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Research article

Sufficiency for singular trajectories in the calculus of variations

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Abstract: For calculus of variations problems of Bolza with variable end-points, nonlinear inequality and equality isoperimetric constraints and nonlinear inequality and equality mixed pointwise constraints, sufficient conditions for strong minima are derived. The main novelty of the new sufficiency results presented in this article concerns their applicability to cases in which the derivatives of the extremals to be optimal solutions are not necessarily continuous nor piecewise continuous but only *essentially bounded* and they do not necessarily satisfy the standard strengthened Legendre condition but only the corresponding *necessary condition*.

Keywords: calculus of variations; inequality and equality isoperimetric constraints; inequality and equality mixed pointwise constraints; free end-points; sufficiency; strong minima; singular extremals
Mathematics Subject Classification: 49K15

1. Introduction

Necessary and sufficient conditions for optimality play a crucial role in solving problems in the calculus of variations. The main necessary conditions for such problems are the well-known conditions of Euler, Weierstrass, Legendre and Jacobi. In many cases, depending on the smoothness of the assumptions, Euler's necessary condition corresponds to a second order differential equation which restricts solutions to lie in a family of trajectories with certain uniformity properties. Also, the necessary condition of Jacobi cannot be applied when the extremal has corners and is not nonsingular. This is an unfortunate feature since, in general, the admissible arcs or trajectories which are candidates for solving the problem are neither nonsingular nor smooth.

On the other hand, one fundamental aspect of the theory of sufficient conditions for optimality consists in slightly strengthening the necessary conditions. Concretely, if an admissible arc satisfies the strengthened conditions of Euler, Legendre and Jacobi, then it is a strict weak minimum. Additionally, if this admissible arc also satisfies the strengthened condition of Weierstrass, then it is a strict strong minimum. Some of the techniques used to obtain sufficiency include the construction of a Mayer field on which the extremals are independent of the path with respect to an invariant integral commonly called the Hilbert integral, the existence of a symmetric solution of the matrix Riccati inequality associated with the problem, a verification function satisfying the Hamilton-Jacobi equation, a quadratic function that satisfies a Hamilton-Jacobi inequality, the nonexistence of conjugate points on the underlying half-closed time interval, or the incorporation of some convexity arguments on the functions delimiting the calculus of variations problem (see for example [1–17] and references therein).

It is important to mention that the smoothness and the nonsingularity assumptions are crucial in the sufficiency theories mentioned above. In other words, the classical sufficiency theory in the calculus of variations, in general, may not give a response when the extremal under consideration is singular or has corners. In fact, as mentioned in [6], there is a gap between the set of necessary and sufficient conditions and [6, Section 3.7] is entirely devoted to the study of some problems for which the nonsingularity assumption fails. There, one finds a method which is only applicable to particular examples and may not hold in general, since "although this algorithm sheds light on the theory, it provides no panacea. Indeed there are no panaceas." Additionally, we refer the reader to [13], where the importance of the nonsingularity assumption in the classical calculus of variations sufficiency theory is fully explained.

In this paper we derive two new sufficiency results which provide sufficient conditions for strong local minima in certain classes of parametric and nonparametric calculus of variations problems of Bolza with variable or free end-points, inequality and equality nonlinear isoperimetric constrains, and nonlinear mixed pointwise inequality and equality constraints. The main novelty of our new sufficient theorems concerns their applicability to cases in which the extremals under consideration may be singular and nonsmooth, that is, the strengthened condition of Legendre and the continuity of the derivative of the proposed extremal are no longer required. More precisely, given an admissible extremal whose derivative is not continuous nor piecewise continuous but only *essentially bounded*, the elements comprising the new sufficiency theorems are the classical transversality condition, a crucial inequality which arises from the original algorithm used to prove the main result of the article, the necessary condition of Legendre, but not its *strengthened version*, the positivity of the second variation over the set of all nonnul admissible variations, and three refined Weierstrass conditions which are related to the functions delimiting the problems.

Another distinguishing characteristic of the main sufficiency result of the nonparametric calculus of variations problem presented in this paper is the fact that the initial and final end-points of the states are completely free, that is, they are not only variable end-points but they may belong to any set which is not necessarily a smooth manifold described by some functions which usually involve some type of equality or inequality conditions.

The paper is organized as follows. In Section 2 we pose the parametric calculus of variations problem we shall deal with together with some basic definitions and the statement of the main result of the article. In Section 3 we enunciate the nonparametric calculus of variations problem we shall study together with some basic definitions and a corollary which is also one of the main results of the paper. Section 4 is devoted to state two auxiliary lemmas in which the proof of the theorem is strongly based. Section 5 is dedicated to the proof of the main theorem of the article. In Section 6 we prove the lemmas given in Section 4 and, in the final section, we provide an example which shows how the sufficient theory developed in this paper widens the range of applicability of the classical calculus of variations theory.

2. Statement of a parametric problem of Bolza and the main result

Suppose we are given an interval $T := [t_0, t_1]$ in $\mathbf{R}$, and functions $l(b)\colon \mathbf{R}^p \to \mathbf{R}$, $l_\gamma(b)\colon \mathbf{R}^p \to \mathbf{R}$ $(\gamma = 1, \ldots, K)$, $\Psi_i(b)\colon \mathbf{R}^p \to \mathbf{R}^n$ $(i = 0, 1)$, $L(t, x, \dot{x})\colon T \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$, $L_\gamma(t, x, \dot{x})\colon T \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ $(\gamma = 1, \ldots, K)$ and $\varphi(t, x, \dot{x})\colon T \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^s$. Set

$$\mathcal{R} := \{(t, x, \dot{x}) \in T \times \mathbf{R}^n \times \mathbf{R}^n \mid \varphi_\alpha(t, x, \dot{x}) \leq 0\,(\alpha \in R),\ \varphi_\beta(t, x, \dot{x}) = 0\,(\beta \in S)\}$$

where $R := \{1, \ldots, r\}$ and $S := \{r + 1, \ldots, s\}$ $(r = 0, 1, \ldots, s)$. If $r = 0$ then $R = \emptyset$ and we disregard statements involving φ_α. Similarly, if $r = s$ then $S = \emptyset$ and we disregard statements involving φ_β.

It will be assumed throughout the paper that L, L_γ $(\gamma = 1, \ldots, K)$ and $\varphi = (\varphi_1, \ldots, \varphi_s)$ have first and second derivatives with respect to x and $\dot{x}$. Moreover, we shall assume that the functions l, l_γ $(\gamma = 1, \ldots, K)$ and Ψ_i $(i = 0, 1)$ are of class C^2 on $\mathbf{R}^p$. Also, if we denote by $c(t, x, \dot{x})$ either $L(t, x, \dot{x})$, $L_\gamma(t, x, \dot{x})$ $(\gamma = 1, \ldots, K)$, $\varphi(t, x, \dot{x})$ or any of its partial derivatives of order less than or equal to two with respect to x and $\dot{x}$, we shall assume that if C is any bounded subset of $T \times \mathbf{R}^n \times \mathbf{R}^n$, then $|c(C)|$ is a bounded subset of $\mathbf{R}$. Additionally, we shall assume that if $\{(\Gamma_q, \Lambda_q)\}$ is any sequence in $AC(T; \mathbf{R}^n) \times L^1(T; \mathbf{R}^n)$ such that for some $U \subset T$ measurable and some $\{(\Gamma_0, \Lambda_0)\} \in AC(T; \mathbf{R}^n) \times L^\infty(T; \mathbf{R}^n)$, $(\Gamma_q(t), \Lambda_q(t)) \to (\Gamma_0(t), \Lambda_0(t))$ uniformly on U, then for all $q \in \mathbf{N}$, $c(t, \Gamma_q(t), \Lambda_q(t))$ is measurable on U and

$$c(t, \Gamma_q(t), \Lambda_q(t)) \to c(t, \Gamma_0(t), \Lambda_0(t)) \text{ uniformly on } U.$$

Note that all conditions above concerning the functions L, L_γ $(\gamma = 1, \ldots, K)$ and φ, are satisfied if the functions L, L_γ $(\gamma = 1, \ldots, K)$, φ and their first and second derivatives with respect to x and $\dot{x}$ are continuous on $T \times \mathbf{R}^n \times \mathbf{R}^n$.

Set

$$\mathcal{X} := AC(T; \mathbf{R}^n), \quad \mathcal{U}_s := L^\infty(T; \mathbf{R}^s), \quad \mathcal{A} := \mathcal{X} \times \mathbf{R}^p.$$

We shall use the notation x_b to denote any element $x_b := (x, b) \in \mathcal{A}$. Let $\mathcal{B}$ any subset of $\mathbf{R}^p$ which we shall call the set of *parameters*. The parametric calculus of variations problem we shall deal with, denoted by (P), is that of minimizing the functional

$$I(x_b) := l(b) + \int_{t_0}^{t_1} L(t, x(t), \dot{x}(t))dt$$

over all $x_b \in \mathcal{A}$ satisfying the constraints

$$\begin{cases} c(t, x(t), \dot{x}(t)) \text{ is integrable on } T. \\ b \in \mathcal{B}. \\ x(t_i) = \Psi_i(b) \text{ for } i = 0, 1. \\ I_i(x_b) := l_i(b) + \int_{t_0}^{t_1} L_i(t, x(t), \dot{x}(t))dt \leq 0 \ (i = 1, \ldots, k). \\ I_j(x_b) := l_j(b) + \int_{t_0}^{t_1} L_j(t, x(t), \dot{x}(t))dt = 0 \ (j = k + 1, \ldots, K). \\ (t, x(t), \dot{x}(t)) \in \mathcal{R} \text{ (a.e. in } T). \end{cases}$$

Elements $b = (b_1, \ldots, b_p)^*$ (the notation $*$ denotes transpose) in $\mathcal{B}$ will be called *parameters*, elements x_b in $\mathcal{A}$ will be called *arcs or trajectories*, and a trajectory x_b is *admissible* if it satisfies the constraints. The notation x_{0b_0} refers to an element $(x_0, b_0) \in \mathcal{A}$.

Let us now introduce some definitions which will be used throughout the paper.

• An arc x_{0b_0} *solves* (P) if it is admissible and $I(x_{0b_0}) \leq I(x_b)$ for all admissible arcs x_b. For strong local minima, an admissible arc x_{0b_0} is called a *strong minimum* of (P) if it is a minimum of I relative to the following norm

$$\|x_b\| := |b| + \sup_{t\in T} |x(t)| = |b| + \|x\|_C,$$

that is, if for some $\epsilon > 0$, $I(x_{0b_0}) \leq I(x_b)$ for all admissible arcs satisfying $\|x_b - x_{0b_0}\| < \epsilon$.

• For all $x \in \mathcal{X}$, we use the notation $(\tilde{x}(t))$ in order to represent $(t, x(t), \dot{x}(t))$. Also, $(\tilde{x}_0(t))$ represents $(t, x_0(t), \dot{x}_0(t))$.

• Given K real numbers $\lambda_1, \ldots, \lambda_K$, for any x_b admissible define the functional I_0 by

$$I_0(x_b) := I(x_b) + \sum_{\gamma=1}^{K} \lambda_\gamma I_\gamma(x_b) = l_0(b) + \int_{t_0}^{t_1} L_0(\tilde{x}(t))dt,$$

where $l_0 \colon \mathbf{R}^p \to \mathbf{R}$ is given by

$$l_0(b) := l(b) + \sum_{\gamma=1}^{K} \lambda_\gamma l_\gamma(b),$$

and $L_0 \colon T \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ is given by

$$L_0(t, x, \dot{x}) := L(t, x, \dot{x}) + \sum_{\gamma=1}^{K} \lambda_\gamma L_\gamma(t, x, \dot{x}).$$

• Given $\lambda_1, \ldots, \lambda_K$, for all $(t, x, \dot{x}, \rho, \mu) \in T \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^s$, define the Hamiltonian of the problem by

$$H(t, x, \dot{x}, \rho, \mu) := \langle \rho, \dot{x} \rangle - L_0(t, x, \dot{x}) - \langle \mu, \varphi(t, x, \dot{x}) \rangle,$$

where $\rho \in \mathbf{R}^n$ denotes the adjoint variable and $\mu \in \mathbf{R}^s$ is the associated multiplier of the mixed constraints.

• Given $(\rho, \mu) \in \mathcal{X} \times \mathcal{U}_s$, and $\lambda_1, \ldots, \lambda_K$, for all $(t, x, \dot{x}) \in T \times \mathbf{R}^n \times \mathbf{R}^n$, define the following function associated to the Hamiltonian,

$$F_0(t, x, \dot{x}) := -H(t, x, \dot{x}, \rho(t), \mu(t)) - \langle \dot{\rho}(t), x \rangle.$$

• Given $(\rho, \mu) \in \mathcal{X} \times \mathcal{U}_s$ and $\lambda_1, \ldots, \lambda_K$, for any x_b admissible define the functional J_0 by

$$J_0(x_b) := \langle \rho(t_1), x(t_1) \rangle - \langle \rho(t_0), x(t_0) \rangle + l_0(b) + \int_{t_0}^{t_1} F_0(\tilde{x}(t))dt.$$

• The notation y_β refers to any element (y, β) in $\mathcal{A}$.

• Given $(\rho, \mu) \in \mathcal{X} \times \mathcal{U}_s$, and $\lambda_1, \ldots, \lambda_K$, for any $x_b \in \mathcal{A}$ with $\dot{x} \in L^\infty(T; \mathbf{R}^n)$ and any $y_\beta \in \mathcal{A}$ consider *the first variations* of J_0 and I_γ $(\gamma = 1, \ldots, K)$ with respect to x_b over y_β which are given, respectively, by

$$J_0'(x_b; y_\beta) := \langle \rho(t_1), y(t_1) \rangle - \langle \rho(t_0), y(t_0) \rangle + l_0'(b)\beta + \int_{t_0}^{t_1} \{F_{0x}(\tilde{x}(t))y(t) + F_{0\dot{x}}(\tilde{x}(t))\dot{y}(t)\}dt,$$

$$I'_\gamma(x_b;y_\beta) := l'_\gamma(b)\beta + \int_{t_0}^{t_1} \{L_{\gamma x}(\tilde{x}(t))y(t) + L_{\gamma\dot{x}}(\tilde{x}(t))\dot{y}(t)\}dt.$$

• For all $(t,x,\dot{x}) \in T \times \mathbf{R}^n \times \mathbf{R}^n$, denote by

$$\mathcal{I}_a(t,x,\dot{x}) := \{\alpha \in R \mid \varphi_\alpha(t,x,\dot{x}) = 0\},$$

the set of active indices of $(t,x,\dot{x})$ with respect to the mixed inequality constraints.

