Mathematics

## Research article

# Existence results for $\phi$-Laplacian impulsive differential equations with periodic conditions 

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#### Abstract

Based on a Manasevich and Mawhin continuation theorem and some analysis skills we obtain sufficient conditions for existence results for $\phi$-Laplacian nonlinear impulsive differential equations with periodic boundary conditions:


$$
\begin{gathered}
\left(\phi\left(y^{\prime}\right)\right)^{\prime}=f\left(t, y(t), y^{\prime}(t)\right), \quad \text { a.e. } t \in[0, b], \\
y\left(t_{k}^{+}\right)-y\left(t_{k}^{-}\right)=I_{k}\left(y\left(t_{k}^{-}\right)\right), \quad k=1, \ldots, m, \\
y^{\prime}\left(t_{k}^{+}\right)-y^{\prime}\left(t_{k}^{-}\right)=\bar{I}_{k}\left(y\left(t_{k}^{-}\right)\right), \quad k=1, \ldots, m, \\
y(0)=y(b), \quad y^{\prime}(0)=y^{\prime}(b),
\end{gathered}
$$

where $0<t_{1}<t_{2}<\cdots<t_{m}<b, f:[0, b] \times \mathbb{R}^{n} \times \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ is a Carathéodory function, $I_{k}, \bar{I}_{k} \in C\left(\mathbb{R}^{n}, \mathbb{R}^{n}\right)$ and $\phi: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ is a suitable monotone homeomorphism.

Keywords: $\phi$-Laplacian; impulsive problem; quasi-linear operator; $M$-compact mapping; topological degree
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## 1. Introduction

Differential equations with impulses were considered for the first time by Milman and Myshkis [19] and then followed by a period of active research which culminated with the monograph
by Halanay and Wexler [14]. Many phenomena and evolution processes in the field of physics, chemical technology, population dynamics, and natural sciences may change state abruptly or be subject to short-term perturbations, (see, for instance [1,16] and the references therein). These short perturbations may be seen as impulses. Impulsive problems arise also in various applications in communications, mechanics (jump discontinuities in velocity), electrical engineering, medicine, and biology. For example, in the periodic treatment of some diseases, impulses correspond to the administration of a drug treatment. In environmental sciences, impulses correspond to seasonal changes of the water level of artificial reservoirs. Their models are described by impulsive differential equations and inclusions. To date, a spectrum of mathematical results (such as existence, asymptotic behavior, $\ldots$ ) have been obtained (see $[2,3,8,9,13,16,28,29]$ and the references therein).

There have been many approaches to the study of the existence of solutions of impulsive differential equations, such as fixed point theory, topological degree theory (including continuation methods and coincidence degree theory), comparison methods and monotone iterative methods (see [15, 23, 27, 31, 33]).

Recently in [20,30], the authors studied the existence and multiplicity of solutions of some classes of second order impulsive problems by variational methods.

In the last few years, $\phi$-Laplacian problems for differential equations such as

$$
\begin{gathered}
\left(\phi\left(y^{\prime}\right)\right)^{\prime}=f\left(t, y, y^{\prime}\right) \\
y(0)=y(b), y^{\prime}(0)=y^{\prime}(b)
\end{gathered}
$$

have been investigated by several authors. Existence and multiplicity results for the nonresonance and resonance cases have been presented in [4,5,24,26]. These types of problems with impulse effects have been considered by [25], in which they are based on the lower and upper functions.

Coincidence degree, introduced by Mawhin in 1972, is a topological tool for the investigation of the semilinear equation $L u+N u=f$, where $L$ is a linear Fredholm operator of index zero (not necessarily invertible) and $N$ is a nonlinear perturbation. Continuation theorems involving these kinds of mappings $(L, N)$ became an effective procedure in proving the existence of solutions of a large variety of boundary value problems. This method is extended in [18] to the case when $L$ is a quasi-linear operator in view of its application to problems involving $p$-Laplacian-like operators.

In this paper we use the continuation methods to prove the existence of solutions to the $\phi$-Laplacian problem for differential, equations with impulse effects and periodic boundary conditions

$$
\begin{gather*}
\left(\phi\left(y^{\prime}(t)\right)\right)^{\prime}=f\left(t, y(t), y^{\prime}(t)\right), t \in J:=[0, b], \quad t \neq t_{k}, \quad k=1, \ldots, m,  \tag{1.1}\\
y\left(t_{k}^{+}\right)-y\left(t_{k}^{-}\right)=I_{k}\left(y\left(t_{k}^{-}\right)\right), \quad t=t_{k}, \quad k=1, \ldots, m,  \tag{1.2}\\
y^{\prime}\left(t_{k}^{+}\right)-y^{\prime}\left(t_{k}^{-}\right)=\bar{I}_{k}\left(y\left(t_{k}^{-}\right)\right), \quad t=t_{k}, \quad k=1, \ldots, m,  \tag{1.3}\\
y(0)=y(b), \quad y^{\prime}(0)=y^{\prime}(b), \tag{1.4}
\end{gather*}
$$

where $0<t_{1}<t_{2}<\cdots<t_{m}<b, f:[0, b] \times \mathbb{R}^{n} \times \mathbb{R}^{n} \times \mathbb{R}^{n}$ is a Carathéodory function, $I_{k}, \bar{I}_{k} \in C\left(\mathbb{R}^{n}, \mathbb{R}^{n}\right)$ and $\phi: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ is a suitable monotone homeomorphism.

Because the $\phi$-Laplacian is nonlinear, the continuation theorem of Mawhin [11] is not applicable, which leads to difficulty for solving the problems (1.1)-(1.4). By using a Manasevich and Mawhin continuation theorem [18] and some analysis techniques, we establish some sufficient conditions for the existence of periodic solutions of the problems (1.1)-(1.4).

## 2. Preliminaries

Let $\phi: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ be a continuous function which satisfies the following two conditions:
$\left(H_{1}\right)$ For any $x_{1}, x_{2} \in \mathbb{R}^{n}, x_{1} \neq x_{2}$,

$$
\left\langle\phi\left(x_{1}\right)-\phi\left(x_{2}\right), x_{1}-x_{2}\right\rangle>0 .
$$

$\left(H_{2}\right)$ There exists a function $\alpha:[0, \infty) \rightarrow[0, \infty), \alpha(s) \rightarrow+\infty$ as $s \rightarrow+\infty$, such that

$$
\langle\phi(x), x\rangle \geq \alpha(\|x\|)\|x\|, \quad \text { for all } x \in \mathbb{R}^{n} .
$$

It is well-known that under these two conditions $\phi$ is an homeomorphism from $\mathbb{R}^{n}$ onto $\mathbb{R}^{n}$, and that $\left\|\phi^{-1}(y)\right\| \rightarrow+\infty$ as $\|y\| \rightarrow+\infty$.

Definition 2.1. A map $f:[p, q] \times \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ is said to be $L^{1}$-Carathéodory if
(i) $t \rightarrow f(t, y)$ is measurable for all $y \in \mathbb{R}^{n}$,
(ii) $y \rightarrow f(t, y)$ is continuous for almost each $t \in[p, q]$,
(iii) for each $r>0$, there exists $h_{r} \in L^{1}\left([p, q], \mathbb{R}_{+}\right)$such that

$$
\|f(t, y)\| \leq h_{r}(t) \text { for almost each } t \in[p, q] \text { and for all }\|y\| \leq r .
$$

Let $X$ and $Z$ be two real Banach spaces with norms $\|\cdot\|_{X}$ and $\|\cdot\|_{Z}$, respectively. A continuous operator $M: X \cap \operatorname{dom} M \rightarrow Z$ is said to be quasi-linear if
(i) $\operatorname{dimKer} M=\operatorname{dim}^{-1}(0)=n<\infty$;
(ii) $\operatorname{Im} M=M(X \cap \operatorname{dom} M)$ is a closed subset of $Z$.

Let $X=X_{1} \oplus X_{2}$ and $Z=Z_{1} \oplus Z_{2}$, where $X_{1}=\operatorname{KerM}, Z_{2}=\operatorname{ImM}$ and $X_{2}, Z_{1}$ are respectively the complementary spaces of $X_{1}$ in $X, Z_{2}$ in $Z$. Assume that $\operatorname{dim} X_{1}=\operatorname{dim} Z_{1}$; then we can define $P: X \rightarrow X_{1}$ and $Q: Z \rightarrow Z_{1}$ as the corresponding orthogonal projections such that

$$
\operatorname{Im} P=\operatorname{Ker} M, \quad \operatorname{Ker} Q=\operatorname{Im} M .
$$

Denote by $J: Z_{1} \rightarrow X_{1}$ a homeomorphism with $J(0)=0$.
Let $\Omega$ be a bounded open subset of $X$, with $0 \in \Omega$ such that $\operatorname{domM\cap \Omega } \neq \emptyset$, and consider a parameter family of nonlinear perturbations (generally) $N_{\lambda}:[0,1] \times \bar{\Omega} \rightarrow Z$ with $N_{1}=N$. The continuous operator $N_{\lambda}$ is said to be $M$-compact in $\bar{\Omega}$ with respect to $M$ if there is an operator $K: \operatorname{Im} M \rightarrow X_{2}$ with $K(0)=0$ such that for $\lambda \in[0,1]$,

$$
\begin{gathered}
(I-Q) N_{\lambda}(\bar{\Omega}) \subset I m M, \\
(I-Q) N_{0}=0, \quad Q N_{\lambda} x=0 \Leftrightarrow Q N x=0, \quad \lambda \in(0,1), \\
K M=I-P, \quad K(I-Q) N_{\lambda}: \bar{\Omega} \rightarrow X_{2} \quad \text { is compact }, \\
M\left[P+K(I-Q) N_{\lambda}\right]=(I-Q) N_{\lambda} .
\end{gathered}
$$

We introduce the intermediate map

$$
\begin{equation*}
F(\lambda, \cdot)=P+K(I-Q) N+J Q N, \tag{2.1}
\end{equation*}
$$

which is clearly compact under the above assumptions.
Consider the abstract equation

$$
\begin{equation*}
M x=N_{\lambda} x, \quad \lambda \in(0,1] . \tag{2.2}
\end{equation*}
$$

Now, we consider the continuation theorem for a quasilinear operator due to Ge and Ren [12].
Lemma 2.1. [12] Let $X$ and $Z$ be Banach spaces, $\Omega \subset X$ be an open and bounded nonempty set, $M$ be a quasi-linear operator and $N_{\lambda}$ be a $M$-compact operator in $\bar{\Omega}$. Then (2.2) has a solution $x \in \bar{\Omega}$ [resp $x \in \partial \Omega$ ] if and only if $x \in \bar{\Omega}$ [resp $x \in \partial \Omega$ ] is a fixed point of $F(\lambda, \cdot)$ defined in (2.1).

