Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 5(1): 376–384.
DOI:10.3934/math.2020025
Received: 24 October 2019
Accepted: 19 November 2019
Published: 22 November 2019

Research article

New inequalities of Wilker's type for hyperbolic functions

Ling Zhu*

Department of Mathematics, Zhejiang Gongshang University, Hangzhou, Zhejiang, China

* **Correspondence:** Email: zhuling0571@163.com; Tel: +86057188802322; Fax: +86057188802322.

Abstract: Using power series expansions of functions and the monotonicity criterion for the quotient of power series, we establish some new Wilker-type inequalities for hyperbolic functions.

Keywords: Wilker-type inequalities; hyperbolic functions
Mathematics Subject Classification: 26D05, 26D15

1. Introduction

Wilker [1] proposed two open problems as the following statements:
(a) If $0 < x < \pi/2$, then

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2. \tag{1.1}$$

(b) There exists a largest constant c such that

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2 + cx^3 \tan x \tag{1.2}$$

for $0 < x < \pi/2$.

Sumner et al. [2] affirmed the truth of two problems above and obtained a further results as follows

$$\frac{16}{\pi^4}x^3 \tan x < \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2 < \frac{8}{45}x^3 \tan x,\ 0 < x < \frac{\pi}{2}, \tag{1.3}$$

where $16/\pi^4$ and $8/45$ are the best constants in (1.3).

Some refreshing proofs of inequalities (1.1) and (1.3) can be found in Pinelis [3]. In 2007, the author of this paper [4] established a new Wilker-type inequality involving hyperbolic functions as follows:

$$\left(\frac{\sinh x}{x}\right)^2+\frac{\tanh x}{x}-2>\frac{8}{45}x^3\tanh x,\ \ x\neq 0, \tag{1.4}$$

where $8/45$ can not be replaced by any larger number.

In 2011, Sun and Zhu [5] showed another new Wilker-type inequality involving hyperbolic functions and obtained the following result.

$$\left(\frac{x}{\sinh x}\right)^2+\frac{x}{\tanh x}-2<\frac{2}{45}x^3\sinh x,\ \ x\neq 0, \tag{1.5}$$

where $2/45$ can not be replaced by any smaller number.

The purpose of this paper is to give three new inequalities of Wilker-type for hyperbolic functions.

Theorem 1. *Let $x\neq 0$. Then*

$$\left(\frac{\sinh x}{x}\right)^2+\frac{\tanh x}{x}-2>\alpha x^4\left(\frac{\tanh x}{x}\right)^{6/7} \tag{1.6}$$

holds if and only if $\alpha\leq 8/45$.

Theorem 2. *The function*

$$G(x)=\frac{\left(\frac{x}{\sinh x}\right)^2+\frac{x}{\tanh x}-2}{x^3\tanh x}$$

has only one maximum point $x_0=1.54471\ldots$ *on* $(0,\infty)$*, so the function* $G(x)$ *has the maximum value*

$$G(x_0)=\max_{x\in(0,\infty)}G(x)=0.050244\ldots=\theta_0.$$

Specifically, for $x\neq 0$,

$$\left(\frac{x}{\sinh x}\right)^2+\frac{x}{\tanh x}-2<\theta x^3\tanh x \tag{1.7}$$

holds if and only if $\theta\geq\theta_0$.

Theorem 3. *Let $x\neq 0$. Then*

$$\left(\frac{x}{\sinh x}\right)^2+\frac{x}{\tanh x}-2<\beta x^4\left(\frac{\tanh x}{x}\right)^{4/7} \tag{1.8}$$

holds if and only if $\beta\geq 2/45$.

2. A lemma

In order to prove Theorem 2, we need the following lemma. We introduce a useful auxiliary function $H_{f,g}$. For $-\infty\leq a<b\leq\infty$, let f and g be differentiable on (a,b) and $g'\neq 0$ on (a,b). Then the function $H_{f,g}$ is defined by

$$H_{f,g}:=\frac{f'}{g'}g-f.$$

The function $H_{f,g}$ has some well properties and play an important role in the proof of a monotonicity criterion for the quotient of power series.

