Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 5(1): 650–672.
DOI:10.3934/math.2020044
Received: 23 September 2019
Accepted: 29 November 2019
Published: 17 December 2019

Research article

Relatively equi-statistical convergence via deferred Nörlund mean based on difference operator of fractional-order and related approximation theorems

B. B. Jena[1], **S. K. Paikray**[1,*], **S. A. Mohiuddine**[2,3] **and Vishnu Narayan Mishra**[4]

[1] Department of Mathematics, Veer Surendra Sai University of Technology, Burla 768018, Odisha, India

[2] Department of General Required Courses, Mathematics, Faculty of Applied Studies, King Abdulaziz University, Jeddah 21589, Saudi Arabia

[3] Operator Theory and Applications Research Group, Department of Mathematics, King Abdulaziz University, Jeddah 21589, Saudi Arabia

[4] Department of Mathematics, Indira Gandhi National Tribal University, Lalpur, Amarkantak, Anuppur, Madhya Pradesh 484 887, India

* **Correspondence:** Email: skpaikray_math@vssut.ac.in.

Abstract: In the proposed paper, we have introduced the notion of point-wise relatively statistical convergence, relatively equi-statistical convergence and relatively uniform statistical convergence of sequences of functions based on the difference operator of fractional order including (p, q)-gamma function via the deferred Nörlund mean. As an application point of view, we have proved a Korovkin type approximation theorem by using the relatively deferred Nörlund equi-statistical convergence of difference sequences of functions and intimated that our theorem is a generalization of some well-established approximation theorems of Korovkin type which was presented in earlier works. Moreover, we estimate the rate of the relatively deferred Nörlund equi-statistical convergence involving a non-zero scale function. Furthermore, we use the modulus of continuity to estimate the rate of convergence of approximating positive linear operators. Finally, we set up various fascinating examples in connection with our results and definitions presented in this paper.

Keywords: deferred Nörlund mean; relatively statistical uniform convergence; relatively $\psi_n^{p,q}$-equi-statistical convergence; Korovkin-type approximation theorem; rate of the relatively $\psi_n^{p,q}$-equi-statistical convergence
Mathematics Subject Classification: 40A05, 41A36, 40G15

1. Introduction and preliminaries

The rapid growth of sequence spaces has been accompanied by the recent works of many researchers in the field of statistical convergence and it has got tremendous importance over conventional convergence. The basic idea of statistical convergence was initially studied in the year 1951 by Fast [18] and Steinhaus [52] independently. Recently, the approximation of functions by positive linear operators based on statistical convergence and statistical summability has become a dynamic area of research. Moreover, very recently the introduction of equi-statistical convergence, point-wise statistical convergence and uniform statistical convergence of sequences of functions by Srivastava *et al.* (see [45, 51]) has enriched the scope of current research in this direction. As of late, statistical convergence has been very wide and fruitful field for the application in several branches of Analysis; in particular, Approximation Theory, Fourier Analysis, Operator Theory, Theory of Orthogonal series, Number Theory and so on. For more various approaches to statistical convergence and statistical summability, we refer to the current works [22, 24, 25, 30, 31, 40, 44].

Let $K \subseteq \mathbb{N}$ and

$$K_n = \{k : k \leqq n \quad \text{and} \quad k \in K\}.$$

The asymptotic (or natural) density (see [18, 52]) of K is given by

$$d(K) = \lim_{n\to\infty} \frac{1}{n} |\{k : k \leqq n \quad \text{and} \quad k \in K\}| = \xi,$$

presuming that the limit exists, where $|K_n|$ be the cardinality of K_n. A given sequence (x_n) is statistically convergent to ℓ if, for every $\epsilon > 0$,

$$K_\epsilon = \{k : k \in \mathbb{N} \quad \text{and} \quad |x_k - \ell| \geq \epsilon\}$$

has asymptotic (or natural) density zero. That is, for every $\epsilon > 0$,

$$d(K_\epsilon) = \lim_{n\to\infty} \frac{1}{n} |\{k : k \in \mathbb{N} \quad \text{and} \quad |x_k - \ell| \geqq \epsilon\}| = 0.$$

Here, we write

$$\text{stat} \lim_{n\to\infty} x_n = \ell.$$

Now, in order to justify the importance of statistical convergence over classical convergence, we present the following example.

Example 1.1. Let $x = (x_n)$ be a sequence defined by

$$x_n = \begin{cases} \frac{1}{4} & (n = m^2; \ \ m \in \mathbb{N}) \\ \\ \frac{1}{n+1} & (\text{otherwise}). \end{cases}$$

Here, the sequence (x_n) is statistically convergent to 0, even if it is not classically convergent.

The preliminary idea of weighted statistical convergence was introduced in 2009 by Karakaya and Chishti [29]. Recently, the concept was modified by Mursaleen *et al.* (see [40]). Suppose that (s_k) be a sequence of non-negative numbers and such that

$$S_n = \sum_{k=0}^{n} s_k; \quad (s_0 > 0, \ \ n \to \infty).$$

Set

$$\varphi_n = \frac{1}{S_n} \sum_{k=0}^{n} s_k x_k \qquad (n \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}),$$

then the given sequence (x_n) is weighted statistically convergent (or $\text{stat}_{\bar{N}}$-convergent) to a number ℓ if, for every $\epsilon > 0$,

$$\{k : k \leqq S_n \quad \text{and} \quad s_k|x_k - \ell| \geqq \epsilon\}$$

has weighted density zero (see [40]). That is, for every $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{S_n} |\{k : k \leqq S_n \quad \text{and} \quad s_k|x_k - \ell| \geqq \epsilon\}| = 0.$$

Here, we write

$$\text{stat}_{\bar{N}} \lim x_n = \ell.$$

In the year 2013, Belen and Mohiuddine [10] established a new technique for weighted statistical convergence in terms of the de la Vallée Poussin mean and it was subsequently investigated further by Braha *et al.* [11] as the Λ_n-Weighted statistical convergence. Recently, Mohiuddine [34] presented another variant of weighted statistical convergence by taking into account non-negative weighted regular matrix. Very recently, a certain class of equi-statistical convergence via deferred Nörlund summability mean and related approximation theorem were studied by Srivastava *et al.* (see, for details, [45]).

Next, the notion of relatively uniform convergence of a sequence of functions was introduced by Moore [38]. Later, in the similar lines it was modified by Chittenden [13] and was defined over a closed interval, $I = [a, b] \subseteq \mathbb{R}$. We recall here, the definition of uniform convergence relative to a scale function as follows. A sequence of functions (f_n), defined over $[a, b]$, converges relatively uniformly to a limit function $f(x)$ if, there exists a scale function $\sigma(x)$ ($|\sigma(x)| > 0$) defined over $[a, b]$, such that for every $\epsilon > 0$, there exists an integer n_ϵ and for every $n > n_\epsilon$,

$$\left| \frac{f_n(x) - f(x)}{\sigma(x)} \right| \leqq \epsilon$$

holds true uniformly for all $x \in [a, b] \subseteq \mathbb{R}$.

Now, to see the importance of relatively uniform convergence over classical uniform convergence, we present the following example (see [15, 37]).

Example 1.2. For each $n \in \mathbb{N}$, define $f_n : [0, 1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} \frac{nx}{1+n^2x^2} & (0 < x \leqq 1) \\ 0 & (x = 0). \end{cases}$$

The sequence (f_n) of functions is not classically uniformly convergent on $[0, 1]$; however, it is converging to $f = 0$ uniformly, relative to a scale function

$$\sigma(x) = \begin{cases} \frac{1}{x} & (0 < x \leqq 1) \\ 1 & (x = 0) \end{cases}$$

on [0, 1]. Here, we write

$$f_n \rightrightarrows f = 0 \quad ([0,1];\sigma)\,.$$

Subsequently, uniform convergence can be viewed as a special case of uniform convergence relative to the non-zero scale function (that is, relatively uniform convergence). Recently, based on the natural density of a set, Demirci and Orhan [15] introduced the notion of relatively uniform statistical convergence of a sequence of functions. Later, they have also established the definitions of relative modular convergence and relative modular statistical convergence for double sequences of measurable real-valued functions in the same year (see [16]). Furthermore, very recently, Kadak *at al.* [28] studied the theory of relatively uniform weighted $\alpha\beta$-statistical convergence and relatively uniform statistical Φ-summability based on linear difference operators of fractional-order.