• For all $x_b \in \mathcal{A}$, denote by

$$i_a(x_b) := \{i = 1,\dots,k \mid I_i(x_b) = 0\},$$

the set of active indices of x_b with respect to the isoperimetric inequality constraints.

• Given $x_b \in \mathcal{A}$, let $\mathcal{Y}(x_b)$ be the set of all $y_\beta \in \mathcal{A}$ with $\dot{y} \in L^2(T;\mathbf{R}^n)$ satisfying

$$\begin{cases} y(t_i) = \Psi'_i(b)\beta \ (i = 0,1), \\ I'_i(x_b;y_\beta) \le 0 \ (i \in i_a(x_b)), \ \ I'_j(x_b;y_\beta) = 0 \ (j = k+1,\dots,K), \\ \varphi_{\alpha x}(\tilde{x}(t))y(t) + \varphi_{\alpha\dot{x}}(\tilde{x}(t))\dot{y}(t) \le 0 \ (\text{a.e. in } T, \alpha \in \mathcal{I}_a(\tilde{x}(t))), \\ \varphi_{\beta x}(\tilde{x}(t))y(t) + \varphi_{\beta\dot{x}}(\tilde{x}(t))\dot{y}(t) = 0 \ (\text{a.e. in } T, \beta \in S). \end{cases}$$

The set $\mathcal{Y}(x_b)$ will be called the set of *admissible variations* along x_b.

• Given $(\rho,\mu) \in \mathcal{X} \times \mathcal{U}_s$, and $\lambda_1,\dots,\lambda_K$, for any $x_b \in \mathcal{A}$ with $\dot{x} \in L^\infty(T;\mathbf{R}^n)$ and any $y_\beta \in \mathcal{A}$ with $\dot{y} \in L^2(T;\mathbf{R}^n)$, we define the *second variation* of J_0 with respect to x_b over y_β, by

$$J''_0(x_b;y_\beta) := \langle l''_0(b)\beta,\beta\rangle + \int_{t_0}^{t_1} 2\Omega_0(x;t,y(t),\dot{y}(t))dt,$$

where for all $(t,y,\dot{y}) \in T \times \mathbf{R}^n \times \mathbf{R}^n$,

$$2\Omega_0(x;t,y,\dot{y}) := \langle y, F_{0xx}(\tilde{x}(t))y\rangle + 2\langle y, F_{0x\dot{x}}(\tilde{x}(t))\dot{y}\rangle + \langle \dot{y}, F_{0\dot{x}\dot{x}}(\tilde{x}(t))\dot{y}\rangle.$$

• Given $(\rho,\mu) \in \mathcal{X} \times \mathcal{U}_s$, $\lambda_1,\dots,\lambda_K$ and $x_{0b_0} \in \mathcal{A}$, we say that x_{0b_0} is *singular*, if for some $\tau \in T$, $|H_{\dot{x}\dot{x}}(\tilde{x}_0(\tau),\rho(\tau),\mu(\tau))| = 0$. It satisfies the *Legendre* condition if

$$F_{0\dot{x}\dot{x}}(\tilde{x}_0(t)) = -H_{\dot{x}\dot{x}}(\tilde{x}_0(t),\rho(t),\mu(t)) \ge 0 \quad (\text{a.e. in } T)$$

and the *strengthened Legendre* condition, if $F_{0\dot{x}\dot{x}}(\tilde{x}_0(t)) > 0$ $(t \in T)$.

• Denote by E_0 the *Weierstrass excess function* of F_0, given by

$$E_0(t,x,\dot{x},u) := F_0(t,x,u) - F_0(t,x,\dot{x}) - F_{0\dot{x}}(t,x,\dot{x})(u-\dot{x}).$$

• Similarly, the *Weierstrass excess function* of L_γ $(\gamma = 1,\dots,K)$, is given by

$$E_\gamma(t,x,\dot{x},u) := L_\gamma(t,x,u) - L_\gamma(t,x,\dot{x}) - L_{\gamma\dot{x}}(t,x,\dot{x})(u-\dot{x}).$$

• For all $\pi = (\pi_1,\dots,\pi_n)^* \in \mathbf{R}^n$, set

$$V(\pi) := (1+|\pi|^2)^{1/2} - 1.$$

• For all $x \in \mathcal{X}$, define

$$D(x) := V(x(t_0)) + \int_{t_0}^{t_1} V(\dot{x}(t))dt.$$

• As we mentioned above the symbol $*$ denotes transpose.

It is well-known that, under certain normality assumptions (see for example [10]), if x_{0b_0} is a strong minimum of (P), then there exist $\rho \in \mathcal{X}$ and $\mu \in \mathcal{U}_s$ with $\mu_\alpha(t) \geq 0$ and $\mu_\alpha(t)\varphi_\alpha(\tilde{x}_0(t)) = 0$ ($\alpha \in R$, a.e. in T) and multipliers $\lambda_1, \ldots, \lambda_K$ with $\lambda_i \geq 0$ and $\lambda_i I_i(x_{0b_0}) = 0$ ($i = 1, \ldots, k$) such that

$$\dot{\rho}(t) = -H_x^*(\tilde{x}_0(t), \rho(t), \mu(t)) \quad \text{and} \quad H_{\dot{x}}^*(\tilde{x}_0(t), \rho(t), \mu(t)) = 0 \quad \text{(a.e. in } T).$$

The relations given above are the Euler-Lagrange equations of the constrained problem (P) and if (x_0, ρ, μ) satisfies the Euler-Lagrange equations, then (x_0, ρ, μ) will be called an *extremal*.

The following theorem is the main result of the article. Given an admissible arc x_{0b_0} with $\dot{x}_0$ neither continuous nor piecewise continuous but only *essentially bounded*, this theorem gives sufficient conditions assuring that x_{0b_0} is a strong minimum of problem (P). Hypothesis (i) of Theorem 2.1 is known as the transversality condition, hypothesis (ii) arises from the original proof of the theorem, condition (iii) is the necessary condition of Legendre, but not its *strengthened version*, hypothesis (iv) is the positivity of the second variation over the set of all nonnull admissible variations and finally, hypothesis (v) involves three conditions related to the Weierstrass excess functions.

2.1 Theorem: *Let x_{0b_0} be an admissible arc with $\dot{x}_0 \in L^\infty(T; \mathbf{R}^n)$. Assume that $\mathcal{I}_a(\tilde{x}_0(\cdot))$ is piecewise constant on T, and there exist $(\rho, \mu) \in \mathcal{X} \times \mathcal{U}_s$ with $\mu_\alpha(t) \geq 0$ and $\mu_\alpha(t)\varphi_\alpha(\tilde{x}_0(t)) = 0$ ($\alpha \in R$, a.e. in T), two positive numbers h, ϵ, and multipliers $\lambda_1, \ldots, \lambda_K$ with $\lambda_i \geq 0$ and $\lambda_i I_i(x_{0b_0}) = 0$ ($i = 1, \ldots, k$) such that (x_0, ρ, μ) is an extremal and the following holds:*

(i) $l_0'(b_0) + \rho^*(t_1)\Psi_1'(b_0) - \rho^*(t_0)\Psi_0'(b_0) = 0.$

(ii) $\rho^*(t_1)\Psi_1''(b_0; \beta) - \rho^*(t_0)\Psi_0''(b_0; \beta) \geq 0$ *for all* $\beta \in \mathbf{R}^p$.

(iii) $H_{\dot{x}\dot{x}}(\tilde{x}_0(t), \rho(t), \mu(t)) \leq 0$ (*a.e. in* T).

(iv) $J_0''(x_{0b_0}; y_\beta) > 0$ *for all nonnull* $y_\beta \in \mathcal{Y}(x_{0b_0})$.

(v) *For all x_b admissible with $\|x - x_0\|_C < \epsilon$,*

a. $E_0(t, x(t), \dot{x}_0(t), \dot{x}(t)) \geq 0$ (*a.e. in* T).

b. $\int_{t_0}^{t_1} E_0(t, x(t), \dot{x}_0(t), \dot{x}(t))dt \geq h \int_{t_0}^{t_1} V(\dot{x}(t) - \dot{x}_0(t))dt.$

c. $\int_{t_0}^{t_1} E_0(t, x(t), \dot{x}_0(t), \dot{x}(t))dt \geq h| \int_{t_0}^{t_1} E_\gamma(t, x(t), \dot{x}_0(t), \dot{x}(t))dt|$ $(\gamma = 1, \ldots, K)$.

Then for some $\theta_1, \theta_2 > 0$ and all admissible trajectories x_b satisfying $\|x_b - x_{0b_0}\| < \theta_1$,

$$I(x_b) \geq I(x_{0b_0}) + \theta_2 \min\{|b - b_0|^2, D(x - x_0)\}.$$

In particular, x_{0b_0} is a strong minimum of (P).

3. Statement of a nonparametric problem of Bolza and a fundamental result

Suppose we are given an interval $T := [t_0, t_1]$ in $\mathbf{R}$, two sets $\mathcal{B}_0, \mathcal{B}_1 \subset \mathbf{R}^n$ and functions $\ell(x_1, x_2)\colon \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$, $\ell_\gamma(x_1, x_2)\colon \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ $(\gamma = 1, \ldots, K)$, $\mathcal{L}(t, x, \dot{x})\colon T \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$,

$\mathcal{L}_\gamma(t, x, \dot{x})\colon T \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ $(\gamma = 1, \ldots, K)$ and $\phi(t, x, \dot{x})\colon T \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^s$. Set

$$\bar{\mathcal{R}} := \{(t, x, \dot{x}) \in T \times \mathbf{R}^n \times \mathbf{R}^n \mid \phi_\alpha(t, x, \dot{x}) \leq 0\, (\alpha \in R),\ \phi_\beta(t, x, \dot{x}) = 0\, (\beta \in S)\}$$

where $R := \{1, \ldots, r\}$ and $S := \{r + 1, \ldots, s\}$ $(r = 0, 1, \ldots, s)$. If $r = 0$ then $R = \emptyset$ and we disregard statements involving φ_α. Similarly, if $r = s$ then $S = \emptyset$ and we disregard statements involving φ_β.

It will be assumed throughout this section that $\mathcal{L}$, $\mathcal{L}_\gamma$ $(\gamma = 1, \ldots, K)$ and $\phi = (\phi_1, \ldots, \phi_s)$ have first and second derivatives with respect to x and $\dot{x}$. Moreover, we shall assume that the functions ℓ, ℓ_γ $(\gamma = 1, \ldots, K)$ are of class C^2 on $\mathbf{R}^n \times \mathbf{R}^n$.

Also, if we denote by $c(t, x, \dot{x})$ either $\mathcal{L}(t, x, \dot{x})$, $\mathcal{L}_\gamma(t, x, \dot{x})$ $(\gamma = 1, \ldots, K)$, $\phi(t, x, \dot{x})$ or any of its partial derivatives of order less than or equal to two with respect to x and $\dot{x}$, we shall assume that all the assumptions posed in Section 2 in the statement of the problem are satisfied.

As in Section 2, $\mathcal{X}$ will denote the space of absolutely continuous functions mapping T to $\mathbf{R}^n$ and $\mathcal{U}_s := L^\infty(T; \mathbf{R}^s)$ the space of essentially bounded functions mapping T to $\mathbf{R}^s$.

The nonparametric calculus of variations problem we shall deal with, denoted by $(\bar{P})$, consists in minimizing the functional

$$\mathcal{J}(x) := \ell(x(t_0), x(t_1)) + \int_{t_0}^{t_1} \mathcal{L}(t, x(t), \dot{x}(t))dt$$

over all $x \in \mathcal{X}$ satisfying the constraints

$$\begin{cases} c(t, x(t), \dot{x}(t)) \text{ is integrable on } T. \\ x(t_i) \in \mathcal{B}_i \text{ for } i = 0, 1. \\ \mathcal{J}_i(x) := \ell_i(x(t_0), x(t_1)) + \int_{t_0}^{t_1} \mathcal{L}_i(t, x(t), \dot{x}(t))dt \leq 0 \ (i = 1, \ldots, k). \\ \mathcal{J}_j(x) := \ell_j(x(t_0), x(t_1)) + \int_{t_0}^{t_1} \mathcal{L}_j(t, x(t), \dot{x}(t))dt = 0 \ (j = k + 1, \ldots, K). \\ (t, x(t), \dot{x}(t)) \in \bar{\mathcal{R}} \text{ (a.e. in } T). \end{cases}$$

Elements x in $\mathcal{X}$ will be called *arcs or trajectories*, and a trajectory x is *admissible* if it satisfies the constraints.

An arc x_0 *solves* $(\bar{P})$ if it is admissible and $\mathcal{J}(x_0) \leq \mathcal{J}(x)$ for all admissible arcs x. For strong minima, an admissible arc x_0 is called a *strong minimum* of $(\bar{P})$ if it is a minimum of $\mathcal{J}$ relative to the norm

$$\|x\| := \sup_{t \in T} |x(t)|,$$

that is, if for some $\epsilon > 0$, $\mathcal{J}(x_0) \leq \mathcal{J}(x)$ for all admissible arcs satisfying $\|x - x_0\| < \epsilon$.

Let $\Psi\colon \mathbf{R}^n \to \mathbf{R}^n \times \mathbf{R}^n$ be any function of class C^2 such that $\mathcal{B}_0 \times \mathcal{B}_1 \subset \Psi(\mathbf{R}^n)$. Associate the nonparametric problem $(\bar{P})$ with the parametric problem of Section 2, which we denote by (P_Ψ), that is, (P_Ψ) will be the parametric problem given in Section 2, with $p = n$, $\mathcal{B} = \Psi^{-1}(\mathcal{B}_0 \times \mathcal{B}_1)$, $l = \ell \circ \Psi$, $l_\gamma = \ell_\gamma \circ \Psi$ $(\gamma = 1, \ldots, K)$, $L = \mathcal{L}$, $L_\gamma = \mathcal{L}_\gamma$ $(\gamma = 1, \ldots, K)$, $\varphi = \phi$ and Ψ_0, Ψ_1 the components of Ψ, that is, $\Psi = (\Psi_0, \Psi_1)$. Recall that the notation x_b means (x, b) where $b \in \mathbf{R}^n$ is a parameter.

