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Research article

Orbital stability of periodic solutions of an impulsive system with a linear continuous-time part

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Abstract: An impulsive system with a linear continuous-time part and a nonlinear discrete-time part is considered. A criterion for exponential orbital stability of its periodic solutions is obtained. The proof is based on linearization by the first approximation of an auxiliary discrete-time system. The formulation of the criterion depends significantly on a number of impulses per period of the solution. The paper provides a mathematical rationale for some results previously examined in mathematical biology by computer simulations.

Keywords: systems with impulses; hybrid systems; periodic solutions; exponential stability; orbital stability
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1. Introduction

The system considered in this paper is a generalization of a hormonal regulation model introduced in [1] and later developed in a number of publications (see, among others, [2–10]). This system belongs to the class of impulsive differential equations widely addressed in mathematical literature (see, e. g., [11–17]). An impulsive release of some hormones results from the fact that it is controlled by neurons of the brain. The amplitudes and frequencies of this release are governed by other hormones secreted in certain organs or tissues. The working regimes of such a hormonal system are periodic oscillations.

Following [14], an impulsive system can be seen as a special class of hybrid systems that includes both continuous-time and discrete-time dynamics. We are interested in the case when the times between impulses are variable and depend on the state vector and the previous impulse moments. In electrical engineering and biology such systems are called pulse-frequency modulated [18].

It is well known, that the conventional stability concept [19, 20] is unsuitable for systems with impulses, since their solutions are discontinuous in time and even for close initial values, solutions may differ significantly in some vicinities of jumping times. Definitions of stability of solutions taking into account specifity of impulsive systems can be found, e. g., in [11, 13, 16]. An alternative approach is an adoption of the orbital stability concept to impulsive systems, see [21–27].

In the present paper we provide a criterion for some kind of exponential orbital stability of periodical solutions. The stability concept pursued here is different from that was addressed in previous works. Its analogues for non-impulsive systems can be found in [28, 29].

When compared with [1], the result of this paper is novel in several aspects. Firstly, in [1] an orbital stability was considered, and here we obtain a more powerful result that ensures some kind of exponential orbital stability. Secondly, in [1] the statement on orbital stability was formulated, however its proof was only sketched because of the lack of space. In the present paper we provide a detailed and rigorous mathematical proof. Additionally, the system treated here is more general than that in [1]. Thirdly, in most publications on impulsive systems the initial impulse moment is considered fixed and common for all the solutions. However it is not suitable for some applications, e. g., when we address synchronization problems [2]. In this paper we examine stability in the augmented phase space including continuous-time and discrete-time coordinates.

The paper is organized as follows. Firstly, a special class of linear systems with a nonlinear impulsive impact is described. Then a characterization of its periodic solutions with the help of an auxiliary discrete-time system is given. We also prove some results on stability of discrete-time systems. Based on these results, a criterion for exponential orbital stability of periodic solutions of the initial impulsive system is established. Finally, a numerical example is provided.

2. Impulsive system

2.1. System description

Consider a system with impulses that describes dynamics of an unknown p-dimensional vector function $x(t)$ and a discrete-time sequence $\{t_n\}_{n=0}^{\infty}$:

$$\dot{x} = Ax(t), \quad t_n < t < t_{n+1}, \tag{2.1}$$

$$x(t_n^+) - x(t_n^-) = \Lambda_n, \tag{2.2}$$

$$t_{n+1} = t_n + T_n, \tag{2.3}$$

$$T_n = \Phi(x(t_n^-)), \quad \Lambda_n = F(x(t_n^-)), \tag{2.4}$$

$n \geq 0$, $t \geq t_0$. Here A is a nonzero constant $p \times p$ matrix, t_n is an increasing scalar sequence and Λ_n is a sequence of p-dimensional vectors. The notation $x(t^-)$, $x(t^+)$ is used for left-sided and right-sided limits of the function $x(\cdot)$ at the point t. The system's solutions have jumps Λ_n at times t_n, and T_n is the duration of the interval between consecutive jumping times t_n, t_{n+1}. The sequences T_n and Λ_n are defined by (2.4), where $\Phi(\cdot)$, $F(\cdot)$ are nonlinear continuous maps, $\Phi : \mathbb{R}^p \mapsto \mathbb{R}, \quad F : \mathbb{R}^p \mapsto \mathbb{R}^p$.

Assume that $\Phi(\cdot)$ has finite lower and upper bounds:

$$0 < \Phi_1 \leq \Phi(x) \leq \Phi_2 \tag{2.5}$$

for all $x \in \mathbb{R}^p$, where Φ_1, Φ_2 are some constants. Additionally, suppose that $\Phi(\cdot)$, $F(\cdot)$ are Lipschitz with some Lipschitz constants L_Φ, L_F, respectively:

$$|\Phi(x) - \Phi(y)| \leq L_\Phi \|x - y\|, \quad \|F(x) - F(y)\| \leq L_F \|x - y\| \quad \text{for all} \quad x, y. \tag{2.6}$$

Here $||\cdot||$ is the Euclidean vector norm.

The state space systems (2.1)–(2.4) is $(p+1)$-dimensional, since it encompasses dynamics not only of $x(t)$, but also of t_n. Initial values for its solutions are given by points $(t_0, x(t_0^-))$.

2.2. Existence and uniqueness of solutions

Integrating (2.1) and taking (2.2) into account, we obtain

$$x(t) = \mathrm{e}^{A(t-t_n)}(x(t_n^-) + \Lambda_n), \quad t_n < t < t_{n+1}. \tag{2.7}$$

Hence, if we know a solution at a point t_n, namely the pair $(t_n, x(t_n^-))$, we can uniquely extend $x(t)$ to the whole interval (t_n, t_{n+1}) and thus obtain the pair $(t_{n+1}, x(t_{n+1}^-))$. Moreover, from (2.5) it follows that $t_{n+1} - t_n \geq \Phi_1 > 0$, so the sequence $\{t_n\}$, $n = 0, 1, \ldots$ is strictly increasing, has no accumulation points, and $t_n \to +\infty$ as $n \to \infty$.