Theorem 2.2. [12] Let $X$ and $Z$ be two Banach spaces and $\Omega \subset X$ be an open and bounded nonempty set. Suppose $M: X \cap \operatorname{dom} M \rightarrow Z$ is a quasilinear operator and $N_{\lambda}: \bar{\Omega} \rightarrow Z, \lambda \in[0,1]$, is $M-$ compact. In addition, if
i) $M x \neq N_{\lambda} x, \quad \lambda \in(0,1), \quad x \in \partial \Omega$,
ii) $\operatorname{deg}\left(\operatorname{JQN}_{1}, \Omega \cap \operatorname{Ker} M, 0\right) \neq 0$,
then the abstract equation $M x=N_{1} x$ has at least one solution in $\bar{\Omega}$.

## 3. Fixed point formulation

In order to define a solution for Problems (1.1)-(1.2), consider the space of piece-wise continuous functions:
$P C\left([0, b], \mathbb{R}^{n}\right)=\left\{y:[0, b] \rightarrow \mathbb{R}^{n}, y_{k} \in C\left(J_{k}, \mathbb{R}^{n}\right), k=0, \ldots, m\right.$, such that
$y\left(t_{k}^{-}\right)$and $y\left(t_{k}^{+}\right)$exist and satisfy $y\left(t_{k}^{-}\right)=y\left(t_{k}\right)$ for $\left.k=1, \ldots, m\right\}$,
endowed with the norm

$$
\|y\|_{0}=\max \left\{\left\|y_{k}\right\|_{\infty}, \quad k=0, \ldots, m\right\},\left\|y_{k}\right\|_{\infty}=\sup _{t \in J_{k}}|y(t)|,
$$

and let

$$
P C_{b}\left([0, b], \mathbb{R}^{n}\right)=\{y \in P C: y(0)=y(b)\}
$$

be a Banach space with the same norm of $P C$.
Also, consider

$$
\begin{aligned}
P C^{1}\left([0, b], \mathbb{R}^{n}\right)= & \left\{y \in P C: y_{k}^{\prime} \in C\left(J_{k}, \mathbb{R}^{n}\right), k=0, \ldots, m,\right. \text { such that } \\
& y^{\prime}\left(t_{k}^{-}\right) \text {and } y^{\prime}\left(t_{k}^{+}\right) \text {exist and satisfy } y^{\prime}\left(t_{k}^{-}\right)=y^{\prime}\left(t_{k}\right), \\
& \text { for } k=1, \ldots, m\},
\end{aligned}
$$

endowed with the norm

$$
\|y\|_{1}=\max \left(\|y\|_{0},\left\|y^{\prime}\right\|_{0}\right), \quad \text { or } \quad\|y\|_{1}=\|y\|_{0}+\left\|y^{\prime}\right\|_{0},
$$

where $y_{k}=\left.y\right|_{J_{k}}$ and $J_{k}=\left(t_{k}, t_{k+1}\right]$, and let

$$
P C_{b}^{1}=\left\{y \in P C^{1}: y(0)=y(b), y^{\prime}(0)=y^{\prime}(b)\right\}
$$

be a Banach space with the same norm of $P C^{1}$. Define $X=P C_{b}^{1}, Z=L^{1}\left([0, b], \mathbb{R}^{n}\right) \times \mathbb{R}^{n m} \times \mathbb{R}^{n m} \times \mathbb{R}^{n}$, and for $z=\left(x, c_{1}, c_{2}, c_{3}\right) \in Z$, we have

$$
\|z\|_{Z}=\max \left\{\|x\|_{L^{1}},\left\|c_{1}\right\|,\left\|c_{2}\right\|,\left\|c_{3}\right\|\right\} .
$$

Let

$$
\begin{gathered}
M: \operatorname{domM} \subseteq X \rightarrow Z \\
y \mapsto\left(\left(\phi\left(y^{\prime}\right)\right)^{\prime}, \Delta y\left(t_{1}\right), \ldots, \Delta y\left(t_{m}\right), \Delta y^{\prime}\left(t_{1}\right), \ldots, \Delta y^{\prime}\left(t_{m}\right), 0\right) .
\end{gathered}
$$

Let $h \in P C\left(J, \mathbb{R}^{n}\right)$. Then we define $y$ by:

$$
y(t)=\left\{\begin{array}{lll}
L_{0}(h)(t), & \text { if } & t \in\left[0, t_{1}\right]  \tag{3.1}\\
L_{1}(h)(t), & \text { if } & t \in\left(t_{1}, t_{2}\right] \\
\cdots & & \\
L_{m}(h)(t), & \text { if } & t \in\left(t_{m}, b\right]
\end{array}\right.
$$

where

$$
\begin{gathered}
L_{0}(h)(t)=c+\int_{0}^{t} \phi^{-1}\left[\phi(d)+\int_{0}^{s} h(\tau) d \tau\right] d s, \\
t \in\left[0, t_{1}\right], c=y(0), \text { and } d=y^{\prime}(0), \\
L_{1}(h)(t)=L_{0}(h)\left(t_{1}\right)+I_{1}\left(L_{0}(h)\left(t_{1}\right)\right)+\int_{t_{1}}^{t} \phi^{-1}\left[\phi\left(L_{0}^{\prime}(h)\left(t_{1}\right)+\bar{I}_{1}\left(L_{0}(h)\left(t_{1}\right)\right)\right)\right. \\
\left.+\int_{t_{1}}^{s} h(\tau) d \tau\right] d s, \quad t \in\left(t_{1}, t_{2}\right], \\
L_{2}(h)(t)=L_{1}(h)\left(t_{2}\right)+I_{2}\left(L_{1}(h)\left(t_{2}\right)\right) \\
\quad+\int_{t_{2}}^{t} \phi^{-1}\left[\phi\left(L_{1}^{\prime}(h)\left(t_{2}\right)+\bar{I}_{2}\left(L_{1}(h)\left(t_{2}\right)\right)\right)\right. \\
\left.+\int_{t_{2}}^{s} h(\tau) d \tau\right] d s, \quad t \in\left(t_{2}, t_{3}\right], \\
\ldots \\
L_{m}(h)(t)=L_{m-1}(h)\left(t_{m}\right)+I_{m}\left(L_{m-1}(h)\left(t_{m}\right)\right)+\int_{t_{m}}^{t} \phi^{-1}\left[\phi \left(L_{m-1}^{\prime}(h)\left(t_{m}\right)\right.\right. \\
\left.\left.+\bar{I}_{m}\left(L_{m-1}(h)\left(t_{m}\right)\right)\right)+\int_{t_{m}}^{s} h(\tau) d \tau\right] d s, \quad t \in\left(t_{m}, b\right] .
\end{gathered}
$$

We will prove that $M$ is a quasi-linear operator.

$$
\operatorname{Ker} M=\{y \in \operatorname{dom} M: y(t)=y(0), t \in J\} .
$$

For $y \in \operatorname{Ker} M$, we have

$$
\left(\phi\left(y^{\prime}(t)\right)^{\prime}=0 \Rightarrow \phi\left(y^{\prime}(t)\right)=c:=\left(\begin{array}{c}
c_{0} \\
c_{1} \\
\ldots \\
c_{m-1}
\end{array}\right)\right.
$$

and then

$$
\begin{aligned}
y(t) & =\left(\begin{array}{c}
L_{0}(h)(t) \\
L_{1}(h)(t) \\
\ldots \\
L_{m-1}(h)(t)
\end{array}\right)=\phi^{-1}(c) t+y(0) \\
& =t\left(\begin{array}{c}
\phi^{-1}\left(c_{0}\right) \\
\phi^{-1}\left(c_{1}\right) \\
\ldots \\
\phi^{-1}\left(c_{m-1}\right)
\end{array}\right)+\left(\begin{array}{c}
L_{0}(h)(0) \\
L_{1}(h)(0) \\
\ldots \\
L_{m-1}(h)(0)
\end{array}\right) .
\end{aligned}
$$

The boundary conditions imply that

$$
\left(\begin{array}{c}
\phi^{-1}\left(c_{0}\right) \\
\phi^{-1}\left(c_{1}\right) \\
\ldots \\
\phi^{-1}\left(c_{m-1}\right)
\end{array}\right)=\left(\begin{array}{c}
0 \\
0 \\
\ldots \\
0
\end{array}\right) \Rightarrow\left(\begin{array}{c}
L_{0}(h)(t) \\
L_{1}(h)(t) \\
\ldots \\
L_{m-1}(h)(t)
\end{array}\right)=\left(\begin{array}{c}
L_{0}(h)(0) \\
L_{1}(h)(0) \\
\ldots \\
L_{m-1}(h)(0)
\end{array}\right) .
$$

Thus

$$
y(t)=y(0)=c \in \mathbb{R}^{n} .
$$

Next, $\operatorname{ImM}=\left\{\left(x, a_{1}, \ldots, a_{m}, \bar{a}_{1}, \ldots, \bar{a}_{m}, d\right) \in Z:\left(\phi\left(y^{\prime}(t)\right)^{\prime}=x(t)\right.\right.$, a.e. $t \in[0, b], \Delta y\left(t_{k}\right)=a_{k}, \Delta y^{\prime}\left(t_{k}\right)=$ $\left.\bar{a}_{k}, k=1, \ldots, m, x(t) \in \operatorname{dom} M\right\}$.