3.1 Lemma: *The following is satisfied:*

(i) *x_b is an admissible arc of (P_Ψ) if and only if x is an admissible arc of $(\bar{P})$ and $b \in \Psi^{-1}(x(t_0), x(t_1))$.*

(ii) *If x_b is an admissible arc of (P_Ψ), then*

$$\mathcal{J}(x) = I(x_b).$$

(iii) *If x_{0b_0} is a solution of (P_Ψ), then x_0 is a solution of $(\bar{P})$.*

Proof: Conditions (i) and (ii) follow from the definitions of the problems. Now, let x an admissible arc of $(\bar{P})$ and let $b \in \Psi^{-1}(x(t_0), x(t_1))$. By (i), x_0 is an admissible arc of $(\bar{P})$ and x_b is an admissible arc of (P_Ψ). Then by (ii),

$$\mathcal{J}(x_0) = I(x_{0b_0}) \leq I(x_b) = \mathcal{J}(x)$$

which shows (iii). ▮

The following corollary which is a consequence of Theorem 2.1 and Lemma 3.1, provides a set of sufficient conditions of problem $(\bar{P})$.

3.2 Corollary: *Let $\Psi\colon \mathbf{R}^n \to \mathbf{R}^n \times \mathbf{R}^n$ be any function of class C^2 such that $\mathcal{B}_0 \times \mathcal{B}_1 \subset \Psi(\mathbf{R}^n)$ and let (P_Ψ) be the parametric problem defined in the previous paragraph of Lemma 3.1. Let x_{0b_0} be an admissible arc of (P_Ψ) with $\dot{x}_0 \in L^\infty(T; \mathbf{R}^n)$. Assume that $\mathcal{I}_a(\tilde{x}_0(\cdot))$ is piecewise constant on T, and there exist $(\rho, \mu) \in \mathcal{X} \times \mathcal{U}_s$ with $\mu_\alpha(t) \geq 0$ and $\mu_\alpha(t)\varphi_\alpha(\tilde{x}_0(t)) = 0$ ($\alpha \in R$, a.e. in T), two positive numbers h, ϵ, and multipliers $\lambda_1, \ldots, \lambda_K$ with $\lambda_i \geq 0$ and $\lambda_i I_i(x_{0b_0}) = 0$ ($i = 1, \ldots, k$) such that (x_0, ρ, μ) is an extremal and the following holds:*

(i) $l_0'(b_0) + \rho^*(t_1)\Psi_1'(b_0) - \rho^*(t_0)\Psi_0'(b_0) = 0.$

(ii) $\rho^*(t_1)\Psi_1''(b_0;\beta) - \rho^*(t_0)\Psi_0''(b_0;\beta) \geq 0$ *for all* $\beta \in \mathbf{R}^n$.

(iii) $H_{\dot{x}\dot{x}}(\tilde{x}_0(t), \rho(t), \mu(t)) \leq 0$ (*a.e. in* T).

(iv) $J_0''(x_{0b_0}; y_\beta) > 0$ *for all nonnull* $y_\beta \in \mathcal{Y}(x_{0b_0})$.

(v) *For all x_b admissible with $\|x - x_0\| < \epsilon$,*

a. $E_0(t, x(t), \dot{x}_0(t), \dot{x}(t)) \geq 0$ (*a.e. in* T).

b. $\int_{t_0}^{t_1} E_0(t, x(t), \dot{x}_0(t), \dot{x}(t))dt \geq h \int_{t_0}^{t_1} V(\dot{x}(t) - \dot{x}_0(t))dt.$

c. $\int_{t_0}^{t_1} E_0(t, x(t), \dot{x}_0(t), \dot{x}(t))dt \geq h| \int_{t_0}^{t_1} E_\gamma(t, x(t), \dot{x}_0(t), \dot{x}(t))dt|$ ($\gamma = 1, \ldots, K$).

Then, x_0 is a strong minimum of $(\bar{P})$.

4. Auxiliary results

In this section we state two auxiliary results which will be used to prove Theorem 2.1. The proof of these results will be given in Section 6. As before $\mathcal{X}$ denotes $AC(T; \mathbf{R}^n)$.

In the following two lemmas, we shall assume that we are given $x_0 \in \mathcal{X}$ and $\{x^q\}$ a sequence in $\mathcal{X}$ such that

$$\lim_{q\to\infty} D(x^q - x_0) = 0 \quad \text{and} \quad d_q := [2D(x^q - x_0)]^{1/2} > 0 \quad (q \in \mathbf{N}).$$

For all $q \in \mathbf{N}$ and $t \in T$, let

$$y_q(t) := \frac{x^q(t) - x_0(t)}{d_q}.$$

We say that $\dot{x}^q(t) \to \dot{x}_0(t)$ almost uniformly on T, if for any $\epsilon > 0$, there exists $U_\epsilon \subset T$ measurable with $m(U_\epsilon) < \epsilon$ such that $\dot{x}^q(t) \to \dot{x}_0(t)$ uniformly on $T \setminus U_\epsilon$.

4.1 Lemma: *For some subsequence of $\{x^q\}$, again denoted by $\{x^q\}$, and some $y_0 \in \mathcal{X}$ with $\dot{y}_0 \in L^2(T;\mathbf{R}^n)$, $\dot{x}^q(t) \to \dot{x}_0(t)$ almost uniformly on T, $y_q(t) \to y_0(t)$ uniformly on T and $\{\dot{y}_q\}$ converges weakly in $L^1(T;\mathbf{R}^n)$ to $\dot{y}_0$.*

4.2 Lemma: *Let $U \subset T$ measurable, $R_0 \in L^\infty(U;\mathbf{R}^{n\times n})$ and $\{R_q\}$ a sequence in $L^\infty(U;\mathbf{R}^{n\times n})$. If $\dot{x}^q(t) \to \dot{x}_0(t)$ uniformly on U, $R_q(t) \to R_0(t)$ uniformly on U and $R_0(t) \geq 0$ ($t \in U$), then*

$$\liminf_{q\to\infty} \int_U \langle R_q(t)\dot{y}_q(t), \dot{y}_q(t)\rangle dt \geq \int_U \langle R_0(t)\dot{y}_0(t), \dot{y}_0(t)\rangle dt.$$

5. Proof of Theorem 2.1

The proof of Theorem 2.1 will be divided in three Lemmas. In Lemmas 5.1, 5.2 and 5.3 below, we shall be assuming that all the hypotheses of Theorem 2.1 are satisfied. Before enunciating the lemmas, we shall introduce some definitions.

First of all, note that given $x = (x_1, \ldots, x_n)^* \in \mathbf{R}^n$ and $b = (b_1, \ldots, b_p)^* \in \mathbf{R}^p$, if we define $x\mathbf{i}, b\mathbf{j} \in \mathbf{R}^{n+p}$ by $x\mathbf{i} := (x_1, \ldots, x_n, 0, \ldots, 0)^*$ and $b\mathbf{j} := (0, \ldots, 0, b_1, \ldots, b_p)^*$, then

$$x\mathbf{i} + b\mathbf{j} = (x_1, \ldots, x_n, b_1, \ldots, b_p)^* = \begin{pmatrix} x \\ b \end{pmatrix} \in \mathbf{R}^{n+p}.$$

Define $\tilde{F}_0\colon T \times \mathbf{R}^{n+p} \times \mathbf{R}^n \to \mathbf{R}$ by

$$\tilde{F}_0(t, \xi, \dot{x}) := \frac{l_0(\xi_{n+1}, \ldots, \xi_{n+p})}{t_1 - t_0} + F_0(t, \xi_1, \ldots, \xi_n, \dot{x}).$$

Observe that the Weierstrass excess function $\tilde{E}_0\colon T \times \mathbf{R}^{n+p} \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ of $\tilde{F}_0$ is given by

$$\tilde{E}_0(t, \xi, \dot{x}, u) := \tilde{F}_0(t, \xi, u) - \tilde{F}_0(t, \xi, \dot{x}) - \tilde{F}_{0\dot{x}}(t, \xi, \dot{x})(u - \dot{x}).$$

It is clear that for all $(t, x, \dot{x}, u) \in T \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n$ and all $b \in \mathbf{R}^p$,

$$\tilde{E}_0(t, x\mathbf{i} + b\mathbf{j}, \dot{x}, u) = E_0(t, x, \dot{x}, u).$$

Define

$$\tilde{J}_0(x_b) := \langle \rho(t_1), x(t_1)\rangle - \langle \rho(t_0), x(t_0)\rangle + \int_{t_0}^{t_1} \tilde{F}_0(t, x(t)\mathbf{i} + b\mathbf{j}, \dot{x}(t))dt.$$

We have that $J_0(x_b) = \tilde{J}_0(x_b)$ for all $x_b \in \mathcal{A}$, and

$$\tilde{J}_0(x_b) = \tilde{J}_0(x_{0b_0}) + \tilde{J}_0'(x_{0b_0}; x_b - x_{0b_0}) + \tilde{K}_0(x_{0b_0}; x_b) + \tilde{\mathcal{E}}_0(x_{0b_0}; x_b) \tag{1}$$

where

$$\begin{aligned}
\tilde{\mathcal{E}}_0(x_{0b_0}; x_b) &:= \int_{t_0}^{t_1} \tilde{E}_0(t, x(t)\mathbf{i} + b\mathbf{j}, \dot{x}_0(t), \dot{x}(t))dt, \\
\tilde{K}_0(x_{0b_0}; x_b) &:= \int_{t_0}^{t_1} \{\tilde{M}_0(t, x(t)\mathbf{i} + b\mathbf{j}) + \langle \dot{x}(t) - \dot{x}_0(t), \tilde{N}_0(t, x(t)\mathbf{i} + b\mathbf{j})\rangle\}dt, \\
\tilde{J}_0'(x_{0b_0}; x_b - x_{0b_0}) &:= \langle \rho(t_1), x(t_1) - x_0(t_1)\rangle - \langle \rho(t_0), x(t_0) - x_0(t_0)\rangle
\end{aligned}$$

$$+\int_{t_0}^{t_1}\{\tilde F_{0\xi}(t,x_0(t)\mathbf{i}+b_0\mathbf{j},\dot x_0(t))([x(t)-x_0(t)]\mathbf{i}+[b-b_0]\mathbf{j})$$
$$+\tilde F_{0\dot x}(t,x_0(t)\mathbf{i}+b_0\mathbf{j},\dot x_0(t))(\dot x(t)-\dot x_0(t))\}dt,$$

and $\tilde M_0$, $\tilde N_0$ are given by

$$\tilde M_0(t,x\mathbf{i}+b\mathbf{j}) := \tilde F_0(t,x\mathbf{i}+b\mathbf{j},\dot x_0(t))-\tilde F_0(t,x_0(t)\mathbf{i}+b_0\mathbf{j},\dot x_0(t))-\tilde F_{0\xi}(t,x_0(t)\mathbf{i}+b_0\mathbf{j},\dot x_0(t))([x-x_0(t)]\mathbf{i}+[b-b_0]\mathbf{j}),$$

$$\tilde N_0(t,x\mathbf{i}+b\mathbf{j}) := \tilde F^*_{0\dot x}(t,x\mathbf{i}+b\mathbf{j},\dot x_0(t))-\tilde F^*_{0\dot x}(t,x_0(t)\mathbf{i}+b_0\mathbf{j},\dot x_0(t)).$$

We have,

$$\tilde M_0(t,x\mathbf{i}+b\mathbf{j}) = \tfrac{1}{2}\langle [x-x_0(t)]\mathbf{i}+[b-b_0]\mathbf{j},\tilde P_0(t,x\mathbf{i}+b\mathbf{j})([x-x_0(t)]\mathbf{i}+[b-b_0]\mathbf{j})\rangle, \tag{2a}$$

$$\tilde N_0(t,x\mathbf{i}+b\mathbf{j}) = \tilde Q_0(t,x\mathbf{i}+b\mathbf{j})([x-x_0(t)]\mathbf{i}+[b-b_0]\mathbf{j}), \tag{2b}$$

where

$$\tilde P_0(t,x\mathbf{i}+b\mathbf{j}) := 2\int_0^1(1-\lambda)\tilde F_{0\xi\xi}(t,[x_0(t)+\lambda(x-x_0(t))]\mathbf{i}+[b_0+\lambda(b-b_0)]\mathbf{j},\dot x_0(t))d\lambda,$$
$$\tilde Q_0(t,x\mathbf{i}+b\mathbf{j}) := \int_0^1\tilde F_{0\dot x\xi}(t,[x_0(t)+\lambda(x-x_0(t))]\mathbf{i}+[b_0+\lambda(b-b_0)]\mathbf{j},\dot x_0(t))d\lambda.$$

5.1 Lemma: *For some $\nu,\kappa>0$ ($\kappa\le\epsilon$) and any admissible arc x_b satisfying $\|x_b-x_{0b_0}\|<\kappa$,*

$$\tilde{\mathcal{E}}_0(x_{0b_0};x_b) \ge h[D(x-x_0)-V(x(t_0)-x_0(t_0))],$$
$$|\tilde K_0(x_{0b_0};x_b)| \le \nu\|x_b-x_{0b_0}\|[1+D(x-x_0)].$$

Proof: By condition (v)(b) of Theorem 2.1, given x_b admissible with $\|x_b-x_{0b_0}\|<\epsilon$,

$$\tilde{\mathcal{E}}_0(x_{0b_0};x_b) = \int_{t_0}^{t_1}\tilde E_0(t,x(t)\mathbf{i}+b\mathbf{j},\dot x_0(t),\dot x(t))dt = \int_{t_0}^{t_1}E_0(t,x(t),\dot x_0(t),\dot x(t))dt$$
$$\ge h\int_{t_0}^{t_1}V(\dot x(t)-\dot x_0(t))dt = h[D(x-x_0)-V(x(t_0)-x_0(t_0))].$$