Thus we conclude that for any initial value $(t_0, x(t_0^-))$ there exists a unique solution $(t_n, x(t))$ of (2.1)–(2.4) for $t \in [t_0, +\infty)$ and $n = 0, 1, 2, \ldots$.

2.3. Reduction to a discrete-time system

For brevity, let us introduce notation $x_n = x(t_n^-)$. From (2.7) it follows that for any solution of (2.1)–(2.4) and for all $n \geq 0$

$$x_{n+1} = \mathrm{e}^{AT_n}(x_n + \Lambda_n) \tag{2.8}$$

(see, e. g., [30]).

Then any solution $(t_n, x(t))$ of (2.1)–(2.4) obeys a system of discrete-time (difference) equations

$$x_{n+1} = \mathrm{e}^{A\Phi(x_n)}(x_n + F(x_n)), \tag{2.9}$$

$$t_{n+1} = t_n + \Phi(x_n), \quad n = 0, 1, \ldots. \tag{2.10}$$

Notice that (2.10) does not influence (2.9), so the dynamics of (2.9) can be studied independently of (2.10). Discrete-time system (2.9) is p-dimensional, while the augmented system (2.9), (2.10) is $(p+1)$-dimensional. Further (2.9), (2.10) will be called the discrete-time system related to (2.1)–(2.4).

Observe that if some sequences $\{x_n\}$, $\{t_n\}$ satisfy (2.9), (2.10), then there exists a unique solution $(t_n, x(t))$ of (2.1)–(2.4) with $x(t_n^-) = x_n$, $n = 0, 1, \ldots$. Indeed, between the sampling times t_n, t_{n+1} the function $x(t)$ can be uniquely recovered as

$$x(t) = \mathrm{e}^{A(t-t_n)}(x_n + \Lambda_n) \quad \text{or} \quad x(t) = \mathrm{e}^{A(t-t_{n+1})}x_{n+1}.$$

Thus there is one-to-one correspondence between solutions of discrete-time systems (2.9), (2.10) and of impulsive systems (2.1)–(2.4).

3. Periodic solutions

3.1. Periodic solutions of the discrete-time system

Let us inroduce a map $Q : \mathbb{R}^p \mapsto \mathbb{R}^p$,

$$Q(x) = \mathrm{e}^{A\Phi(x)}(x + F(x)). \tag{3.1}$$

Then (2.9) can be rewritten as

$$x_{n+1} = Q(x_n). \tag{3.2}$$

Consider the discrete-time equation (2.9), where the map $Q(\cdot)$ is defined by (3.1). Solutions of (2.9) are infinite sequences $\{x_n\}_{n=0}^{\infty} = \{x_0, \ldots, x_k, \ldots\}$.

For an integer $k \geq 1$ consider the kth iteration of the map Q:

$$Q^{(k)}(x) = \underbrace{Q(Q(\ldots(Q(}_{k}x))\ldots)).$$

Then any solution of (2.9) can be found as $x_n = Q^{(n)}(x_0)$, $n \geq 1$, where x_0 is an initial value.

A solution $\{x_n\}$ of (2.9) is called 1-*periodic* or 1-*cycle*, if $x_n \equiv x_0$ for all $n \geq 0$. The initial value x_0 of a 1-cycle is a fixed point of the map Q, i. e., $Q(x_0) = x_0$.

A solution $\{x_n\}$ of (2.9) is called *m-periodic* ($m > 1$) or *m-cycle*, if $x_{n+m} = x_n$ for all $n \geq 0$, and the vectors $x_0, \ldots, x_{m-1}$ are all different. Obviously, the initial value x_0 of an m-cycle is a fixed point of the mth iterate, but not a fixed point of lower iterates:

$$Q^{(m)}(x_0) = x_0, \quad Q^{(k)}(x_0) \neq x_0 \quad \text{for} \quad 1 \leq k < m. \tag{3.3}$$

Let $\{x_n, t_n\}$ be a solution of discrete-time systems (2.9), (2.10), where the sequence $\{x_n\}$ is m-periodic ($m \geq 1$). Then

$$x_{n+m} = x_n, \quad t_{n+m} = t_n + T \quad \text{for} \quad n \geq 0 \tag{3.4}$$

with

$$T = \Phi(x_0) + \ldots + \Phi(x_{m-1}). \tag{3.5}$$

The second equality (3.4) follows from periodicity of x_n and the relationships

$$t_{n+m} = t_n + \Phi(x_n) + \ldots + \Phi(x_{n+m-1}) = t_n + \Phi(x_0) + \ldots + \Phi(x_{m-1}) = t_n + T.$$

3.2. Periodic solutions of the continuous-time impulsive system

Let T be some positive real number. The solution $(t_n, x(t))$ of impulsive systems (2.1)–(2.4) will be called *T-periodic* if the function $x(t)$ is T-periodic for $t \geq t_0$ with the least period T.

Theorem 1. *Establish a correspondence between periodic solutions* $(t_n, x(t))$ *of impulsive systems* (2.1)–(2.4) *and periodic solutions* $\{x_n\}$ *of discrete-time system* (2.9).

(i) Let a solution $\{x_n\}$ *of* (2.9) *be an m-cycle with some integer* $m \geq 1$. *Then the solution* $(x(t), t_n)$ *of the systems* (2.1)–(2.4) *with an arbitrary initial value* t_0 *and the initial condition* $x(t_0^-) = x_0$ *is periodic with the least period T defined by* (3.5).

