(t)}{k}\right|\tfrac{\rho_\iota(\nu_\iota-a)^{m-p}}{\Gamma(m-p+1)}\Big]\\
&\le\|\mu\|\Big[\tfrac{(T-a)^m}{\Gamma(m+1)}+\tfrac{1}{|\alpha_1|}\Big(\left|\gamma_3\right|\tfrac{(\zeta-a)^{m+1}}{\Gamma(m+2)}+\left|\gamma_2\right|\tfrac{(T-a)^m}{\Gamma(m+1)}+\sum_{\iota=1}^{e}\left|\gamma_4\right|\tfrac{\vartheta_\iota(\nu_\iota-a)^m}{\Gamma(m+1)}\Big)\\
&\quad+\tfrac{|\alpha_1(T-a)-\alpha_2|}{k}\Big(\left|\varrho_3\right|\tfrac{(\zeta-a)^{m-p+1}}{\Gamma(m-p+2)}+\left|\varrho_2\right|\tfrac{(T-a)^{m-p}}{\Gamma(m-p+1)}+\left|\varrho_4\right|\tfrac{\rho_\iota(\nu_\iota-a)^{m-p}}{\Gamma(m-p+1)}\Big)\Big]\\
&\le\|\mu\|\,c\le r.
\end{aligned}$$

Thus $Q_1u+Q_2v\subset B_r$, which verifies assumption (i) in Theorem 2.4. For $u, v\in B_r$, consider

$$\begin{aligned}
&\|Q_2u-Q_2v\|\\
&=\sup_{t\in[a,T]}\Big\|\Big[\tfrac{1}{K}\int_a^\zeta\left(\beta\gamma_3\tfrac{(\zeta-r)^p}{\Gamma(m+1)}+\tfrac{\varrho_3}{\Gamma(m-p+1)}\eta(t)\right)(\zeta-r)^{m-p}F(r,u(r))dr\\
&\quad-\tfrac{1}{K}\int_a^T\left(\beta\gamma_2\tfrac{(T-r)^p}{\Gamma(m)}+\tfrac{\varrho_2}{\Gamma(m-p)}\eta(t)\right)(T-r)^{m-p-1}F(r,u(r))dr\\
&\quad+\tfrac{1}{K}\sum_{\iota=1}^{e}\int_a^{\nu_\iota}\left(\beta\gamma_4\vartheta_\iota\tfrac{(\nu_\iota-r)^p}{\Gamma(m)}+\tfrac{\varrho_4\rho_\iota}{\Gamma(m-p)}\eta(t)\right)(\nu_\iota-r)^{m-p-1}F(r,u(r))dr\Big]\\
&\quad-\Big[\tfrac{1}{K}\int_a^\zeta\left(\beta\gamma_3\tfrac{(\zeta-r)^p}{\Gamma(m+1)}+\tfrac{\varrho_3}{\Gamma(m-p+1)}\eta(t)\right)(\zeta-r)^{m-p}F(r,v(r))dr\\
&\quad-\tfrac{1}{K}\int_a^T\left(\beta\gamma_2\tfrac{(T-r)^p}{\Gamma(m)}+\tfrac{\varrho_2}{\Gamma(m-p)}\eta(t)\right)(T-r)^{m-p-1}F(r,v(r))dr\\
&\quad+\tfrac{1}{K}\sum_{\iota=1}^{e}\int_a^{\nu_\iota}\left(\beta\gamma_4\vartheta_\iota\tfrac{(\nu_\iota-r)^p}{\Gamma(m)}+\tfrac{\varrho_4\rho_\iota}{\Gamma(m-p)}\eta(t)\right)(\nu_\iota-r)^{m-p-1}F(r,v(r))dr\Big]\Big\|
\end{aligned}$$