Lemma 1. *([6]) Let* $f(t) = \sum_{k=0}^{\infty} a_k t^k$ *and* $g(t) = \sum_{k=0}^{\infty} b_k t^k$ *be two real power series converging on* $(-r, r)$ *and* $b_k > 0$ *for all* k*. Suppose that for certain* $m \in \mathbb{N}$*, the non-constant sequence* $\{a_k/b_k\}$ *is increasing (resp. decreasing) for* $0 \le k \le m$ *and decreasing (resp. increasing) for* $k \ge m$*. Then the function* f/g *is strictly increasing (resp. decreasing) on* $(0, r)$ *if and only if* $H_{f,g}(r^-) \ge 0$ *(resp.* ≤ 0*). Moreover, if* $H_{f,g}(r^-) < 0$ *(resp.* > 0*), then there exists* $t_0 \in (0, r)$ *such that the function* f/g *is strictly increasing (resp. decreasing) on* $(0, t_0)$ *and strictly decreasing (resp. increasing) on* (t_0, r)*.*

3. Proofs of Theorems 1–3

3.1. Proof of Theorem 1

Since

$$\left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} - 2 > \frac{8}{45} x^4 \left(\frac{\tanh x}{x}\right)^{6/7}$$

$$\begin{aligned} \Leftrightarrow \quad & \left[\left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} - 2\right]^7 > \left[\frac{8}{45} x^4 \left(\frac{\tanh x}{x}\right)^{6/7}\right]^7 \\ = \quad & \frac{2097\,152}{373\,669\,453\,125} x^{22} \tanh^6 x, \end{aligned}$$

we can let

$$F(x) = 7 \ln\left[\left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} - 2\right] - \ln\left[\frac{2097\,152}{373\,669\,453\,125} x^{22} \tanh^6 x\right], \ x > 0.$$

Then

$$F'(x) = \frac{h(x)}{8x^3 (\tanh x)\left(\cosh^3 x\right)\left[\left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} - 2\right]},$$

where

$$\begin{aligned} h(x) \quad = \quad & 18 \sinh 3x - 18 \sinh 5x + 36 \sinh x + 88x^2 \sinh 3x + 56x \cosh x \\ & -63x \cosh 3x + 7x \cosh 5x + 96x^3 \cosh x + 96x^2 \sinh x. \end{aligned}$$

By substituting the power series expansions of all hyperbolic functions involved in the above formula into $h(x)$, we obtain that

$$\begin{aligned} h(x) \quad = \quad & 18 \sum_{n=0}^{\infty} \frac{(3x)^{2n+1}}{(2n+1)!} - 18 \sum_{n=0}^{\infty} \frac{(5x)^{2n+1}}{(2n+1)!} + 36 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \\ & +88x^2 \sum_{n=0}^{\infty} \frac{(3x)^{2n+1}}{(2n+1)!} + 56x \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} - 63x \sum_{n=0}^{\infty} \frac{(3x)^{2n}}{(2n)!} \\ & +7x \sum_{n=0}^{\infty} \frac{(5x)^{2n}}{(2n)!} + 96x^3 \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} + 96x^2 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \end{aligned}$$