(ii) Let $(t_n, x(t))$ *be a T-periodic soluton of* (2.1)–(2.4) *with some initial values* t_0 *and* $x(t_0^-) = x_0$. *Take the solution* $\{x_n\}$ *of* (2.9) *with the same initial value* x_0. *Then there exists an integer* $m \geq 1$ *such that* $\{x_n\}$ *is an m-cycle and* (3.5) *is satisfied.*

Proof. Consider a solution of (2.1)–(2.4) with the initial condition $x(t_0^-) = x_0$. Then $x(t_n^-)$ satisfies (2.9) and T_n, Λ_n are defined by (2.4). Thus

$$x(t_n^-) = x_n, \quad T_n = \Phi(x_n), \quad \Lambda_n = F(x_n) \quad \text{for all} \quad n \geq 0. \tag{3.6}$$

Since $\{x_n\}$ is an m-cycle, we have

$$x(t_{n+m}^-) = x(t_n^-), \quad T_{n+m} = T_n, \quad \Lambda_{n+m} = \Lambda_n \quad \text{for all} \quad n \geq 0. \tag{3.7}$$

Let $t_n < t < t_{n+1}$ for some $n \geq 0$. From (3.4) we obtain $t_{n+m} < t + T < t_{n+m+1}$, hence

$$x(t) = \mathrm{e}^{A(t-t_n)}(x_n + \Lambda_n), \quad x(t+T) = \mathrm{e}^{A(t+T-t_{n+m})}(x_{n+m} + \Lambda_{n+m}).$$

This evidently implies $x(t+T) = x(t)$ for $t_n^+ \leq t < t_{n+1}$. Since n is taken arbitrarily, $x(t)$ is T-periodic.

Let us prove that T is the least period of $x(t)$. Assume that there exists some number $0 < \tilde{T} < T$ such that $x(t+\tilde{T}) = x(t)$ for $t \geq t_0^-$. Then

$$x(t_0^- + \tilde{T}) = x(t_0^-) = x_0, \quad x(t_0^+ + \tilde{T}) = x(t_0^+) = x_0 + \Lambda_0.$$

Thus $t_0 + \tilde{T}$ is a jumping point of the vector function $x(t)$. Since $x(t)$ has no jumping points but $t_0, t_1, t_2, \ldots$, there exists some $k \geq 1$ such that $t_0 + \tilde{T} = t_k$. Hence $x_k = x_0$ and $k < m$ that contradicts the supposition that m is the least period for the sequence $\{x_n\}$. Thus statement (i) is proved.

Consider a T-periodic soluton $(t_n, x(t))$ of (2.1)–(2.4) with some initial values t_0 and $x(t_0^-) = x_0$. then the sequences $\{x_n\}$, $\{t_n\}$, where $x_n = x(t_n^-)$, satisfy (2.9), (2.10) and $T_n = \Phi(x_n)$, $\Lambda_n = F(x_n)$.

As it was shown above, then $t_0 + T$ is a jumping point and there exists some $m \geq 1$ such that $t_0 + T = t_m$. Here

$$T = t_m - t_0 = (t_m - t_{m-1}) + \ldots + (t_1 - t_0) = T_{m-1} + \ldots + T_0,$$

so (3.5) holds. Moreover, due to T-periodicity of $x(t)$ we have

$$x_m = x(t_m^-) = x(t_0^- + T) = x(t_0^-) = x_0.$$

Hence $Q^{(m)}(x_0) = x_0$ and the sequence $\{x_n\}$ is m-periodic.

Let us prove that m is the least period of $\{x_n\}$. If there exists some integer k such that $0 < k < m$ and $\{x_n\}$ is k-periodic, then from the first part of Theorem 1 we obtain that the corresponding solution of (2.1)–(2.4) has a period $\tilde{T} = T_0 + \ldots + T_{k-1}$ and $\tilde{T} < T$. This contradicts the supposition that T is the least period. □

In particular, Theorem 1 claims that for every periodic solution of a least period T there exists an integer m such that any interval $[t_n,\ t_n + T)$ contains exactly m jumping times. Such a solution will also be called an *m-cycle*.

4. General statements on stability of discrete-time systems

We will follow the definitions from [31]. Consider a discrete-time system

$$x_{n+1} = Q(x_n), \quad n = 0, 1, \ldots, \tag{4.1}$$

where x_n is a p-dimentional vector and Q is a continuous map, $Q : \mathbb{R}^p \mapsto \mathbb{R}^p$. (Here we do not require that $Q(\cdot)$ be of form (3.1).)

Let us take a solution $\{x_n\}_{n=0}^{\infty}$ of (4.1) with some initial value x_0.

The solution $\{x_n\}_{n=0}^{\infty}$ is called *stable* if for any given $\varepsilon > 0$ there exists $\delta > 0$ such that all solutions $\{\tilde{x}_n\}_{n=0}^{\infty}$ of (4.1) with $\|x_0 - \tilde{x}_0\| < \delta$ are such that $\|x_n - \tilde{x}_n\| < \varepsilon$ for all $n \geq 0$.

A solution $\{\tilde{x}_n\}_{n=0}^{\infty}$ with perturbed initial data will be called a *perturbed* solution.

The solution $\{x_n\}_{n=0}^{\infty}$ is called *exponentially stable* if there exist numbers $\delta > 0$, $M \geq 1$ and $0 < r < 1$ such that all solutions $\{\tilde{x}_n\}_{n=0}^{\infty}$ of (4.1) with $\|x_0 - \tilde{x}_0\| < \delta$ satisfy

$$\|x_n - \tilde{x}_n\| \leq Mr^n \|x_0 - \tilde{x}_0\| \tag{4.2}$$

for all $n \geq 0$.