using $(H1)$

$$\begin{aligned}
&\le L\Big[\left|\tfrac{1}{\alpha_1}\right|\Big(\left|\gamma_3\right|\tfrac{(\zeta-a)^{m+1}}{\Gamma(m+2)}+\left|\gamma_2\right|\tfrac{(T-a)^m}{\Gamma(m+1)}+\left|\gamma_4\right|\sum_{\iota=1}^{e}\vartheta_\iota\tfrac{(\nu_\iota-a)^m}{\Gamma(m+1)}\Big)\\
&+\left|\tfrac{\alpha_1(T-a)-\alpha_2}{K}\right|\Big(\left|\varrho_3\right|\tfrac{(\zeta-a)^{m-p+1}}{\Gamma(m-p+2)}+\left|\varrho_2\right|\tfrac{(T-a)^{m-p}}{\Gamma(m-p+1)}+\left|\varrho_4\right|\sum_{\iota=1}^{e}\tfrac{\rho_\iota(\nu_\iota-a)^{m-p}}{\Gamma(m-p+1)}\Big)\Big]\|u-v\|.\\
&=Ld\,\|u-v\|.
\end{aligned}$$

From (2.18), we have $Ld<1$, so Q_2 is a contraction.
Next, we show that Q_1 is compact and continuous. The continuity of F implies the continuity of Q_1

and since $\|Q_1 u\| \leq \|\mu\| \frac{(T-a)^m}{\Gamma(m+1)}$, therefore Q_1 is uniformly bounded on B_r.
Set $\sup_{t\in[a,T]\times\beta_r} |F(t,u)| = \hat{F}$ and for $a \leq t_1 \leq t_2 \leq T$, consider

$$\begin{aligned}
&\|Q_1 u(t_2) - Q_1 u(t_1)\| \\
= &\left\| \int_a^{t_2} \frac{(t_2-r)^{m-1}}{\Gamma(m)} F(r,u(r))dr - \int_a^{t_1} \frac{(t_1-r)^{m-1}}{\Gamma(m)} F(r,u(r))dr \right\| \\
\leq &\hat{F} \left\| \int_a^{t_1} \frac{(t_2-r)^{m-1}}{\Gamma(m)} dr + \int_{t_1}^{t_2} \frac{(t_2-r)^{m-1}}{\Gamma(m)} dr - \int_a^{t_1} \frac{(t_1-r)^{m-1}}{\Gamma(m)} dr \right\| \\
= &\left[\frac{(t_1-a)^m - (t_2-a)^m}{\Gamma(m+1)} \right] \hat{F}.
\end{aligned}$$

As $t_1 \to t_2$, the above expression tends to zero independent of $u \in B_r$. This implies that Q_1 is relatively compact on B_r. Hence, it follows by the Arzelá Ascoli theorem that the operator Q_1 is compact on B_r. Thus all the hypothesis of the Theorem 2.4, are satisfied. Therefore the problems (2.1)–(1.3) has at least one solution. □

3. Uniqueness of solution via Banach contraction

Theorem 3.1 (Banach). Let (M,ρ) be a complete metric space and $T : M \to M$ be a self mapping. If there exists $\delta \in (0,1)$ such that

$$\rho(Tx, Ty) \leq \delta\rho(x,y)$$

for all $x, y \in M$. Then T has a unique fixed point.

Now we state and prove our result regarding uniqueness of solution.

Theorem 3.2. Suppose $F : [a.T] \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfying $(H1)$. If $L < c^{-1}$, where c is defined in (2.15), then there exists a unique solution of the problems (1.2)–(1.3).

Proof. Set

$$\sup_{t\in[a,T]} F(t,0) = L_2 \tag{3.1}$$

and choose $\epsilon \geq \frac{cL_2}{1-Lc} > 0$. We show that the mapping Q defined in (2.14) satisfies $Q(B_\epsilon) \subset B_\epsilon$. For $u \in B_\epsilon$ and $t \in [a,T]$, we consider

$$\begin{aligned}
|F(t,u(t))| &= |[F(t,u(t)) - F(t,0)] + F(t,0)| \\
&\leq |F(t,u(t)) - F(t,0)| + |F(t,0)| \\
&\leq L\|u\| + L_2.
\end{aligned}$$