$$
\begin{aligned}
&= 18\sum_{n=5}^{\infty}\frac{3^{2n+1}}{(2n+1)!}x^{2n+1} - 18\sum_{n=5}^{\infty}\frac{5^{2n+1}}{(2n+1)!}x^{2n+1} + 36\sum_{n=5}^{\infty}\frac{1}{(2n+1)!}x^{2n+1} \\
&\quad +88\sum_{n=4}^{\infty}\frac{3^{2n+1}}{(2n+1)!}x^{2n+3} + 56\sum_{n=5}^{\infty}\frac{1}{(2n)!}x^{2n+1} - 63\sum_{n=5}^{\infty}\frac{3^{2n}}{(2n)!}x^{2n+1} \\
&\quad +7\sum_{n=5}^{\infty}\frac{5^{2n}}{(2n)!}x^{2n+1} + 96\sum_{n=4}^{\infty}\frac{1}{(2n)!}x^{2n+3} + 96\sum_{n=4}^{\infty}\frac{1}{(2n+1)!}x^{2n+3}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{n=5}^{\infty}\frac{18\left(3^{2n+1}-5^{2n+1}+2\right)+(2n+1)\left(56-63\cdot 3^{2n}+7\cdot 5^{2n}\right)}{(2n+1)!}x^{2n+1} \\
&\quad +\sum_{n=4}^{\infty}\frac{24\left(11\cdot 3^{2n}+4\right)+96\,(2n+1)}{(2n+1)!}x^{2n+3} \\
&= \sum_{n=5}^{\infty}\frac{18\left(3^{2n+1}-5^{2n+1}+2\right)+(2n+1)\left(56-63\cdot 3^{2n}+7\cdot 5^{2n}\right)}{(2n+1)!}x^{2n+1} \\
&\quad +\sum_{n=5}^{\infty}\frac{24\left(11\cdot 3^{2n-2}+4\right)+96\,(2n-1)}{(2n-1)!}x^{2n+1} \\
&= \sum_{n=5}^{\infty}\frac{l\,(n)}{3\,(2n+1)!}x^{2n+1},
\end{aligned}
$$

where

$$
l\,(n) = 3\,(14n-83)\,5^{2n} + \left(352n^2-202n-27\right)3^{2n} + \left(2304n^3+1152n^2+336n+276\right).
$$

Since $l\,(5) = 77\,856\,768 > 0$ and for $n \geq 6$,

$$
\begin{aligned}
14n-83 &\geq 14\cdot 6-83 = 1 > 0, \\
352n^2-202n-27 &\geq 352\cdot 6^2-202\cdot 6-27 = 11\,433 > 0, \\
2304n^3+1152n^2+336n+276 &> 0,
\end{aligned}
$$

we have that $l\,(n) > 0$ for $n \geq 5$. So $h(x) > 0$ and $F'(x) > 0$ for $x > 0$. Then the function $F(x)$ is strictly increasing on $(0, \infty)$. Therefore, $F(x) > F(0^+) = 0$. At the same time, we find that

$$
\lim_{x\to 0^+}\frac{\left(\frac{\sinh x}{x}\right)^2+\frac{\tanh x}{x}-2}{x^4\left(\frac{\tanh x}{x}\right)^{6/7}} = \frac{8}{45}.
$$

Then the proof of Theorem 1 is complete.

3.2. *Proof of Theorem 2*

Let

$$G(x)=\frac{(\frac{x}{\sinh x})^2+\frac{x}{\tanh x}-2}{x^3\tanh x},\ 0<x<\infty.$$

Rewrite $G(x)$ as

$$\begin{aligned}
G(x) &= \frac{\cosh x}{x^3\sinh x}\left(\frac{x^2}{\sinh^2 x}+\frac{x\cosh x}{\sinh x}-2\right)\\
&= \frac{\left(x^2+x\cosh x\sinh x-2\sinh^2 x\right)(\cosh x)}{x^3\sinh^3 x}\\
&= \frac{\frac{1}{4}\left(x\sinh 3x-2\cosh 3x+2\cosh x+x\sinh x+4x^2\cosh x\right)}{\frac{1}{4}x^3(\sinh 3x-3\sinh x)}\\
&= \frac{x\sinh 3x-2\cosh 3x+2\cosh x+x\sinh x+4x^2\cosh x}{x^3(\sinh 3x-3\sinh x)}\\
&:= \frac{f(x)}{g(x)}.
\end{aligned}$$

By substituting the power series expansions of the hyperbolic functions involved in the above formula into $f(x)$ and $g(x)$ we have