From the following problem

$$
\left\{\begin{array}{l}
\left(\phi\left(y^{\prime}(t)\right)\right)^{\prime}=x(t), \quad \text { a.e. } t \in[0, b]  \tag{3.2}\\
\Delta y\left(t_{k}\right)=a_{k}, \quad k=1, \ldots, m \\
\Delta y^{\prime}\left(t_{k}\right)=\bar{a}_{k}, \quad k=1, \ldots, m
\end{array}\right.
$$

we have

$$
y(t)=\left\{\begin{array}{lll}
L_{0}(x)(t), & \text { if } & t \in\left[0, t_{1}\right], \\
L_{1}(x)(t), & \text { if } & t \in\left(t_{1}, t_{2}\right], \\
\cdots & & \\
L_{m}(x)(t), & \text { if } & t \in\left(t_{m}, b\right]
\end{array}\right.
$$

Since $y(0)=y(b)$, we also have

$$
\begin{aligned}
& L_{0}(x)(0)=L_{m}(x)(b), \\
& c= L_{m-1}(x)\left(t_{m}\right)+a_{m}\left(L_{m-1}(x)\left(t_{m}\right)\right)+\int_{t_{m}}^{b} \phi^{-1}\left[\phi \left(L_{m-1}^{\prime}(x)\left(t_{m}\right)\right.\right. \\
&\left.\left.+\bar{a}_{m}\left(L_{m-1}(x)\left(t_{m}\right)\right)\right)+\int_{t_{m}}^{s} x(\tau) d \tau\right] d s, \\
& c= L_{m-2}(x)\left(t_{m-1}\right)+a_{m-1}\left(L_{m-2}(x)\left(t_{m}\right)\right)+\int_{t_{m-1}}^{t_{m}} \phi^{-1}\left[\phi \left(L_{m-2}^{\prime}(x)\left(t_{m}\right)\right.\right. \\
&\left.\left.+\bar{a}_{m-1}\left(L_{m-2}(x)\left(t_{m}\right)\right)\right)+\int_{t_{m-1}}^{s} x(\tau) d \tau\right] d s+a_{m}\left(L_{m-1}(x)\left(t_{m}\right)\right) \\
&+\int_{t_{m}}^{b} \phi^{-1}\left[\phi\left(L_{m-1}^{\prime}(x)\left(t_{m}\right)+\bar{a}_{m}\left(L_{m-1}(x)\left(t_{m}\right)\right)\right)+\int_{t_{m}}^{s} x(\tau) d \tau\right] d s, \\
& \quad \ldots \\
& c=+\sum_{k=1}^{m} a_{k}+\int_{0}^{t_{1}} \phi^{-1}\left[\phi(d)+\int_{0}^{s} x(\tau) d \tau\right] d s+\int_{t_{1}}^{t_{2}} \phi^{-1}\left[\phi \left(L_{0}^{\prime}(x)\left(t_{1}\right)\right.\right. \\
&+\left.\left.\bar{a}_{1}\left(L_{0}(x)\left(t_{1}\right)\right)\right)+\int_{t_{1}}^{s} x(\tau) d \tau\right] d s+\ldots+\int_{t_{m-1}}^{t_{m}} \phi^{-1}\left[\phi \left(L_{m-2}^{\prime}(x)\left(t_{m-1}\right)\right.\right. \\
&\left.\left.+\bar{a}_{m-1}\left(L_{m-2}(x)\left(t_{m-1}\right)\right)\right)+\int_{t_{m-1}}^{s} x(\tau) d \tau\right] d s+\int_{t_{m}}^{b} \phi^{-1}\left[\phi \left(L_{m-1}^{\prime}(x)\left(t_{m}\right)\right.\right. \\
&\left.\left.+\bar{a}_{m}\left(L_{m-1}(x)\left(t_{m}\right)\right)\right)+\int_{t_{m}}^{s} x(\tau) d \tau\right] d s,
\end{aligned}
$$

so that

$$
\sum_{k=1}^{m} a_{k}+\int_{0}^{b} \phi^{-1}\left[\phi\left(d+\sum_{k=1}^{m} \bar{a}_{k}\right)+\int_{0}^{t} x(s) d s\right] d t=0
$$

and then

$$
\begin{aligned}
\operatorname{ImM}= & \left\{\left(x, a_{1}, \ldots, a_{m}, \bar{a}_{1}, \ldots, \bar{a}_{m}, d\right) \in Z:\right. \\
& \left.\sum_{k=1}^{m} a_{k}+\int_{0}^{b} \phi^{-1}\left[\phi\left(d+\sum_{k=1}^{m} \bar{a}_{k}\right)+\int_{0}^{t} x(s) d s\right] d t=0\right\} .
\end{aligned}
$$

Using the projections,

$$
\begin{gathered}
P: P C_{b}^{1} \rightarrow P C_{b}^{1}, y \mapsto y(0), \\
Q: Z \rightarrow Z, z \rightarrow Q(z), \\
z=\left(x, a_{1}, \ldots, a_{m}, \bar{a}_{1}, \ldots, \bar{a}_{m}, d\right) \\
\mapsto\left(0, \ldots, 0, \frac{1}{b}\left(\sum_{k=1}^{m} a_{k}+\int_{0}^{b} \phi^{-1}\left[\phi\left(d+\sum_{k=1}^{m} \bar{a}_{k}\right)+\int_{0}^{t} x(s) d s\right] d t\right)\right),
\end{gathered}
$$

we obtain that

$$
\operatorname{Im} P=\operatorname{Ker} M, \operatorname{Ker} Q=\operatorname{Im} M .
$$

Thus

$$
\operatorname{dim} \operatorname{Ker} M=n=\operatorname{dim}(Z / \operatorname{Im} M),
$$

and moreover, $\operatorname{Im} M$ is a closed subspace of $Z$. Therefore $M$ is a quasi-linear.
For any $\Omega \subset d o m M$, define the family

$$
\begin{gathered}
N_{\lambda}: \bar{\Omega} \rightarrow Z, \\
y \mapsto\left(\lambda f\left(t, y, y^{\prime}\right), I_{1}\left(y\left(t_{1}\right)\right), \ldots, I_{m}\left(y\left(t_{m}\right)\right), \bar{I}_{1}\left(y\left(t_{1}\right)\right), \ldots, \bar{I}_{m}\left(y\left(t_{m}\right)\right), 0\right) .
\end{gathered}
$$

The problems (1.1)-(1.4) is equivalent to the operator equation

$$
\begin{equation*}
M y=N_{1} y . \tag{3.3}
\end{equation*}
$$

We will prove that $N_{\lambda}$ is $M$-compact in $\bar{\Omega}$. It is easy to show that

$$
(I-Q) N_{\lambda}(\bar{\Omega}) \subset I m M \text { and } Q N_{\lambda}(\bar{\Omega})=0
$$

Let

$$
\begin{gathered}
K: \operatorname{ImM} \rightarrow \operatorname{dom} M \cap \operatorname{Ker} P, \\
z \mapsto \sum_{0<t_{k}<t} a_{k}+\int_{0}^{t} \phi^{-1}\left[\phi\left(d+\sum_{0<t_{k}<s} \bar{a}_{k}\right)+\int_{0}^{s} x(\tau) d \tau\right] d s,
\end{gathered}
$$

and the homeomorphism

$$
J: \operatorname{Im} Q \rightarrow \operatorname{Ker} M
$$

is given by

$$
(0, d) \rightarrow J(0, d)=d
$$

We supposed that $f\left(t, y, y^{\prime}\right)$ is a Carathéodory function and $\phi^{-1}$ is continuous. Then we have the following result.

Lemma 3.1. Suppose $\Omega \subset P^{1}\left([0, b], \mathbb{R}^{n}\right)$ is a bounded open set. Then $N_{\lambda}$ is $M$-compact in $\bar{\Omega}$.
Proof. We will prove that

- $Q N_{\lambda}: X \rightarrow Z$ is continuous and sends bounded sets into bounded sets.
- $K(I-Q) N_{\lambda}: \bar{\Omega} \rightarrow X$ is completely continuous.

Step 1: $Q N_{\lambda}$ sends bounded sets into bounded sets and it is continuous. First,

$$
Q N_{\lambda}(y)(t)=\left(\begin{array}{c}
0 \\
0 \\
\cdots \\
\frac{1}{b}\left(\sum_{k=1}^{m} I_{k}\left(y\left(t_{k}\right)\right) \quad+\int_{0}^{b} \phi^{-1}\left[\phi\left(\sum_{k=1}^{m} \bar{I}_{k}\left(y\left(t_{k}\right)\right)\right)\right.\right. \\
\left.\left.+\int_{0}^{t} \lambda f\left(s, y(s), y^{\prime}(s)\right) d s\right] d t\right)
\end{array}\right)
$$

$\circ Q N_{\lambda}$ sends bounded sets into bounded sets in $P C^{1}\left([0, b], \mathbb{R}^{n}\right)$.
Let $y \in \bar{\Omega}=\left\{y \in P C^{1}\left([0, b], \mathbb{R}^{n}\right):\|y\|_{1} \leq r\right\}$.

$$
\begin{aligned}
\left\|Q N_{\lambda}(y)(t)\right\| \leq & \left\|\sum_{k=1}^{m} I_{k}\left(y\left(t_{k}\right)\right)\right\|+\int_{0}^{b} \| \phi^{-1}\left[\phi\left(\sum_{k=1}^{m} \bar{I}_{k}\left(y\left(t_{k}\right)\right)\right)\right. \\
& \left.+\int_{0}^{t} \lambda f\left(s, y(s), y^{\prime}(s)\right) d s\right] \| d t
\end{aligned}
$$

Clearly $\|y\|_{1} \leq r$, and then, there exists $r_{*}>0$ such that

$$
\left\|\bar{I}_{k}\left(y\left(t_{k}\right)\right)\right\| \leq r_{*} \text { and }\left\|I_{k}\left(y\left(t_{k}\right)\right)\right\| \leq r_{*}, k=1, \ldots, m
$$

Since $I_{k}$ and $\bar{I}_{k}$ are continuous and $\bar{B}\left(0, r_{*}\right)$ is compact in $\mathbb{R}^{n}$, we have

$$
r_{1}=\sum_{k=1}^{m} \sup _{x \in \bar{B}\left(0, r_{z}\right)}\left\|I_{k}(x)\right\|<\infty \text { and } r_{2}=\sum_{k=1}^{m} \sup _{x \in \bar{B}\left(0, r_{z}\right)}\left\|\bar{I}_{k}(x)\right\|<\infty .
$$

Using the fact that $f$ is a Carathédory function, we obtain

$$
\begin{aligned}
\left\|\phi\left(\sum_{k=1}^{m} \bar{I}_{k}\left(y\left(t_{k}\right)\right)\right)+\int_{0}^{t} \lambda f\left(s, y(s), y^{\prime}(s)\right) d s\right\| & \leq\left\|\phi\left(\sum_{k=1}^{m} \bar{I}_{k}\left(y\left(t_{k}\right)\right)\right)\right\|+\int_{0}^{t}\left\|f\left(s, y(s), y^{\prime}(s)\right)\right\| d s \\
& \leq \| \phi\left(\sum_{k=1}^{m} \sup _{x \in \bar{B}\left(0, r_{2}\right.} \bar{I}_{k}(x)\left\|+\int_{0}^{t}\right\| h_{q}(s) \| d s\right. \\
& \leq \sup _{x \in B\left(0, r_{2}\right)}\|\phi(x)\|+\left\|h_{q}\right\|_{L^{1}}:=l .
\end{aligned}
$$

Since $\phi^{-1}$ is continuous, we also have

$$
\sup _{x \in \bar{B}(0, l)}\left|\phi^{-1}(x)\right|<\infty .
$$

Thus

$$
\left\|Q N_{\lambda}\right\|_{Z} \leq r_{1}+b \sup _{x \in \bar{B}(0, l)}\left|\phi^{-1}(x)\right|:=r_{3} .
$$

- $Q N_{\lambda}$ is continuous.

