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*Research article*

## New classes of strongly exponentially preinvex functions

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**Abstract:** In this paper, some new classes of the strongly exponentially generalized preinvex functions involving an auxiliary non-negative function and a bifunction are introduced. New relationships among various concepts of strongly exponentially generalized preinvex functions are established. It is shown that the optimality conditions of differentiable strongly exponentially generalized preinvex functions can be characterized by exponentially variational-like inequalities. As special cases, one can obtain various new and known results from our results. Results obtained in this paper can be viewed as refinement and improvement of previously known results.

**Keywords:** preinvex functions; monotone operators; invex functions; exponentially preinvex functions

**Mathematics Subject Classification:** 26D10, 49J40

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### 1. Introduction

Recently, several extensions and generalizations have been considered for classical convexity. Strongly convex functions were introduced and studied by Polyak [45], which play an important part in the optimization theory and related areas. Karmardian [15] used the strongly convex functions to discuss the unique existence of a solution of the nonlinear complementarity problems. Strongly convex functions are used to investigate the convergence analysis for solving variational inequalities and equilibrium problems, see Zu and Marcotte [51]. See also Nikodem and Pales [20] and Qu and Li [46]. Awan et al. [7–9] have derived Hermite-Hadamard type inequalities, which provide upper and lower estimate for the integrand. For more applications and properties of the strongly convex functions. See, for example, [1, 2, 4, 9, 10, 14–16, 18–20, 25, 29, 30, 33–35, 40, 42, 45, 46, 52].

Hanson [13] introduced the concept of invex function for the differentiable functions, which played significant part in the mathematical programming. Ben-Israel and Mond [11] introduced the concept of invex set and preinvex functions. It is known that the differentiable preinvex function are invex functions. The converse also holds under certain conditions, see [17]. Noor [22] proved that the

minimum of the differentiable preinvex functions on the invex set can be characterized the variational-like inequality. Noor [26–28] proved that a function  $f$  is preinvex function, if and only if, it satisfies the Hermite-Hadamard type integral inequality. Noor et al. [29–31, 37–39] investigated the applications of the strongly preinvex functions and their variant forms. See also [48–50] and the references therein.

It is known that more accurate and inequalities can be obtained using the logarithmically convex functions than the convex functions. Closely related to the log-convex functions, we have the concept of exponentially convex(concave) functions, the origin of exponentially convex functions can be traced back to Bernstein [12]. Avriel [6] introduced and studied the concept of  $r$ -convex functions, where as the  $(r, p)$ -convex functions were studied by Antczak [3]. For further properties of the  $r$ -convex functions, see Zhao et al. [51] and the references therein. Exponentially convex functions have important applications in information theory, big data analysis, machine learning and statistic. See [2, 3, 5, 6, 41] and the references therein. Noor and Noor [32–34] considered the concept of exponentially convex functions and discussed the basic properties. It is worth mentioning that these exponentially convex functions [13–15] are distinctly different from the exponentially convex functions considered and studied by Bernstein [12], Awan et al. [7] and Pecaric et al. [43, 44]. We would like to point out that The definition of exponential convexity in Noor and Noor [32–34] is quite different from Bernstein [12]. Noor and Noor [36] studied the properties of the exponentially preinvex functions and their variant forms. They have shown that the exponentially preinvex functions enjoy the same interesting properties which exponentially convex functions have. See [32–34] and the references therein for more details.

Inspired by the research work going in this field, we introduce and consider some new classes of nonconvex functions with respect to an arbitrary non-negative function and arbitrary bifunction, which is called the strongly exponentially generalized preinvex functions. Several new concepts of monotonicity are introduced. We establish the relationship between these classes and derive some new results under some mild conditions. Optimality conditions for differentiable strongly exponentially generalized preinvex functions are investigated. As special cases, one can obtain various new and refined versions of known results. It is expected that the ideas and techniques of this paper may stimulate further research in this field.

## 2. Preliminary results

Let  $K_\eta$  be a nonempty closed set in a real Hilbert space  $H$ . We denote by  $\langle \cdot, \cdot \rangle$  and  $\| \cdot \|$  be the inner product and norm, respectively. Let  $F : K_\eta \rightarrow R$  be a continuous function and  $\eta(., .) : K_\eta \times K_\eta \rightarrow R$  be an arbitrary continuous bifunction. Let  $\xi(.)$  be a non-negative function.