We will need the following auxiliary statement on stability by the first approximation (see Theorem 2.10 [31] for the proof).

Lemma 1. *Consider system* (4.1) *and let its solution* $\{x_n\}_{n=0}^{\infty}$ *be a 1-cycle, i. e.,* $x_n \equiv x_0$ *and* x_0 *is a fixed point of the map* $Q(\cdot)$*:* $Q(x_0) = x_0$*. Assume that* $Q(\cdot)$ *is continuously differentiable in a neighborhood of* x_0 *and its Jacobian matrix* $J(x_0)$ *is Schur stable [32], i. e., all its eigenvalues are within the unit circle of the complex plane. Then this 1-cycle is exponentially stable.*

Let us extend the previous statement to the case of an m-cycle with $m > 1$. Again, let $J(x)$ be the Jacobian matrix of $Q(\cdot)$ at a point x.

Lemma 2. *Consider a solution* $\{x_n\}$ *that is an m-cycle,* $m > 1$*. Assume that the vector valued function* $Q(\cdot)$ *is continuously differentiable in some neighborhoods of the points* $x_0, \ldots x_{m-1}$*. If the matrix product of Jacobian matrices* $J(x_{m-1}) \cdots J(x_0)$ *is Schur stable, then the m-cycle is exponentially stable.*

Proof. For brevity, denote $J_n = J(x_n)$. Since the sequence x_n is m-periodic, J_n is also m-periodic and $J_{n+m} = J_n$ for all $n \geq 0$.

Consider the m-cycle $\{x_n\}_{n=0}^{\infty}$ and a perturbed solution $\{\tilde{x}_n\}_{n=0}^{\infty}$. Define the sequences $y_n = x_{mn}$ and $\tilde{y}_n = \tilde{x}_{mn}$, $n = 0, 1, \ldots$. Obviously y_n and $\tilde{y}_n$ satisfy the discrete-time equations

$$y_{n+1} = Q^{(m)}(y_n), \tag{4.3}$$

$$\tilde{y}_{n+1} = Q^{(m)}(\tilde{y}_n), \quad n \geq 0, \tag{4.4}$$

with the initial values $y_0 = x_0$, $\tilde{y}_0 = \tilde{x}_0$. Here $Q^{(m)}$ is the mth iteration of the map Q.

As it was mentioned previously, an initial value x_0 of an m-cycle is a fixed point of the iteration $Q^{(m)}$, i. e., $Q^{(m)}(x_0) = x_0$. By the chain rule, the Jacobian matrix of $Q^{(m)}$ at the point x_0 is the product $J_{m-1} \cdots J_0$. By the supposition, this product is Schur stable. Thus $\{y_n\}$ is a 1-cycle of (4.3) with a Schur stable Jacobian matrix. From Lemma 1, this 1-cycle is exponentially stable. Hence there exist numbers $M_0 \geq 1$, $0 < r_0 < 1$ and $h_0 > 0$ such that if $\|x_0 - \tilde{x}_0\| < h_0$ then $\|x_{nm} - \tilde{x}_{nm}\| \leq M_0 r_0^n \|x_0 - \tilde{x}_0\|$ for all $n \geq 0$.

It is easily seen that for any two square matrices A_1, A_2 of the same sizes the products $A_1 A_2$ and $A_2 A_1$ are simultaneously Schur stable or unstable. If the product $J_{m-1} \cdots J_0$ is Schur stable, from the previous statement we get that any matrix product obtained from $J_{m-1} \cdots J_0$ by a cyclic permutation will be also Schur stable. Then from m-periodicity of J_n it follows that the matrix product $J_{n+m-1} \cdots J_n$ is Schur stable for any $n \geq 0$.

Let us fix some integer s, $1 \leq s < m$, and define the sequences $y_n = x_{mn+s}$ and $\tilde{y}_n = \tilde{x}_{mn+s}$, $n = 0, 1, \ldots$. They satisfy the same equations (4.3), (4.4), however with the initial data $y_0 = x_s$, $\tilde{y}_0 = \tilde{x}_s$.

Moreover, y_n is a 1-cycle of (4.3). Since the product $J_{m+s-1} \cdots J_s$ is Schur stable, arguing as above we conclude that there exist there exist $M_s \geq 1$, $0 < r_s < 1$ and $h_s > 0$ such that if $\|x_s - \tilde{x}_s\| < h_s$ then $\|x_{nm+s} - \tilde{x}_{nm+s}\| \leq M_s r_s^n \|x_s - \tilde{x}_s\|$ for all $n \geq 0$. Define

$$M = \max_{0 \leq s \leq m-1} M_s, \quad r = \max_{0 \leq s \leq m-1} r_s, \quad h = \min_{0 \leq s \leq m-1} h_s.$$

Thus, if $\max_{0 \leq s \leq m-1} \|x_s - \tilde{x}_s\| < h$, then

$$\|x_n - \tilde{x}_n\| \leq M r^n \max_{0 \leq s \leq m-1} \|x_s - \tilde{x}_s\| \tag{4.5}$$

for all $n \geq 0$.