Let $\left(y_{\alpha}\right)_{\alpha \in \mathbb{N}}$ be a sequence such that $y_{n} \rightarrow y$ in $P C^{1}\left([0, b], \mathbb{R}^{n}\right)$. Then there exists $r>0$ such that

$$
\|y\|_{1} \leq r \text { for all } \alpha \in \mathbb{N} .
$$

Hence there exists $h_{r} \in L^{1}\left([0, b], \mathbb{R}^{n}\right)$ such that

$$
\left\|f\left(t, y_{\alpha}(t), y_{\alpha}^{\prime}(t)\right)\right\| \leq h_{r}(t) \text { a.e. } t \in J .
$$

From the definition of $Q N_{\lambda}$, we have

$$
\begin{aligned}
\left\|Q N_{\lambda}\left(y_{\alpha}\right)(t)-Q N_{\lambda}(y)(t)\right\| \leq & \| \int_{0}^{b} \phi^{-1}\left[\phi\left(\sum_{k=1}^{m} \bar{I}_{k}\left(y_{\alpha}\left(t_{k}\right)\right)\right)\right. \\
& \left.+\int_{0}^{t} \lambda f\left(s, y_{\alpha}(s), y_{\alpha}^{\prime}(s)\right) d s\right] \\
& -\phi^{-1}\left[\phi\left(\sum_{k=1}^{m} \bar{I}_{k}(y)\left(t_{k}\right)\right)\right) \\
& \left.+\int_{0}^{t} \lambda f\left(s, y(s), y^{\prime}(s)\right) d s\right] d t \| .
\end{aligned}
$$

By the dominated convergence theorem, and since $I_{k}, \bar{I}_{k}, \phi$ and $\phi^{-1}$ are continuous functions, we get

$$
\begin{aligned}
\left\|Q N_{\lambda}\left(y_{\alpha}\right)-Q N_{\lambda}(y)\right\|_{Z} \leq & \frac{1}{b} \sum_{k=1}^{m}\left\|I_{k}\left(y_{\alpha}\left(t_{k}\right)\right)-I_{k}\left(y\left(t_{k}\right)\right)\right\| \\
& +\frac{1}{b} \int_{0}^{b} \| \phi^{-1}\left[\phi\left(\sum_{k=1}^{m} \bar{I}\left(y_{\alpha}\left(t_{k}\right)\right)\right)\right. \\
& \left.+\int_{0}^{t} \lambda f\left(s, y_{\alpha}(s), y_{\alpha}^{\prime}(s)\right) d s\right] \\
& -\phi^{-1}\left[\phi\left(\sum_{k=1}^{m} \bar{I}_{k}\left(y\left(t_{k}\right)\right)\right)\right. \\
& \left.+\int_{0}^{t} \lambda f\left(s, y(s), y^{\prime}(s)\right) d s\right] \| d t .
\end{aligned}
$$

Thus

$$
\left\|Q N_{\lambda}\left(y_{\alpha}\right)-Q N_{\lambda}(y)\right\|_{Z} \rightarrow 0 \text { as } \alpha \rightarrow \infty .
$$

Hence, $Q N_{\lambda}$ is continuous.
Step 2: $K(I-Q) N_{\lambda}$ is completely continuous.

$$
\left(K N_{\lambda} y\right)(t)=\sum_{0<t_{k}<t} I_{k}\left(u\left(t_{k}\right)\right)+\int_{0}^{t} \phi^{-1}\left[\phi\left(\sum_{0<t_{k}<s} \bar{I}_{k}\left(y\left(t_{k}\right)\right)\right)+\int_{0}^{s} f\left(\tau, y(\tau), y^{\prime}(\tau)\right) d \tau\right] d t .
$$

For this we prove that $(I-Q) N_{\lambda}$ sends bounded sets into bounded sets and $K N_{\lambda}$ is completely continuous. The first part is immediate. For the second part, as in Step 1, we can prove that $K N_{\lambda}$ is bounded and continuous.

It remains to show that $K N_{\lambda}$ is equicontinuous, then by using the Arzelà-Ascoli theorem $K N_{\lambda}$ is compact. Indeed Let $l_{1}, l_{2} \in[0, b], l_{1}<l_{2}$ and $\Omega$ be a bounded set in $P C^{1}\left([0, b], \mathbb{R}^{n}\right)$, and let $y \in \Omega$. Then

$$
\begin{aligned}
\left\|\left(K N_{\lambda} y\right)^{\prime}(t)\right\| & =\left\|\phi^{-1}\left[\phi\left(\sum_{0<t_{k}<I_{I}} \bar{I}_{k}\left(y\left(t_{k}\right)\right)\right)+\int_{0}^{t} \lambda f(s, y(s), y(s)) d s\right]\right\| \\
& \leq \sup _{x \in \bar{B}(0, l)}\left\|\phi^{-1}(x)\right\|:=r^{\prime},
\end{aligned}
$$

where $l$ is defined in Step 1. By the mean value theorem, we obtain

$$
\left\|\left(K N_{\lambda} y\right)\left(l_{2}\right)-\left(K N_{\lambda} y\right)\left(l_{1}\right)\right\|=\left\|\left(K N_{\lambda} y\right)^{\prime}(\xi)\left(l_{2}-l_{1}\right)\right\| \leq r^{\prime}\left|l_{2}-l_{1}\right|
$$

As $l_{2} \rightarrow l_{1}$ the right hand side of the above inequality tends to zero. Also we have

$$
\begin{aligned}
\left\|\phi\left(\left(K N_{\lambda} y\right)^{\prime}\left(l_{2}\right)\right)-\phi\left(\left(K N_{\lambda} y\right)^{\prime}\left(l_{1}\right)\right)\right\| \leq & \sum_{l_{1} \leq t_{k}<l_{2}} \sup _{x \in B\left(0, r_{x}\right)}\left\|\bar{I}_{k}(x)\right\| \\
& +\int_{l_{1}}^{l_{2}} h_{r}(t) d t \rightarrow 0 \text { as } l_{1} \rightarrow l_{2} .
\end{aligned}
$$

Since $\phi^{-1}$ is continuous function, we conclude that $\overline{K(\Omega)}$ is compact.

- Using the fact that $f$ is a Carathéodory function and that $I_{k}, \bar{I}_{k}, \phi$ and $\phi^{-1}$ are continuous functions, we can easily prove that $K N_{\lambda}$ is continuous.

Let us now consider the abstract differential periodic problem

$$
\begin{equation*}
M y=N_{\lambda} y, \tag{3.4}
\end{equation*}
$$

and define

$$
F(y, \lambda):=P y+J Q N_{\lambda} y+K(I-Q) N_{\lambda} y, y \in P C^{1} .
$$

We obtain that $F$ is a completely continuous operator. Furthermore, the operator equation (3.4) is equivalent to the following fixed point equation:

$$
y=F(y, \lambda), \quad y \in P C^{1}\left(J, \mathbb{R}^{n}\right) .
$$

Let us now consider the simple periodic boundary value problem

$$
\begin{gather*}
\left(\phi\left(y^{\prime}(t)\right)\right)^{\prime}=h(t), t \in[0, b], \quad t \neq t_{k}, \quad k=1, \ldots, m  \tag{3.5}\\
y\left(t_{k}^{+}\right)-y\left(t_{k}^{-}\right)=I_{k}\left(y\left(t_{k}^{-}\right)\right), \quad t=t_{k}, \quad k=1, \ldots, m  \tag{3.6}\\
y^{\prime}\left(t_{k}^{+}\right)-y^{\prime}\left(t_{k}^{-}\right)=\bar{I}_{k}\left(y\left(t_{k}^{-}\right)\right), \quad t=t_{k}, \quad k=1, \ldots, m  \tag{3.7}\\
y(0)=y(b), \quad y^{\prime}(0)=y^{\prime}(b) \tag{3.8}
\end{gather*}
$$

where $h \in L^{1}$ is such that $\int_{0}^{b} h(s) d s=0$, and let us recall that $y$ is a continuous solution to (3.5)-(3.8) defined by

$$
y(t)=\left\{\begin{array}{lll}
L_{0}(h)(t), & \text { if } \quad t \in\left[0, t_{1}\right]  \tag{3.9}\\
L_{1}(h)(t), & \text { if } t \in\left(t_{1}, t_{2}\right], \\
\cdots & & \\
L_{m}(h)(t), & \text { if } & t \in\left(t_{m}, b\right],
\end{array}\right.
$$

where

$$
\begin{gathered}
L_{0}(h)(t)=c+\int_{0}^{t} \phi^{-1}\left[\phi(d)+H_{0}(h)(s)\right] d s, \quad t \in\left[0, t_{1}\right], \\
L_{1}(h)(t)= \\
+L_{0}(h)\left(t_{1}\right)+I_{1}\left(L_{0}(h)\left(t_{1}\right)\right)+\int_{t_{1}}^{t} \phi^{-1}\left[\phi\left(L_{0}^{\prime}(h)\left(t_{1}\right)+\bar{I}_{1}\left(L_{0}(h)\left(t_{1}\right)\right)\right)\right. \\
+ \\
\left.L_{1}(h)(s)\right] d s, \quad t \in\left(t_{1}, t_{2}\right], \\
\\
\left.+H_{2}(h)(s)\right] d s, \quad t \in\left(t_{2}, t_{3}\right], \\
\quad \ldots \\
L_{m}(h)(t)=L_{m-1}(h)\left(t_{m}\right)+I_{m}\left(L_{m-1}(h)\left(t_{m}\right)\right)+\int_{t_{m}}^{t} \phi^{-1}\left[\phi\left(L_{m-1}^{\prime}(h)\right)+\int_{t_{m}}^{t} \phi^{-1}\left[\phi\left(L_{1}^{\prime}(h)\left(t_{2}\right)+\bar{I}_{2}\left(L_{1}(h)\left(t_{2}\right)\right)\right)\right.\right. \\
\left.\left.\quad+\bar{I}_{m}\left(L_{m-1}(h)\left(t_{m}\right)\right)\right)+H_{m}(h)(s)\right] d s, \quad t \in\left(t_{m}, b\right],
\end{gathered}
$$

and

$$
H_{0}(h)(s)=\int_{0}^{s} h(\tau) d \tau, \quad \text { if } \quad t \in\left[0, t_{1}\right]
$$

$$
\begin{array}{rlr}
H_{1}(h)(s)=\int_{t_{1}}^{s} h(\tau) d \tau, & \text { if } & t \in\left(t_{1}, t_{2}\right] \\
\ldots \\
H_{m}(h)(s)=\int_{t_{m}}^{s} h(\tau) d \tau, & \text { if } & t \in\left(t_{m}, b\right] .
\end{array}
$$