**Definition 2.1** ([11]). *The set  $K_\eta$  in  $H$  is said to be invex set with respect to an arbitrary bifunction  $\eta(., .)$ , if*

$$u + \lambda\eta(v, u) \in K_\eta, \quad \forall u, v \in K_\eta, \lambda \in [0, 1].$$

The invex set  $K_\eta$  is also called  $\eta$ -connected set. If  $\eta(v, u) = v - u$ , then the invex set  $K_\eta$  is a convex set, but the converse is not true. For example, the set  $K_\eta = R - (-\frac{1}{2}, \frac{1}{2})$  is an invex set with respect to  $\eta$ , where

$$\eta(v, u) = \begin{cases} v - u, & \text{for } v > 0, u > 0 \quad \text{or} \quad v < 0, u < 0 \\ u - v, & \text{for } v < 0, u > 0 \quad \text{or} \quad v < 0, u < 0. \end{cases}$$

It is clear that  $K_\eta$  is not a convex set.

We now introduce some new classes of strongly exponentially preinvex functions.

**Definition 2.2.** The function  $F$  on the invex set  $K_\eta$  is said to be strongly exponentially generalized preinvex with respect to the bifunction  $\xi(\cdot)$ , if there exists a constant  $\mu > 0$ , such that

$$e^{F(u+\lambda\eta(v,u))} \leq (1-\lambda)e^{F(u)} + \lambda e^{F(v)} - \mu\lambda(1-\lambda)\xi(\eta(v,u)), \quad \forall u, v \in K_\eta, \lambda \in [0, 1].$$

The function  $F$  is said to be strongly exponentially generalized preconcave, if and only if,  $-F$  is strongly exponentially generalized preinvex. Note that every strongly exponentially generalized convex function is a strongly exponentially generalized preinvex, but the converse is not true.

We now discuss some special cases of the strongly exponentially preinvex functions:

**I.** If  $h(\eta(v, u)) = \|\eta(v, u)\|^2$ , then the strongly exponentially generalized preinvex function becomes strongly generalized preinvex functions, that is,

$$e^{F(u+\lambda\eta(v,u))} \leq (1-\lambda)e^{F(u)} + \lambda e^{F(v)} - \mu\lambda(1-\lambda)\|\eta(v, u)\|^2, \quad \forall u, v \in K_\eta, \lambda \in [0, 1].$$

For the properties of the strongly preinvex functions in variational inequalities and equilibrium problems, see Noor [29–31].

**II.** If  $\eta(v, u) = v - u$ , then Definition 2.2 becomes:

**Definition 2.3.** The function  $F$  on the convex set  $K$  is said to be strongly exponentially generalized convex with respect to a non-negative function  $\xi(\cdot)$ , if there exists a constant  $\mu > 0$ , such that

$$e^{F(u+\lambda(v-u))} \leq (1-\lambda)e^{F(u)} + \lambda e^{F(v)} - \mu\lambda(1-\lambda)\xi(v-u), \quad \forall u, v \in K, \lambda \in [0, 1].$$

**III.** If  $\xi(\eta(v, u)) = W(v, u)$ , where  $W(v, u)$  is any arbitrary bifunction, then Definition 2.2 becomes:

**Definition 2.4.** The function  $F$  on the convex set  $K$  is said to be strongly exponentially generalized convex with respect to a non-negative bifunction  $W(\cdot, \cdot)$ , if there exists a constant  $\mu > 0$ , such that

$$e^{F(u+\lambda(v-u))} \leq (1-\lambda)e^{F(u)} + \lambda e^{F(v)} - \mu\lambda(1-\lambda)W(v, u), \quad \forall u, v \in K, \lambda \in [0, 1],$$

which has been studied by Noor [21].

In brief, for appropriate choice of the arbitrary function  $\xi(\cdot)$  and the bifunction  $\eta(\cdot, \cdot)$ , one can obtain a wide class of exponentially preinvex convex functions and their variant forms, see Noor and Noor [33, 34].

**Definition 2.5.** The function  $F$  on the invex set  $K_\eta$  is said to be strongly exponentially affine generalized preinvex with respect to the bifunction  $\xi(\cdot)$ , if there exists a constant  $\mu > 0$ , such that

$$e^{F(u+\lambda\eta(v,u))} = (1-\lambda)e^{F(u)} + \lambda e^{F(v)} - \mu\lambda(1-\lambda)\xi(\eta(v, u)), \quad \forall u, v \in K_\eta, \xi \in [0, 1].$$

If  $t = \frac{1}{2}$ , then the function  $F$  satisfies

$$e^{F(\frac{2u+\eta(v,u)}{2})} = \frac{1}{2}\{e^{F(u)} + e^{F(v)}\} - \frac{1}{4}\mu\xi(\eta(v, u)), \quad \forall u, v \in K_\eta,$$

and is called strongly exponentially affine Jensen generalized preinvex function.