On the other hand, by (2) and using $[t,b]$ in order to denote $(t,x(t)\mathbf{i}+b\mathbf{j})$, observe that for some constants $c_0,c_1>0$, for all x_b admissible with $\|x_b-x_{0b_0}\|<1$ and almost all $t\in T$,

$$|\tilde M_0[t,b]+\langle\dot x(t)-\dot x_0(t),\tilde N_0[t,b]\rangle|$$
$$= |\tfrac{1}{2}\langle[x(t)-x_0(t)]\mathbf{i}+[b-b_0]\mathbf{j},\tilde P_0[t,b]([x(t)-x_0(t)]\mathbf{i}+[b-b_0]\mathbf{j})+2\tilde Q_0^*[t,b](\dot x(t)-\dot x_0(t))\rangle|$$
$$\le \tfrac{1}{2}|[x(t)-x_0(t)]\mathbf{i}+[b-b_0]\mathbf{j}|\cdot[|\tilde P_0[t,b]||[x(t)-x_0(t)]\mathbf{i}+[b-b_0]\mathbf{j}|+2|\tilde Q_0^*[t,b]||\dot x(t)-\dot x_0(t)|]$$
$$\le c_0|[x(t)-x_0(t)]\mathbf{i}+[b-b_0]\mathbf{j}|\cdot[|[x(t)-x_0(t)]\mathbf{i}+[b-b_0]\mathbf{j}|+|\dot x(t)-\dot x_0(t)|]$$

$$\begin{aligned}
&\le c_0|[x(t)-x_0(t)]\mathbf{i}+[b-b_0]\mathbf{j}|\cdot[|x(t)-x_0(t)|+|b-b_0|+|\dot{x}(t)-\dot{x}_0(t)|]\\
&\le c_0|[x(t)-x_0(t)]\mathbf{i}+[b-b_0]\mathbf{j}|\cdot[\|x_b-x_{0b_0}\|+|\dot{x}(t)-\dot{x}_0(t)|]\\
&\le c_0|[x(t)-x_0(t)]\mathbf{i}+[b-b_0]\mathbf{j}|\cdot[1+|\dot{x}(t)-\dot{x}_0(t)|]\\
&\le c_1|[x(t)-x_0(t)]\mathbf{i}+[b-b_0]\mathbf{j}|\cdot[1+|\dot{x}(t)-\dot{x}_0(t)|^2]^{1/2}.
\end{aligned}$$

Setting $\nu := \max\{c_1, c_1(t_1-t_0)\}$, for all x_b admissible with $\|x_b - x_{0b_0}\| < 1$,

$$\begin{aligned}
|\tilde{K}_0(x_{0b_0}; x_b)| &\le c_1\|x_b-x_{0b_0}\|\int_{t_0}^{t_1}[1+V(\dot{x}(t)-\dot{x}_0(t))]dt\\
&\le \nu\|x_b-x_{0b_0}\|[1+D(x-x_0)-V(x(t_0)-x_0(t_0))]\\
&\le \nu\|x_b-x_{0b_0}\|[1+D(x-x_0)]
\end{aligned}$$

and so the conclusion of the lemma is obtained with $\kappa = \min\{\epsilon, 1\}$. ∎

5.2 Lemma: *If conclusion of Theorem 2.1 is false, then there exists a subsequence $\{x_{b_q}^q\}$ of admissible arcs such that*

$$\lim_{q\to\infty} D(x^q - x_0) = 0 \quad \textit{and} \quad d_q := [2D(x^q - x_0)]^{1/2} > 0 \quad (q \in \mathbf{N}).$$

Proof: If conclusion of Theorem 2.1 is false, then for all $\theta_1, \theta_2 > 0$, there exists an admissible arc x_b such that

$$\|x_b - x_{0b_0}\| < \theta_1 \quad \text{and} \quad I(x_b) < I(x_{0b_0}) + \theta_2 \min\{|b-b_0|^2, D(x-x_0)\}. \tag{3}$$

Since

$$\mu_\alpha(t) \ge 0\ (\alpha \in R,\ \text{a.e. in}\ T) \quad \text{and} \quad \lambda_i \ge 0\ (i = 1, \ldots, k),$$

if x_b is admissible, then $I(x_b) \ge J_0(x_b)$. Also, since

$$\mu_\alpha(t)\varphi_\alpha(\tilde{x}_0(t)) = 0\ (\alpha \in R, \text{a.e. in}\ T) \quad \text{and} \quad \lambda_i I_i(x_{0b_0}) = 0\ (i = 1, \ldots, k),$$

then $I(x_{0b_0}) = J_0(x_{0b_0})$. Therefore (3) implies that, for all $\theta_1, \theta_2 > 0$, there exists x_b admissible with

$$\|x_b - x_{0b_0}\| < \theta_1 \quad \text{and} \quad J_0(x_b) < J_0(x_{0b_0}) + \theta_2 \min\{|b-b_0|^2, D(x-x_0)\}.$$

Let κ and ν be the positive numbers given in Lemma 5.1. Thus, if conclusion of Theorem 2.1 is false, then for all $q \in \mathbf{N}$, there exists $x_{b_q}^q$ admissible such that

$$\|x_{b_q}^q - x_{0b_0}\| < \min\{\kappa, 1/q\}, \quad J_0(x_{b_q}^q) - J_0(x_{0b_0}) < \min\left\{\frac{|b_q-b_0|^2}{q}, \frac{D(x^q-x_0)}{q}\right\}. \tag{4}$$

Clearly, $D(x^q - x_0) = 0$ if and only if $x^q = x_0$. Then, by the second relation of (4),

$$D(x^q - x_0) = 0 \Longrightarrow b_q \ne b_0.$$

Suppose $D(x^q - x_0) = 0$ for infinitely many q's. For $i = 0, 1$, we have

$$0 = x^q(t_i) - x_0(t_i) = \Psi_i(b_q) - \Psi_i(b_0) = \int_0^1 \Psi_i'(b_0 + \lambda[b_q - b_0])(b_q - b_0)d\lambda, \tag{5}$$

$$0 = \Psi_i(b_q) - \Psi_i(b_0) = \Psi_i'(b_0)(b_q - b_0) + \int_0^1 (1-\lambda)\Psi_i''(b_0 + \lambda[b_q - b_0]; b_q - b_0)d\lambda. \tag{6}$$

Denoting by (b_q, b_0) the line segment in $\mathbf{R}^p$ joining the points b_q and b_0, by the second relation of (4), by condition (i) of Theorem 2.1, by (6), and the mean value theorem, there exists $\Xi_q \in (b_q, b_0)$ such that

$$\begin{aligned} 0 &> J_0(x_{0b_q}) - J_0(x_{0b_0}) \\ &= l_0(b_q) - l_0(b_0) \\ &= l_0'(b_0)(b_q - b_0) + \tfrac{1}{2}\langle l_0''(\Xi_q)(b_q - b_0), b_q - b_0\rangle \\ &= \rho^*(t_0)\Psi_0'(b_0)(b_q - b_0) - \rho^*(t_1)\Psi_1'(b_0)(b_q - b_0) + \tfrac{1}{2}\langle l_0''(\Xi_q)(b_q - b_0), b_q - b_0\rangle \\ &= \sum_{i=0}^{1}(-1)^{i+1}\int_0^1 (1-\lambda)\rho^*(t_i)\Psi_i''(b_0 + \lambda[b_q - b_0]; b_q - b_0)d\lambda + \tfrac{1}{2}\langle l_0''(\Xi_q)(b_q - b_0), b_q - b_0\rangle. \end{aligned} \tag{7}$$

Choose an appropriate subsequence of $\{(b_q - b_0)/|b_q - b_0|\}$ (without relabeling), such that

$$\lim_{q\to\infty} \frac{b_q - b_0}{|b_q - b_0|} = \beta_0 \tag{8}$$

for some $\beta_0 \in \mathbf{R}^p$ with $|\beta_0| = 1$. By (5),

$$\Psi_i'(b_0)\beta_0 = 0 \quad (i = 0, 1). \tag{9}$$

For all $(t, \xi, \dot{x}) \in T \times \mathbf{R}^{n+p} \times \mathbf{R}^n$ and $\gamma = 1, \ldots, K$, if we set

$$\tilde{L}_\gamma(t, \xi, \dot{x}) := \frac{l_\gamma(\xi_{n+1}, \ldots, \xi_{n+p})}{t_1 - t_0} + L_\gamma(t, \xi_1, \ldots, \xi_n, \dot{x}),$$

and for all $(t, \xi, \dot{x}, u) \in T \times \mathbf{R}^{n+p} \times \mathbf{R}^n \times \mathbf{R}^n$ and $\gamma = 1, \ldots, K$, if we set

$$\tilde{E}_\gamma(t, \xi, \dot{x}, u) := \tilde{L}_\gamma(t, \xi, u) - \tilde{L}_\gamma(t, \xi, \dot{x}) - \tilde{L}_{\gamma\dot{x}}(t, \xi, \dot{x})(u - \dot{x}),$$

we have that for all $x_b \in \mathcal{A}$ and $\gamma = 1, \ldots, K$,

$$\tilde{I}_\gamma(x_b) = \tilde{I}_\gamma(x_{0b_0}) + \tilde{I}_\gamma'(x_{0b_0}; x_b - x_{0b_0}) + \tilde{K}_\gamma(x_{0b_0}; x_b) + \tilde{\mathcal{E}}_\gamma(x_{0b_0}; x_b)$$

where

$$\begin{aligned} \tilde{\mathcal{E}}_\gamma(x_{0b_0}; x_b) &:= \int_{t_0}^{t_1} \tilde{E}_\gamma(t, x(t)\mathbf{i} + b\mathbf{j}, \dot{x}_0(t), \dot{x}(t))dt, \\ \tilde{K}_\gamma(x_{0b_0}; x_b) &:= \int_{t_0}^{t_1} \{\tilde{M}_\gamma(t, x(t)\mathbf{i} + b\mathbf{j}) + \langle \dot{x}(t) - \dot{x}_0(t), \tilde{N}_\gamma(t, x(t)\mathbf{i} + b\mathbf{j})\rangle\}dt, \end{aligned}$$

$$\begin{aligned}\tilde{I}'_\gamma(x_{0b_0}; x_b - x_{0b_0}) &:= \int_{t_0}^{t_1} \{\tilde{L}_{\gamma\xi}(t, x_0(t)\mathbf{i} + b_0\mathbf{j}, \dot{x}_0(t))([x(t) - x_0(t)]\mathbf{i} + [b - b_0]\mathbf{j}) \\ &\qquad + \tilde{L}_{\gamma\dot{x}}(t, x_0(t)\mathbf{i} + b_0\mathbf{j}, \dot{x}_0(t))(\dot{x}(t) - \dot{x}_0(t))\}dt, \\ \tilde{I}_\gamma(x_b) &:= \int_{t_0}^{t_1} \tilde{L}_\gamma(t, x(t)\mathbf{i} + b\mathbf{j}, \dot{x}(t))dt,\end{aligned}$$

and $\tilde{M}_\gamma$, $\tilde{N}_\gamma$ are given by

$$\tilde{M}_\gamma(t, x\mathbf{i}+b\mathbf{j}) := \tilde{L}_\gamma(t, x\mathbf{i}+b\mathbf{j}, \dot{x}_0(t)) - \tilde{L}_\gamma(t, x_0(t)\mathbf{i}+b_0\mathbf{j}, \dot{x}_0(t)) - \tilde{L}_{\gamma\xi}(t, x_0(t)\mathbf{i}+b_0\mathbf{j}, \dot{x}_0(t))([x - x_0(t)]\mathbf{i}+[b - b_0]\mathbf{j}),$$

$$\tilde{N}_\gamma(t, x\mathbf{i} + b\mathbf{j}) := \tilde{L}^*_{\gamma\dot{x}}(t, x\mathbf{i} + b\mathbf{j}, \dot{x}_0(t)) - \tilde{L}^*_{\gamma\dot{x}}(t, x_0(t)\mathbf{i} + b_0\mathbf{j}, \dot{x}_0(t)).$$

We have

$$\tilde{M}_\gamma(t, x\mathbf{i} + b\mathbf{j}) = \tfrac{1}{2}\langle [x - x_0(t)]\mathbf{i} + [b - b_0]\mathbf{j}, \tilde{P}_\gamma(t, x\mathbf{i} + b\mathbf{j})([x - x_0(t)]\mathbf{i} + [b - b_0]\mathbf{j})\rangle,$$

$$\tilde{N}_\gamma(t, x\mathbf{i} + b\mathbf{j}) = \tilde{Q}_\gamma(t, x\mathbf{i} + b\mathbf{j})([x - x_0(t)]\mathbf{i} + [b - b_0]\mathbf{j}),$$

where

$$\begin{aligned}\tilde{P}_\gamma(t, x\mathbf{i} + b\mathbf{j}) &:= 2\int_0^1 (1 - \lambda)\tilde{L}_{\gamma\xi\xi}(t, [x_0(t) + \lambda(x - x_0(t))]\mathbf{i} + [b_0 + \lambda(b - b_0)]\mathbf{j}, \dot{x}_0(t))d\lambda, \\ \tilde{Q}_\gamma(t, x\mathbf{i} + b\mathbf{j}) &:= \int_0^1 \tilde{L}_{\gamma\dot{x}\xi}(t, [x_0(t) + \lambda(x - x_0(t))]\mathbf{i} + [b_0 + \lambda(b - b_0)]\mathbf{j}, \dot{x}_0(t))d\lambda.\end{aligned}$$

Since x_{0b_q} and x_{0b_0} are admissible, for all $i \in i_a(x_{0b_0})$, we have

$$\begin{aligned}0 &\geq I_i(x_{0b_q}) \\ &= I_i(x_{0b_q}) - I_i(x_{0b_0}) \\ &= \tilde{I}_i(x_{0b_q}) - \tilde{I}_i(x_{0b_0}) \\ &= \tilde{I}'_i(x_{0b_0}; x_{0b_q} - x_{0b_0}) + \tilde{K}_i(x_{0b_0}; x_{0b_q}) + \tilde{\mathcal{E}}_i(x_{0b_0}; x_{0b_q}) \\ &= I'_i(x_{0b_0}; x_{0b_q} - x_{0b_0}) + \tilde{K}_i(x_{0b_0}; x_{0b_q}) \\ &= l'_i(b_0)(b_q - b_0) + \tilde{K}_i(x_{0b_0}; x_{0b_q}).\end{aligned}$$

As one readily verifies, for all $\gamma = 1, \ldots, K$,

$$\lim_{q\to\infty} \frac{\tilde{K}_\gamma(x_{0b_0}; x_{0b_q})}{|b_q - b_0|} = 0.$$

Then, for $i \in i_a(x_{0b_0})$,

$$0 \geq l'_i(b_0)\beta_0 = I'_i(x_{0b_0}; 0_{\beta_0}). \tag{10}$$

On the other hand, once again since x_{0b_q} and x_{0b_0} are admissible, for all $j = k + 1, \ldots, K$, we have

$$0 = I_j(x_{0b_q}) - I_j(x_{0b_0})$$

$$\begin{aligned}
&= \tilde{I}_j(x_{0b_q}) - \tilde{I}_j(x_{0b_0}) \\
&= \tilde{I}'_j(x_{0b_0}; x_{0b_q} - x_{0b_0}) + \tilde{K}_j(x_{0b_0}; x_{0b_q}) + \tilde{\mathcal{E}}_j(x_{0b_0}; x_{0b_q}) \\
&= I'_j(x_{0b_0}; x_{0b_q} - x_{0b_0}) + \tilde{K}_j(x_{0b_0}; x_{0b_q}) \\
&= l'_j(b_0)(b_q - b_0) + \tilde{K}_j(x_{0b_0}; x_{0b_q}).
\end{aligned}$$

Then, for $j = k+1, \ldots, K$,

$$0 = l'_j(b_0)\beta_0 = I'_j(x_{0b_0}; 0_{\beta_0}). \tag{11}$$

Consequently, by (9), (10) and (11), $0_{\beta_0} \in \mathcal{Y}(x_{0b_0})$.