The boundary conditions imply that

$$
\sum_{0<t_{k}<b} I_{k}\left(L_{k-1}(h)\left(t_{k}\right)\right)+\int_{0}^{b} \phi^{-1}[a+H(h)(t)] d t=0
$$

with

$$
a=\phi\left(y^{\prime}(0)+\sum_{0<t_{k}<b} \bar{I}_{k}\left(L_{k-1}(h)\left(t_{k}\right)\right)\right), \quad H(h)(t)=\int_{0}^{t} h(s) d s
$$

For fixed $l \in P C\left([0, b], \mathbb{R}^{n}\right)$ and $c \in \mathbb{R}^{n}$, let us define

$$
\begin{equation*}
G_{c, l}(a)=\sum_{0<t_{k}<b} I_{k}\left(L_{k-1}(h)\left(t_{k}\right)\right)+\int_{0}^{b} \phi^{-1}[a+l(t)] d t . \tag{3.10}
\end{equation*}
$$

Proposition 3.2. If $\phi$ satisfies conditions $\left(H_{1}\right)$ and $\left(H_{2}\right)$, then the function $G_{c, l}$ has the following properties:
(i) For any fixed $l \in P C\left([0, b], \mathbb{R}^{n}\right)$ and $c \in \mathbb{R}^{n}$, the equation

$$
\begin{equation*}
G_{c, l}(a)=0 \tag{3.11}
\end{equation*}
$$

has unique solution $\widetilde{a}(l)$.
(ii) If $\left\langle\sum_{k=1}^{m} I_{k}(x), \widetilde{a}(l)\right\rangle>0, x \in \mathbb{R}^{n}$, where the function $\widetilde{a}: P C\left([0, b], \mathbb{R}^{n}\right) \rightarrow \mathbb{R}^{n}$ is defined in ( $i$ ), then $\widetilde{a}$ is continuous and sends bounded sets into bounded sets.

Proof. (i) By $\left(H_{1}\right)$, it is immediate that

$$
\left\langle G_{c, l}\left(a_{1}\right)-G_{c, l}\left(a_{2}\right), a_{1}-a_{2}\right\rangle>0, \quad \text { for } a_{1} \neq a_{2},
$$

and hence if (3.11) has a solution, it is unique. To prove its existence we will show that $\left\langle G_{c, l}(a), a\right\rangle>0$ for $\|a\|$ sufficiently large. Indeed we have

$$
\begin{aligned}
\left\langle G_{c, l}(a), a\right\rangle= & \left\langle\sum_{0<t_{k}<b} I_{k}\left(L_{k-1}(h)\left(t_{k}\right)\right)+\int_{0}^{b} \phi^{-1}(a+l(t)) d t, a\right\rangle \\
= & \left\langle\sum_{0<t_{k}<b} I_{k}\left(L_{k-1}(h)\left(t_{k}\right)\right), a\right\rangle+\int_{0}^{b}\left\langle\phi^{-1}(a+l(t)), a\right\rangle d t \\
= & \left\langle\sum_{0<t_{k}<b} I_{k}\left(L_{k-1}(h)\left(t_{k}\right)\right), a\right\rangle+\int_{0}^{b}\left\langle\phi^{-1}(a+l(t)), a+l(t)\right\rangle d t \\
& -\int_{0}^{b}\left\langle\phi^{-1}(a+l(t)), l(t)\right\rangle d t,
\end{aligned}
$$

and thus

$$
\begin{align*}
\left\langle G_{c, l}(a), a\right\rangle \geq & \left\langle\sum_{0<t_{k}<b} I_{k}\left(L_{k-1}(h)\left(t_{k}\right)\right), a\right\rangle+\int_{0}^{b}\left\langle\phi^{-1}(a+l(t)), a+l(t)\right\rangle d t  \tag{3.12}\\
& -\|l\|_{0} \int_{0}^{b}\left\|\phi^{-1}(a+l(t))\right\| d t .
\end{align*}
$$

From $\left(H_{2}\right), \forall y \in \mathbb{R}^{n}$, we have that

$$
\begin{equation*}
\left\langle\phi^{-1}(y), y\right\rangle \geq \alpha\left(\left\|\phi^{-1}(y)\right\|\right)\left\|\phi^{-1}(y)\right\| . \tag{3.13}
\end{equation*}
$$

Thus from (3.12) and (3.13),

$$
\begin{align*}
\left\langle G_{c, l}(a), a\right\rangle \geq & \left\langle\sum_{0<t_{k}<b} I_{k}\left(L_{k-1}(h)\left(t_{k}\right)\right), a\right\rangle+\int_{0}^{b} \alpha\left(\left\|\phi^{-1}(a+l(t))\right\|\right)\left\|\phi^{-1}(a+l(t))\right\| \\
& -\|l\|_{0} \int_{0}^{b}\left\|\phi^{-1}(a+l(t))\right\| d t  \tag{3.14}\\
\geq & \left\langle\sum_{0<t_{k}<b} I_{k}\left(L_{k-1}(h)\left(t_{k}\right)\right), a\right\rangle \\
& +\int_{0}^{b}\left(\alpha\left(\left\|\phi^{-1}(a+l(t))\right\|\right)-\|l\|_{0}\right)\left\|\phi^{-1}(a+l(t))\right\| d t .
\end{align*}
$$

Since $\|a\| \rightarrow \infty$ implies that $\left\|\phi^{-1}(a+l(t))\right\| \rightarrow \infty$, uniformly for $t \in[0, b]$, we find from (3.14) that there exists an $r>0$ such that

$$
\left\langle G_{c, l}(a), a\right\rangle>0 \quad \text { for all } a \in \mathbb{R}^{n} \text { with } \quad\|a\|=r .
$$

It follows by an elementary topological degree argument that the equation $G_{c, l}(a)=0$ has a solution for each $l \in P C$, which by our previous argument is unique. In this way we define a function $\widetilde{a}: P C \rightarrow \mathbb{R}^{n}$ which satisfies

$$
\begin{equation*}
\sum_{0<t_{k}<b} I_{k}\left(L_{k-1}(h)\left(t_{k}\right)\right)+\int_{0}^{b} \phi^{-1}[\widetilde{a}+l(t)] d t=0 . \tag{3.15}
\end{equation*}
$$

To prove (ii) let $B$ be a bounded subset of $P C$ and let $l \in B$. From (3.15)

$$
\left\langle\sum_{0<t_{k}<b} I_{k}\left(L_{k-1}(h)\left(t_{k}\right)\right)+\int_{0}^{b} \phi^{-1}[\widetilde{a}(l)+l(t)] d t, \widetilde{a}(l)\right\rangle=0 .
$$

Then

$$
\left\langle\sum_{0<t_{k}<b} I_{k}\left(L_{k-1}(h)\left(t_{k}\right)\right), \widetilde{a}(l)\right\rangle+\int_{0}^{b}\left\langle\phi^{-1}[\widetilde{a}(l)+l(t)] d t, \widetilde{a}(l)\right\rangle=0,
$$

and hence

$$
\begin{align*}
\left\langle\sum_{0<t_{k}<b} I_{k}\left(L_{k-1}(h)\left(t_{k}\right)\right), \widetilde{a}(l)\right\rangle & +\int_{0}^{b}\left\langle\phi^{-1}[\widetilde{a}(l)+l(t)] d t, \widetilde{a}(l)+l(t)\right\rangle  \tag{3.16}\\
& =\int_{0}^{b}\left\langle\phi^{-1}[\widetilde{a}(l)+l(t)] d t, l(t)\right\rangle .
\end{align*}
$$

Assume that $\{\widetilde{a}(l), l \in B\}$ is not bounded. Then, for an arbitrary $A>0$, there is an $l \in B$, with $\|l\|_{0}$ sufficiently large, so that

$$
A \leq \alpha\left(\left|\phi^{-1}(\widetilde{a}(l)+l(t))\right|\right) .
$$

Hence by using (3.16) and (3.13), we find that

$$
\begin{aligned}
\left.A \int_{0}^{b} \| \phi^{-1} \widetilde{a}(l)+l(t)\right) \| \leq & \left.\int_{0}^{b} \alpha\left(\| \phi^{-1} \widetilde{a}(l)+l(t)\right) \|\right)\left\|\phi^{-1}(\widetilde{a}(l)+l(t))\right\| d t \\
\leq & \left.\int_{0}^{b}\left\langle\phi^{-1} \widetilde{a}(l)+l(t)\right), \widetilde{a}(l)+l(t)\right\rangle d t \\
= & -\left\langle\sum_{k=1}^{m} I_{k}\left(L_{k-1}(h)\left(t_{k}\right)\right), \widetilde{a}(l)\right\rangle \\
& +\int_{0}^{b}\left\langle\phi^{-1}(\widetilde{a}(l)+l(t)), l(t)\right\rangle d t \\
\leq & \int_{0}^{b}\left\langle\phi^{-1}(\widetilde{a}(l)+l(t)), l(t)\right\rangle \\
\leq & \left.\|l l\|_{0} \int_{0}^{b} \| \phi^{-1} \widetilde{a}(l)+l(t)\right) \| .
\end{aligned}
$$

Thus $A \leq\|l\|_{0}$, which is a contradiction. Therefore $\widetilde{a}$ sends bounded sets in PC into bounded sets in $\mathbb{R}^{n}$.