**Definition 2.6.** The function  $F$  on the invex set  $K_\eta$  is said to be strongly exponentially generalized quasi preinvex with respect to a non-negative bifunction  $\xi(\cdot)$ , if there exists a constant  $\mu > 0$  such that

$$e^{F(u+\lambda\eta(v,u))} \leq \max\{e^{F(u)}, e^{F(v)}\} - \mu\lambda(1-\lambda)\xi(\eta(v,u)), \quad \forall u, v \in K_\eta, \lambda \in [0, 1].$$

**Definition 2.7.** The function  $F$  on the invex set  $K_\eta$  is said to be strongly exponentially generalized log-preinvex with respect to  $\xi(\cdot)$ , if there exists a constant  $\mu > 0$  such that

$$e^{F(u+\lambda\eta(v,u))} \leq (e^{F(u)})^{1-\lambda}(e^{F(v)})^\lambda - \mu\lambda(1-\lambda)\xi(\eta(v,u)), \quad \forall u, v \in K_\eta, \lambda \in [0, 1],$$

where  $F(\cdot) > 0$ .

From the above definition, we have

$$\begin{aligned} e^{F(u+t\eta(v,u))} &\leq (e^{F(u)})^{1-t}(e^{F(v)})^t - \mu\lambda(1-\lambda)\xi(\eta(v,u)) \\ &\leq (1-t)e^{F(u)} + te^{F(v)} - \mu\lambda(1-\lambda)\xi(\eta(v,u)) \\ &\leq \max\{e^{F(u)}, e^{F(v)}\} - \mu\lambda(1-\lambda)\xi(\eta(v,u)). \end{aligned}$$

It is observed that every strongly exponentially generalized log-preinvex function is strongly exponentially generalized preinvex function and every strongly exponentially generalized preinvex function is a strongly exponentially generalized quasi-preinvex function. However, the converse is not true.

For  $\lambda = 1$ , Definitions 2.2 and 2.7 reduce to the following condition.

**Condition A.**

$$e^{F(u+\eta(v,u))} \leq e^{F(v)}, \quad \forall u, v \in K_\eta.$$

**Definition 2.8.** The differentiable function  $F$  on the invex set  $K_\eta$  is said to be strongly exponentially generalized invex function with respect to an arbitrary non-negative function  $\xi(\cdot)$  and the bifunction  $\eta(\cdot, \cdot)$ , if there exists a constant  $\mu > 0$  such that

$$e^{F(v)} - e^{F(u)} \geq \langle e^{F(u)} F'(u), \eta(v, u) \rangle + \mu\xi(\eta(v, u)), \quad \forall u, v \in K_\eta,$$

where  $F'(u)$ , is the differential of  $F$  at  $u$ .

**Remark 2.1.** If  $\mu = 0$ , then the Definitions 2.2–2.7 reduce to the ones in Noor and Noor [36].

**Definition 2.9.** A differential operator  $F' : K \rightarrow H$  is said to be:

1. strongly exponentially generalized  $\eta$ -monotone, iff, there exists a constant  $\alpha > 0$  such that

$$\langle e^{F(u)} F'(u), \eta(v, u) \rangle + \langle e^{F(v)} F'(v), \eta(u, v) \rangle \leq -\alpha\{\xi(\eta(v, u)) + \xi(\eta(u, v))\}, \quad u, v \in K_\eta.$$

2. exponentially  $\eta$ -monotone, iff,

$$\langle e^{F(u)} F'(u), \eta(v, u) \rangle + \langle e^{F(v)} F'(v), \eta(u, v) \rangle \leq 0, \quad u, v \in K_\eta.$$

3. strongly exponentially generalized  $\eta$ -pseudomonotone, iff, there exists a constant  $\nu > 0$  such that

$$\langle e^{F(u)} F'(u), \eta(v, u) \rangle + \nu\xi(\eta(v, u)) \geq 0 \Rightarrow -\langle e^{F(v)} F'(v), \eta(u, v) \rangle \geq 0, \quad u, v \in K_\eta.$$

4. *strongly exponentially relaxed generalized  $\eta$ -pseudomonotone, iff, there exists constant  $\mu > 0$  such that*

$$\langle e^{F(u)}F'(u), \eta(v, u) \rangle \geq 0 \Rightarrow -\langle e^{F(v)}F'(v), \eta(u, v) \rangle + \mu\xi(\eta(u, v)) \geq 0, \quad u, v \in K_\eta.$$

5. *strictly exponentially  $\eta$ -monotone, iff,*

$$\langle e^{F(u)}F'(u), \eta(v, u) \rangle + \langle e^{F(v)}F'(v), \eta(u, v) \rangle < 0, \quad u, v \in K_\eta.$$