By (7), (8) and condition (ii) of Theorem 2.1, it follows that

$$\begin{aligned}
0 &\geq \tfrac{1}{2}[\rho^*(t_1)\Psi''_1(b_0;\beta_0) - \rho^*(t_0)\Psi''_0(b_0;\beta_0) + \langle l''_0(b_0)\beta_0, \beta_0\rangle] \\
&\geq \tfrac{1}{2}\langle l''_0(b_0)\beta_0, \beta_0\rangle \\
&= \tfrac{1}{2}J''_0(x_{0b_0}; 0_{\beta_0})
\end{aligned}$$

which contradicts (iv) of Theorem 2.1. Therefore, we may assume that for all $q \in \mathbf{N}$,

$$d_q = [2D(x^q - x_0)]^{1/2} > 0.$$

Since (x_0, ρ, μ) is an extremal, for all $q \in \mathbf{N}$,

$$\tilde{J}'_0(x_{0b_0}; x^q_{b_q} - x_{0b_0}) = \langle \rho(t_1), x^q(t_1) - x_0(t_1)\rangle - \langle \rho(t_0), x^q(t_0) - x_0(t_0)\rangle + l'_0(b_0)(b_q - b_0)$$

and thus

$$\lim_{q\to\infty} \tilde{J}'_0(x_{0b_0}; x^q_{b_q} - x_{0b_0}) = 0. \tag{12}$$

By (1), the first relation of (4) and Lemma 5.1, for all $q \in \mathbf{N}$,

$$\begin{aligned}
\tilde{J}_0(x^q_{b_q}) - \tilde{J}_0(x_{0b_0}) &= \tilde{J}'_0(x_{0b_0}; x^q_{b_q} - x_{0b_0}) + \tilde{K}_0(x_{0b_0}; x^q_{b_q}) + \tilde{\mathcal{E}}_0(x_{0b_0}; x^q_{b_q}) \\
&\geq \tilde{J}'_0(x_{0b_0}; x^q_{b_q} - x_{0b_0}) - \nu\|x^q_{b_q} - x_{0b_0}\| \\
&\quad + D(x^q - x_0)(h - \nu\|x^q_{b_q} - x_{0b_0}\|) - hV(x^q(t_0) - x_0(t_0)),
\end{aligned}$$

then, by (4), for all $q \in \mathbf{N}$,

$$D(x^q - x_0)\Big(h - \frac{\nu}{q} - \frac{1}{q}\Big) < \frac{\nu}{q} + hV(x^q(t_0) - x_0(t_0)) - \tilde{J}'_0(x_{0b_0}; x^q_{b_q} - x_{0b_0}).$$

By (12),

$$\lim_{q\to\infty} D(x^q - x_0) = 0. \blacksquare$$

5.3 Lemma: *If conclusion of Theorem 2.1 is false, then condition (iv) of Theorem 2.1 is false.*

Proof: Let $\{x^q_{b_q}\}$ be the sequence of admissible arcs given in Lemma 5.2. Then,

$$\lim_{q\to\infty} D(x^q - x_0) = 0 \quad \text{and} \quad d_q = [2D(x^q - x_0)]^{1/2} > 0 \quad (q \in \mathbf{N}).$$

Case (1): Suppose first that the sequence $\{(b_q - b_0)/d_q\}$ is bounded in $\mathbf{R}^p$.

For all $q \in \mathbf{N}$ and $t \in T$, define

$$y_q(t) := \frac{x^q(t) - x_0(t)}{d_q} \quad \text{and} \quad \omega_q(t) := y_q(t)\mathbf{i} + \frac{b_q - b_0}{d_q}\mathbf{j}.$$

By Lemma 4.1, for some $y_0 \in \mathcal{X}$ with $\dot y_0 \in L^2(T;\mathbf{R}^n)$ and a subsequence of $\{x^q\}$, again denoted by $\{x^q\}$, $\{\dot y_q\}$ converges weakly in $L^1(T;\mathbf{R}^n)$ to $\dot y_0$. Once again, by Lemma 4.1,

$$\lim_{q\to\infty} y_q(t) = y_0(t) \text{ uniformly on } T. \tag{13}$$

Since the sequence $\{(b_q - b_0)/d_q\}$ is bounded in $\mathbf{R}^p$, then we may assume that there exists some $\beta_0 \in \mathbf{R}^p$ such that

$$\lim_{q\to\infty} \frac{b_q - b_0}{d_q} = \beta_0. \tag{14}$$

First, we are going to show that for $i = 0, 1$,

$$y_0(t_i) = \Psi_i'(b_0)\beta_0. \tag{15}$$

Note first that for $i = 0, 1$ and all $q \in \mathbf{N}$, we have that

$$y_q(t_i) = \int_0^1 \Psi_i'(b_0 + \lambda[b_q - b_0])\frac{(b_q - b_0)}{d_q}d\lambda. \tag{16}$$

By (13), (14) and (16), we obtain (15). Now, we claim that

$$J_0''(x_{0b_0}; y_{0\beta_0}) \le 0 \quad \text{and} \quad y_{0\beta_0} \not\equiv (0, 0). \tag{17}$$

To prove it, observe that by (2), (13) and (14),

$$\begin{aligned}
\lim_{q\to\infty} \frac{\tilde M_0(t, x^q(t)\mathbf{i} + b_q\mathbf{j})}{d_q^2} &= \lim_{q\to\infty} \tfrac{1}{2}\langle \omega_q(t), \tilde P_0(t, x^q(t)\mathbf{i} + b_q\mathbf{j})\omega_q(t)\rangle \\
&= \tfrac{1}{2}\langle y_0(t)\mathbf{i} + \beta_0\mathbf{j}, \tilde F_{0\xi\xi}(t, x_0(t)\mathbf{i} + b_0\mathbf{j}, \dot x_0(t))[y_0(t)\mathbf{i} + \beta_0\mathbf{j}]\rangle,
\end{aligned}$$

$$\lim_{q\to\infty} \frac{\tilde N_0(t, x^q(t)\mathbf{i} + b_q\mathbf{j})}{d_q} = \lim_{q\to\infty} \tilde Q_0(t, x^q(t)\mathbf{i} + b_q\mathbf{j})\omega_q(t) = \tilde F_{0\dot x\xi}(t, x_0(t)\mathbf{i} + b_0\mathbf{j}, \dot x_0(t))[y_0(t)\mathbf{i} + \beta_0\mathbf{j}]$$

both uniformly on T. With this in mind and since $\{\dot y_q\}$ converges weakly in $L^1(T;\mathbf{R}^n)$ to $\dot y_0$, we have

$$\lim_{q\to\infty} \frac{\tilde K_0(x_{0b_0}; x^q_{b_q})}{d_q^2} = \frac{1}{2}\int_{t_0}^{t_1} \{\langle y_0(t)\mathbf{i} + \beta_0\mathbf{j}, \tilde F_{0\xi\xi}(t, x_0(t)\mathbf{i} + b_0\mathbf{j}, \dot x_0(t))[y_0(t)\mathbf{i} + \beta_0\mathbf{j}]\rangle$$

$$+2\langle \dot y_0(t), \tilde F_{0\dot x\xi}(t, x_0(t)\mathbf{i} + b_0\mathbf{j}, \dot x_0(t))[y_0(t)\mathbf{i} + \beta_0\mathbf{j}]\rangle\}dt. \quad (18)$$

Since (x_0, ρ, μ) is an extremal and by condition (i) of Theorem 2.1, it follows that

$$\begin{aligned}
&\lim_{q\to\infty} \frac{\tilde J_0'(x_{0b_0}; x^q_{b_q} - x_{0b_0})}{d_q^2} \\
= &\lim_{q\to\infty} \frac{1}{d_q^2}[\langle \rho(t_1), x^q(t_1) - x_0(t_1)\rangle - \langle \rho(t_0), x^q(t_0) - x_0(t_0)\rangle + l_0'(b_0)(b_q - b_0)] \\
= &\lim_{q\to\infty} \frac{1}{d_q^2}[\rho^*(t_1)(\Psi_1(b_q) - \Psi_1(b_0) - \Psi_1'(b_0)(b_q - b_0)) \\
&\qquad -\rho^*(t_0)(\Psi_0(b_q) - \Psi_0(b_0) - \Psi_0'(b_0)(b_q - b_0))] \\
= &\lim_{q\to\infty} \frac{1}{d_q^2} \int_0^1 \sum_{i=0}^{1} (-1)^{i+1}(1-\lambda)\rho^*(t_i)\Psi_i''(b_0 + \lambda[b_q - b_0]; b_q - b_0)d\lambda \\
= &\tfrac{1}{2}[\rho^*(t_1)\Psi_1''(b_0;\beta_0) - \rho^*(t_0)\Psi_0''(b_0;\beta_0)]. \quad (19)
\end{aligned}$$

Consequently, by (1), (4), (19), and condition (ii) of Theorem 2.1,

$$0 \ge \lim_{q\to\infty} \frac{\tilde K_0(x_{0b_0}; x^q_{b_q})}{d_q^2} + \liminf_{q\to\infty} \frac{\tilde{\mathcal{E}}_0(x_{0b_0}; x^q_{b_q})}{d_q^2}. \quad (20)$$

Now, let us show that

$$\liminf_{q\to\infty} \frac{\tilde{\mathcal{E}}_0(x_{0b_0}; x^q_{b_q})}{d_q^2} \ge \frac{1}{2}\int_{t_0}^{t_1} \langle \dot y_0(t), \tilde F_{0\dot x\dot x}(t, x_0(t)\mathbf{i} + b_0\mathbf{j}, \dot x_0(t))\dot y_0(t)\rangle dt. \quad (21)$$

To this end, let U a measurable subset of T such that $\dot x^q(t) \to \dot x_0(t)$ uniformly on U. For all $q \in \mathbf{N}$ and $t \in U$, we have that

$$\frac{1}{d_q^2}\tilde E_0(t, x^q(t)\mathbf{i} + b_q\mathbf{j}, \dot x_0(t), \dot x^q(t)) = \tfrac{1}{2}\langle \dot y_q(t), R_q(t)\dot y_q(t)\rangle,$$

where

$$R_q(t) := 2\int_0^1 (1-\lambda)\tilde F_{0\dot x\dot x}(t, x^q(t)\mathbf{i} + b_q\mathbf{j}, \dot x_0(t) + \lambda[\dot x^q(t) - \dot x_0(t)])d\lambda.$$

Clearly,

$$\lim_{q\to\infty} R_q(t) = R_0(t) := \tilde F_{0\dot x\dot x}(t, x_0(t)\mathbf{i} + b_0\mathbf{j}, \dot x_0(t)) \text{ uniformly on } U.$$

By condition (iii) of Theorem 2.1, we have

$$\tilde F_{0\dot x\dot x}(t, x_0(t)\mathbf{i} + b_0\mathbf{j}, \dot x_0(t)) = R_0(t) \ge 0 \quad (t \in U).$$

With this in mind, and since by (v)(a) of Theorem 2.1 for all $q \in \mathbf{N}$,

$$E_0(t, x^q(t), \dot x_0(t), \dot x^q(t)) \ge 0 \quad \text{(a.e. in } T),$$

by Lemma 4.2,

$$\begin{aligned}
\liminf_{q\to\infty} \frac{\tilde{\mathcal{E}}_0(x_{0b_0}; x^q_{b_q})}{d_q^2} &= \liminf_{q\to\infty} \frac{1}{d_q^2} \int_{t_0}^{t_1} \tilde{E}_0(t, x^q(t)\mathbf{i} + b_q\mathbf{j}, \dot{x}_0(t), \dot{x}^q(t))dt \\
&= \liminf_{q\to\infty} \frac{1}{d_q^2} \int_{t_0}^{t_1} E_0(t, x^q(t), \dot{x}_0(t), \dot{x}^q(t))dt \\
&\geq \liminf_{q\to\infty} \frac{1}{d_q^2} \int_U E_0(t, x^q(t), \dot{x}_0(t), \dot{x}^q(t))dt \\
&= \liminf_{q\to\infty} \frac{1}{d_q^2} \int_U \tilde{E}_0(t, x^q(t)\mathbf{i} + b_q\mathbf{j}, \dot{x}_0(t), \dot{x}^q(t))dt \\
&= \frac{1}{2} \liminf_{q\to\infty} \int_U \langle \dot{y}_q(t), R_q(t)\dot{y}_q(t)\rangle dt \geq \frac{1}{2} \int_U \langle \dot{y}_0(t), R_0(t)\dot{y}_0(t)\rangle dt.
\end{aligned}$$

As U can be chosen to differ from T by a set of an arbitrarily small measure and the function

$$t \mapsto \langle \dot{y}_0(t), R_0(t)\dot{y}_0(t)\rangle$$

belongs to $L^1(T; \mathbf{R})$, this inequality holds when $U = T$, and this establishes (21). With this in mind, by (18) and (20), we have

$$\begin{aligned}
0 &\geq \int_{t_0}^{t_1} \{\langle \dot{y}_0(t), \tilde{F}_{0\dot{x}\dot{x}}(t, x_0(t)\mathbf{i} + b_0\mathbf{j}, \dot{x}_0(t))\dot{y}_0(t)\rangle + 2\langle \dot{y}_0(t), \tilde{F}_{0\dot{x}\xi}(t, x_0(t)\mathbf{i} + b_0\mathbf{j}, \dot{x}_0(t))[y_0(t)\mathbf{i} + \beta_0\mathbf{j}]\rangle \\
&\qquad + \langle y_0(t)\mathbf{i} + \beta_0\mathbf{j}, \tilde{F}_{0\xi\xi}(t, x_0(t)\mathbf{i} + b_0\mathbf{j}, \dot{x}_0(t))[y_0(t)\mathbf{i} + \beta_0\mathbf{j}]\rangle\}dt \\
&= \langle l_0''(b_0)\beta_0, \beta_0\rangle \\
&\qquad + \int_{t_0}^{t_1} \{\langle \dot{y}_0(t), F_{0\dot{x}\dot{x}}(\tilde{x}_0(t))\dot{y}_0(t)\rangle + 2\langle \dot{y}_0(t), F_{0\dot{x}x}(\tilde{x}_0(t))y_0(t)\rangle + \langle y_0(t), F_{0xx}(\tilde{x}_0(t))y_0(t)\rangle\}dt \\
&= \langle l_0''(b_0)\beta_0, \beta_0\rangle + \int_{t_0}^{t_1} 2\Omega_0(x_0; t, y_0(t), \dot{y}_0(t))dt = J_0''(x_{0b_0}; y_{0\beta_0}).
\end{aligned}$$