Finally, to show the continuity of $\widetilde{a}$, let $\left(l_{\alpha}\right)$ be a convergent sequence in $P C$, say $l_{\alpha} \rightarrow l$, as $\alpha \rightarrow \infty$. Since $\left(\widetilde{a}\left(l_{\alpha}\right)\right)$ is a bounded sequence, any subsequence of it contains a convergent subsequence, denoted by $\left(a\left(l_{\alpha_{j}}\right)\right.$. Let $a\left(l_{\alpha_{j}}\right) \rightarrow \widehat{a}$, as $j \rightarrow \infty$. By letting $j \rightarrow \infty$ in

$$
\int_{0}^{b} \phi^{-1}\left(\widetilde{a}\left(l_{\alpha_{j}}\right)+l_{\alpha_{j}}(t)\right) d t=0
$$

we find that

$$
\int_{0}^{b} \phi^{-1}(\widehat{a}+l(t)) d t=0
$$

and hence $\widetilde{a}(l)=\widehat{a}$, which shows the continuity of $\widetilde{a}$.

## 4. Existence results

We will assume in this section that $\phi: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ is continuous and satisfies conditions $\left(H_{1}\right)-\left(H_{2}\right)$ of Section 2. Our aim in this part is to apply the Manasevich and Mawhin continuation theorem [18] for quasilinear equations to the quasilinear problem with impulse effects (1.1)-(1.4).

Theorem 4.1. Assume that $\Omega$ is an open bounded set in $\operatorname{PC}\left([0, b], \mathbb{R}^{n}\right)$ such that the following conditions hold:
$\left(H_{3}\right)$ For each $\lambda \in(0,1)$ the problem

$$
\left\{\begin{array}{l}
\left(\phi\left(y^{\prime}\right)\right)^{\prime}=\lambda f\left(t, y, y^{\prime}\right)  \tag{4.1}\\
\Delta y\left(t_{k}\right)=\lambda I_{k}\left(y\left(t_{k}\right)\right) \\
\Delta y^{\prime}\left(t_{k}\right)=\lambda \bar{I}_{k}\left(y\left(t_{k}\right)\right) \\
y(0)=y(b), y^{\prime}(0)=y^{\prime}(b)
\end{array}\right.
$$

has no solution on $\partial \Omega$.
$\left(H_{4}\right)$ The equation

$$
\begin{align*}
G_{c, d}(a) & :=\frac{1}{b}\left[\sum_{k=1}^{m} I_{k}\left(L_{k-1}(h)\left(t_{k}\right)\right)+\int_{0}^{b} \phi^{-1}[\phi(d)\right.  \tag{4.2}\\
& \left.\left.\left.+\sum_{k=1}^{m} \bar{I}_{k}\left(l_{k-1}(h)\left(t_{k}\right)\right)\right)+\int_{0}^{t} f(s, a, 0) d s\right] d t\right]=0
\end{align*}
$$

where $h=f(t, a, 0)$ and $L_{0}(h)(t)=c+\int_{0}^{t} \phi^{-1}\left[\left(\phi(d)+\int_{0}^{s} f(\tau, a, 0) d \tau\right] d s, t \in\left[0, t_{1}\right]\right.$ has no solution on $\partial \Omega \cap \mathbb{R}^{n}$.
$\left(H_{5}\right)$ The Brouwer degree

$$
d_{B}\left[G, \Omega \cap \mathbb{R}^{n}, 0\right] \neq 0
$$

Then the problems (1.1)-(1.4) has a solution in $\Omega$.

Proof. We transform the problems (1.1)-(1.4) into the one parameter family of problems

$$
\left\{\begin{align*}
\left(\phi\left(y^{\prime}\right)\right)^{\prime}= & \lambda f\left(t, y, y^{\prime}\right)+(1-\lambda)\left(\frac { 1 } { b } \left[\sum_{k=1}^{m} I_{k}\left(L_{k-1}(h)\left(t_{k}\right)\right)\right.\right.  \tag{4.3}\\
& +\int_{0}^{b} \phi^{-1}\left[\phi\left(y^{\prime}(0)+\sum_{k=1}^{m} \bar{I}_{k}\left(L_{k-1}(h)\left(t_{k}\right)\right)\right)\right. \\
& \left.\left.\left.+\int_{0}^{t} f\left(s, y(s), y^{\prime}(s)\right) d s\right] d t\right]\right) \\
\Delta y\left(t_{k}\right)= & \lambda I_{k}\left(y\left(t_{k}\right)\right) \\
\Delta y^{\prime}\left(t_{k}\right)= & \lambda \bar{I}_{k}\left(y\left(t_{k}\right)\right) \\
y(0)=y(b), \quad & y^{\prime}(0)=y^{\prime}(b)
\end{align*}\right.
$$

For $\lambda \in(0,1]$, observe that in both cases, $y$ is a solution of problem (4.1) or $y$ is a solution of problem (4.3). We have necessarily

$$
\frac{1}{b}\left(\sum_{k=1}^{m} I_{k}\left(L_{k-1}(h)\left(t_{k}\right)\right)+\int_{0}^{b} \phi^{-1}\left[\phi\left(y^{\prime}(0)+\sum_{k=1}^{m} \bar{I}_{k}\left(L_{k-1}(h)\left(t_{k}\right)\right)\right)+\int_{0}^{t} f\left(s, y(s), y^{\prime}(s)\right) d s\right] d t\right)=0
$$

It follows that, for $\lambda \in(0,1]$, problems (4.1) and (4.3) have the same solutions. Furthermore it is easy to see that $f$ Carathéodory implies that $N: P C^{1}\left([0, b], \mathbb{R}^{n}\right) \times[0,1] \rightarrow Z$ defined by

$$
\begin{aligned}
N(y, \lambda)= & \left(\lambda J N_{f}(y)+(1-\lambda) \operatorname{UQN}_{f}(y), I_{1}\left(y\left(t_{1}\right)\right),\right. \\
& \left.\ldots, I_{m}\left(y\left(t_{m}\right)\right), \bar{I}_{1}\left(y\left(t_{1}\right)\right), \ldots, \bar{I}_{m}\left(y\left(t_{m}\right)\right), 0\right),
\end{aligned}
$$

where

$$
N_{f}(y)=N(y, 1), \quad \text { for each } y \in P C^{1}\left([0, b], \mathbb{R}^{n}\right),
$$

is continuous and sends bounded sets into bounded sets. Problem (4.3) can be written in the equivalent form

$$
\begin{equation*}
y=F_{f}(y, \lambda) \tag{4.4}
\end{equation*}
$$

with

$$
F_{f}(y, \lambda)=P y+J Q N_{f}(y)+K \circ\left[\lambda J(I-Q) N_{f}\right](y) .
$$

We assume that for $\lambda=1$, (4.4) does not have a solution on $\partial \Omega$. By hypothesis $\left(H_{3}\right)$, it follows that (4.4) has no solution for $(y, \lambda) \in \partial \Omega \times(0,1]$. For $\lambda=0$, (4.3) is equivalent to the problem

$$
\left\{\begin{array}{l}
\left(\phi\left(y^{\prime}\right)\right)^{\prime}=\frac{1}{b}\left[\sum_{k=1}^{m} I_{k}\left(L_{k-1}(h)\left(t_{k}\right)\right)+\int_{0}^{b} \phi^{-1}\left[\phi \left(y^{\prime}(0)\right.\right.\right.  \tag{4.5}\\
\left.\left.\left.+\sum_{k=1}^{m} \bar{I}_{k}\left(l_{k-1}(h)\left(t_{k}\right)\right)\right)+\int_{0}^{t} f\left(s, y(s), y^{\prime}(s)\right) d s\right] d t\right] \\
y(0)=y(b), y^{\prime}(0)=y^{\prime}(b)
\end{array}\right.
$$

And thus if $y$ is a solution of this problem, we must have

$$
\begin{align*}
\sum_{k=1}^{m} I_{k}\left(L_{k-1}(h)\left(t_{k}\right)\right)+ & \int_{0}^{b} \phi^{-1}\left[\phi\left(y^{\prime}(0)+\sum_{k=1}^{m} \bar{I}_{k}\left(L_{k-1}(h)\left(t_{k}\right)\right)\right)\right.  \tag{4.6}\\
& \left.+\int_{0}^{r} f\left(s, y(s), y^{\prime}(s)\right) d s\right] d r=0
\end{align*}
$$

Hence

$$
\left(\phi\left(y^{\prime}(t)\right)\right)^{\prime}=0,
$$

which implies that

$$
y^{\prime}(t)=\phi^{-1}(c)
$$

where $c \in \mathbb{R}^{n}$ is a constant. Integrating this last equation over $[0, b]$, we obtain $\phi^{-1}(c)=0$, and thus $y(t)=e$, a constant, and by (4.6)

$$
\begin{aligned}
\sum_{k=1}^{m} I_{k}\left(L_{k-1}(h)\left(t_{k}\right)\right)+ & \int_{0}^{b} \phi^{-1}\left[\phi\left(y^{\prime}(0)+\sum_{k=1}^{m} \bar{I}_{k}\left(l_{k-1}(h)\left(t_{k}\right)\right)\right)\right. \\
& \left.+\int_{0}^{r} f(s, e, 0) d s\right] d r=0
\end{aligned}
$$

which, together with hypothesis $\left(H_{4}\right)$, implies that $y=e \notin \partial \Omega$. Thus we prove that (4.4) has no solution $(y, \lambda) \in \partial \Omega \times[0,1]$. Then we have that for each $\lambda \in[0,1]$ the Leray-Schauder degree $d_{L S}[I-$ $\left.F_{f}(., \lambda), \Omega, 0\right]$ is well defined and by properties of that degree, that

$$
\begin{equation*}
d_{L S}\left[I-F_{f}(\cdot, 1), \Omega, 0\right]=d_{L S}\left[I-F_{f}(\cdot, 0), \Omega, 0\right] . \tag{4.7}
\end{equation*}
$$

Now it is clear that the problem

$$
\begin{equation*}
y=F_{f}(y, 1) \tag{4.8}
\end{equation*}
$$

is equivalent to the problem (1.1)-(1.4), and will have a solution, if we can prove that

$$
d_{L S}\left[I-F_{f}(\cdot, 0), \Omega, 0\right] \neq 0
$$

We have that

$$
F_{f}(y, 0)=P y+\operatorname{JQN}_{f}(y)
$$

Thus we obtain

$$
\begin{aligned}
y-F_{f}(y, 0)= & y-P y-\frac{1}{b}\left[\sum_{k=1}^{m} I_{k}\left(L_{k-1}(h)\left(t_{k}\right)\right)+\right. \\
& \int_{0}^{b} \phi^{-1}\left[\phi\left(y^{\prime}(0)+\sum_{k=1}^{m} \bar{I}_{k}\left(l_{k-1}(h)\left(t_{k}\right)\right)\right)\right. \\
& \left.\left.+\int_{0}^{t} f\left(s, y(s), y^{\prime}(s)\right) d s\right] d t\right] .
\end{aligned}
$$

Hence by the properties of Leray-Schauder degree (proved in [7]), we have that

$$
d_{L S}\left(I-F_{f}(., 0), \Omega, 0\right)=(-1)^{N} d_{B}\left(G, \Omega \cap \mathbb{R}^{n}, 0\right)
$$

where the function $G$ is defined in Proposition 3.2, and $d_{B}$ denotes the Brouwer degree. Since by the Hypothesis $\left(H_{5}\right)$ this last degree is different from zero, the theorem is proved.