Now for $u \in B_\epsilon$, consider

$$\begin{aligned}
\|Q(u)\| = &\sup_{t\in[a,T]} \left| \int_a^t \frac{(t-r)^{m-1}}{\Gamma(m)} F(r,u(r))\,dr \right. \\
&+ \frac{1}{K} \int_a^\zeta \left(\beta\gamma_3 \frac{(\zeta-r)^p}{\Gamma(m+1)} + \frac{\varrho_3}{\Gamma(m-p+1)}\eta(t) \right) (\zeta-r)^{m-p} F(r,u(r))\,dr
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{K}\int_a^T\left(\beta\gamma_2\frac{(T-r)^p}{\Gamma(m)}+\frac{\varrho_2}{\Gamma(m-p)}\eta(t)\right)(T-r)^{m-p-1}F(r,u(r))\,dr\\
&+\frac{1}{K}\sum_{\iota=1}^{e}\int_a^{v_\iota}\left(\beta\gamma_4\vartheta_\iota\frac{(v_\iota-r)^p}{\Gamma(m)}+\frac{\varrho_4\rho_\iota}{\Gamma(m-p)}\eta(t)\right)(v_\iota-r)^{m-p-1}F(r,u(r))\,dr\Bigg|.\\
\leq\;& (L\epsilon+L_2)\,c\leq\epsilon.
\end{aligned}
$$

Thus $Q(B_\epsilon)\subset B_\epsilon$.
Now we show that Q is a contraction. For $u,v\in H$, we have

$$
\begin{aligned}
\|Qu-Qv\| &= \sup_{t\in[a,T]}\Bigg|\int_a^t\frac{(t-r)^{m-1}}{\Gamma(m)}\left(F(r,u(r))-F(r,v(r))\right)dr\\
&+\frac{1}{K}\int_a^\zeta\left(\beta\gamma_3\frac{(\zeta-r)^p}{\Gamma(m+1)}+\frac{\varrho_3}{\Gamma(m-p+1)}\eta(t)\right)(\zeta-r)^{m-p}\left(F(r,u(r))-F(r,v(r))\right)dr\\
&-\frac{1}{K}\int_a^T\left(\beta\gamma_2\frac{(T-r)^p}{\Gamma(m)}+\frac{\varrho_2}{\Gamma(m-p)}\eta(t)\right)(T-r)^{m-p-1}\left(F(r,u(r))-F(r,v(r))\right)dr\\
&+\frac{1}{K}\sum_{\iota=1}^{e}\int_a^{v_\iota}\left(\beta\gamma_4\vartheta_\iota\frac{(v_\iota-r)^p}{\Gamma(m)}+\frac{\varrho_4\rho_\iota}{\Gamma(m-p)}\eta(t)\right)(v_\iota-r)^{m-p-1}\left(F(r,u(r))-F(r,v(r))\right)dr\Bigg|.\\
&\leq Lc\,\|u-v\|,
\end{aligned}
\tag{3.2}
$$

Since $Lc<1$, therefore Q is contraction, so there exists a unique solution w of Q which is a unique solution of (1.2)–(1.3). □

The following is the example that illustrate the above theorem to ensure the existence of a unique solution.

Example 3.3. Consider the following non-separated multi-point fractional boundary value problem

$$
\begin{aligned}
{}^cD^{1.75}u(t) &= \frac{1}{\sqrt{t+14}}\tan^{-1}\left(u(t)^{0.5}+e^t\cos t\right)+\sin t,\ 2<t<3,\\
\gamma_1u(2)+\gamma_2u(3) &= \gamma_3\int_a^\zeta u(\tau)d\tau+\gamma_4\sum_{\iota=1}^{4}\vartheta_\iota u(v_\iota),\\
\varrho_1\,{}^cD^{0.5}u(2)+\varrho_2\,{}^cD^{0.5}u(3) &= \varrho_3\int_a^\zeta {}^cD^{0.5}u(\tau)d\tau+\varrho_4\sum_{\iota=1}^{e}\rho_\iota\,{}^cD^{0.5}u(v_\iota),
\end{aligned}
$$