Let $E \subset \mathbb{R}$ be compact and (f_n) be a sequence of functions defined on E. The sequence (f_n) is relatively uniform statistically converges to a limit function f defined on E, if there exists a scale function $\sigma(x)$ $(|\sigma(x)| > 0)$ on E such that, for each $\epsilon > 0$,

$$\left\{k : k \leqq n \;\text{ and }\; \sup_{x\in E}\left|\frac{f_k(x) - f(x)}{\sigma(x)}\right| \geqq \epsilon\right\}$$

has zero natural density, that is, for every $\epsilon > 0$, we have

$$\lim_{n\to\infty}\left|\left\{k : k \leqq n \;\text{ and }\; \sup_{x\in E}\left|\frac{f_k(x) - f(x)}{\sigma(x)}\right| \geqq \epsilon\right\}\right| = 0.$$

Here, we write

$$\text{stat} \quad f_n(x) \rightrightarrows f(E;\sigma).$$

The notion of fractional-order difference operator was firstly introduced by Chapman [12] and later on, it has been used by many researchers. For more details, see the current works [8,9,26]. In the year 2016, some new classes of fractional-order difference sequence spaces were studied by Baliarsingh in [7]. Very recently, Kadak [25] has developed the concept of weighted statistical convergence based on generalized difference sequence including (p,q)-integers. Again, Kadak [24] extended it for (p,q)-gamma function.

Balcerzak *et al.* [6] introduced the concept of equi-statistical convergence in the year 2007 and later by Aktuğlu and Gezer [4] and Mohiuddine and Alamri [35] by using the idea of lacunary. Subsequently, it was extended to the λ_n-equi-statistical convergence and accordingly some associated approximation theorems were proved by Srivastava *et al.* [51]. Furthermore, based on the equi-statistical convergence via deferred Nörlund summability mean, Srivastava *et al.* [45] has also proved certain Korovkin type approximation theorems.

Motivated essentially by the above-mentioned works, here in this paper, we introduce the concept of relatively $\psi_n^{p,q}$-equi-statistical convergence, point-wise relatively $\psi_n^{p,q}$-statistical convergence and relatively uniform $\psi_n^{p,q}$-statistical convergence for a sequence of real valued functions and show that the relatively $\psi_n^{p,q}$-equi-statistical convergence lies between the point-wise relatively $\psi_n^{p,q}$-statistical convergence and relatively uniform $\psi_n^{p,q}$-statistical convergence. Moreover, as an application point of view, we have proved a Korovkin type approximation theorem by using the relatively deferred Nörlund equi-statistical convergence of difference sequences of functions including (p,q)-gamma function and exhibited that our theorem is a generalization of some well-established approximation theorems of

Korovkin type which were presented in earlier works. Also, we estimate the rate of the relatively deferred Nörlund equi-statistical convergence involving a non-zero scale function. Furthermore, we use the modulus of continuity to estimate the rate of convergence of approximating linear positive operators. Finally, we set up various remarks and observations in connection with our results and definitions.

2. Deferred Nörlund-type mean

We now recall and introduce some definitions which are required for the present investigation. Let ω be the set of all real valued sequences and suppose any subspace of ω be the sequence space. Suppose ℓ_∞ be the class of all bounded linear spaces and let c, c_0 be the respective classes for convergent and null sequences with real and complex terms. Let (x_k) be any sequence in ω and h be a positive constant. For real numbers a, b and c. Recently, a new class of difference sequence space of fractional order has been introduced by Baliarsingh [7] and is as follows:

$$(\Delta_h^{a,b,c} x) = \sum_{i=0}^{\infty} \frac{(-a)_i(-b)_i}{i!(-c)_i h^{a+b-c}} x_{k-i} \quad (k \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}) \tag{2.1}$$

where $(\alpha)_k$ denote the *Pochhammer symbol* or *shifted factorial* of the real number α and moreover, it is defined by using the familiar Euler's gamma function as

$$(\alpha)_k = \begin{cases} 1 & (\alpha = 0 \text{ or } k = 0) \\ \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} = \alpha(\alpha+1)(\alpha+2)...(\alpha+k-1) & (k \in \mathbb{N}). \end{cases}$$

Note that the series defined in (2.1) converges for all $c > a + b$ (see [19]).

In the similar lines, he has also introduced the difference sequence space of fractional order for differentiable functions.

Let $h \to 0$ and $f(x)$ be a differentiable function with fractional order. Associated with this function $f(x)$, define the sequence

$$f_h(x) = f(x - ih)_{k \in \mathbb{N}_0}$$

and the sequence spaces $\Delta_{h,x}^{a,b,c}(f_h(x))$ via the difference operator $(\Delta_{h,x}^{a,b,c})$ as

$$\Delta_{h,x}^{a,b,c}(f_h(x)) = \sum_{i=0}^{\infty} \frac{(-a)_i(-b)_i}{i!(-c)_i h^{a+b-c}} f(x - ih). \tag{2.2}$$

Subsequently, with the development of q-calculus, various researchers worked on certain new generalizations of positive linear operators based on q-integers (see [32]). Recently, Mursaleen *et al.* [39] developed the (p, q)-analogue of Bernstein operators in connection with (p, q)-integers and later on, some approximation results for Baskakov operators and Bernstein-Schurer operators are studied for (p, q)-integers by [1, 41].

We, next recall some definitions and basic notations on (p, q)-integers for our present study. For any $(n \in \mathbb{N})$ the (p, q)-integer $[n]_{p,q}$ is defined by,

$$[n]_{p,q} = \begin{cases} \frac{p^n-q^n}{p-q} & (n \geq 1) \\ 0 & (n = 0) \end{cases}$$

where $0 < q < p \leqq 1$. The (p, q)-factorial is defined by

$$[n]!_{p,q} = \begin{cases} [1]_{p,q}[2]_{p,q}...[n]_{p,q} & (n \geqq 1) \\ 1 & (n = 0). \end{cases}$$

The (p, q)-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]!_{p,q}}{[k]!_{p,q}\,[n-k]!_{p,q}} \qquad \text{for all} \quad n, k \in \mathbb{N} \ \text{ and } \quad n \geqq k.$$

The formula for (p, q)-binomial expansion is as follows:

$$(ax + by)^n_{p,q} = \sum_{k=0}^{n} p^{\frac{(n-k)(n-k+1)}{2}} q^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} a^{n-k} b^k x^{n-k} y^k,$$

$$(x + y)^n_{p,q} = (x + y)(px + qy)(p^2x + q^2y) \cdots (p^{n-1}x + q^{n-1}y),$$
$$(1 - x)^n_{p,q} = (1 - x)(p - qx)(p^2 - q^2y) \cdots (p^{n-1} - q^{n-1}x).$$

In the year 1738, Euler first studied the gamma function in order to extend the factorial function to the real and complex numbers. The gamma function $\Gamma(x)$ is defined by an improper integral

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt \ \ (x > 0).$$

Integration by parts gives the important functional relation of the gamma function,

$$\Gamma(x + 1) = x\Gamma(x). \tag{2.3}$$

If x is positive say k, then (2.3) gives

$$\Gamma(k + 1) = k!.$$

In the year 1904, Jackson [20] studied the notion of q-gamma function, as an extension of ordinary gamma function. It is defined by

$$\Gamma_q(x) = (1 - q)^{1-x} \prod_{n=0}^{\infty} \frac{1 - q^{n+1}}{1 - q^{n+x}} = (1 - q)^{1-x} \frac{(q; q)_\infty}{(q^x; q)_\infty} \quad (0 < q < 1),$$

where the symbol $(.; .)$ is called the infinite q-Pochhammer and also called the q-shifted factorial, it is defined by

$$(a; q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n).$$

If limit $q \to 1$, $\Gamma_q(x)$ reduces to the general gamma function $\Gamma(x)$ and it also fulfils the property that

$$\Gamma_q(x+1) = \frac{1-q^x}{1-q}\Gamma_q(x) \ \ (x>0)$$

and for any positive integers k such that $\Gamma_q(k) = [k-1]!_q$, where the symbol $[.]!_q$ is the q-factorial function.

Subsequently, the (p,q)-gamma function has been introduced as an extension of q-gamma function (see [43]). For $0 < q < p \leqq 1$, the (p,q)-gamma function is defined by

$$\Gamma_{p,q}(x) = (p-q)^{1-x}\prod_{n=0}^{\infty}\frac{p^{n+1}-q^{n+1}}{p^{n+x}-q^{n+x}} = (p-q)^{1-x}\frac{(p;q)^{\infty}_{p,q}}{(p^x;q^x)^{\infty}_{p,q}} \ \ (0<q<p\leqq 1).$$

where

$$(a;b)^{\infty}_{p,q} = \prod_{n=0}^{\infty}(ap^n - bq^n) \ \ (0 < a, b < 1)$$

and the symbol $(.;.)$ is called the (p,q)-Pochhammer or (p,q)-shifted factorial. Here if $p = 1$, then the $\Gamma_{p,q}(x)$ reduces to the $\Gamma_q(x)$.