Our next theorem is a consequence of Theorem 4.1. We need first the following definition.
Let $f=\left(f_{1}, \ldots, f_{n}\right): J \times \mathbb{R}^{n} \times \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ be a Carathéodory function. We will say that $f$ satisfies a generalized Villari condition if there is an $\rho_{0}>0$ such that for all $y \in P C_{b}^{1}, y=\left(y_{1}, \ldots, y_{n}\right)$, with

$$
\min _{t \in J}\left|y_{j}\right|>\rho_{0},
$$

for some $j \in\{1, \ldots, n\}$, it holds that

$$
\begin{equation*}
\int_{0}^{b} f_{i}\left(t, y(t), y^{\prime}(t)\right) d t \neq 0 \tag{4.9}
\end{equation*}
$$

for some $i \in\{1, \ldots, n\}$.
Let $B(R)$ denote the open ball in $\mathbb{R}^{n}$ with center zero and radius $R$.
Theorem 4.2. Assume that the following conditions hold:
$\left(K_{1}\right)$ There exist $v \in C^{1}\left(\mathbb{R}^{n}, \mathbb{R}^{n}\right)$ and $h \in L^{1}\left(J, \mathbb{R}_{+}\right)$such that

$$
\left\langle\phi(y), v^{\prime}(x) y\right\rangle \geq 0,\langle x, v(y)-v(z)\rangle \leq 0, \text { for any } x, y, z \in \mathbb{R}^{n},
$$

and

$$
|f(t, x, y)| \leq\langle f(t, x, y), v(x)\rangle+h(t)
$$

for all $x, y \in \mathbb{R}^{n}$ and a.e $t \in J$.
$\left(K_{2}\right) f$ satisfies a generalized Villari condition.
$\left(K_{3}\right)$ There exist positive constants $c_{k}$, such that for each $k=1, \ldots, m$, we have

$$
\left|I_{k}(x)\right| \leq c_{k}, \quad \forall x \in \mathbb{R}^{n},
$$

and

$$
\begin{aligned}
\sum_{k=1}^{m}\langle & \left.\phi\left(x+\bar{I}_{k}(y)\right), v(z)-v\left(y+I_{k}(y)\right)\right\rangle \leq 0, \\
& \sum_{k=1}^{m}\left\langle\int_{t_{k-1}}^{t_{k}} f(t, x, y) d t, v(y)\right\rangle \leq 0,
\end{aligned}
$$

for any $x, y, z \in \mathbb{R}^{n}$.
( $K_{4}$ ) There is an $R_{0}>0$, such that all the possible solutions to the equation

$$
\begin{align*}
G(a) & :=\frac{1}{b}\left[\sum_{k=1}^{m} I_{k}\left(L_{k-1}(h)\left(t_{k}\right)\right)+\int_{0}^{b} \phi^{-1}\left[\phi \left(y^{\prime}(0)\right.\right.\right.  \tag{4.10}\\
& \left.\left.\left.+\sum_{k=1}^{m} \bar{I}_{k}\left(l_{k-1}(h)\left(t_{k}\right)\right)\right)+\int_{0}^{s} f(s, a, 0) d s\right] d t\right]=0,
\end{align*}
$$

belong $B\left(0, R_{0}\right)$.
${ }_{( } K_{5}$ ) The Brouwer degree

$$
d_{B}\left[G, B\left(0, R_{0}\right), 0\right] \neq 0
$$

Then the problems (1.1)-(1.4) has at least one solution.
Proof. Let $(y, \lambda), y \in P C_{b}^{1}, \lambda \in(0,1)$, be a solution to (4.1), then using (4.2), we have

$$
\begin{aligned}
0 \geq & -\int_{0}^{b}\left\langle\phi\left(y^{\prime}(t)\right), v^{\prime}(y(t)) y^{\prime}(t)\right\rangle \\
= & \int_{0}^{b}\left\langle\left(\phi\left(y^{\prime}(t)\right)\right)^{\prime}, v(y(t))\right\rangle-\left\langle L_{0}^{\prime}(k)(0), v\left(L_{0}(k)\left(t_{1}\right)\right)-v\left(L_{0}(k)(0)\right)\right\rangle \\
& -\sum_{k=1}^{m}\left\langle\phi\left(L_{k-1}^{\prime}(k)\left(t_{k}\right)+\bar{I}_{k}\left(L_{k-1}(k)\left(t_{k}\right)\right)\right)\right),
\end{aligned}
$$

$$
\begin{aligned}
& \left.n\left(L_{k}(k)\left(t_{k+1}\right)\right)-v\left(L_{k-1}(k)\left(t_{k}\right)+I_{k}\left(L_{k-1}(k)\left(t_{k}\right)\right)\right)\right\rangle \\
& -\left\langle\int_{t_{k-1}}^{t_{k}} f(t, x, y) d t, v\left(L_{k-1}(k)\left(t_{k}\right)\right)\right\rangle \\
= & \lambda \int_{0}^{b}\left\langle f\left(t, y(t), y^{\prime}(t)\right), v(y(t))\right\rangle-\left\langle L_{0}^{\prime}(k)(0), v\left(L_{0}(k)\left(t_{1}\right)\right)-v\left(L_{0}(k)(0)\right)\right\rangle \\
& -\sum_{k=1}^{m}\left\langle\phi\left(L_{k-1}^{\prime}(k)\left(t_{k}\right)+\bar{I}_{k}\left(L_{k-1}(k)\left(t_{k}\right)\right)\right)\right) \\
& \left.n\left(L_{k}(k)\left(t_{k+1}\right)\right)-v\left(L_{k-1}(k)\left(t_{k}\right)+I_{k}\left(L_{k-1}(k)\left(t_{k}\right)\right)\right)\right\rangle \\
& -\left\langle\int_{t_{k-1}}^{t_{k}} f(t, x, y) d t, v\left(L_{k-1}(k)\left(t_{k}\right)\right)\right\rangle .
\end{aligned}
$$

Then, from hypothesis $\left(K_{1}\right),\left(K_{2}\right)$ and $\left(K_{3}\right)$, we get

$$
\begin{equation*}
\lambda \int_{0}^{b}\left\langle f\left(t, y(t), y^{\prime}(t)\right), v(y(t))\right\rangle \leq 0 \tag{4.11}
\end{equation*}
$$

Let us set $\phi\left(y^{\prime}(t)\right)=\widetilde{b}(t)+\bar{b}$, with $\int_{0}^{b} \widetilde{b}(t) d t=0$, and $\bar{b}=\frac{1}{b} \int_{0}^{b} \phi\left(y^{\prime}(t)\right) d t$. From (4.11) and (4.2), we get

$$
\widetilde{b}(t)=\lambda f\left(t, y(t), y^{\prime}(t)\right) .
$$

Then,

$$
\begin{align*}
\left\|\widetilde{b^{\prime}}\right\|_{L^{1}} & \leq \int_{0}^{b}\left|f\left(t, y(t), y^{\prime}(t)\right)\right| d t  \tag{4.12}\\
& \leq \int_{0}^{b}\left\langle f\left(t, y(t), y^{\prime}(t)\right), v(y(t))\right\rangle+\|h\|_{L^{1}} \leq\|h\|_{L^{1}}
\end{align*}
$$

which yields

$$
\|\widetilde{b}\|_{0} \leq b\|h\|_{L^{1}}
$$

We next find an a priori bound for $\bar{b}$. We have that

$$
\begin{equation*}
y^{\prime}(t)=\phi^{-1}(\widetilde{b}(t)+\bar{b}) \tag{4.13}
\end{equation*}
$$

Hence, by integrating on each $J_{k}=\left(t_{k-1}, t_{k}\right]$ and using the boundary conditions, we obtain

$$
y(t)=y(0)+\sum_{0<t_{k}<t} I_{k}\left(L_{k-1}(h)\left(t_{k}\right)\right)+\int_{0}^{t} \phi^{-1}(\widetilde{b}(s)+\bar{b}) d s, \quad t \in[0, b] .
$$

Then,

$$
\begin{equation*}
I_{k}\left(L_{k-1}(h)\left(t_{k}\right)\right)+\int_{0}^{b} \phi^{-1}(\widetilde{b}(t)+\bar{b}) d t=0 \tag{4.14}
\end{equation*}
$$

By Proposition 3.2, it follows that $\bar{b}=\widetilde{a}(\widetilde{b})$, where the function $\widetilde{a}$ is defined in that proposition. Recalling that $\widetilde{a}$ sends bounded sets into bounded sets, we have that there is a positive constant $C_{1}$ such that

$$
|\bar{b}| \leq C_{1}
$$

Hence, from (4.13) and the fact that $\phi^{-1}$ is continuous, we obtain a positive constant $C_{2}$ such that

$$
\begin{equation*}
\left\|y^{\prime}\right\|_{0} \leq C_{2} \tag{4.15}
\end{equation*}
$$

Hence for $t \in\left[0, t_{1}\right]$, we get

$$
\left\|\int_{0}^{t} y^{\prime}(s) d s\right\| \leq \int_{0}^{t_{1}}\left\|y^{\prime}(t)\right\| d t \leq C_{2} t_{1}:=M_{1}
$$

For $t \in\left(t_{1}, t_{2}\right]$

$$
\left\|\int_{t_{1}}^{t} y^{\prime}(s) d s\right\| \leq \int_{t_{1}}^{t_{2}}\left\|y^{\prime}(t)\right\| d t \leq C_{2}\left(t_{2}-t_{1}\right):=M_{2}
$$