Continuous differentiability implies that $Q(\cdot)$ is locally Lipschitz. Then for every x_k, $0 \leq k \leq m-1$, there exist a sufficiently small number $\delta_k >$ and a number $L_k > 0$ such that if $\|\tilde{x}_k - x_k\| < \delta_k$, then

$$\|\tilde{x}_{k+1} - x_{k+1}\| = \|Q(\tilde{x}_k) - Q(x_k)\| \leq L_k \|\tilde{x}_k - x_k\|.$$

Let us take

$$\delta = \min_{0 \leq k \leq m-1} \delta_k, \quad L = \max_{0 \leq k \leq m-1} L_0 \cdots L_k.$$

Hence if $\|\tilde{x}_0 - x_0\| < \delta$ then

$$\max_{0 \leq k \leq m-1} \|\tilde{x}_k - x_k\| \leq L \|\tilde{x}_0 - x_0\|. \tag{4.6}$$

Combining (4.5) and (4.6) we obtain that if $\|\tilde{x}_0 - x_0\| < \min\{\delta, h\}$ then

$$\|x_n - \tilde{x}_n\| \leq M L r^n \|x_0 - \tilde{x}_0\|, \quad n \geq 0.$$

Thus we arrive at the exponential stability of the m-cycle $\{x_n\}_{n=0}^{\infty}$. □

5. Stability of cycles of the augmented discrete-time system

Augment discrete-time system (4.1) with system (2.10). This results in system

$$x_{n+1} = Q(x_n), \tag{5.1}$$

$$t_{n+1} = t_n + \Phi(x_n). \tag{5.2}$$

that is considered in the $(p+1)$-dimensional space $\mathbb{R}^{p+1}$. As in the previous section, the function $Q(\cdot)$ may be rather general, not only of form (3.1).

Equation (5.1) (a reduced system) can be treated independently of (5.2). If $Q(\cdot)$ is continuously differentiable, Lemmas 1, 2 provide conditions for exponential stability of periodic solutions of (5.1).

However, if we consider augmented system (5.1), (5.2) it is easily seen that its solutions cannot be exponentially stable. Indeed, let $\{t_n, x_n\}$ and $\{\tilde{t}_n, \tilde{x}_n\}$ be two solutions of (5.1), (5.2) such that $\tilde{x}_0 = x_0$, but $\tilde{t}_0 \neq t_0$. Then $\tilde{t}_n - t_n \equiv \tilde{t}_0 - t_0$ and $\tilde{t}_n - t_n$ does not vanish as $n \to \infty$. Nevertheless, solutions of the augmented system can be just stable.

Theorem 2. *Assume that the function* $\Phi(\cdot)$ *is Lipschitz with some Lipschitz constant* L_Φ. *Suppose that* $\{t_n, x_n\}_{n=0}^{\infty}$ *is a solution of systems* (5.1), (5.2) *such that* $\{x_n\}$ *is an m-cycle of* (5.1). *Let* $\{x_n\}$ *be exponentially stable, i. e., there exist numbers* $\delta > 0$, $M \geq 1$ *and* $0 < r < 1$ *such that all solutions* $\{\tilde{x}_n\}$ *of* (5.1) *with* $\|x_0 - \tilde{x}_0\| < \delta$ *satisfy* (4.2) *for all* $n \geq 0$. *Then inequality* $\|x_0 - \tilde{x}_0\| < \delta$ *also implies*

$$|t_n - \tilde{t}_n| \leq |t_0 - \tilde{t}_0| + \frac{L_\Phi M}{1 - r} \|x_0 - \tilde{x}_0\|. \tag{5.3}$$

Proof. From (2.6) we get

$$|\Phi(x_k) - \Phi(\tilde{x}_k)| \le L_\Phi \|x_k - \tilde{x}_k\|$$

for all $k \ge 0$. Then $\|x_0 - \tilde{x}_0\| < \delta$ implies

$$|\Phi(x_k) - \Phi(\tilde{x}_k)| \le L_\Phi M r^k \|x_0 - \tilde{x}_0\| \quad \text{for all} \quad k \ge 0. \tag{5.4}$$

We have

$$t_n = t_0 + \sum_{k=0}^{n-1}(t_{k+1} - t_k) = t_0 + \sum_{k=0}^{n-1} \Phi(x_k), \quad \tilde{t}_n = \tilde{t}_0 + \sum_{k=0}^{n-1} \Phi(\tilde{x}_k),$$

hence

$$t_n - \tilde{t}_n = t_0 - \tilde{t}_0 + \sum_{k=0}^{n-1} (\Phi(x_k) - \Phi(\tilde{x}_k)). \tag{5.5}$$

If $\|x_0 - \tilde{x}_0\| < \delta$ then from (5.4), (5.5) we come to (5.3). □

Obviously, (4.2) and (5.3) imply (non-exponential) stability of $\{t_n, x_n\}_{n=0}^{\infty}$ considered in $\mathbb{R}^{p+1}$.

6. Orbital stability of periodic solutions of the impulsive system

In this section we will introduce a "hybrid" approach to exponential orbital stability for periodic solutions of (2.1)–(2.4) (for a non-impulsive case see [28, 29]). This approach is slightly different from the "geometrical" one [21–24, 26].

As previously, let $\|\cdot\|$ denote the Euclidean norm for vectors and its induced (spectral) norm for matrices [33]. Let $\{t_n, x(t)\}$ be a periodic solution of impulsive system (2.1)–(2.4) defined for $t \ge t_0$, $n = 0, 1, \ldots$. Assume that $\mathcal{O}$ is a *periodic orbit* of $x(t)$, namely the set

$$\mathcal{O} = \{x \in \mathbb{R}^p \ : \ (\exists t)\ t \ge t_0^- \ \wedge \ x = x(t)\}.$$

For any point $x \in \mathbb{R}^p$ define the distance between x and $\mathcal{O}$ as

$$\operatorname{dist}(x, \mathcal{O}) = \inf_{\xi \in \mathcal{O}} \|x - \xi\|. \tag{6.1}$$

Assume that the functions $\Phi(\cdot)$, $F(\cdot)$ are continuously differentiable and Lipschitz, satisfying (2.6). Let $J(x)$ be the Jacobian matrix of the right-hand side of (3.1) at a point x and estimates (2.5) be valid. For brevity, we will use notation $x_n = x(t_n^-)$, $\tilde{x}_n = x(\tilde{t}_n^-)$, $n = 0, 1, \ldots$ for solutions $\{t_n, x(t)\}$, $\{\tilde{t}_n, \tilde{x}(t)\}$ of (2.1)–(2.4).