Definition 2.1. Let the sequence $(f_h(x))$ in ω, $0 < q < p \leqq 1$ and a, b, c be a non-negative integer, and let h is positive constant. We define a generalized difference sequence under the difference operator

$$\Delta^{a,b,c}_{h,p,q} : \omega \to \omega$$

by

$$\Delta^{a,b,c}_{h,p,q}(f_h(x)) = \sum_{i=0}^{\infty}(-1)^i\frac{(a)^i_{p,q}\,(b)^i_{p.q}}{(i)_{p,q}!\,(c)^i_{p,q}\,h^{a+b-c}}f(x-ih) \ \ (k\in\mathbb{N}), \tag{2.4}$$

where $(s)^i_{p,q}$ is (p,q)-shifted factorial of non-negative integers s which is being defined (p,q)-gamma function as

$$(s)^i_{p,q} = \begin{cases} 1 & (s=0 \text{ or } i=0) \\ \frac{\Gamma_{p,q}(s+1)}{\Gamma_{p,q}(s-i+1)} = [s]_{p,q}[s-1]_{p,q}[s-2]_{p,q}...[s-i+1]_{p,q} & (s\in\mathbb{N}). \end{cases}$$

That is,

$$\begin{aligned}\Delta^{a,b,c}_{h,p,q}(f_h(x)) &= \sum_{i=0}^{\infty}(-1)^i\frac{(a)^i_{p,q}\,(b)^i_{p.q}}{(i)_{p,q}!\,(c)^i_{p,q}\,h^{a+b-c}}f(x-ih)\\ &= \frac{1}{h^{a+b-c}}f(x) - \frac{[a]_{p,q}[b]_{p,q}}{[c]_{p,q}h^{a+b-c}}f(x-h) + \frac{[a]_{p,q}[a-1]_{p,q}[b]_{p,q}[b-1]_{p,q}}{[c]_{p,q}[c-1]_{p,q}[2]_{p,q}!h^{a+b-c}}f(x-2h) - ...\\ &\quad +(-1)^r\frac{(a)^r_{p,q}(b)^r_{p,q}}{r!(c)^r_{p,q}h^{a+b-c}}f(x-rh) + ...\end{aligned}$$

$$= \frac{1}{h^{a+b-c}} f(x) - \frac{(p^a - q^a)(p^b - q^b)}{(p^c - q^c)(p - q)h^{a+b-c}} f(x - h)$$

$$+ \frac{(p^a - q^a)(p^{a-1} - q^{a-1})(p^b - q^b)(p^{b-1} - q^{b-1})}{(p^c - q^c)(p^{c-1} - q^{c-1})(p - q)^2(p + q)h^{a+b-c}} f(x - 2h) - \cdots.$$

Throughout the paper, we assume that the series defined in (2.4) converges for all $c > a + b$ (see [19]).

Example 2.2. Let $f(x) = x$, $(\forall\, x \in \mathbb{N} \subset \mathbb{R})$, and we choose $a = 2$, $b = c$, we have

$$\Delta^{2,b,b}_{1,p,q} f(x) = \sum_{i=0}^{2} (-1)^i \frac{(2)^i_{p,q}}{(i)_{p,q}!} f(x - i).$$

Clearly, depending on the choice of the values of p and q, the difference sequence $\Delta^{2,b,b}_{1,p,q}(f(x))$ of third order has different limits. This situation is due to the definition of (p, q)-integers. However, in order to obtain a convergence criterion for all values of p and q, belonging to the operator $\Delta^{2,b,b}_{1,p,q}$, we must have to overcome this difficulty. This type of difficulties can be avoided in the following two ways. The first one is taking $p = q = 1$ and thus the operator reduces to the usual difference sequence. Next, the second way is to replace $p = p_n$ and $q = q_n$ under the limits, $\lim_n q_n = \alpha$ and $\lim_n p_n = \beta$ $(0 \leq \alpha, \beta \leq 1)$ where $0 < q_n < p_n \leqq 1$, for all $(n \in \mathbb{N})$. Afterwards, the difference sequence $\Delta^{a,b,c}_{1,p,q}(f(x))$ of third order 2, converges to the value $3 - (\beta^2 + \alpha\beta + \alpha^2)$. Thus, if we take $q_n = \left(\frac{n}{n+s}\right) < \left(\frac{n}{n+t}\right) = p_n$ such that $0 < q_n < p_n \leqq 1$ $(s > t > 0)$, then $\lim_n q_n = 1 = \lim_n p_n$ and hence $\Delta^{2,b.b}_{1,p,q}(f(x)) \to 0$.

Remark 2.3. If $\lim_n q_n = 1$ and $\lim_n p_n = 1$, then the difference operator $\Delta^{a,b,c}_{h,p,q}$ reduces to the $\Delta^{a,b,c}_{h}$ (see [28]).

Next, before presenting the notion of the deferred Nörlund mean under the generalized difference operator of fractional order, including (p, q)-gamma function, we recall the regularity condition as follows: Let (a_n) and (b_n) be sequences of non-negative integers fulfilling the conditions: (a) $a_n < b_n$ $(n \in \mathbb{N})$ and (b) $\lim\limits_{n\to\infty} b_n = \infty$. Note that, the conditions (a) and (b) are regularity conditions for the proposed deferred Nörlund mean (see Agnew [3]).

We, now assume that (s_n) and (t_n) be the sequences of non-negative real numbers such that

$$S_n = \sum_{m=a_n+1}^{b_n} s_m \quad \text{and} \quad T_n = \sum_{m=a_n+1}^{b_n} t_m.$$

The convolution of the above sequences can be presented as (see [45]),

$$R^{b_n}_{a_n+1} = (S * T)_{b_n} = \sum_{v=a_n+1}^{b_n} s_v t_{b_n - v}.$$

Now, for defining the sequence (f_n) of function as deferred Nörlund $(D^b_a(N, s, t)$ mean by the fractional difference operator $(\Delta^{a,b,c}_{h,p,q})$ including (p, q)-gamma function, we first set,

$$\psi^{p,q}_n = \frac{1}{R^{b_n}_{a_n+1}} \sum_{m=a_n+1}^{b_n} s_{b_n - m} t_m \left(\Delta^{a,b,c}_{h,p,q} f_m(x)\right). \tag{2.5}$$

We say that the sequence (f_n) of functions is deferred Nörlund summable to a number ℓ by the operator $\Delta^{a,b,c}_{h,p,q}$ if,

$$\lim_{n\to\infty} \psi_n^{p,q} = \ell.$$

3. Definitions and associated inclusion relations

In the earlier works of Srivastava *et al.* [45] the definitions of point-wise statistical convergence, equi-statistical convergence and uniform statistical convergence via the deferred Nörlund mean were introduced. In the similar lines, here we introduce the definitions of point-wise relatively statistical convergence, relatively equi-statistical convergence and relatively uniform statistical convergence via the deferred Nörlund mean. Let $C(E)$ is the space of continuous real-valued functions defined over E. Also, it is a Banach space.

We, now introduce the following new definitions which are required for the present investigation.

Definition 3.1. Let $0 < q_n < p_n \leq 1$ such that $\lim_n q_n = \alpha$ and $\lim_n p_n = \beta$ $(0 < \alpha, \beta \leq 1)$ and let r is a non-negative integer. Also, let (a_n) and (b_n) be sequences of non-negative integers. A sequence (f_n) of functions is point-wise relatively $\psi_n^{p,q}$-statistical convergent to a function f on E if, there exists a scale function $\sigma(x)$ $(|\sigma(x)| > 0)$ on E such that, for every $\epsilon > 0$ and for every $x \in E$,

$$\lim_{n\to\infty} \frac{\Lambda_n^{p,q}(x; \epsilon\sigma)}{R^{b_n}_{a_n+1}} = 0,$$

where

$$\Lambda_n^{p,q}(x; \epsilon\sigma) = \left|\left\{ m : m \leqq R^{b_n}_{a_n+1} \text{ and } s_{b_n-m}t_m \left| \frac{\Delta^{a,b,c}_{h,p,q} f_m(x) - f(x)}{\sigma(x)} \right| \geqq \epsilon \right\}\right|.$$

Here, we can write

$$f_n \to f \ \ (\psi_n^{p,q}\text{-pointwise-relatively-stat}).$$

Definition 3.2. Let $0 < q_n < p_n \leq 1$ such that $\lim_n q_n = \alpha$ and $\lim_n p_n = \beta$ $(0 < \alpha, \beta \leq 1)$ and let r is a non-negative integer. Also, let (a_n) and (b_n) be sequences of non-negative integers. A sequence (f_n) of functions is relatively $\psi_n^{p,q}$-equi-statistical convergent to a function f defined on E if, there exists a scale function $\sigma(x)$ $(|\sigma(x)| > 0)$ on E such that, for every $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{\Lambda_n^{p,q}(x; \epsilon\sigma)}{R^{b_n}_{a_n+1}} = 0,$$

uniformly relatively with respect to $x \in E$, that is,

$$\lim_{n\to\infty} \frac{\|\Lambda_n^{p,q}(x; \epsilon\sigma)\|_{C(E)}}{R^{b_n}_{a_n+1}} = 0,$$

where

$$\Lambda_n^{p,q}(x; \epsilon\sigma) = \left|\left\{ m : m \leqq R^{b_n}_{a_n+1} \text{ and } s_{b_n-m}t_m \left| \frac{\Delta^{a,b,c}_{h,p,q}(f_m(x)) - f(x)}{\sigma(x)} \right| \geqq \epsilon \right\}\right|.$$