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty}\frac{3^{2n+1}x^{2n+2}}{(2n+1)!}-2\sum_{n=0}^{\infty}\frac{3^{2n}x^{2n}}{(2n)!}+2\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}+\sum_{n=0}^{\infty}\frac{x^{2n+2}}{(2n+1)!}\\
&\quad +4\sum_{n=0}^{\infty}\frac{x^{2n+2}}{(2n)!}\\
&= \sum_{n=2}^{\infty}\frac{3^{2n+1}x^{2n+2}}{(2n+1)!}-2\sum_{n=3}^{\infty}\frac{3^{2n}x^{2n}}{(2n)!}+2\sum_{n=3}^{\infty}\frac{x^{2n}}{(2n)!}+\sum_{n=2}^{\infty}\frac{x^{2n+2}}{(2n+1)!}\\
&\quad +4\sum_{n=2}^{\infty}\frac{x^{2n+2}}{(2n)!}\\
&= \sum_{n=2}^{\infty}\frac{3^{2n+1}+1+4(2n+1)}{(2n+1)!}x^{2n+2}-\sum_{n=3}^{\infty}\frac{2\left(3^{2n}-1\right)}{(2n)!}x^{2n}\\
&= \sum_{n=2}^{\infty}\frac{3^{2n+1}+1+4(2n+1)}{(2n+1)!}x^{2n+2}-\sum_{n=2}^{\infty}\frac{2\left(3^{2n+2}-1\right)}{(2n+2)!}x^{2n+2}\\
&= \sum_{n=2}^{\infty}\frac{(2n+2)\left(3^{2n+1}+1+4(2n+1)\right)-2\left(3^{2n+2}-1\right)}{(2n+2)!}x^{2n+2}\\
&= \sum_{n=2}^{\infty}\frac{2\left[(3n-6)3^{2n}+8n^2+13n+6\right]}{(2n+2)!}x^{2n+2}:=\sum_{n=2}^{\infty}a_n x^{2n+2},
\end{aligned}$$

and

$$\begin{aligned}
g(x) &= x^3\left(\sum_{n=0}^{\infty}\frac{(3x)^{2n+1}}{(2n+1)!}-3\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}\right)=\sum_{n=0}^{\infty}\frac{\left(3^{2n+1}-3\right)x^{2n+4}}{(2n+1)!}\\
&= \sum_{n=1}^{\infty}\frac{\left(3^{2n+1}-3\right)x^{2n+4}}{(2n+1)!}=\sum_{n=2}^{\infty}\frac{\left(3^{2n-1}-3\right)}{(2n-1)!}x^{2n+2}:=\sum_{n=2}^{\infty}b_nx^{2n+1}.
\end{aligned}$$

Let

$$c_n=\frac{a_n}{b_n}=\frac{\frac{2\left[(3n-6)3^{2n}+8n^2+13n+6\right]}{(2n+2)!}}{\frac{\left(3^{2n-1}-3\right)}{(2n-1)!}}=\frac{2\left[(3n-6)\,3^{2n}+8n^2+13n+6\right]}{(2n+2)\,(2n+1)\,2n\,(3^{2n-1}-3)}.$$

We can calculate that

$$c_2=\frac{2}{45}<c_3=\frac{2}{35}>c_4=\frac{206}{4095},$$

and show that $\{c_n\}_{n\geq 4}$ is decreasing:

$$\begin{aligned}
c_n &\geq c_{n+1}\\
\Leftrightarrow\ & \frac{C}{D}:=\frac{2\left((3n-6)\,3^{2n}+8n^2+13n+6\right)}{(2n+2)\,(2n+1)\,2n\,(3^{2n-1}-3)}\\
\geq\ & \frac{2\left((3n-3)\,3^{2n+2}+8\,(n+1)^2+13\,(n+1)+6\right)}{(2n+4)\,(2n+3)\,(2n+2)\,(3^{2n+1}-3)}:=\frac{E}{F}.
\end{aligned}$$

In fact,

$$CF-DE=\frac{2}{3}p(n),$$

where

$$\begin{aligned}
p(n) &= \left(108n^2-189n-324\right)3^{4n}\\
&\quad+\left(128n^4+1104n^3+952n^2+1026n+648\right)3^{2n}\\
&\quad-\left(144n^3+612n^2+837n+324\right).
\end{aligned}$$