Theorem 3. *Consider a solution $\{t_n, x(t)\}$ of (2.1)–(2.4) with initial values $\{t_0, x(t_0^-) = x_0\}$ that is an m-cycle for some $m \ge 1$. Let the matrix product $J(x_{m-1}) \cdots J(x_0)$ be Schur stable. Then their exist constant numbers $C_0 > 0$, $C > 0$, $0 < r < 1$, $\Delta_0 > 0$, $\Delta > 0$ such that for any other solution $\{\tilde{t}_n, \tilde{x}(t)\}$ of (2.1)–(2.4) with initial values $\{\tilde{t}_0, \tilde{x}(\tilde{t}_0^-) = \tilde{x}_0\}$ satisfying inequalities*

$$|\tilde{t}_0 - t_0| < \Delta_0, \quad \|\tilde{x}_0 - x_0\| < \Delta \tag{6.2}$$

it follows that

$$|\tilde{t}_n - t_n| \le |\tilde{t}_0 - t_0| + C_0 \|\tilde{x}_0 - x_0\|, \tag{6.3}$$

$$\sup_{t \in D_n} \operatorname{dist}(\tilde{x}(t), O) \le C r^n \|\tilde{x}_0 - x_0\|, \quad D_n = [\tilde{t}_n, \tilde{t}_{n+1}), \tag{6.4}$$

for all $n \ge 0$.

Proof. Consider discrete-time systems (2.9), (2.10) related to (2.1)–(2.4). From Lemmas 1, 2 we conclude that the sequence $\{x_n\}$ is an exponentially stable solution of (2.9).

Thus there exist numbers $\delta > 0$, $M \ge 1$ and $0 < r < 1$ such that all solutions $\{\tilde{x}_n\}$ of (2.9) with $\|x_0 - \tilde{x}_0\| < \delta$ satisfy (4.2) for all $n \ge 0$. Then from Theorem 2 we get that inequality (6.3) is valid with

$$C_0 = \frac{L_\Phi M}{1 - r}.$$

Let us choose Δ, Δ_0 so small that

$$\Delta \le \delta, \quad \Delta_0 + C_0 \Delta < \tfrac{1}{2}\Phi_1. \tag{6.5}$$

If inequalities (6.2) are valid, the second inequality (6.5) implies $2|\tilde{t}_n - t_n| \le \Phi_1$ for all $n \ge 0$. It follows

$$t_{n+1} - t_n > (\tilde{t}_{n+1} - t_{n+1}) + (t_n - \tilde{t}_n) \tag{6.6}$$

for all $n \ge 0$. Inequality (6.6) can be also rewritten as

$$\tilde{t}_{n+1} + t_n - t_{n+1} < t_{n+1} - t_n + \tilde{t}_n, \tag{6.7}$$

which implies

$$(\tilde{t}_n,\ \tilde{t}_{n+1}) \subset (\tilde{t}_n,\ t_{n+1} - t_n + \tilde{t}_n) \cup (\tilde{t}_{n+1} + t_n - t_{n+1},\ \tilde{t}_{n+1}). \tag{6.8}$$

Let n be an arbitrarily integer, $n \ge 0$ and t is a number, $\tilde{t}_n < t < \tilde{t}_{n+1}$. The following two cases will be considered.

(i) Firstly, let $\tilde{t}_n < t < t_{n+1} - t_n + \tilde{t}_n$, which follows

$$t_n < t + t_n - \tilde{t}_n < t_{n+1}.$$

Thus

$$\tilde{x}(t) = \mathrm{e}^{A(t-\tilde{t}_n)}(\tilde{x}_n + F(\tilde{x}_n)), \qquad x(t + t_n - \tilde{t}_n) = \mathrm{e}^{A(t-\tilde{t}_n)}(x_n + F(x_n))$$

and

$$\|\tilde{x}(t) - x(t + t_n - \tilde{t}_n)\| \le \mathrm{e}^{\|A\|\,|t-\tilde{t}_n|}(\|\tilde{x}_n - x_n)\| + \|F(\tilde{x}_n) - F(x_n)\|) \le \mathrm{e}^{\|A\|\,\Phi_2}(1 + L_F)\|x_n - \tilde{x}_n\|.$$

Hence

$$\operatorname{dist}(\tilde{x}(t), O) \le \mathrm{e}^{\|A\|\,\Phi_2}(1 + L_F)\|x_n - \tilde{x}_n\|. \tag{6.9}$$

(ii) Let $\tilde{t}_{n+1} + t_n - t_{n+1} < t < \tilde{t}_{n+1}$. Then

$$t_n < t + t_{n+1} - \tilde{t}_{n+1} < t_{n+1}$$

and

$$\tilde{x}(t) = \mathrm{e}^{A(t-\tilde{t}_{n+1})}\tilde{x}_{n+1}, \qquad x(t + t_{n+1} - \tilde{t}_{n+1}) = \mathrm{e}^{A(t-\tilde{x}_{n+1})}x_{n+1}.$$

Hence

$$\|\tilde{x}(t) - x(t + t_{n+1} - \tilde{t}_{n+1})\| \le \mathrm{e}^{\|A\|\,\Phi_2}\|\tilde{x}_{n+1} - x_{n+1}\|$$

and

$$\operatorname{dist}(\tilde{x}(t), O) \le \mathrm{e}^{\|A\|\,\Phi_2}\|x_{n+1} - \tilde{x}_{n+1}\|. \tag{6.10}$$