Here, we write

$$f_n \to f \ (\psi_n^{p,q}\text{-relatively-equi-stat}).$$

Definition 3.3. Let $0 < q_n < p_n \leq 1$ such that $\lim_n q_n = \alpha$ and $\lim_n p_n = \beta$ $(0 < \alpha, \beta \leq 1)$ and let r is a non-negative integer. Also, let (a_n) and (b_n) be sequences of non-negative integers. A sequence (f_n) of functions is relatively uniform $\psi_n^{p,q}$-statistical convergent to a function f on E if, there exists a scale function $\sigma(x)$ $(|\sigma(x)| > 0)$ on E such that, for every $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{\Lambda_n^{p,q}(x;\epsilon\sigma)}{R_{a_n+1}^{b_n}} = 0,$$

where

$$\Lambda_n^{p,q}(x;\epsilon\sigma) = \left|\left\{m : m \leqq R_{a_n+1}^{b_n} \text{ and } s_{b_n-m}t_m \left\|\frac{\Delta_{h,p,q}^{a,b,c}(f_m(x)) - f(x)}{\sigma(x)}\right\|_{C(E)} \geqq \epsilon\right\}\right|.$$

Here, we write

$$f_n \rightrightarrows f \ (\psi_n^{p,q}\text{-relatively-uniform-stat}).$$

As a consequence of the above definitions, we present the following inclusion relations (without proof in the form of a Lemma) in the similar lines of our earlier work (see [4, 35, 45]).

Lemma 3.4. *The implications,*

$$f_n \rightrightarrows f \ (\psi_n^{p,q}\text{-}\textit{relatively-uniform-stat}) \Longrightarrow f_n \to f \ (\psi_n^{p,q}\text{-}\textit{relatively-equi-stat})$$

$$\Longrightarrow \ f_n \to f \ (\psi_n^{p,q}\text{-}\textit{pointwise-relatively-stat}) \tag{3.1}$$

are fairly true.

Moreover, reverse the implications of (3.1) are not necessarily true, that is, the above inclusions are strict.

The following examples justifies the strictness of the implications of our Lemma 3.4.

Example 3.5. The present example shows the strictness of the first inclusion under (3.1). We have, for

$$a_n = 0, \ \ b_n = n, \ \ s_{b_n-m}t_m = 1, \ \ \sigma(x) = 1;$$

let $f_n : E \to \mathbb{R}$, $n \in \mathbb{N}$ be the real sequence of continuous functions defined on $E = [0, 1]$ (see [4]) is given as,

$$\Delta_{h,p,q}^{a,b,c} f_n(x) = \begin{cases} -4n^2(n+1)^2\left(x - \frac{1}{n}\right)\left(x - \frac{1}{n+1}\right) & \left(x \in \left(\frac{1}{n+1}, \frac{1}{n}\right]\right) \\ \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

Then, for every $\epsilon > 0$,

$$\frac{1}{R_{a_n+1}^{b_n}} \left|\left\{m : m \leqq R_{a_n+1}^{b_n} \text{ and } s_{b_n-m}t_m \left|\frac{\Delta_{h,p,q}^{a,b,c} f_m(x) - f(x)}{\sigma(x)}\right| \geqq \epsilon\right\}\right| \leqq \frac{1}{R_{a_n+1}^{b_n}} \to 0 \ \ (n \to \infty)$$

uniformly on E. This implies

$$f_n \to 0 \ (\psi_n^{p,q}\text{-relatively-equi-stat}).$$

But since,

$$\sup_{x\in[0,1]}\left|\frac{\Delta^{a,b,c}_{h,p,q}f_n(x)}{\sigma(x)}\right| = 1 \ \ (n \in \mathbb{N}),$$

we see that

$$f_n \rightrightarrows 0 \ \ (\psi_n^{p,q}\text{-relatively-uniform-stat})$$

does not fairly true.

Example 3.6. The present example shows the strictness of the first inclusion under (3.1). We have, for

$$a_n = 0, \ \ b_n = n, \ \ s_{b_n-m}t_m = 1;$$

let $f_n : E \to \mathbb{R}, \ n \in \mathbb{N}$ be the real sequence of continuous functions defined over $E = [0, 1]$ is given as

$$\Delta^{a,b,c}_{h,p,q}f_n(x) = x^{n-1} \ \ \textit{and} \ \ \sigma(x) = \frac{1}{x}.$$

We can see that (f_n) converges relatively point-wise to f, where

$$f(x) = \begin{cases} 0 & (0 \leqq x < 1) \\ 1 & (x = 1). \end{cases}$$

Hence

$$f_n(x) \to f(x) \quad (\psi_n^{p,q}\text{-pointwise-relatively-stat}).$$

If we take $\epsilon = \frac{1}{3}$ then $\forall \ n \in \mathbb{N}$, there exists $r > n$ such that, for every $m \in [3^{r-1}, 3^r)$ and $x \in \left(\sqrt[3r]{\frac{1}{3}}, 1\right)$, we have

$$\left|\frac{\Delta^{a,b,c}_{h,p,q}f_m(x)}{\sigma(x)}\right| = |x^m| > \left|\left(\sqrt[3r]{\frac{1}{3}}\right)^m\right| > \left|\left(\frac{1}{3}\right)^{\frac{3r}{3r}}\right| = \frac{1}{3}. \tag{3.3}$$

Thus, clearly we have the following condition,

$$f_n \to 0 \quad (\psi_n^{p,q}\text{-relativey-equi-stat})$$

does not fairly true.

4. A Korovkin-type theorem for relatively equi-statistical convergence via $\psi_n^{p,q}$-mean

Recently, a few researchers worked on extending the approximation theorems of Korovkin-type in several ways on the basis of various aspects, involving (for instance) function spaces, abstract Banach lattices, and so on. This well established theory is highly valuable in analysis and many other fields. The main concern of this paper is to introduce the notion of point-wise relatively $\psi_n^{p,q}$-statistical convergence, relatively $\psi_n^{p,q}$-equi-statistical convergence and relatively uniform $\psi_n^{p,q}$-statistical convergence with respect to the difference operator of fractional order, including (p, q)-gamma function, and then to establish some associated approximation type results in relevance to our presumably defined new concept of relatively $\psi_n^{p,q}$-equi-statistical convergence, that will effectively extend and improve most (if not all) of the existing results depending on the choice of deferred Nörlund $\psi_n^{p,q}$-mean. Furthermore, based upon the proposed methods, we wish to approximate the order of convergence and to investigate a Korovkin type approximation result for a sequence of functions. In fact, we extend here the result of Srivastava *et al.* [45] by using the notion of the relatively $\psi_n^{p,q}$-equi-statistical convergence for the generalized difference operator of fractional order, including (p,q)-gamma function and prove the following theorem. For more various approaches in this direction, we refer to the current works [2,11,14–17,21,23–25,27,30,33,36,42,44–49].

Let $\mathfrak{L} : C[a,b] \to C[a,b]$ be a positive linear operator, where $C[a,b]$ is the linear space of all continuous real-valued functions defined over $[a,b]$. Also, $C[a,b]$ is a Banach space and for $f \in C[a,b]$, the sup norm of f, is given by

$$\|f\|_\infty = \sup_{x\in[a,b]} f(x).$$

In the similar lines of the well established approximation theorems of Korovkin-type (see [31,45]), here we present the following theorem for relatively equi-statical convergence via our proposed mean.

Theorem 4.1. *Let* $\mathfrak{L}_m : C[a,b] \to C[a,b]$ $(m \in \mathbb{N})$ *is a sequence of positive linear operators and let* $f \in C[E]$ $(E = [a,b])$*, then*

$$\mathfrak{L}_m(f;x) \to f(x) \quad (\psi_n^{p,q}\text{-equi-stat}) \ \text{ on } \ (E;\sigma) \tag{4.1}$$

if and only if

$$\mathfrak{L}_m(f_i,x) \to f_i(x), \ (\psi_n^{p,q}\text{-equi-stat}) \ \text{ on } \ (E;\sigma_i) \ (i=0,1,2), \tag{4.2}$$

where

$$\sigma(x) = \max\{|\sigma_i(x)| : |\sigma_i(x)| > 0 \ \text{ and } \ \sigma_i(x) \text{ is unbounded } (i=0,1,2)\},$$

and

$$f_0(x) = 1, \ \ f_1(x) = x \ \text{ and } \ f_2(x) = x^2.$$

Proof. Since each of the functions given by

$$f_i(x) = x^i \in C(E) \ \ (i = 0,1,2)$$

are continuous, the implication,

$$(4.1) \Longrightarrow (4.2)$$

holds true. Now for the competition of the proof, we first assume that (4.2) is true. Further, suppose $f \in C[E]$, then there exists a constant $\mathcal{K} > 0$ such that $|f(x)| \leq \mathcal{K}$ $(\forall\, x \in E)$, for which

$$|f(y) - f(x)| \leq 2\mathcal{K} \ \ (x, y \in X).$$

Certainly, for given $\epsilon > 0$, there exists $\delta > 0$, such that

$$|f(y) - f(x)| < \epsilon \ \text{ whenever } \ |y - x| < \delta \tag{4.3}$$

for all $x, y \in X$.