Since for $n\geq 4$, we have $108n^2-189n-324\geq 648>0$. By mathematical induction it is easy to prove that

$$3^{2n}>\frac{144n^3+612n^2+837n+324}{128n^4+1104n^3+952n^2+1026n+648}$$

holds for $n\geq 4$. So $p(n)>0$ for $n\geq 4$. This leads to $\{c_n\}_{n\geq 4}$ is decreasing. Therefore

$$c_2<c_3>c_4>c_5>\cdots.$$

We compute to get

$$H_{f,g}(\infty)=\lim_{x\to\infty}\left(\frac{f'}{g'}g-f\right)=-\infty,$$

By Lemma 1 we obtain that there exists $x_0\in(0,\infty)$ such that the function $G(x)=f/g$ is strictly increasing on $(0,x_0)$ and strictly decreasing on (x_0,∞). That is, x_0 is the only maximum point of the function $G(x)$ on $(0,\infty)$. Let us determine the maximum point x_0 and the maximum value $G(x_0)$. We compute

$$G'(x) = -2\frac{r(x)}{x^4\left(15\cosh 2x - 6\cosh 4x + \cosh 6x - 10\right)},$$

where

$$\begin{aligned} r(x) \;=\; & -10x - 15\sinh 2x + 12\sinh 4x - 3\sinh 6x + 8x^3\cosh 2x + 4x^3\cosh 4x \\ & -12x^2\sinh 2x + 6x^2\sinh 4x + 15x\cosh 2x - 6x\cosh 4x + x\cosh 6x - 12x^3. \end{aligned}$$

We find out that

$$G'(1.54471) = 1.108\,6\times 10^{-7},\; G'(1.54472) = -2.229\,2\times 10^{-9}.$$

So $x_0 = 1.54471\ldots$ and

$$G(x_0) = \max_{x\in(0,\infty)} G(x) = 0.050244\ldots.$$

Considering the reasons

$$\lim_{x\to 0^+} G(x) = \frac{2}{45} = 0.044444\ldots,\;\; \lim_{x\to\infty} G(x) = 0,$$

we have

$$\min_{x\in(0,\infty)} G(x) = 0.$$

The proof of Theorem 2 is completed.

3.3. Proof of Theorem 3

Let

$$H(x) = \ln\left[\frac{128}{373\,669\,453\,125}\frac{x^{24}}{\cosh^4 x}\sinh^4 x\right] - 7\ln\left[\left(\frac{x}{\sinh x}\right)^2 + \frac{x}{\tanh x} - 2\right],$$

where $0 < x < \infty$. Then

$$H'(x) = \frac{q(x)}{8x\left(\cosh x\sinh^3 x\right)\left[\left(\frac{x}{\sinh x}\right)^2 + \frac{x}{\tanh x} - 2\right]},$$

where

$$\begin{aligned} q(x) \;=\; & 15x + 96\sinh 2x - 48\sinh 4x + 56x^3\cosh 2x + 84x^2\sinh 2x \\ & -32x\cosh 2x + 17x\cosh 4x + 88x^3. \end{aligned}$$

By substituting the power series expansions of the hyperbolic functions involved in the above formula into $q(x)$, we have

$$q(x) \;=\; 15x + 96\sum_{n=0}^{\infty}\frac{2^{2n+1}x^{2n+1}}{(2n+1)!} - 48\sum_{n=0}^{\infty}\frac{4^{2n+1}x^{2n+1}}{(2n+1)!} + 56x^3\sum_{n=0}^{\infty}\frac{2^{2n}x^{2n}}{(2n)!}$$

$$
\begin{aligned}
& +84x^2 \sum_{n=0}^{\infty} \frac{2^{2n+1} x^{2n+1}}{(2n+1)!} - 32x \sum_{n=0}^{\infty} \frac{2^{2n} x^{2n}}{(2n)!} + 17x \sum_{n=0}^{\infty} \frac{4^{2n} x^{2n}}{(2n)!} + 88x^3 \\
= \; & 96 \sum_{n=5}^{\infty} \frac{2^{2n+1} x^{2n+1}}{(2n+1)!} - 48 \sum_{n=5}^{\infty} \frac{4^{2n+1} x^{2n+1}}{(2n+1)!} - 32 \sum_{n=5}^{\infty} \frac{2^{2n} x^{2n+1}}{(2n)!} \\
& +17 \sum_{n=5}^{\infty} \frac{4^{2n} x^{2n+1}}{(2n)!} + 56 \sum_{n=4}^{\infty} \frac{2^{2n} x^{2n+3}}{(2n)!} + 84 \sum_{n=4}^{\infty} \frac{2^{2n+1} x^{2n+3}}{(2n+1)!} \\
& +56 \sum_{n=4}^{\infty} \frac{2^{2n} x^{2n+3}}{(2n)!} + 84 \sum_{n=4}^{\infty} \frac{2^{2n+1} x^{2n+3}}{(2n+1)!}
\end{aligned}
$$