Now, let us show that $y_{0\beta_0} \not\equiv (0, 0)$. By (20), the first conclusion of Lemma 5.1, the fact that $V(\pi) \leq |\pi|^2/2$ for all $\pi \in \mathbf{R}^n$,

$$\begin{aligned}
0 &\geq \lim_{q\to\infty} \frac{\tilde{K}_0(x_{0b_0}; x^q_{b_q})}{d_q^2} + \frac{h}{2} - \frac{h}{2} \limsup_{q\to\infty} \frac{|x^q(t_0) - x_0(t_0)|^2}{d_q^2} \\
&= \lim_{q\to\infty} \frac{\tilde{K}_0(x_{0b_0}; x^q_{b_q})}{d_q^2} + \frac{h}{2} - \frac{h}{2} \limsup_{q\to\infty} \frac{|\Psi_0(b_q) - \Psi_0(b_0)|^2}{d_q^2} \\
&= \lim_{q\to\infty} \frac{\tilde{K}_0(x_{0b_0}; x^q_{b_q})}{d_q^2} + \frac{h}{2} - \frac{h}{2} \limsup_{q\to\infty} \left| \int_0^1 \Psi_0'(b_0 + \lambda[b_q - b_0])\left(\frac{b_q - b_0}{d_q}\right)d\lambda \right|^2 \\
&= \lim_{q\to\infty} \frac{\tilde{K}_0(x_{0b_0}; x^q_{b_q})}{d_q^2} + \frac{h}{2} - \frac{h}{2}|\Psi_0'(b_0)\beta_0|^2
\end{aligned}$$

$$= \lim_{q\to\infty} \frac{\tilde{K}_0(x_{0b_0}; x^q_{b_q})}{d_q^2} + \frac{h}{2} - \frac{h}{2}|y_0(t_0)|^2.$$

With this in mind and (18), the fact that $y_{0\beta_0} \equiv (0,0)$ contradicts the positivity of h and this establishes (17).

Let us now show that

$$I_i'(x_{0b_0}; y_{0\beta_0}) \leq 0 \quad (i \in i_a(x_{0b_0})). \tag{22}$$

To this end, note that, for all $\gamma = 0, 1, \ldots, K$,

$$\lim_{q\to\infty} \frac{\tilde{M}_\gamma(t, x^q(t)\mathbf{i} + b_q\mathbf{j})}{d_q} = \lim_{q\to\infty} \tfrac{1}{2}\langle [x^q(t) - x_0(t)]\mathbf{i} + [b_q - b_0]\mathbf{j}, \tilde{P}_\gamma(t, x^q(t)\mathbf{i} + b_q\mathbf{j})\omega_q(t)\rangle = 0,$$

$$\lim_{q\to\infty} \tilde{N}_\gamma(t, x^q(t)\mathbf{i} + b_q\mathbf{j}) = \lim_{q\to\infty} \tilde{Q}_\gamma(t, x^q(t)\mathbf{i} + b_q\mathbf{j})([x^q(t) - x_0(t)]\mathbf{i} + [b_q - b_0]\mathbf{j}) = 0,$$

all uniformly on T and $\{\dot{y}_q\}$ converges weakly in $L^1(T; \mathbf{R}^n)$ to $\dot{y}_0$, then for all $\gamma = 0, 1, \ldots, K$,

$$\lim_{q\to\infty} \frac{\tilde{K}_\gamma(x_{0b_0}; x^q_{b_q})}{d_q} = 0. \tag{23}$$

As in (19), we have

$$\begin{aligned} &\lim_{q\to\infty} \frac{\tilde{J}_0'(x_{0b_0}; x^q_{b_q} - x_{0b_0})}{d_q} \\ &= \lim_{q\to\infty} \frac{1}{d_q} \sum_{i=0}^{1} (-1)^{i+1} \int_0^1 (1-\lambda)\rho^*(t_i)\Psi_i''(b_0 + \lambda[b_q - b_0]; b_q - b_0)d\lambda \\ &= 0. \end{aligned} \tag{24}$$

By (4), (23) and (24),

$$0 \geq \limsup_{q\to\infty} \frac{\tilde{J}_0(x^q_{b_q}) - \tilde{J}_0(x_{0b_0})}{d_q} = \limsup_{q\to\infty} \frac{\tilde{\mathcal{E}}_0(x_{0b_0}; x^q_{b_q})}{d_q}.$$

Since $\tilde{\mathcal{E}}_0(x_{0b_0}; x^q_{b_q}) \geq 0$ $(q \in \mathbf{N})$, then

$$\lim_{q\to\infty} \frac{\tilde{\mathcal{E}}_0(x_{0b_0}; x^q_{b_q})}{d_q} = 0.$$

Therefore, by condition (v)(c) of Theorem 2.1, for all $\gamma = 1, \ldots, K$,

$$\lim_{q\to\infty} \frac{\tilde{\mathcal{E}}_\gamma(x_{0b_0}; x^q_{b_q})}{d_q} = 0. \tag{25}$$

As for all $q \in \mathbf{N}$ and $i \in i_a(x_{0b_0})$,

$$\begin{aligned} 0 &\geq I_i(x^q_{b_q}) \\ &= I_i(x^q_{b_q}) - I_i(x_{0b_0}) \\ &= \tilde{I}_i(x^q_{b_q}) - \tilde{I}_i(x_{0b_0}) \\ &= \tilde{I}'_i(x_{0b_0}; x^q_{b_q} - x_{0b_0}) + \tilde{K}_i(x_{0b_0}; x^q_{b_q}) + \tilde{\mathcal{E}}_i(x_{0b_0}; x^q_{b_q}) \\ &= I'_i(x_{0b_0}; x^q_{b_q} - x_{0b_0}) + \tilde{K}_i(x_{0b_0}; x^q_{b_q}) + \tilde{\mathcal{E}}_i(x_{0b_0}; x^q_{b_q}), \end{aligned}$$

then, by (23) and (25), for all $i \in i_a(x_{0b_0})$,

$$0 \geq \lim_{q\to\infty} \frac{I'_i(x_{0b_0}; x^q_{b_q} - x_{0b_0})}{d_q}.$$

Therefore, since $y_q(t) \to y_0(t)$ uniformly on T, $\{\dot{y}_q\}$ converges weakly in $L^1(T; \mathbf{R}^n)$ to $\dot{y}_0$ and $(b_q - b_0)/d_q \to \beta_0$, then for all $i \in i_a(x_{0b_0})$,

$$0 \geq \lim_{q\to\infty} \frac{I'_i(x_{0b_0}; x^q_{b_q} - x_{0b_0})}{d_q} = I'_i(x_{0b_0}; y_{0\beta_0})$$

which establishes (22).

Now, let us show that

$$I'_j(x_{0b_0}; y_{0\beta_0}) = 0 \quad (j = k+1, \ldots, K). \tag{26}$$

Indeed, as for all $q \in \mathbf{N}$ and $j = k+1, \ldots, K$,

$$\begin{aligned} 0 &= I_j(x^q_{b_q}) - I_j(x_{0b_0}) \\ &= \tilde{I}_j(x^q_{b_q}) - \tilde{I}_j(x_{0b_0}) \\ &= \tilde{I}'_j(x_{0b_0}; x^q_{b_q} - x_{0b_0}) + \tilde{K}_j(x_{0b_0}; x^q_{b_q}) + \tilde{\mathcal{E}}_j(x_{0b_0}; x^q_{b_q}) \\ &= I'_j(x_{0b_0}; x^q_{b_q} - x_{0b_0}) + \tilde{K}_j(x_{0b_0}; x^q_{b_q}) + \tilde{\mathcal{E}}_j(x_{0b_0}; x^q_{b_q}), \end{aligned}$$

by (23) and (25), for all $j = k+1, \ldots, K$,

$$0 = \lim_{q\to\infty} \frac{I'_j(x_{0b_0}; x^q_{b_q} - x_{0b_0})}{d_q} = I'_j(x_{0b_0}; y_{0\beta_0})$$

which is precisely (26).

We claim that, for all $\alpha \in \mathcal{I}_a(\tilde{x}_0(t))$,

$$\varphi_{\alpha x}(\tilde{x}_0(t))y_0(t) + \varphi_{\alpha\dot{x}}(\tilde{x}_0(t))\dot{y}_0(t) \leq 0 \quad \text{(a.e. in } T\text{)}. \tag{27}$$

Indeed, for all $\alpha \in R$, $q \in \mathbf{N}$, almost all $t \in T$ and $\lambda \in [0, 1]$, define

$$G^\alpha_q(t; \lambda) := \varphi_\alpha(t, x_0(t) + \lambda[x^q(t) - x_0(t)], \dot{x}_0(t) + \lambda[\dot{x}^q(t) - \dot{x}_0(t)]),$$

$$\mathcal{W}_q^\alpha(t) := [-\varphi_\alpha(\tilde{x}^q(t))]^{1/2},$$

$$Z_0^\alpha(t) := -\varphi_{\alpha x}(\tilde{x}_0(t))y_0(t) - \varphi_{\alpha\dot{x}}(\tilde{x}_0(t))\dot{y}_0(t),$$

where as usual $(\tilde{x}^q(t)) := (t, x^q(t), \dot{x}^q(t))$ (a.e. in T). Given $t \in [t_0, t_1)$ a point of continuity of $\mathcal{I}_a(\tilde{x}_0(\cdot))$ and $\alpha \in \mathcal{I}_a(\tilde{x}_0(t))$, since $\mathcal{I}_a(\tilde{x}_0(\cdot))$ is piecewise constant on T, there exists an interval $[t, \bar{t}] \subset T$ with $t < \bar{t}$ such that $\varphi_\alpha(\tilde{x}_0(\tau)) = 0$ for all $\tau \in [t, \bar{t}]$. Using the notation

$$[\tau] := (\tau, x_0(\tau) + \lambda[x^q(\tau) - x_0(\tau)], \dot{x}_0(\tau) + \lambda[\dot{x}^q(\tau) - \dot{x}_0(\tau)]),$$

we have

$$\begin{aligned}
0 &\le \lim_{q\to\infty} \int_{[t,\bar{t}]\cap U} \frac{(\mathcal{W}_q^\alpha(\tau))^2}{d_q} d\tau = \lim_{q\to\infty} \frac{1}{d_q} \int_{[t,\bar{t}]\cap U} \{-\varphi_\alpha(\tilde{x}^q(\tau)) + \varphi_\alpha(\tilde{x}_0(\tau))\} d\tau \\
&= -\lim_{q\to\infty} \frac{1}{d_q} \int_{[t,\bar{t}]\cap U} \{G_q^\alpha(\tau; 1) - G_q^\alpha(\tau; 0)\} d\tau = -\lim_{q\to\infty} \frac{1}{d_q} \int_{[t,\bar{t}]\cap U} \int_0^1 \frac{\partial}{\partial\lambda} G_q^\alpha(\tau; \lambda) d\lambda d\tau \\
&= -\lim_{q\to\infty} \frac{1}{d_q} \int_{[t,\bar{t}]\cap U} \int_0^1 \{\varphi_{\alpha x}[\tau](x^q(\tau) - x_0(\tau)) + \varphi_{\alpha\dot{x}}[\tau](\dot{x}^q(\tau) - \dot{x}_0(\tau))\} d\lambda d\tau \\
&= -\lim_{q\to\infty} \int_{[t,\bar{t}]\cap U} \int_0^1 \{\varphi_{\alpha x}[\tau] y_q(\tau) + \varphi_{\alpha\dot{x}}[\tau] \dot{y}_q(\tau)\} d\lambda d\tau \\
&= \int_{[t,\bar{t}]\cap U} \{-\varphi_{\alpha x}(\tilde{x}_0(\tau))y_0(\tau) - \varphi_{\alpha\dot{x}}(\tilde{x}_0(\tau))\dot{y}_0(\tau)\} d\tau = \int_{[t,\bar{t}]\cap U} Z_0^\alpha(\tau) d\tau.
\end{aligned}$$

Since U can be chosen to differ from T by a set of an arbitrarily small measure, then

$$0 \le \int_t^{\bar{t}} Z_0^\alpha(\tau) d\tau.$$

If $Z_0^\alpha(\tau) < 0$ on a measurable set Θ such that $\Theta \subset [t, \bar{t}]$ and $m(\Theta) > 0$, then

$$0 > \int_{\Theta\cap U} Z_0^\alpha(\tau) d\tau = \lim_{q\to\infty} \int_{\Theta\cap U} \frac{(\mathcal{W}_q^\alpha(\tau))^2}{d_q} d\tau \ge 0$$

which is a contradiction. Therefore, $Z_0^\alpha(\tau) \ge 0$ a.e. in $[t, \bar{t}]$ with $t \in [t_0, t_1)$ an arbitrary point of continuity of $\mathcal{I}_a(\tilde{x}_0(\cdot))$ and, therefore, $Z_0^\alpha(t) \ge 0$ for almost all $t \in T$ which shows (27).