Let us choose

$$\phi(y,x) = (y-x)^2.$$

Next, if

$$|y-x| \geqq \delta, \ \ \forall \ \ x,y \in X,$$

then, we clearly get

$$|f(y)-f(x)| < \frac{2\mathcal{K}}{\delta^2}\phi(y,x). \tag{4.4}$$

From equation (4.3) and (4.4), we have

$$|f(y)-f(x)| < \epsilon + \frac{2\mathcal{K}}{\delta^2}\phi(y,x)$$

$$\Longrightarrow -\epsilon - \frac{2\mathcal{K}}{\delta^2}\cdot\phi(y,x) \leqq (f(y)-f(x)) \leqq \epsilon + \frac{2\mathcal{K}}{\delta^2}\phi(y,x). \tag{4.5}$$

Now, since the operator $\mathfrak{L}_m(1,x)$ is linear and monotone, by using this in (4.5), we obtain

$$\mathfrak{L}_m(1,x)\left(-\epsilon - \frac{2\mathcal{K}}{\delta^2}\phi(y,x)\right) \leqq \mathfrak{L}_m(1,x)(f(y)-f(x)) \leqq \mathfrak{L}_m(1,x)\left(\epsilon + \frac{2\mathcal{K}}{\delta^2}\phi(y,x)\right). \tag{4.6}$$

We note here that x is fixed and so $f(x)$ is a constant number. Therefore, we get

$$\begin{aligned}-\epsilon\mathfrak{L}_m(1,x) - \frac{2\mathcal{K}}{\delta^2}\mathfrak{L}_m(\phi,x) &\leqq \mathfrak{L}_m(f,x) - f(x)\mathfrak{L}_m(1,x)\\ &\leqq \epsilon\mathfrak{L}_m(1,x) + \frac{2\mathcal{K}}{\delta^2}\mathfrak{L}_m(\phi,x);\end{aligned} \tag{4.7}$$

and moreover in association with the following identity:

$$\mathfrak{L}_m(f,x) - f(x) = [\mathfrak{L}_m(f,x) - f(x)\mathfrak{L}_m(1,x)] + f(x)[\mathfrak{L}_m(1,x)-1], \tag{4.8}$$

yields

$$\mathfrak{L}_m(f,x) - f(x) < \epsilon\mathfrak{L}_m(1,x) + \frac{2\mathcal{K}}{\delta^2}\mathfrak{L}_m(\phi,x) + f(x)[\mathfrak{L}_m(1,x)-1]. \tag{4.9}$$

Further, computing $\mathfrak{L}_m(\phi,x)$ as,

$$\begin{aligned}\mathfrak{L}_m(\phi,x) &= \mathfrak{L}_m((y-x)^2,x) = \mathfrak{L}_m(y^2-2xy+x^2,x)\\ &= \mathfrak{L}_m(y^2,x) - 2x\mathfrak{L}_m(y,x) + x^2\mathfrak{L}_m(1,x)\\ &= [\mathfrak{L}_m(y^2,x) - x^2] - 2x[\mathfrak{L}_m(y,x)-x] + x^2[\mathfrak{L}_m(1,x)-1]\end{aligned}$$

and using (4.9), we get

$$\mathfrak{L}_m(f,x) - f(x) < \epsilon \mathfrak{L}_m(1,x) + \frac{2\mathcal{K}}{\delta^2}\{[\mathfrak{L}_m(y^2,x) - x^2] - 2x[\mathfrak{L}_m(y,x) - x] + x^2[\mathfrak{L}_m(1,x) - 1]\}$$

$$+ f(x)[\mathfrak{L}_m(1,x) - 1]$$

$$= \epsilon[\mathfrak{L}_m(1,x) - 1] + \epsilon + \frac{2\mathcal{K}}{\delta^2}\{[\mathfrak{L}_m(y^2,x) - x^2] - 2x[\mathfrak{L}_m(y,x) - x]$$

$$+ x^2[\mathfrak{L}_m(1,x) - 1]\} + f(x)[\mathfrak{L}_m(1,x) - 1].$$

Then,

$$|\mathfrak{L}_m(f,x) - f(x)| \leqq \epsilon + \left(\epsilon + \frac{2\mathcal{K}}{\delta^2} + \mathcal{K}\right)|\mathfrak{L}_m(1,x) - 1|$$

$$+ \frac{4\mathcal{K}}{\delta^2}|\mathfrak{L}_m(y,x) - x| + \frac{2\mathcal{K}}{\delta^2}|\mathfrak{L}_m(y^2,x) - x^2|$$

$$\leqq \mathcal{M}\left(|\mathfrak{L}_m(1,x) - 1| + |\mathfrak{L}_m(y,x) - x| + |\mathfrak{L}_m(y^2,x) - x^2|\right), \tag{4.10}$$

so that, by writing $\Delta^{a,b,c}_{h,p,q}\mathfrak{L}_m(.,x)$ instead of $\mathfrak{L}_m(.,x)$,

$$s_{b_n-m}t_m \left\|\Delta^{a,b,c}_{h,p,q}\mathfrak{L}_m(f,x) - f(x)\right\|_{C(E)} \leqq \epsilon + \mathcal{M}\Big\{ s_{b_n-m}t_m \left|\Delta^{a,b,c}_{h,p,q}\mathfrak{L}_m(1,x) - 1\right|$$

$$+ s_{b_n-m}t_m \left|\Delta^{a,b,c}_{h,p,q}\mathfrak{L}_m(y,x) - x\right|$$

$$+ s_{b_n-m}t_m \left|\Delta^{a,b,c}_{h,p,q}\mathfrak{L}_m(y^2,x) - x^2\right| \Big\}, \tag{4.11}$$

where

$$\mathcal{M} = \max\left(\epsilon + \frac{2\mathcal{K}}{\delta^2} + \mathcal{K},\ \frac{4\mathcal{K}}{\delta^2},\ \frac{2\mathcal{K}}{\delta^2}\right).$$

Now, for given $r' > 0$, choose $\epsilon > 0$ such that, $\frac{\epsilon}{\sigma(x)} < r'$. Moreover, for

$$\Lambda_m(x; r\sigma) = \left|\left\{ m : m \leqq R^{b_n}_{a_n+1} \text{ and } s_{b_n-m}t_m \left|\frac{\Delta^{a,b,c}_{h,p,q}\mathfrak{L}_m(f,x) - f(x)}{\sigma(x)}\right| \geqq r' \right\}\right|$$

and

$$\Lambda_{i,m}(x, r\sigma) = \left|\left\{ m : m \leqq R^{b_n}_{a_n+1} \text{ and } s_{b_n-m}t_m \left|\frac{\Delta^{a,b,c}_{h,p,q}\mathfrak{L}_m(f_i,x) - f_i(x)}{\sigma_i(x)}\right| \geqq \frac{r' - \frac{\epsilon}{\sigma(x)}}{3\mathcal{M}} \right\}\right|,$$

we easily obtain from (4.11) that

$$\Lambda_m(x, r\sigma) \leqq \sum_{i=0}^{2} \Lambda_{i,m}(x, r\sigma).$$

Thus, we fairly have

$$\frac{\|\Lambda_m(x, r\sigma)\|_{C(E)}}{R_{a_n+1}^{b_n}} \leqq \sum_{i=0}^{2} \frac{\|\Lambda_{i,m}(x, r\sigma)\|_{C(E)}}{R_{a_n+1}^{b_n}}. \tag{4.12}$$

Consequently, by Definition 3.2 and under the above assumption for the implication in (4.2), the right-hand side of (4.12) tends to zero ($n \to \infty$). We, thus get

$$\lim_{n\to\infty} \frac{\|\Lambda_m(x, r\sigma)\|_{C(E)}}{R_{a_n+1}^{b_n}} = 0 \quad (r > 0).$$

Hence, the implication in (4.1) is true. Which completes the proof of the Theorem. □

Example 4.2. Suppose $E = [0, 1]$ and let the (p, q)-analogue of Bernstein operators $\mathfrak{B}_{n,p,q}(f; x)$ on $C[E]$ is considered as (see [39]),

$$\mathfrak{B}_{n,p,q}(f; x) = \frac{1}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{k(k-1)}{2}} x^k \prod_{s=0}^{n-k-1} (p^s - q^s x) f\left(\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}}\right) \quad (x \in [E]).$$