Since $0 < r < 1$, estimates (6.9), (6.10) and (4.2) imply (6.4) with

$$C = M(1 + L_F)\,\mathrm{e}^{\|A\|\,\Phi_2}.$$

The proof is complete. □

Notice that (6.4) implies

$$\operatorname{dist}(\tilde{x}(t), O) \to 0 \quad \text{as} \quad t \to +\infty.$$

7. Example

Periodic solutions of dynamical systems are of major interest for characterization of physiological rhythms [34–37]. Many of these rhythms result from regulatory processes in biological organs. If neurons of the brain are involved into this mechanism, the processes become impulsive (see, e. g., [38, 39]) and impulsive differential equations seem to be an adequate mathematical tool for describing them. To reduce the model's complexity and to elucidate regulatory mechanisms lying behind the model, an ensemble of synchronized neurons is sometimes replaced by a consolidated impulsive oscillator [38, 40]. This idea was realized in a endocrinological model proposed in [1].

Consider a mathematical model of a hypothalamus-driven hormonal system that is given by impulsive equations [1]

$$\begin{aligned} \dot{x}_1 &= -\alpha x_1(t), \quad t_n < t < t_{n+1},\\ \dot{x}_2 &= -\beta x_2(t) + g_1 x_1(t),\\ \dot{x}_3 &= -\gamma x_3(t) + g_2 x_2(t), \end{aligned} \tag{7.1}$$

where $x_1(t)$, $x_2(t)$, $x_3(t)$ are scalar functions. Assume that the functions $x_2(t)$, $x_3(t)$ are continuous and $x_1(t)$ has jumps at some times $\{t_n\}_{n=0}^{\infty}$ with

$$x_1(t_n^+) - x_1(t_n^-) = F_0(x_3(t_n)), \quad t_{n+1} = t_n + \Phi_0(x_3(t_n)). \tag{7.2}$$

Here $F_0(\cdot)$, $\Phi_0(\cdot)$ are Hill functions [41]

$$F_0(y) = k_1 + k_2\frac{(y/h)^q}{1 + (y/h)^q}, \quad \Phi_0(y) = k_3 + \frac{k_4}{1 + (y/h)^q}, \tag{7.3}$$

all the parameters k_1, k_2, k_3, k_4, h, q are positive, q is integer.

The biological sense of this model is the following. The function $x_1(t)$ describes the concentration in the blood of a hypothalamic hormone, which is released impulsively from an ensemble of neurons of hypothalamus. The function $x_2(t)$ is the concentrations of a pituitary hormone, and $x_3(t)$ is the concentration of a hormone released from a controlled gland involved in regulation (such as liver, pancreas or gonads). The coefficients α, β, γ characterize hormonal clearance rates, g_1, g_2 — secretion rates. All the parameters are positive. The hypothalamic hormone stimulates the secretion of the pituitary hormone, the pituitary hormone stimulates the secretion of the controlled gland's hormone. The latter hormone inhibits the neural activity and the impulsive release of the hypothalamic hormone — the impulses become sparser and their amplitudes decrease. (The inhibitory effect is seen from the fact that $F_0(y)$ is decreasing and $\Phi_0(y)$ is increasing for $y > 0$.) Thus the system attains a dynamical equilibrium (homeostasis) that is characterized by some periodic oscillation.

Equations (7.1), (7.2) can be rewritten in the form of (2.1)–(2.4) with

$$A = \begin{bmatrix} -\alpha & 0 & 0 \\ g_1 & -\beta & 0 \\ 0 & g_2 & -\gamma \end{bmatrix}, \quad F(x) = BF_0(Cx), \quad \Phi(x) = \Phi_0(Cx), \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad C = [0\ 0\ 1].$$

Thus the map $Q(x)$ defined by (3.1) becomes

$$Q(x) = \mathrm{e}^{A\Phi_0(Cx)}(x + BF_0(Cx))$$

with the Jacobian matrix

$$J(x) = \mathrm{e}^{A\Phi_0(Cx)}(I + F_0'(Cx)BC) + \Phi_0'(Cx)AQ(x)C.$$

(Here I is the identity matrix of order 3.) For a 1-cycle we have

$$x_0 = Q(x_0),$$

i. e., the initial value is a fixed point of the map $Q(\cdot)$, while for a 2-cycle we obtain two equations

$$x_1 = Q(x_0), \quad x_0 = Q(x_1), \quad x_0 \neq x_1. \tag{7.4}$$

Let us consider a numerical example with

$$\alpha = 0.006, \quad \beta = 0.15, \quad \gamma = 0.2. \quad g_1 = 2, \quad g_2 = 0.5.$$

Assume that

$$k_1 = 60, \quad k_2 = 40, \quad k_3 = 3, \quad k_4 = 2, \quad h = 2.7, \quad q = 2\,.$$

Then Eqs. (7.4) have a solution

$$x_0 = \begin{bmatrix} 0.010128064492541 \\ 0.225030919954899 \\ 0.803444772662005 \end{bmatrix}, \quad x_1 = \begin{bmatrix} 0.108959325614995 \\ 2.413175576714000 \\ 8.568352684689630 \end{bmatrix}$$

and

$$J(x_0) = \begin{bmatrix} -0.000963588946497 & -0.000000423878714 & 0.002413366480393 \\ -0.021404733495303 & -0.000009416058032 & 0.053609427810324 \\ -0.076392335501037 & -0.000033606710143 & 0.191329052639657 \end{bmatrix},$$

$$J(x_1) = \begin{bmatrix} -0.013853946141307 & -0.000000297418098 & 0.002627810354290 \\ -0.305492835883428 & -0.000006558568407 & 0.057945744483940 \\ -1.076639814491772 & -0.000023115469451 & 0.204216552344842 \end{bmatrix}.$$

The product of the Jacobian matrices $J(x_1)J(x_0)$ is Schur stable with the eigenvalues

$$\lambda_1 = 0.190356046517081, \quad \lambda_2 = 0.000000001118046, \quad \lambda_3 = 0.000000000000000.$$

Thus we arrived at an orbitally stable 2-cycle that is visualized in Figure 1. This type of dynamical behavior is observed, e. g., in growth hormone's profiles [42].