Now, let us show that for all $\beta \in S$,

$$\varphi_{\beta x}(\tilde{x}_0(t))y_0(t) + \varphi_{\beta\dot{x}}(\tilde{x}_0(t))\dot{y}_0(t) = 0 \text{ (a.e. in } T). \tag{28}$$

Indeed, for all $\beta \in S$, $q \in \mathbf{N}$, almost all $t \in T$ and $\lambda \in [0, 1]$, define

$$\mathcal{H}_q^\beta(t; \lambda) := \varphi_\beta(t, x_0(t) + \lambda[x^q(t) - x_0(t)], \dot{x}_0(t) + \lambda[\dot{x}^q(t) - \dot{x}_0(t)]).$$

For all $\beta \in S$, $q \in \mathbf{N}$ and almost all $t \in T$, we have

$$0 = \mathcal{H}_q^\beta(t; 1) - \mathcal{H}_q^\beta(t; 0) = \int_0^1 \frac{\partial}{\partial\lambda} \mathcal{H}_q^\beta(t; \lambda) d\lambda$$

$$= \int_0^1 \{\varphi_{\beta x}[t](x^q(t) - x_0(t)) + \varphi_{\beta \dot{x}}[t](\dot{x}^q(t) - \dot{x}_0(t))\}d\lambda.$$

Therefore, for all $\beta \in S$, $q \in \mathbf{N}$ and almost all $t \in T$,

$$0 = \int_0^1 \{\varphi_{\beta x}[t]y_q(t) + \varphi_{\beta \dot{x}}[t]\dot{y}_q(t)\}d\lambda. \tag{29}$$

By (29), for all $t \in T$ and $\beta \in S$,

$$0 = \int_{[t_0,t]\cap U} \{\varphi_{\beta x}(\tilde{x}_0(\tau))y_0(\tau) + \varphi_{\beta \dot{x}}(\tilde{x}_0(\tau))\dot{y}_0(\tau)\}d\tau.$$

Since, as before, U can be chosen to differ from T by a set of an arbitrarily small measure, then for all $t \in T$ and $\beta \in S$,

$$0 = \int_{t_0}^{t} \{\varphi_{\beta x}(\tilde{x}_0(\tau))y_0(\tau) + \varphi_{\beta \dot{x}}(\tilde{x}_0(\tau))\dot{y}_0(\tau)\}d\tau$$

and so (28) is verified. Consequently, from (15), (22), (26), (27) and (28), $y_{0_{\beta_0}} \in \mathcal{Y}(x_{0_{b_0}})$. This fact together with (17) contradicts condition (iv) of Theorem 2.1.

Case (2): Now, suppose that the sequence $\{(b_q - b_0)/d_q\}$ is not bounded. Then,

$$\lim_{q\to\infty}\left|\frac{b_q - b_0}{d_q}\right| = +\infty. \tag{30}$$

Choose an appropriate subsequence of $\{(b_q - b_0)/|b_q - b_0|\}$ (without relabeling), and $\bar{\beta}_0 \in \mathbf{R}^p$ with $|\bar{\beta}_0| = 1$, such that

$$\lim_{q\to\infty}\frac{b_q - b_0}{|b_q - b_0|} = \bar{\beta}_0. \tag{31}$$

For all $q \in \mathbf{N}$ and $t \in T$, define

$$\bar{\omega}_q(t) := \frac{x^q(t) - x_0(t)}{|b_q - b_0|}\mathbf{i} + \frac{b_q - b_0}{|b_q - b_0|}\mathbf{j}.$$

By Lemma 4.1 and (30),

$$\lim_{q\to\infty}\frac{x^q(t) - x_0(t)}{|b_q - b_0|} = \lim_{q\to\infty} y_q(t)\cdot\frac{d_q}{|b_q - b_0|} = y_0(t)\cdot 0 = 0 \text{ uniformly on } T. \tag{32}$$

For $i = 0, 1$ and all $q \in \mathbf{N}$, we have

$$\frac{x^q(t_i) - x_0(t_i)}{|b_q - b_0|} = \int_0^1 \Psi_i'(b_0 + \lambda[b_q - b_0])\Big(\frac{b_q - b_0}{|b_q - b_0|}\Big)d\lambda. \tag{33}$$

By (31), (32) and (33), for $i = 0, 1$,

$$\Psi_i'(b_0)\bar{\beta}_0 = 0. \tag{34}$$

Now, by (2), (31) and (32),

$$\begin{aligned}\lim_{q\to\infty}\frac{\tilde{M}_0(t,x^q(t)\mathbf{i}+b_q\mathbf{j})}{|b_q-b_0|^2} &= \lim_{q\to\infty}\tfrac{1}{2}\langle\bar{\omega}_q(t),\tilde{P}_0(t,x^q(t)\mathbf{i}+b_q\mathbf{j})\bar{\omega}_q(t)\rangle\\ &= \tfrac{1}{2}\langle 0_{\bar{\beta}_0},\tilde{F}_{0\xi\xi}(t,x_0(t)\mathbf{i}+b_0\mathbf{j},\dot{x}_0(t))0_{\bar{\beta}_0}\rangle\\ &= \frac{\langle\bar{\beta}_0,l_0''(b_0)\bar{\beta}_0\rangle}{2(t_1-t_0)},\end{aligned}$$

$$\begin{aligned}\lim_{q\to\infty}\frac{\tilde{N}_0(t,x^q(t)\mathbf{i}+b_q\mathbf{j})}{|b_q-b_0|} &= \lim_{q\to\infty}\tilde{Q}_0(t,x^q(t)\mathbf{i}+b_q\mathbf{j})\bar{\omega}_q(t)\\ &= \tilde{F}_{0\dot{x}\xi}(t,x_0(t)\mathbf{i}+b_0\mathbf{j},\dot{x}_0(t))0_{\bar{\beta}_0}\\ &= 0\end{aligned}$$

both uniformly on T. Together with Lemma 4.1, this implies that

$$\begin{aligned}\lim_{q\to\infty}\frac{\tilde{K}_0(x_{0b_0};x^q_{b_q})}{|b_q-b_0|^2} &= \tfrac{1}{2}\langle\bar{\beta}_0,l_0''(b_0)\bar{\beta}_0\rangle+\lim_{q\to\infty}\int_{t_0}^{t_1}\left\langle\frac{d_q}{|b_q-b_0|}\cdot\dot{y}_q(t),\frac{\tilde{N}_0(t,x^q(t)\mathbf{i}+b_q\mathbf{j})}{|b_q-b_0|}\right\rangle dt\\ &= \tfrac{1}{2}\langle\bar{\beta}_0,l_0''(b_0)\bar{\beta}_0\rangle.\end{aligned} \tag{35}$$

As in (19), we have

$$\lim_{q\to\infty}\frac{\tilde{J}_0'(x_{0b_0};x^q_{b_q}-x_{0b_0})}{|b_q-b_0|^2}=\tfrac{1}{2}[\rho^*(t_1)\Psi_1''(b_0;\bar{\beta}_0)-\rho^*(t_0)\Psi_0''(b_0;\bar{\beta}_0)]. \tag{36}$$

Even more, by (1), (4), (36), and condition (ii) of Theorem 2.1,

$$0\geq\lim_{q\to\infty}\frac{\tilde{K}_0(x_{0b_0};x^q_{b_q})}{|b_q-b_0|^2}+\liminf_{q\to\infty}\frac{\tilde{\mathcal{E}}_0(x_{0b_0};x^q_{b_q})}{|b_q-b_0|^2}. \tag{37}$$

Consequently, since $\tilde{\mathcal{E}}_0(x_{0b_0};x^q_{b_q})\geq 0$ ($q\in\mathbf{N}$), by (35) and (37),

$$0\geq\tfrac{1}{2}\langle\bar{\beta}_0,l_0''(b_0)\bar{\beta}_0\rangle=\tfrac{1}{2}J_0''(x_{0b_0};0_{\bar{\beta}_0}). \tag{38}$$

Let us now show that for all $i\in i_a(x_{0b_0})$,

$$I_i'(x_{0b_0};0_{\bar{\beta}_0})\leq 0. \tag{39}$$

To prove it, note that since

$$\lim_{q\to\infty}\frac{x^q(t)-x_0(t)}{|b_q-b_0|}=0\quad\text{uniformly on } T,$$

and $\{\dot{y}_q\}$ converges weakly in $L^1(T;\mathbf{R}^n)$ to $\dot{y}_0$, for all $\gamma = 0, 1, \ldots, K$,

$$\lim_{q\to\infty} \frac{\tilde{K}_\gamma(x_{0b_0}; x^q_{b_q})}{|b_q - b_0|} = 0. \tag{40}$$

As in (24),

$$\lim_{q\to\infty} \frac{\tilde{J}'_0(x_{0b_0}; x^q_{b_q} - x_{0b_0})}{|b_q - b_0|} = 0. \tag{41}$$

Then, by (4), (40) and (41),

$$0 \geq \limsup_{q\to\infty} \frac{\tilde{J}_0(x^q_{b_q}) - \tilde{J}_0(x_{0b_0})}{|b_q - b_0|} = \limsup_{q\to\infty} \frac{\tilde{\mathcal{E}}_0(x_{0b_0}; x^q_{b_q})}{|b_q - b_0|} \geq 0.$$

Thus, by condition (v)(c) of Theorem 2.1, for all $\gamma = 1, \ldots, K$,

$$\lim_{q\to\infty} \frac{\tilde{\mathcal{E}}_\gamma(x_{0b_0}; x^q_{b_q})}{|b_q - b_0|} = 0. \tag{42}$$

As for all $q \in \mathbf{N}$ and $i \in i_a(x_{0b_0})$,

$$\begin{aligned} 0 &\geq I_i(x^q_{b_q}) \\ &= I_i(x^q_{b_q}) - I_i(x_{0b_0}) \\ &= \tilde{I}_i(x^q_{b_q}) - \tilde{I}_i(x_{0b_0}) \\ &= \tilde{I}'_i(x_{0b_0}; x^q_{b_q} - x_{0b_0}) + \tilde{K}_i(x_{0b_0}; x^q_{b_q}) + \tilde{\mathcal{E}}_i(x_{0b_0}; x^q_{b_q}) \\ &= I'_i(x_{0b_0}; x^q_{b_q} - x_{0b_0}) + \tilde{K}_i(x_{0b_0}; x^q_{b_q}) + \tilde{\mathcal{E}}_i(x_{0b_0}; x^q_{b_q}), \end{aligned}$$

then, by (40) and (42), for all $i \in i_a(x_{0b_0})$,

$$0 \geq \lim_{q\to\infty} \frac{I'_i(x_{0b_0}; x^q_{b_q} - x_{0b_0})}{|b_q - b_0|}.$$

Hence, since

$$\{\dot{y}_q\} \text{ converges weakly to } \dot{y}_0 \text{ in } L^1(T;\mathbf{R}^n), \quad \frac{d_q}{|b_q - b_0|} \to 0,$$

$$\lim_{q\to\infty} \frac{x^q(t) - x_0(t)}{|b_q - b_0|} = 0 \text{ uniformly on } T, \quad \text{and} \quad \frac{b_q - b_0}{|b_q - b_0|} \to \bar{\beta}_0,$$

for all $i \in i_a(x_{0b_0})$,

$$0 \geq \lim_{q\to\infty} \frac{I'_i(x_{0b_0}; x^q_{b_q} - x_{0b_0})}{|b_q - b_0|} = l'_i(b_0)\bar{\beta}_0 = I'_i(x_{0b_0}; 0_{\bar{\beta}_0})$$

which establishes (39). Finally, let us show that for all $j = k+1, \ldots, K$,

$$I'_j(x_{0b_0}; 0_{\bar{\beta}_0}) = 0. \tag{43}$$

Indeed, as for all $q \in \mathbf{N}$ and $j = k+1, \ldots, K$,

$$\begin{aligned}
0 &= I_j(x^q_{b_q}) - I_j(x_{0b_0}) \\
&= \tilde{I}_j(x^q_{b_q}) - \tilde{I}_j(x_{0b_0}) \\
&= \tilde{I}'_j(x_{0b_0}; x^q_{b_q} - x_{0b_0}) + \tilde{K}_j(x_{0b_0}; x^q_{b_q}) + \tilde{\mathcal{E}}_j(x_{0b_0}; x^q_{b_q}) \\
&= I'_j(x_{0b_0}; x^q_{b_q} - x_{0b_0}) + \tilde{K}_j(x_{0b_0}; x^q_{b_q}) + \tilde{\mathcal{E}}_j(x_{0b_0}; x^q_{b_q}),
\end{aligned}$$

then, by (40) and (42), for all $j = k+1, \ldots, K$,

$$0 = \lim_{q\to\infty} \frac{I'_j(x_{0b_0}; x^q_{b_q} - x_{0b_0})}{|b_q - b_0|} = I'_j(x_{0b_0}; 0_{\bar{\beta}_0})$$

which is precisely (43). Consequently, (34), (38), (39) and (43) contradicts condition (iv) of Theorem 2.1 and this completes the proof of Theorem 2.1. ∎

6. Proof of Lemmas 4.1 and 4.2

Proof of Lemma 4.1: For all $q \in \mathbf{N}$ and almost all $t \in T$, define

$$c_q := [1 + \tfrac{1}{2}V(x^q(t_0) - x_0(t_0))]^{1/2} \quad \text{and} \quad W_q(t) := [1 + \tfrac{1}{2}V(\dot{x}^q(t) - \dot{x}_0(t))]^{1/2}.$$

For all $q \in \mathbf{N}$, note that

$$\begin{aligned}
&\frac{|y_q(t_0)|^2}{c_q^2} + \int_{t_0}^{t_1} \frac{|\dot{y}_q(t)|^2}{W_q^2(t)} dt \\
&= \frac{|x^q(t_0) - x_0(t_0)|^2}{d_q^2[1 + \frac{1}{2}V(x^q(t_0) - x_0(t_0))]} + \frac{1}{d_q^2} \int_{t_0}^{t_1} \frac{|\dot{x}^q(t) - \dot{x}_0(t)|^2}{1 + \frac{1}{2}V(\dot{x}^q(t) - \dot{x}_0(t))} dt \\
&= \frac{|x^q(t_0) - x_0(t_0)|^2}{D(x^q - x_0)[2 + V(x^q(t_0) - x_0(t_0))]} + \frac{1}{D(x^q - x_0)} \int_{t_0}^{t_1} \frac{|\dot{x}^q(t) - \dot{x}_0(t)|^2}{2 + V(\dot{x}^q(t) - \dot{x}_0(t))} dt \\
&= \frac{1}{D(x^q - x_0)} \Big(V(x^q(t_0) - x_0(t_0)) + \int_{t_0}^{t_1} V(\dot{x}^q(t) - \dot{x}_0(t)) dt \Big) \\
&= \frac{D(x^q - x_0)}{D(x^q - x_0)} = 1.
\end{aligned}$$

Clearly $\lim_{q\to\infty} c_q = 1$. Then, there exist some subsequence of $\{x^q\}$, again denoted by $\{x^q\}$, some $\bar{y}_0 \in \mathbf{R}^n$ and some $\sigma_0 \in L^2(T; \mathbf{R}^n)$ such that

$$\lim_{q\to\infty} \frac{y_q(t_0)}{c_q} = \lim_{q\to\infty} y_q(t_0) = \bar{y}_0,$$

$$\{\dot{y}_q / W_q\} \text{ converges weakly in } L^2(T; \mathbf{R}^n) \text{ to } \sigma_0.$$

As $W_q^2(t) \geq W_q(t) \geq 1$ for all $q \in \mathbf{N}$ and for almost all $t \in T$, we have

$$0 \leq \int_{t_0}^{t_1} [W_q(t) - 1]dt \leq \int_{t_0}^{t_1} [W_q^2(t) - 1]dt = \frac{1}{2}\int_{t_0}^{t_1} V(\dot{x}^q(t) - \dot{x}_0(t))dt \leq D(x^q - x_0).$$