Also, observe that

$$\mathfrak{B}_{n,p,q}(f_0; x) = 1, \quad \mathfrak{B}_{n,p,q}(f_1; x) = x \text{ and } \mathfrak{B}_{n,p,q}(f_2; x) = \frac{p^{n-1}}{[n]_{p,q}} x + \frac{q[n-1]_{p,q}}{[n]_{p,q}} x^2.$$

Further, we consider $\mathfrak{L}_m : C[E] \to C[E]$ as the sequence of operators (positive linear) of the form:

$$\mathfrak{L}_n(f; x) = [1 + f_n(x)]x\left(1 + x\frac{d}{dx}\right)\mathfrak{B}_{n,p,q}(f; x) \quad (f \in C[E]). \tag{4.13}$$

Also, earlier a part of our operator, that is $x\left(1 + x\frac{d}{dx}\right)$ was used by Al-Salam [5]. For several families of operators of this type we refer [53] and [50]. Next, on choosing the sequence $f_n(x)$ of functions as in Example 3.5, we can have

$$\mathfrak{L}_n(f_0; x) = [1 + f_n(x)]x(1 + x\mathcal{D}) \cdot \mathfrak{B}_{n,p,q}(f_0; x)$$

$$= [1 + f_n(x)]x(1 + x\mathcal{D}) \cdot 1 = [1 + f_n(x)]x,$$

$$\mathfrak{L}_n(f_1; x) = [1 + f_n(x)]x(1 + x\mathcal{D}) \cdot \mathfrak{B}_{n,p,q}(f_1; x)$$

$$= [1 + f_n(x)]x(1 + x\mathcal{D}) \cdot x = [1 + f_n(x)]x(1 + x),$$

and

$$\mathfrak{L}_n(f_2;x) = [1+f_n(x)]x(1+x\mathcal{D})\cdot\mathfrak{B}_{n,p,q}(f_2;x)$$

$$= [1+f_n(x)]x(1+x\mathcal{D})\cdot\left\{\frac{p^{n-1}}{[n]_{p,q}}x+\frac{q[n-1]_{p,q}}{[n]_{p,q}}x^2\right\}$$

$$= [1+f_n(x)]\left\{x^2\left[\left(\frac{q[n-1]_{p,q}}{[n]_{p,q}}\right)x+2\left(\frac{p^{n-1}}{[n]_{p,q}}\right)+2x\left(\frac{q[n-1]_{p,q}}{[n]_{p,q}}\right)\right]\right\}.$$

Further, since

$$f_n \to f = 0 \ \ (\psi_n^{p,q}\text{-equi-stat}) \ \text{ on } \ (E;\sigma),$$

so in similar lines as in Example 3.5, we can conclude that

$$\mathfrak{L}_n(f_i;x) \to f_i \ \ (\psi_n^{p,q}\text{-equi-stat}) \ \text{ on } \ (E;\sigma)$$

for each $i = 0,1,2$. Thus, by Theorem 4.1, we see that

$$\mathfrak{L}_n(f;x) \to f \ \ (\psi_n^{p,q}\text{-equi-stat}) \ \text{ on } \ (E;\sigma)$$

for all $f \in C[E]$

Furthermore, as the sequence of functions (f_n) is not relatively uniform $\psi_n^{p,q}$-statistical convergent to $f = 0$ on E, we can certainly say that the outcomes of Srivastava *et al.* ([45], p. 10, Theorem 1) and also the outcomes in [15, 16] are not fairly true under the operators defined by us in (4.13). Also, the sequence of functions (f_n) is not being converge uniformly to the function $f = 0$ on E, the traditional Korovkin type theorem (see [31]) here also does not work. Moreover, for the operators defined under (4.13), our Theorem 4.1 fairly holds true.

5. Rate of the relatively $\psi_n^{p,q}$-eqiu-statistical convergence

We intend here to investigate the order of the relatively $\psi_n^{p,q}$-equi-statistical convergence of a sequence of positive linear operators defined on $C(E)$ into itself under the modulus of continuity.

Definition 5.1. Let $0 < q_n < p_n \leq 1$ such that $\lim_n q_n = \alpha$ and $\lim_n p_n = \beta$ $(0 < \alpha, \beta \leq 1)$, r is a non-negative integer and let (a_n) and (b_n) be sequences of non-negative integers, and also let (u_n) be a positive non-increasing sequence. A sequence (f_n) of functions is relatively $\psi_n^{p,q}$-equi-statistical convergent to a function f on E with rate $o(u_n)$ if, for every $\epsilon > 0$,

$$\lim_{n\to\infty}\frac{\Lambda_n^{p,q}(x;\epsilon\sigma)}{s_n R_{a_n+1}^{b_n}} = 0$$

uniformly relatively with respect to $x \in E$ or, otherwise, if

$$\lim_{n\to\infty}\frac{\left\|\Lambda_n^{p,q}(x;\epsilon\sigma)\right\|_{C(X)}}{s_n R_{a_n+1}^{b_n}} = 0,$$

where

$$\Lambda_n(x,\epsilon\sigma)=\left|\left\{m: m\leqq R_{a_n+1}^{b_n}\ \text{ and }\ s_{b_n-m}t_m\left|\frac{\Delta_{h,p,q}^{a,b,c}f_m(x)-f(x)}{\sigma(x)}\right|\geqq\epsilon\right\}\right|.$$

We, write here

$$f_n(x)-f(x)=o(u_n)\ \ (\psi_n^{p,q}\text{-equi-stat) on }(E;\sigma).$$

We now need to prove the following Lemma.

Lemma 5.2. *Suppose the sequences (f_n) and $(g_n)\in C[E]$ satisfy the conditions:*

$$f_n(x)-f(x)=o(u_n)\ \ (\psi_n^{p,q}\text{-equi-stat) on }(E;\sigma_0)$$

and

$$g_n(x)-g(x)=o(v_n)\ \ (\psi_n^{p,q}\text{-equi-stat) on }(E;\sigma_1),$$

where $\sigma_0>0$ and $\sigma_1>0$; then all the following assertions are true:

(i) $(f_n(x)+g_n(x))-(f(x)+g(x))=o(w_n)$ *($\psi_n^{p,q}$-equi-stat) on* $(E;\max\{\sigma_0,\sigma_1\})$;

(ii) $(f_n(x)-f(x))(g_n(x)-g(x))=o(u_nv_n)$ *($\psi_n^{p,q}$-equi-stat) on* $(E;\{\sigma_0,\sigma_1\})$*;*

(iii) $\gamma(f_n(x)-f(x))=o(u_n)$ *($\psi_n^{p,q}$-equi-stat) on* $(E;\sigma_0)$, *for any scalar* γ;

(iv) $\{|f_n(x)-f(x)|\}^{\frac{1}{2}}=o(u_n)$ *($\psi_n^{p,q}$-equi-stat) on* $(E;\ \sqrt{|\sigma_0(x)|})$, *where* $w_n=\max\{u_n,v_n\}$.

Proof. For proving the assertion (i) of Lemma 5.2, we consider the following sets for $\epsilon>0$ and $x\in E$:

$$\mathfrak{A}_n(x,\epsilon\sigma)=\left|\left\{m: m\leqq R_{a_n+1}^{b_n}\ \text{ and }\ s_{b_n-m}t_m\left|\frac{\left(\Delta_{h,p,q}^{a,b,c}f_m+\Delta_{h,p,q}^{a,b,c}g_m\right)(x)-(f+g)(x)}{\sigma(x)}\right|\geqq\epsilon\right\}\right|,$$

$$\mathfrak{A}_{0,n}(x,\epsilon\sigma)=\left|\left\{m: m\leqq R_{a_n+1}^{b_n}\ \text{ and }\ s_{b_n-m}t_m\left|\frac{\Delta_{h,p,q}^{a,b,c}f_m(x)-f(x)}{\sigma_0(x)}\right|\geqq\frac{\epsilon}{2}\right\}\right|$$

and

$$\mathfrak{A}_{1,n}(x,\epsilon\sigma)=\left|\left\{m: m\leqq R_{a_n+1}^{b_n}\ \text{ and }\ s_{b_n-m}t_m\left|\frac{\Delta_{h,p,q}^{a,b,c}g_m(x)-g(x)}{\sigma_1(x)}\right|\geqq\frac{\epsilon}{2}\right\}\right|,$$

where

$$\sigma(x)=\max\{|\sigma_i(x)|: i=0,1\}.$$

Clearly, we have

$$\mathfrak{A}_n(x,\epsilon\sigma)\subseteq\mathfrak{A}_{0,n}(x,\epsilon\sigma)\cup\mathfrak{A}_{1,n}(x,\epsilon\sigma).$$

Moreover, since

$$w_n=\max\{u_n,v_n\},\tag{5.1}$$

by using the assertion (4.1) of Theorem 4.1, we obtain

$$\frac{\|\mathfrak{A}_n(x,\epsilon\sigma)\|_{C(E)}}{w_n R_{a_n+1}^{b_n}} \leqq \frac{\|\mathfrak{A}_{0,n}(x,\epsilon\sigma)\|_{C(E)}}{u_n R_{a_n+1}^{b_n}} + \frac{\|\mathfrak{A}_{1,n}(x,\epsilon\sigma)\|_{C(E)}}{v_n R_{a_n+1}^{b_n}}. \tag{5.2}$$

Also, by using the assertion (4.2) of Theorem 4.1, we obtain

$$\frac{\|\mathfrak{A}_n(x,\epsilon\sigma)\|_{C(E)}}{w_n R_{a_n+1}^{b_n}} = 0. \tag{5.3}$$

Thus, the assertion (i) of Lemma 5.2 is proved.