We continue this process and we obtain that, for $t \in\left(t_{m}, b\right]$,

$$
\left\|\int_{t_{m}}^{t} y^{\prime}(s) d s\right\| \leq \int_{t_{m}}^{b}\left\|y^{\prime}(t)\right\| d t \leq C_{2}\left(b-t_{m}\right):=M_{m}
$$

Then

$$
\begin{equation*}
\left\|\int_{0}^{t} y^{\prime}(s) d s\right\| \leq \int_{0}^{b}\left\|y^{\prime}(t)\right\| d t \leq \max \left(M_{1}, M_{2}, \ldots, M_{m}\right):=M . \tag{4.16}
\end{equation*}
$$

Next the solution $y$ satisfies

$$
\begin{align*}
0 & =\frac{1}{b} \int_{0}^{b} f\left(t, y(t), y^{\prime}(t)\right) d t  \tag{4.17}\\
& =\frac{1}{b} f\left(t, y(0)+\sum_{0<t_{k}<t} I_{k}\left(L_{k-1}(h)\left(t_{k}\right)\right)+\int_{0}^{t} y^{\prime}(s) d s, y^{\prime}(t)\right) d t
\end{align*}
$$

By the generalized Villari condition, we have, for each $j_{1} \in\{1, \ldots, n\}$, there exists $t_{j_{1}} \in\left[0, t_{1}\right]$ such that

$$
\left|y_{j_{1}}\left(t_{j_{1}}\right)\right| \leq \rho_{1}
$$

Since

$$
y_{j_{1}}(t)=y_{j_{1}}\left(t_{j_{1}}\right)+\int_{t_{j_{1}}}^{t} y_{j_{1}}^{\prime}(s) d s
$$

we get

$$
\left|y_{j_{1}}(t)\right| \leq\left|y_{j_{1}}\left(t_{j_{1}}\right)\right|+M_{1} \leq \rho_{1}+M_{1}
$$

For each $j_{2} \in\{1, \ldots, n\}$, there exists a $t_{j_{2}} \in\left(t_{1}, t_{2}\right]$ such that

$$
\left|y_{j_{2}}\left(t_{j_{2}}\right)\right| \leq \rho_{2} .
$$

Since

$$
y_{j_{2}}(t)=y_{j_{2}}\left(t_{j_{2}}\right)+I_{1}\left(L_{0}(h)\left(t_{1}\right)\right)+\int_{t_{j_{2}}}^{t} y_{j_{1}}^{\prime}(s) d s
$$

we get

$$
\left|y_{j_{2}}(t)\right| \leq\left|y_{j_{2}}\left(t_{j_{2}}\right)\right|+c_{1}+M_{2} \leq \rho_{2}+c_{1}+M_{2} .
$$

We continue this process, and we get that, for each $j_{m} \in\{1, \ldots, n\}$, there exists a $t_{j_{m}} \in\left(t_{m}, b\right]$ such that

$$
\left|y_{j_{m}}\left(t_{j_{m}}\right)\right| \leq \rho_{m} .
$$

Since

$$
y_{j_{m}}(t)=y_{j_{m}}\left(t_{j_{m}}\right)+I_{m}\left(L_{m-1}(h)\left(t_{m}\right)\right)+\int_{t_{j_{m}}}^{t} y_{j_{m}}^{\prime}(s) d s,
$$

we get

$$
\left|y_{j_{m}}(t)\right| \leq \rho_{m}+c_{m}+M_{m} .
$$

Then for each $j \in\{1, \ldots, n\}$, there exists $t_{j} \in J$ such that

$$
\left|y_{j}(t)\right| \leq \max \rho_{1}, \ldots, \rho_{m}+\sum_{k=1}^{m} c_{k}+M:=\bar{M}
$$

Thus there is a constant $C_{3}$ such that $\|y\|_{0} \leq C_{3}$. It follows that we can find $R_{0}>0$ such that, if $(y, \lambda)$ is a solution to (4.1), then $\|y\|_{1} \leq R_{0}$. We define next the set $\Omega \subset P C_{b}^{1}$ that appears in Theorem 4.1 as $\Omega=B\left(R_{0}\right)$, the open ball in $P C_{b}^{1}$ of center 0 and radius $R_{0}$. Thus condition $\left(H_{3}\right)$ of Theorem 4.1 is satisfied with $\Omega=B\left(R_{0}\right)$, and since the rest of the conditions of Theorem 4.1 are also satisfied, the proof is complete.

Corollary 4.3. Suppose that the following conditions are satisfied.
$\left(M_{1}\right)$ There is a mapping $v \in C^{1}\left(\mathbb{R}^{n}, \mathbb{R}^{n}\right)$ such that conditions $\left(K_{1}\right)$ and $\left(K_{3}\right)$ hold.
$\left(M_{2}\right)$ There exist a function $\widetilde{h} \in L^{1}\left(J, \mathbb{R}_{+}\right)$and continuous function $\eta:[0, \infty) \rightarrow[0, \infty)$, such that

$$
\eta(s) \rightarrow \infty \quad \text { as } \quad s \rightarrow \infty
$$

and

$$
\begin{equation*}
\eta(|x|)-\widetilde{h}(t) \leq|f(t, x, y)| \tag{4.18}
\end{equation*}
$$

for almost all $t \in J$, and all $x, y \in \mathbb{R}^{n}$.
$\left(M_{3}\right)$ Condition ( $K_{5}$ ) holds.
Then the problems (1.1)-(1.4) has at least one solution.
Proof. Let $(y, \lambda), \lambda \in(0,1)$ be a solution to (4.1). As in the proof of Theorem 4.2, it follows from conditions $\left(K_{1}\right)$ and $\left(K_{3}\right)$, that there is a positive constant $C_{2}$ such that $\left\|y^{\prime}\right\|_{0} \leq C_{2}$. We claim that conditions ( $K_{3}$ ) and ( $M_{2}$ ) imply that there is a constant $C_{3}$ such that $\|y\|_{0} \leq C_{3}$. Indeed, from (4.12), we have that

$$
\int_{0}^{b}\left|f\left(t, y(t), y^{\prime}(t)\right)\right| d t \leq\|h\|_{L^{1}}
$$

Then by (4.18)

$$
\begin{equation*}
\int_{0}^{b} \eta(|y(t)|) d t \leq\|h\|_{L^{1}}+\|\widetilde{h}\|_{L^{1}} \tag{4.19}
\end{equation*}
$$

Since (4.15) holds by the reasoning of the previous theorem and $\eta(s) \rightarrow \infty$ as $s \rightarrow \infty$, from (4.19) we find the required bound for $\|y\|_{0}$.

Now let $a \in \mathbb{R}^{n}$ be such that

$$
\sum_{k=1}^{m} I_{k}\left(L_{k-1}(h)\left(t_{k}\right)\right)+\int_{0}^{b} \phi^{-1}\left[\phi\left(y^{\prime}(0)+\sum_{k=1}^{m} \bar{I}_{k}\left(l_{k-1}(h)\left(t_{k}\right)\right)\right)+\int_{0}^{s} f(s, a, 0) d s\right] d t=0
$$

Then (4.18) implies that $\eta(|a|) \leq C_{2}$, and hence $|a| \leq C_{3}$, for some positive constants $C_{2}$ and $C_{3}$. Thus there is $R_{0}>0$ such that every solutions of (4.10) belongs to $B\left(R_{0}\right)$, and since condition ( $K_{5}$ ) holds, then all condition of Theorem 4.1 are satisfied. Therefore, (1.1)-(1.4) has at least one solution.

Example 4.1. We consider the following

$$
\begin{gather*}
\left(\left|y^{\prime}(t)\right|^{p-2} y^{\prime}(t)\right)^{\prime}=-\left(1+y^{2}(t)\right)+h(t), \quad \text { a.e. } t \in J:=[0, \pi],  \tag{4.20}\\
 \tag{4.21}\\
y\left(\frac{\pi^{+}}{4}\right)-y\left(\frac{\pi}{4}\right)=I\left(y\left(\frac{\pi}{4}\right)\right),  \tag{4.22}\\
y^{\prime}\left(\frac{\pi^{+}}{4}\right)-y^{\prime}\left(\frac{\pi}{4}\right)=\bar{I}\left(y\left(\frac{\pi}{4}\right)\right),  \tag{4.23}\\
y(0)=y(\pi), \quad y^{\prime}(0)=y^{\prime}(\pi),
\end{gather*}
$$

where $h: J \rightarrow \mathbb{R}$ defined by

$$
h(t)=\cos t, t \in[0, \pi] .
$$

Let $v=c \in C^{1}(\mathbb{R}, \mathbb{R}), c \in \mathbb{R}_{+}$, then

$$
\left\langle\phi(y), \nu^{\prime}(x) y\right\rangle \geq 0,\langle x, v(y)-v(z)\rangle \leq 0, \quad \text { for any } x, y, z \in \mathbb{R}^{n}
$$

and

$$
|f(t, x, y)| \leq\langle f(t, x, y), v(x)\rangle+h(t),
$$

for all $x, y \in \mathbb{R}$ and a.e $t \in J$.
$\left(\bar{K}_{1}\right)$ There exist positive constants $\alpha_{1}, \alpha_{2}$, such that

$$
I(x)=\alpha_{1}, \bar{I}(x)=\alpha_{2}, \quad \forall x \in \mathbb{R}
$$

and

$$
\langle\phi(x+\bar{I}(y)), v(z)-v(y+I(y))\rangle=0
$$

and

$$
\begin{aligned}
c \int_{0}^{\frac{\pi}{4}} f(t, x, y) d t+c \int_{\frac{\pi}{4}}^{\pi} f(t, x, y) d t & =c \int_{0}^{\pi} f(t, x, y) d t \\
& =-c \pi\left(1+x^{2}\right) \leq 0
\end{aligned}
$$

for any $x, y, z \in \mathbb{R}$.
Define

$$
G(a):=\frac{1}{b}\left[\alpha_{1}+\int_{0}^{b} \phi^{-1}\left[d+\alpha_{2}+\int_{0}^{t} f(s, a, 0) d s\right] d t\right] .
$$

It is clear that $G: \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function. We can show that there exists $R_{0}>0$ such that $G\left(B\left(0, R_{0}\right)\right) \subset B\left(0, R_{0}\right)$. Then $d(G, B(0, R), 0) \neq 0$. So all the conditions of Theorem 4.2 are satisfied. Hence the above problem has at least one solution.

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## Conflict of interest

The authors declare no conflict of interest.

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