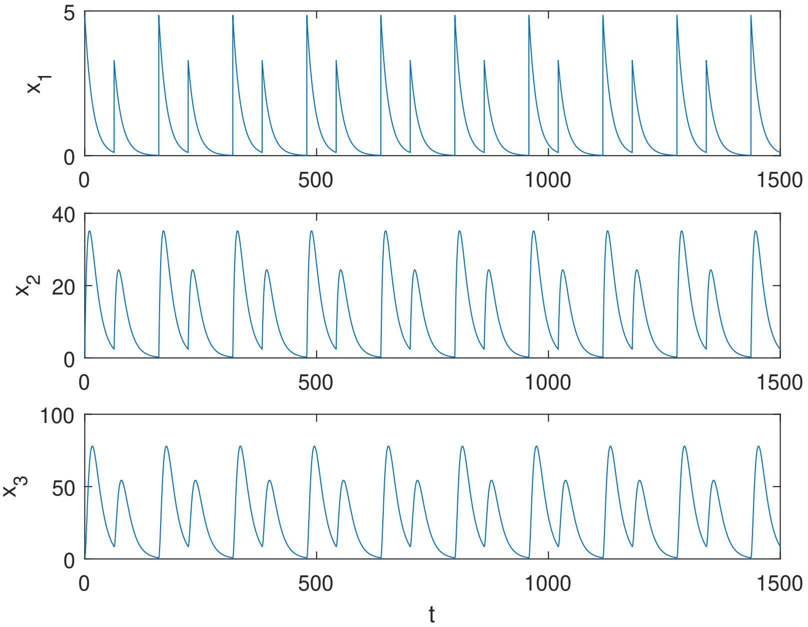

Figure 1. Graphs of $x_1(t)$, $x_2(t)$, $x_3(t)$ corresponding to a stable 2-cycle for $\alpha = 0.006$.

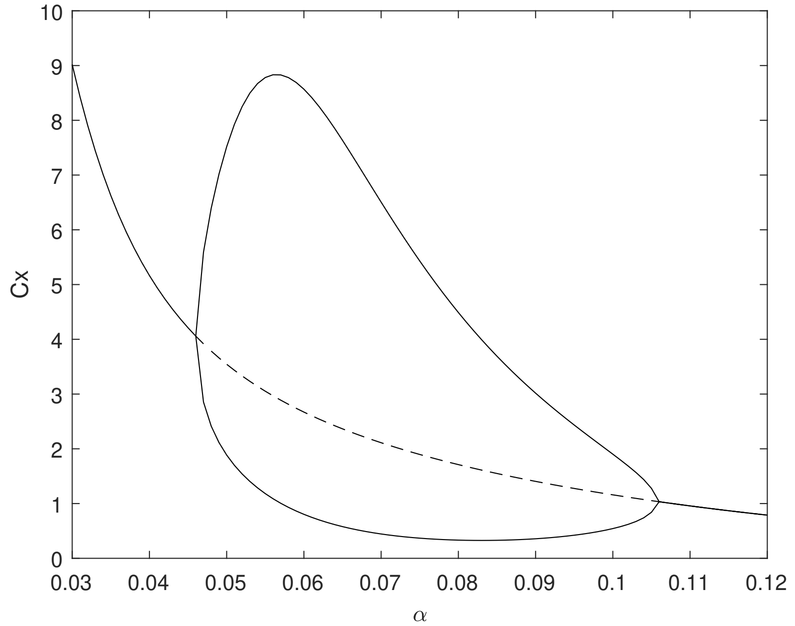

Figure 2. A bifurcation diagram for varying α.

When α varies in the range $0 < \alpha < 0.12$ we get a one-parametric bifurcation diagram shown in Figure 2 [1]. For $\alpha < 0.045$ and $\alpha > 0.11$ we obtain an orbitally stable 1-cycle (the value $Cx = Cx_0$), while for $0.05 < \alpha < 0.10$ there exists an orbitally stable 2-cycle (the values Cx correspond to Cx_0 and Cx_1). The dashed line indicates an unstable 1-cycle. Thus when α increases, it passes through two bifurcation points — for period doubling and for period halving. Such bubble-type diagrams seem to be conventional for physiological oscillations (see [36]). In [43] it was demonstrated that with a suitable choice of parameters, system (7.1)–(7.3) can possess m-cycles for higher values of m and attain a deterministic chaos.

An understanding of hormonal rhythms opens the way to their administration. In review [37] this problem was named one of the main challenges for the interaction between nonlinear dynamics and medicine.

8. Conclusion

The paper considers a stability problem for an impulsive system with a linear dynamics between impulses. The magnitudes and times of jumps are variable and depend on the system's states. We are interested in stable periodical solutions of these impulsive systems. The main characteristic of such a solution is a number of jumps per period, which can be obtained by the study of fixed points of auxiliary discrete-time maps. In this paper we propose a new notion of exponential orbital stability that combines the known stability notions for impulsive and discrete-time systems. An easily verifiable criterion for such a stability is provided.

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Conflict of interest

The author declares that there is no conflict of interests regarding the publication of this paper.

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