Next, since all other assertions ((ii) - (iv)) of Lemma 5.2 are similar as in the assertion (i), so these can be proved along similar lines to complete the proof Lemma 5.2. □

We now recall the modulus of continuity of a function $f \in C[E]$ as,

$$\omega(f,\delta) = \sup\{|f(y)-f(x)| : |y-x| \leqq \delta \ \ (x,y \in E;\ \delta > 0)\}. \tag{5.4}$$

We now introduce a theorem to obtain the rates of relatively deferred Nörlund equi-statistical convergence based on difference operator of fractional order, including (p,q)-gamma function under the support of the modulus of continuity in (5.4).

Theorem 5.3. *Let $E \subset \mathbb{R}$ be compact and let $\mathfrak{L}_m : C(E) \to C(E)$ be a sequence of positive linear operators. Suppose that the following conditions hold true:*

(a) $\mathfrak{L}_m(1,x) - 1 = o(u_n)$ *($\psi_n^{p,q}$-equi-stat) on $(E;\sigma_0)$;*

(b) $\omega(f,\delta_n) = o(v_n)$ *($\psi_n^{p,q}$-equi-stat) on $(E;\sigma_1)$,*

where

$$\delta_n(x) = \left\{\mathfrak{L}_m(\phi^2;x)\right\}^{\frac{1}{2}} \ \ \text{and} \ \ \phi(y) = y - x;$$

then, for each $f \in C(E)$, the assertion as below holds true:

$$\mathfrak{L}_m(f,x) - f = o(w_n) \ \ (\psi_n^{p,q}\text{-equi-stat}) \text{ on } (E;\sigma), \tag{5.5}$$

where w_n is given by (5.1) and

$$\sigma(x) = \max\{|\sigma_0(x)|, |\sigma_1(x)|, |\sigma_0(x)\sigma_1(x)| : \sigma_i(x) > 0 \ \ (i=0,1)\}.$$

Proof. Suppose $E \subset \mathbb{R}$ be compact and let $f \in C[E]$, $x \in E$. Then, it is obvious that,

$$|\mathfrak{L}_m(f;x) - f(x)| \leqq \mathcal{N}|\mathfrak{L}_m(1;x) - 1| + \left(\mathfrak{L}_m(1;x) + \sqrt{\mathfrak{L}_m(1;x)}\right)\omega(f,\delta_n),$$

where

$$\mathcal{N} = \|f\|_{C[E]}.$$

Which yields

$$\left|\frac{\mathfrak{L}_m(f;x) - f(x)}{\sigma(x)}\right| \leqq \mathcal{N}\left|\frac{\mathfrak{L}_m(1;x) - 1}{\sigma_0(x)}\right| + 2\frac{\omega(f,\delta_n)}{\sigma_1(x)}$$

$$+\frac{\omega(f,\delta_n)}{\sigma_1(x)}\left|\frac{\mathfrak{L}_m(1;x) - 1}{\sigma_0(x)}\right| + \frac{\omega(f,\delta_n)}{\sigma_1(x)}\sqrt{\left|\frac{\mathfrak{L}_m(1;x) - 1}{\sigma_0(x)}\right|}. \tag{5.6}$$

Finally, for the conditions (a) and (b) (of Theorem 5.3) along with Lemma 5.2, the last inequality (5.6) helps us to the achieve the assertion (5.5). Therefore, proof of Theorem 5.3 is completed. □

6. Observations and concluding remarks

Here, in the last section of our study, we put forth some further concluding remarks and observations connecting to different outcomes which we have demonstrated here.

Remark 6.1. If we consider the sequence of functions $(f_n)_{n\in\mathbb{N}}$ as given in Example 3.5. Then, since

$$f_n \to 0 \ \ (\psi_n^{p,q}\text{-equi-stat}) \text{ on } (E;\sigma),$$

we, fairly have

$$\mathfrak{L}_m(f_i, x) \to f_i \quad (\psi_n^{p,q}\text{-equi-stat}) \text{ on } (E;\sigma_i) \ \ (i = 0, 1, 2). \tag{6.1}$$

Thus, by using Theorem 4.1, we can write

$$\mathfrak{L}_m(f, x) \to f \ (\psi_n^{p,q}\text{-equi-stat}) \text{ on } (E;\sigma) \ \ (\forall f \in E). \tag{6.2}$$

Moreover, since (f_n) is not relatively uniform $\psi_n^{p,q}$-statistically convergent to $f = 0$ on E, and also since (f_n) is not relatively convergent uniformly to the function $f = 0$ on E, the well established theorem of Korovkin type fairly does not hold under the operators defined in (4.13). Thus, clearly this outcome indicates that our Theorem 1 is a generalization of traditional and statistical versions of Korovkin-type theorems (see [31], [45]).

Remark 6.2. Suppose in Theorem 5.3, we substitute the conditions (a) and (b) by the following condition:

$$\mathfrak{L}_m(f_i, x) - f_i = o(u_{n_i}) \ \ (\psi_n^{p,q}\text{-equi-stat}) \text{ on } (E;\sigma_i), \ \ (i = 0, 1, 2). \tag{6.3}$$

Then, since

$$\mathfrak{L}_m(\phi^2; x) = \mathfrak{L}_m(y^2; x) - 2x\mathfrak{L}_m(y; x) + x^2\mathfrak{L}_m(1; x),$$

we can write

$$\mathfrak{L}_m(\phi^2; x) \leqq \mu \sum_{i=0}^{2} |\mathfrak{L}_m(f_i; x) - f_i(x)|, \tag{6.4}$$

where

$$\mu = 1 + 2\|f_1\|_{C[E]} + \|f_2\|_{C[E]}.$$

Clearly, from (6.3), (6.4) and Lemma 5.2, it follows that

$$\delta_n = \sqrt{\mathfrak{L}_m(\psi^2)} = o(u_n) \ \ (\psi_n^{p,q}\text{-equi-stat}) \text{ on } (E;\sigma), \tag{6.5}$$

where

$$o(u_n) = \max\{u_{n_0}, u_{n_1}, u_{n_2}\}.$$

Thus, definitely, we obtain

$$\omega(f, \delta) = o(u_n) \ (\psi_n^{p,q}\text{-equi-stat}) \text{ on } (E;\sigma).$$

By applying (6.5) in Theorem 5.3, we instantly see that, $\forall\, f \in C[E]$,

$$\mathfrak{L}_m(f;x) - f(x) = o(u_n)\ (\psi_n^{p,q}\text{-equi-stat) on } (E;\sigma). \tag{6.6}$$

Therefore, instead of conditions (a) and (b) of Theorem 5.3, if we use the condition (6.3), then we certainly find the rates of the relatively $\psi_n^{p,q}$-equi-statistical convergence for the sequence $(\mathfrak{L}_m)$ of positive linear operators in our Theorem 4.1.

Remark 6.3. In our present investigation, we have considered a number of fascinating special cases and illustrative examples in relevance to our definitions and also of the outcomes which have been established here.

Acknowledgments

The authors would like to express their heart-felt thanks to the editors and anonymous referees for their most valuable comments and constructive suggestions which leads to the significant improvement of the earlier version of the manuscript.

Conflict of interest

The authors declare no conflict of interest.

References

1. T. Acar, A. Aral, S. A. Mohiuddine, *On Kantorovich modifications of (p, q)-Baskakov operators*, J. Inequal. Appl., **2016** (2016), 1–14.

2. T. Acar, S. A. Mohiuddine, *Statistical $(C, 1)(E, 1)$ summability and Korovkin's theorem*, Filomat, **30** (2016), 387–393.

3. R. P. Agnew, *On deferred Cesàro means*, Ann. Math., **33** (1932), 413–421.

4. H. Aktuğlu, H. Gezer, *Lacunary equi-statistical convergence of positive linear operators*, Cent. Eur. J. Math., **7**(2009), 558–567.

5. W. A. Al-Salam, *Operational representations for the Laguerre and other polynomials*, Duke Math. J., **31** (1964), 127–142.

6. M. Balcerzak, K. Dems, A. Komisarski, *Statistical convergence and ideal convergence for sequences of functions*, J. Math. Anal. Appl., **328** (2007), 715–729.