where $\gamma_1=1,\gamma_2=\frac{1}{2},\gamma_3=1,\gamma_4=\frac{1}{3},\varrho_1=1=\varrho_2=\varrho_3,\varrho_4=\frac{1}{3},\ \vartheta_1=\frac{1}{10},\vartheta_2=\frac{3}{20},\vartheta_3=\frac{1}{4},\vartheta_4=1,$ $v_1=\frac{11}{5},v_2=\frac{12}{5},\ v_3=\frac{13}{5},v_4=\frac{14}{5},\rho_1=\rho_2=\frac{17}{19},\rho_3=\rho_4=\frac{9}{11}$. Note that $|F(t,u(t))|\leq\frac{\pi}{4}+1=L$ and $|F(t,u)-F(t,v)|\leq L|u-v|$, by assuming $L=\frac{1}{4}$. Since $Lc<1$, so by Theorem 3.1, it has a unique solution.

4. Hyers-Ulam stability

In this section we discuss the criteria for Ulam stability of the problems (1.2)–(1.3). The following remarks and definitions will be crucial for proof of our result.

Remark 4.1. From Lemma 2.1 we can write the solution $u \in H$ of fractional BVP (1.2)–(1.3) as;

$$u(t) = \int_a^T G(t,r)F(r,u(r))\,dr, \tag{4.1}$$

where

$$a \leq \zeta \leq v_\iota \leq T,$$

and

$$G(t,r) = \begin{cases} \frac{(t-r)^{m-1}}{\Gamma(m)} + \frac{1}{K}\left(\beta\gamma_3\frac{(\zeta-r)^p}{\Gamma(m+1)} + \frac{\varrho_3}{\Gamma(m-p+1)}\eta(t)\right)(\zeta-r)^{m-p} \\ -\frac{1}{K}\left(\beta\gamma_2\frac{(T-r)^p}{\Gamma(m)} + \frac{\varrho_2}{\Gamma(m-p)}\eta(t)\right)(T-r)^{m-p-1}, \quad a \leq r \leq t \\ \frac{1}{K}\left(\beta\gamma_3\frac{(\zeta-r)^p}{\Gamma(m+1)} + \frac{\varrho_3}{\Gamma(m-p+1)}\eta(t)\right)(\zeta-r)^{m-p} \\ -\frac{1}{K}\left(\beta\gamma_2\frac{(T-r)^p}{\Gamma(m)} + \frac{\varrho_2}{\Gamma(m-p)}\eta(t)\right)(T-r)^{m-p-1} \\ +\frac{1}{K}\sum_{\iota=1}^{e}\left(\beta\gamma_4\vartheta_\iota\frac{(v_\iota-r)^p}{\Gamma(m)} + \frac{\varrho_4}{\Gamma(m-p)}\rho_\iota\eta(t)\right)(v_\iota-r)^{m-p-1}, \quad t \leq r \leq \zeta \\ \frac{-1}{K}\left(\beta\gamma_2\frac{(T-r)^p}{\Gamma(m)} + \frac{\varrho_2}{\Gamma(m-p)}\eta(t)\right)(T-r)^{m-p-1} \\ +\frac{1}{K}\sum_{\iota=1}^{e}\left(\beta\gamma_4\vartheta_\iota\frac{(v_\iota-r)^p}{\Gamma(m)} + \frac{\varrho_4}{\Gamma(m-p)}\rho_\iota\eta(t)\right)(v_\iota-r)^{m-p-1}, \quad \zeta \leq r \leq v_\iota \\ \frac{-1}{K}\left(\beta\gamma_2\frac{(T-r)^p}{\Gamma(m)} + \frac{\varrho_2}{\Gamma(m-p)}\eta(t)\right)(T-r)^{m-p-1}, \quad v_\iota \leq r \leq T. \end{cases} \tag{4.2}$$