$$
\begin{aligned}
= \; & \sum_{n=5}^{\infty} \frac{(34n-175)\,4^{2n} - (64n-160)\,2^{2n}}{(2n+1)!} x^{2n+1} + \sum_{n=4}^{\infty} \frac{112\,(n+2)\,2^{2n}}{(2n+1)!} x^{2n+3} \\
= \; & \sum_{n=5}^{\infty} \frac{(34n-175)\,4^{2n} - (64n-160)\,2^{2n}}{(2n+1)!} x^{2n+1} + \sum_{n=5}^{\infty} \frac{28\,(n+1)\,2^{2n}}{(2n-1)!} x^{2n+1} \\
= \; & \sum_{n=5}^{\infty} \left[\frac{(34n-175)\,4^{2n} - (64n-160)\,2^{2n}}{(2n+1)!} + \frac{28\,(n+1)\,2^{2n}}{(2n-1)!} \right] x^{2n+1} \\
= \; & \sum_{n=5}^{\infty} \frac{(34n-175)\,4^{2n} - (64n-160)\,2^{2n} + 28\,(2n+1)\,(2n)\,(n+1)\,2^{2n}}{(2n+1)!} x^{2n+1} \\
= \; & \sum_{n=5}^{\infty} \frac{(34n-175)\,4^{2n} + 8\left(14n^3 + 21n^2 - n + 20\right) 2^{2n}}{(2n+1)!} x^{2n+1} \\
= \; & \sum_{n=5}^{\infty} \frac{d_n}{(2n+1)!} x^{2n+1},
\end{aligned}
$$

where

$$d_n = (34n-175)\,4^{2n} + 814n^3 + 21n^2 - n + 20, \;\; n \geq 5.$$

We can find $d_5 = 13\,516\,800 > 0$ and $d_n > 0$ holds for $n \geq 6$ due to

$$
\begin{aligned}
34n - 175 & \geq 34 \times 6 - 175 = 29 > 0, \\
14n^3 + 21n^2 - n + 20 & \geq 14 \cdot 6^3 + 21 \cdot 6^2 - 6 + 20 = 3794 > 0.
\end{aligned}
$$

So $H'(x)$ >0. Then $H(x)$ is increasing on $(0, \infty)$. Therefore, $H(x) > H(0^+) = 0$. At the same time, we find that

$$\lim_{x \to 0^+} \frac{\left(\frac{x}{\sinh x}\right)^2 + \frac{x}{\tanh x} - 2}{x^4 \left(\frac{\tanh x}{x}\right)^{4/7}} = \frac{2}{45}.$$

Then the proof of Theorem 3 is complete.

Acknowledgments

This paper is supported by the Natural Science Foundation of China grants No.61772025.

Conflict of interest

The author declares no conflict of interest in this paper.

References

1. J. B. Wilker, *Problem E 3306*, Amer. Math. Monthly, **96** (1989), 55.
2. J. S. Sumner, A. A. Jagers, M. Vowe, et al. *Inequalities involving trigonometric functions*, Amer. Math. Monthly, **98** (1991), 264–267.
3. I. Pinelis, *L'Hospital rules for monotonicity and the Wilker-Anglesio inequality*, Amer. Math. Monthly, **111** (2004), 905–909.
4. L. Zhu, *On Wilker–type inequalities*, Math. Inequal. Appl., **10** (2007), 727–731.
5. Z. J. Sun and L. Zhu, *On New Wilker-type inequalities*, ISRN Mathematical Analysis, **2011** (2011), 1–7.
6. Zh. H. Yang, Y. M. Chu and M. K. Wang, *Monotonicity criterion for the quotient of power series with applications*, J. Math. Anal. Appl., **428** (2015), 587–604.