Thus, it follows that

$$\lim_{q\to\infty} \int_{t_0}^{t_1} [W_q(t) - 1]dt = \lim_{q\to\infty} \int_{t_0}^{t_1} [W_q^2(t) - 1]dt = 0.$$

Note also that

$$\int_{t_0}^{t_1} [W_q(t) - 1]^2 dt = \int_{t_0}^{t_1} [W_q^2(t) - 1]dt - 2\int_{t_0}^{t_1} [W_q(t) - 1]dt.$$

Then for any $h \in L^\infty(T; \mathbf{R}^n)$,

$$\lim_{q\to\infty} \|hW_q - h\|_2 = 0$$

and so

$$\lim_{q\to\infty} \int_{t_0}^{t_1} \langle h(t), \dot{y}_q(t)\rangle dt = \lim_{q\to\infty} \int_{t_0}^{t_1} \left\langle h(t)W_q(t), \frac{\dot{y}_q(t)}{W_q(t)}\right\rangle dt = \int_{t_0}^{t_1} \langle h(t), \sigma_0(t)\rangle dt.$$

Therefore, $\{\dot{y}_q\}$ converges weakly in $L^1(T; \mathbf{R}^n)$ to σ_0. Hence, $\{\dot{y}_q\}$ is equi-integrable on T and therefore the sequence $\{y_q\}$ is equi-continuous on T. Thus, if $y_0(t) := \bar{y}_0 + \int_{t_0}^{t} \sigma_0(\tau)d\tau$, then, $y_0 \in \mathcal{X}$ with $\dot{y}_0 \in L^2(T; \mathbf{R}^n)$ and

$$\lim_{q\to\infty} y_q(t) = \lim_{q\to\infty} y_q(t_0) + \lim_{q\to\infty} \int_{t_0}^{t} \dot{y}_q(\tau)d\tau = y_0(t) \text{ uniformly on } T,$$

$$\{\dot{y}_q\} \text{ converges weakly in } L^1(T; \mathbf{R}^n) \text{ to } \dot{y}_0 = \sigma_0.$$

Now, let us show that $\dot{x}^q(t) \to \dot{x}_0(t)$ almost uniformly on T. For almost all $t \in T$, define

$$W(t) := [1 + \tfrac{1}{2}V(\dot{x}(t))]^{1/2}.$$

Observe that

$$\int_{t_0}^{t_1} \frac{|\dot{x}(t)|^2}{2W^2(t)}dt = \int_{t_0}^{t_1} \frac{|\dot{x}(t)|^2}{2 + V(\dot{x}(t))}dt = \int_{t_0}^{t_1} V(\dot{x}(t))dt \leq D(x),$$

$$\int_{t_0}^{t_1} 2W^2(t)dt = 2t_1 - 2t_0 + \int_{t_0}^{t_1} V(\dot{x}(t))dt \leq 2t_1 - 2t_0 + D(x).$$

From these relations, we have

$$\|\dot{x}\|_1^2 \leq \int_{t_0}^{t_1} \frac{|\dot{x}(t)|^2}{2W^2(t)}dt \int_{t_0}^{t_1} 2W^2(t)dt \leq D(x)[2t_1 - 2t_0 + D(x)].$$

Consequently, $\|\dot{x}^q - \dot{x}_0\|_1 \to 0$ and so some subsequence of $\{\dot{x}^q\}$ converges almost uniformly to $\dot{x}_0$ on T. ∎

Proof of Lemma 4.2: Recall the definition of W_q given in the proof of Lemma 4.1. As $W_q(t) \to 1$ uniformly on U, then for all $h \in L^2(U; \mathbf{R}^n)$,

$$\lim_{q\to\infty} \int_U \langle \dot{y}_q(t), h(t)\rangle dt = \lim_{q\to\infty} \int_U \left\langle \frac{\dot{y}_q(t)}{W_q(t)}, W_q(t)h(t)\right\rangle dt = \int_U \langle \dot{y}_0(t), h(t)\rangle dt,$$

that is, $\{\dot{y}_q\}$ converges weakly in $L^2(U;\mathbf{R}^n)$ to $\dot{y}_0$. As $R_0(t) \geq 0$ $(t \in U)$, the function

$$\dot{y} \mapsto \int_U \langle R_0(t)\dot{y}(t), \dot{y}(t)\rangle dt$$

is convex in $L^2(U;\mathbf{R}^n)$ and since this function is strongly continuous on $L^2(U;\mathbf{R}^n)$, then this function is weakly lower semicontinuous in $L^2(U;\mathbf{R}^n)$. Thus,

$$\liminf_{q\to\infty} \int_U \langle R_0(t)\dot{y}_q(t), \dot{y}_q(t)\rangle dt \geq \int_U \langle R_0(t)\dot{y}_0(t), \dot{y}_0(t)\rangle dt.$$

Since $R_q(t) \to R_0(t)$ uniformly on U, it follows that

$$\liminf_{q\to\infty} \int_U \langle R_q(t)\dot{y}_q(t), \dot{y}_q(t)\rangle dt \geq \int_U \langle R_0(t)\dot{y}_0(t), \dot{y}_0(t)\rangle dt. \blacksquare$$

7. Example

In this section, we show with an example how our sufficiency theory is able to detect optimality even when the proposed extremal to be a strong minimum is singular and its derivative is only essentially bounded. It is worthwhile observing that the initial and final end-points of the states of admissible trajectories are not restricted to belong to any manifold described by any smooth function, in contrast, these boundary points must only lie in the set of real numbers, that is, these boundary points are completely free.

In Example 7.1 since no isoperimetric constraints occur l_0, L_0, F_0, E_0 and J_0'' correspond simply to l, L, F, E and J'' respectively.

7.1 Example: *Let* $x_0\colon [0,1] \to \mathbf{R}$ *be any absolutely continuous function with* $\dot{x}_0 \in L^\infty([0,1];\mathbf{R})$ *and* $x_0(0) = x_0(1) = 0$*. Consider the nonparametric problem* $(\bar{P})$ *of minimizing*

$$\mathcal{J}(x) := x^2(0) - x(1) + \int_0^1 \{\exp(t(\dot{x}(t) - \dot{x}_0(t))) + x(t)\}dt$$

over all arcs $x \in \mathcal{X}$ *satisfying the constraints*

$$\begin{cases} c(t,x(t),\dot{x}(t)) \ \textit{is integrable on } [0,1]. \\ x(0) \in \mathbf{R}, \ \ x(1) \in \mathbf{R}. \\ (t,x(t),\dot{x}(t)) \in \bar{\mathcal{R}} \ (\textit{a.e. in } [0,1]) \end{cases}$$

where

$$\bar{\mathcal{R}} := \{(t,x,\dot{x}) \in [0,1]\times\mathbf{R}\times\mathbf{R} \mid \phi_1(t,x,\dot{x}) \leq 0\},$$

$$\phi_1(t,x,\dot{x}) := (\dot{x} - \dot{x}_0(t))^2 - \exp(t(\dot{x} - \dot{x}_0(t))) + t(\dot{x} - \dot{x}_0(t)) + 1,$$

$\mathcal{X} := AC([0,1];\mathbf{R})$ *and* $c(t,x,\dot{x})$ *denotes either*

$$\mathcal{L}(t,x,\dot{x}) := \exp(t(\dot{x} - \dot{x}_0(t))) + x,$$

$\phi_1(t,x,\dot{x})$*, or any of its partial derivatives of order less than or equal to two with respect to* x *and* $\dot{x}$*.*

For this problem we shall consider the data of the nonparametric problem given in Section 3 which are given by $T = [0,1]$, $n = 1$, $r = 1$, $s = 1$, $k = K = 0$, $\mathcal{B}_0 = \mathbf{R}$, $\mathcal{B}_1 = \mathbf{R}$, $\ell(x_1, x_2) = x_1^2 - x_2$, $\mathcal{L}(t, x, \dot{x}) = \exp(t(\dot{x} - \dot{x}_0(t))) + x$, $\phi_1(t, x, \dot{x}) = (\dot{x} - \dot{x}_0(t))^2 - \exp(t(\dot{x} - \dot{x}_0(t))) + t(\dot{x} - \dot{x}_0(t)) + 1$.

Clearly, all the assumptions posed in the statement of the problem are easily verified.

Also, it is evident that the trajectory x_0 is admissible of $(\bar{P})$. Let $\Psi\colon \mathbf{R} \to \mathbf{R} \times \mathbf{R}$ be defined by $\Psi(b) := (b, b)$. Clearly, Ψ is C^2 in $\mathbf{R}$ and $\mathcal{B}_0 \times \mathcal{B}_1 \subset \Psi(\mathbf{R})$. The associated parametric problem of Section 2 denoted by (P_Ψ) has the following data, $p = 1$, $\mathcal{B} = \Psi^{-1}(\mathcal{B}_0 \times \mathcal{B}_1) = \mathbf{R}$, $l = \ell \circ \Psi$, $L = \mathcal{L}$, $\varphi = \varphi_1 = \phi_1$ and Ψ_0, Ψ_1 the components of Ψ, that is, $\Psi_0(b) = b$, $\Psi_1(b) = b$ ($b \in \mathbf{R}$). Recall that the notation x_b means (x, b) where $b \in \mathbf{R}$ is a parameter.

Observe that if we set $b_0 := 0$, then x_{0b_0} is admissible of (P_Ψ). Also, clearly $\mathcal{I}_a(\tilde{x}_0(\cdot))$ is constant on T. Let $(\rho(t), \mu_1(t)) := (t, 0)$ ($t \in T$) and note that $(\rho, \mu) \in \mathcal{X} \times \mathcal{U}_1$, $\mu_1(t) \geq 0$, $\mu_1(t)\varphi_1(\tilde{x}_0(t)) = 0$ (a.e. in T).

Now, observe that the Hamiltonian H is given by

$$H(t, x, \dot{x}, \rho(t), \mu(t)) = t\dot{x} - \exp(t(\dot{x} - \dot{x}_0(t))) - x.$$

Also, note that

$$H_x(\tilde{x}_0(t), \rho(t), \mu(t)) = -1, \quad H_{\dot{x}}(\tilde{x}_0(t), \rho(t), \mu(t)) = 0 \quad \text{(a.e. in } T),$$

and so (x_0, ρ, μ) satisfies the Euler-Lagrange equations which are given by

$$\dot{\rho}(t) = -H_x(\tilde{x}_0(t), \rho(t), \mu(t)), \quad H_{\dot{x}}(\tilde{x}_0(t), \rho(t), \mu(t)) = 0 \quad \text{(a.e. in } T).$$

Thus, (x_0, ρ, μ) is an extremal. As $\Psi_0(b) = b$, $\Psi_1(b) = b$, $l(b) = b^2 - b$ ($b \in \mathbf{R}$), then

$$l'(b_0) + \rho(1)\Psi_1'(b_0) - \rho(0)\Psi_0'(b_0) = 0$$

and so condition (i) of Corollary 3.2 is satisfied. Also, as one readily verifies,

$$\rho(1)\Psi_1''(b_0; \beta) - \rho(0)\Psi_0''(b_0; \beta) = 0 \text{ for all } \beta \in \mathbf{R}$$

and so condition (ii) of Corollary 3.2 is fulfilled.

Additionally, as

$$H_{\dot{x}\dot{x}}(\tilde{x}_0(t), \rho(t), \mu(t)) = -t^2 \quad \text{(a.e. in } T),$$

it follows that (iii) of Corollary 3.2 is satisfied and (x_0, ρ, μ) is singular since

$$H_{\dot{x}\dot{x}}(\tilde{x}_0(0), \rho(0), \mu(0)) = 0.$$

As $\varphi_{1x}(\tilde{x}_0(t)) = \varphi_{1\dot{x}}(\tilde{x}_0(t)) = 0$ (a.e. in T), then $\mathcal{Y}(x_{0b_0})$ is given by all $y_\beta \in \mathcal{X} \times \mathbf{R}$ with $\dot{y} \in L^2(T; \mathbf{R})$ satisfying $y(0) = \beta$, $y(1) = \beta$. We have,

$$J''(x_{0b_0}; y_\beta) = 2\beta^2 + \int_0^1 t^2 \dot{y}^2(t)dt > 0$$

for all $y_\beta \in \mathcal{Y}(x_{0b_0})$, $y_\beta \neq (0, 0)$, and hence (iv) of Corollary 3.2 is satisfied. Now, observe that for all $(t, x, \dot{x}) \in T \times \mathbf{R} \times \mathbf{R}$,

$$F(t, x, \dot{x}) = -H(t, x, \dot{x}, \rho(t), \mu(t)) - \dot{\rho}(t)x = -t\dot{x} + \exp(t(\dot{x} - \dot{x}_0(t))).$$

Hence for almost all $t \in T$, if x_b is admissible,

$$\begin{aligned} E(t, x(t), \dot{x}_0(t), \dot{x}(t)) &= F(t, x(t), \dot{x}(t)) - F(t, x(t), \dot{x}_0(t)) - F_{\dot{x}}(t, x(t), \dot{x}_0(t))(\dot{x}(t) - \dot{x}_0(t)) \\ &= -t\dot{x}(t) + \exp(t(\dot{x}(t) - \dot{x}_0(t))) + t\dot{x}_0(t) - 1 \\ &= \exp(t(\dot{x}(t) - \dot{x}_0(t))) - t(\dot{x}(t) - \dot{x}_0(t)) - 1 \geq 0 \end{aligned}$$

which implies that (v)(a) of Corollary 3.2 is verified with any $\epsilon > 0$. Finally, for all x_b admissible, we have

$$\begin{aligned} \int_0^1 E(t, x(t), \dot{x}_0(t), \dot{x}(t))dt &= \int_0^1 \{\exp(t(\dot{x}(t) - \dot{x}_0(t))) - t(\dot{x}(t) - \dot{x}_0(t)) - 1\}dt \\ &\geq \int_0^1 (\dot{x}(t) - \dot{x}_0(t))^2 dt \geq 2\int_0^1 V(\dot{x}(t) - \dot{x}_0(t))dt. \end{aligned}$$

Therefore, (v)(b) of Corollary 3.2 is satisfied with any $\epsilon > 0$ and $h = 2$. Since $k = K = 0$, it is evident that (v)(c) of Corollary 3.2 is also verified with any $\epsilon > 0$ and $h = 2$. By Corollary 3.2, x_0 is a strong minimum of $(\bar{P})$.

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Conflict of interest

The author declares no conflict of interest.

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