Moreover

$$|G(t,r)| \leq \left|\frac{T^m}{\Gamma(m)}\right| + 6\left|\frac{T^m}{K}\right|\left|\beta\max\{\gamma_3,\gamma_2\}\frac{(\zeta)^p}{\Gamma(m+1)}\right| + \left|\frac{\max\{\varrho_3,\varrho_2\}}{\Gamma(m-p+1)}\eta^*\right|$$
$$+ 2\frac{1}{K}\sum_{\iota=1}^{e}\left|\beta\gamma_4\vartheta_\iota\frac{(v_\iota)^p}{\Gamma(m)} + \frac{\varrho_4}{\Gamma(m-p)}\rho_\iota\eta^*\right| := \Lambda, \tag{4.3}$$

where

$$\eta^* = \max_{t\in[a,T]}|\eta(t)|. \tag{4.4}$$

Definition 4.2. The fractional BVP (1.2)–(1.3) is said to be Hyers-Ulam stable if there exist constants $\lambda > 0$, such that for each $\epsilon > 0$ and for each solution $v \in H$ of

$$|{}^cD^m v(t) - F(t,v(t))| \leq \epsilon, \ t \in [a,T] \tag{4.5}$$

there exists a solution $u \in H$ of (1.2)–(1.3) such that

$$|v(t) - u(t)| \leq \lambda\epsilon, \ t \in [a,T]. \tag{4.6}$$

Remark 4.3. A function $v \in H$ is a solution of inequality (4.5) if and only if there exists a function $\omega \in H$ such that
(i) $|\omega(t)| \leq \epsilon$, $t \in [a,T]$,
(ii) ${}^cD^m v(t) = F(t,v(t)) + \omega(t)$, $t \in [a,T]$.

In the following result we obtain the criteria under which the problem (1.2)–(1.3) is Hyers-Ulam stable.

Theorem 4.4. Suppose $F : [a.T] \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfying $(H1)$. If $1 \neq L\Lambda(T-a)$, then the problems (1.2)–(1.3) is Hyers-Ulam stable.

Proof. Let $F : [a.T] \times \mathbb{R} \to \mathbb{R}$ be continuous and satisfying $(H1)$. Let $u \in H$ be any solution of inequality (4.5) then by Remark 4.3, we have

$$^{c}D^{m}u(t) = F(t, u(t)) + \omega(t), \text{ for all } t \in [a, T].$$

Using Remark 4.1, we can write

$$u(t) = \int_a^T G(t,r)F(r, u(r))\,dr + \int_a^T G(t,r)\omega(r)\,dr,$$

which gives

$$\left| u(t) - \int_a^T G(t,r)F(r, u(r))\,dr \right| \le \Lambda (T-a)\,\epsilon, \tag{4.7}$$

where $G(t, r)$ and Λ are defined in (4.2) and (4.3). Now let $v \in H$ be a unique solution of fractional BVP (1.2)–(1.3) , consider

$$\begin{aligned} |u(t) - v(t)| &= \left| u(t) - \int_a^T G(t,r)F(r, v(r))\,dr \right| \\ &\le \left| u(t) - \int_a^T G(t,r)F(r, u(r))\,dr \right| \\ &\quad + \left| \int_a^T G(t,r)F(r, u(r))\,dr - \int_a^T G(t,r)F(r, v(r))\,dr \right| \end{aligned}$$

from (4.7) we have

$$|u(t) - v(t)| \le \Lambda (T-a)\,\epsilon + \left| \int_a^T G(t,r)F(r, u(r))\,dr - \int_a^T G(t,r)F(r, v(r))\,dr \right|$$

using $(H1)$ we get

$$\|u - v\| \le \Lambda (T-a)\,\epsilon + L\Lambda (T-a)\,\|u - v\|$$

which further implies

$$\|u - v\| \le \lambda\epsilon,$$

where $\lambda = \frac{\Lambda(T-a)}{1-L\Lambda(T-a)}$. Since $1 \neq L\Lambda(T-a)$, therefore the problems (1.2)–(1.3) is Hyers-Ulam stable. This proves the theorem. □

Remark 4.5. In Example 3.3, the system is Hyers-Ulam stable.

Acknowledgement

Authors are grateful to the reviewers and editors for their suggestions and comments to improve the manuscript.

Conflict of interest

The authors declare no conflict of interest.

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