7. P. Baliarsingh, *On a fractional difference operator*, Alex. Eng. J., **55** (2016), 1811–1816.

8. F. Başar, *Summability Theory and Its Applications*, Bentham Science Publishers, Istanbul, 2012.

9. C. A. Bektaş, M. Et, R. Çolak, *Generalized difference sequence spaces and their dual spaces*, J. Math. Anal. Appl., **292** (2004), 423–432.

10. C. Belen, S. A. Mohiuddine, *Generalized statistical convergence and application*, Appl. Math. Comput., **219** (2013), 9821–9826.

11. N. L. Braha, H. M. Srivastava, S. A. Mohiuddine, *A Korovkin's type approximation theorem for periodic functions via the statistical summability of the generalized de la Vallée Poussin mean*, Appl. Math. Comput., **228** (2014), 162–169.

12. S. Chapman, *On non-integral orders of summability of series and integrals*, Proc. Lond. Math. Soc., **2** (1911), 369–409.

13. E. W. Chittenden, *On the limit functions of sequences of continuous functions converging relatively uniformly*, Trans. AMS, **20** (1919), 179–184.

14. A. A. Das, B. B. Jena, S. K. Paikray, et al. *Statistical deferred weighted summability and associated Korovokin-type approximation theorem*, Nonlinear Sci. Lett. A, **9** (2018), 238–245.

15. K. Demirci, S. Orhan, *Statistically relatively uniform convergence of positive linear operators*, Results Math., **69** (2016), 359–367.

16. K. Demirci, S. Orhan, *Statistical relative approximation on modular spaces*, Results Math., **71** (2017), 1167–1184.

17. O. H. H. Edely, S. A. Mohiuddine, A. K. Noman, *Korovkin type approximation theorems obtained through generalized statistical convergence*, Appl. Math. Lett., **23** (2010), 1382–1387.

18. H. Fast, *Sur la convergence statistique*, Colloq. Math., **2** (1951), 241–244.

19. G. Gasper, M. Rahman, *Basic Hypergeometric Series*, Camb. Univ. Press, 2004.

20. F. H. Jackson, *Ageneralization of the functions* $\Gamma(n)$ *and* x^n, Proc. R. Soc. Lond., **74** (1904), 64–72.

21. B. B. Jena, S. K. Paikray, *Product of statistical probability convergence and its applications to Korovkin-type theorem*, Miskolc Math. Notes, **20** (2019), 1–16.

22. B. B. Jena, S. K. Paikray, U. K. Misra, *Statistical deferred Cesàro summability and its applications to approximation theorems*, Filomat, **32** (2018), 2307–2319.

23. B. B. Jena, S. K. Paikray, U. K. Misra, *Inclusion theorems on general convergence and statistical convergence of* $(L, 1, 1)$*-summability using generalized Tauberian conditions*, Tamsui Oxf. J. Inf. Math. Sci., **31** (2017), 101–115.

24. U. Kadak, *Weighted statistical convergence based on generalized difference operator involving* (p, q)*-gamma function and its applications to approximation theorems*, J. Math. Anal. Appl., **448** (2017), 1633–1650.

25. U. Kadak, *On weighted statistical convergence based on* (p, q)*-integers and related approximation theorems for functions of two variables*, J. Math. Anal. Appl., **443** (2016), 752–764.

26. U. Kadak, P. Baliarsingh, *On certain Euler difference sequence spaces of fractional order and related dual properties*, J. Nonlinear Sci. Appl., **8** (2015), 997–1004.

27. U. Kadak, S. A. Mohiuddine, *Generalized statistically almost convergence based on the difference operator which includes the* (p, q)*-gamma function and related approximation theorems*, Results Math., **73** (2018), Article 9.

28. U. Kadak, H. M. Srivastava, M. Mursaleen, *Relatively uniform weighted summability based on fractional-order difference operator*, Bull. Malays. Math. Sci. Soc., **42** (2019), 2453–2480.

29. V. Karakaya, T. A. Chishti, *Weighted statistical convergence*, Iranian. J. Sci. Technol. Trans. A, **33** (A3)(2009), 219–223.

30. S. Karakuş, K. Demirci, O. Duman, *Equi-statistical convergence of positive linear operators*, J. Math. Anal. Appl., **339** (2008), 1065–1072.

31. P. P. Korovkin, *Linear operators and approximation theory*, Hindustan Publ. Co., Delhi, 1960.

32. A. Lupaş, A q-analogue of the Bernstein operator. In: *Seminar on Numerical and Statistical Calculus*, University of Cluj-Napoca, **9** (1987), 85–92.

33. S. A. Mohiuddine, *An application of almost convergence in approximation theorems*, Appl. Math. Lett., **24** (2011), 1856–1860.

34. S. A. Mohiuddine, *Statistical weighted A-summability with application to Korovkin's type approximation theorem*, J. Inequal. Appl., **2016** (2016).

35. S. A. Mohiuddine, B. A. S. Alamri, *Generalization of equi-statistical convergence via weighted lacunary sequence with associated Korovkin and Voronovskaya type approximation theorems*, Rev. R. Acad. Cienc. Exactas Fs. Nat., Ser. A Mat., RACSAM, **113**(2019), 1955–1973.

36. S. A. Mohiuddine, A. Asiri, B. Hazarika, *Weighted statistical convergence through difference operator of sequences of fuzzy numbers with application to fuzzy approximation theorems*, Int. J. Gen. Syst., **48**((2019), 492–506.

37. S. A. Mohiuddine, B. Hazarika, M. A. Alghamdi, *Ideal relatively uniform convergence with Korovkin and Voronovskaya types approximation theorems*, Filomat, **33**((2019), 4549–4560.

38. E. H. Moore, *An Introduction to a Form of General Analysis*, The New Haven Mathematical Colloquium, Yale University Press, New Haven 1910.

39. M. Mursaleen, K. J. Ansari, A. Khan, *On* (p, q)*-analogue of Bernstein operators*, Appl. Math. Comput., **266** (2015), 874–882.

40. M. Mursaleen, V. Karakaya, M. Ertürk, et al. *Weighted statistical convergence and its application to Korovkin type approximation theorem*, Appl. Math. Comput., **218** (2012), 9132–9137.

41. M. Mursaleen, Md. Nasiuzzaman, A. Nurgali, *Some approximation results on Bernstein-Schurer operators defined by* (p, q)*-integers*, J. Inequal. Appl., **2015** (2015), 1–12.

42. T. Pradhan, S. K. Paikray, B. B. Jena, et al. *Statistical deferred weighted* $\mathcal{B}$*-summability and its applications to associated approximation theorems*, J. Inequal. Appl., **2018** (2018), 1–21.

43. P. N. Sadjang, *On the* (p, q)*-gamma and* (p, q)*-beta function*, arXiv: 1506. 07394v1.

44. H. M. Srivastava, B. B. Jena, S. K. Paikray, et al. *A certain class of weighted statistical convergence and associated Korovkin type approximation theorems for trigonometric functions*, Math. Methods Appl. Sci., **41** (2018), 671–683.

45. H. M. Srivastava, B. B. Jena, S. K. Paikray, et al. *Generalized equi-statistical convergence of the deferred Nörlund summability and its applications to associated approximation theorems*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. (RACSAM), **112** (2018), 1487–1501.

46. H. M. Srivastava, B. B. Jena, S. K. Paikray, et al. *Deferred weighted* $\mathcal{A}$*-statistical convergence based upon the* (p, q)*-Lagrange polynomials and its applications to approximation theorems*, J. Appl. Anal., **24** (2018), 1–16.

47. H. M. Srivastava, B. B. Jena, S. K. Paikray, et al. *Statistically and relatively modular deferred-weighted summability and Korovkin-type approximation theorems*, Symmetry, **11** (2019), 1–20.

48. H. M. Srivastava, B. B. Jena, S. K. Paikray, *Deferred Cesàro statistical probability convergence and its applications to approximation theorems*, J. Nonlinear Convex Anal., **20** (2019), 1777–1792.

49. H. M. Srivastava, B. B. Jena, S. K. Paikray, *A certain class of statistical probability convergence and its applications to approximation theorems*, Appl. Anal. Discrete Math. (in press) 2019.

50. H. M. Srivastava, H. L. Manocha, *Treatise on Generating Functions, Halsted Press (Ellis Horwood Limited, Chichester)*, John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1984.

51. H. M. Srivastava, M. Mursaleen, A. Khan, *Generalized equi-statistical convergence of positive linear operators and associated approximation theorems*, Math. Comput. Model. **55** (2012), 2040–2051.

52. H. Steinhaus, *Sur la convergence ordinaire et la convergence asymptotique*, Colloq. Math., **2** (1951), 73–74.

53. O. V. Viskov, H. M. Srivastava, *New approaches to certain identities involving differential operators*, J. Math. Anal. Appl., **186** (1994), 1–10.