6. *exponentially  $\eta$ -pseudomonotone, iff,*

$$\langle e^{F(u)}F'(u), \eta(v, u) \rangle \geq 0 \Rightarrow \langle e^{F(v)}F'(v), \eta(u, v) \rangle \leq 0, \quad u, v \in K_\eta.$$

7. *exponentially quasi  $\eta$ -monotone, iff,*

$$\langle e^{F(u)}F'(u), \eta(v, u) \rangle > 0 \Rightarrow \langle e^{F(v)}F'(v), \eta(u, v) \rangle \leq 0, \quad u, v \in K_\eta.$$

8. *strictly exponentially  $\eta$ -pseudomonotone, iff,*

$$\langle e^{F(u)}F'(u), \eta(v, u) \rangle \geq 0 \Rightarrow \langle e^{F(v)}F'(v), \eta(u, v) \rangle < 0, \quad u, v \in K_\eta.$$

**Definition 2.10.** *A differentiable function  $F$  on the invex set  $K_\eta$  is said to be strongly exponentially pseudo generalized  $\eta$ -invex function, iff, if there exists a constant  $\mu > 0$  such that*

$$\langle e^{F(u)}F'(u), \eta(v, u) \rangle + \mu h(\eta(u, v)) \geq 0 \Rightarrow e^{F(v)} - e^{F(u)} \geq 0, \quad \forall u, v \in K_\eta.$$

**Definition 2.11.** *A differentiable function  $F$  on  $K_\eta$  is said to be strongly exponentially generalized quasi-invex function, iff, if there exists a constant  $\mu > 0$  such that*

$$e^{F(v)} \leq e^{F(u)} \Rightarrow \langle e^{F(u)}F'(u), \eta(v, u) \rangle + \mu\xi(\eta(u, v)) \leq 0, \quad \forall u, v \in K_\eta.$$

**Definition 2.12.** *The function  $F$  on the set  $K_\eta$  is said to be exponentially pseudo-invex, if*

$$\langle e^{F(u)}F'(u), \eta(v, u) \rangle \geq 0 \Rightarrow e^{F(v)} \geq e^{F(u)}, \quad \forall u, v \in K_\eta.$$

**Definition 2.13.** *The differentiable function  $F$  on the  $K_\eta$  is said to be exponentially quasi-invex function, if such that*

$$e^{F(v)} \leq e^{F(u)} \Rightarrow \langle e^{F(u)}F'(u), \eta(v, u) \rangle \leq 0, \quad \forall u, v \in K_\eta.$$

If  $\eta(v, u) = -\eta(v, u), \forall u, v \in K$ , then Definitions 2.9–2.13 reduce to the known ones. All these new concepts may play important and fundamental part in the mathematical programming and optimization.

We also need the following assumption regarding the bifunction  $\eta(\cdot, \cdot)$ , which plays an important in the derivation of the main results.

**Condition C** [17]. Let  $\eta(\cdot, \cdot) : K_\eta \times K_\eta \rightarrow H$  satisfy thw assumptions

$$\begin{aligned} \eta(u, u + \lambda\eta(v, u)) &= -\lambda\eta(v, u) \\ \eta(v, u + \lambda\eta(v, u)) &= (1 - \lambda)\eta(v, u), \quad \forall u, v \in K_\eta, \lambda \in [0, 1]. \end{aligned}$$

### 3. Main results

Through out this section we assume that the non-negative function  $\xi$  is even and homogeneous of degree two, that is,  $\xi(-u) = \xi(u)$ ,  $\xi(\gamma u) = \gamma^2 \xi(u)$ ,  $\forall u \in K_\eta, \gamma \in R$ , unless otherwise specified.

**Theorem 3.1.** *Let  $F$  be a differentiable function on the invex set  $K_\eta$  in  $H$  and Condition C hold. Let the function  $\xi$  be even and exponentially homogeneous of degree 2. If the function  $F$  is strongly exponentially generalized preinvex function, if and only if,  $F$  is a strongly exponentially generalized invex function.*

*Proof.* Let  $F$  be a strongly exponentially generalized preinvex function. Then

$$e^{F(u+\lambda\eta(v,u))} \leq (1-\lambda)e^{F(u)} + \lambda e^{F(v)} - \lambda(1-\lambda)\mu\xi(\eta(v,u)), \quad \forall u, v \in K_\eta,$$

which can be written as

$$e^{F(v)} - e^{F(u)} \geq \left\{ \frac{e^{F(u+\lambda\eta(v,u))} - e^{F(u)}}{\lambda} \right\} + (1-\lambda)\mu\xi(\eta(v,u)).$$

Taking the limit in the above inequality as  $\lambda \rightarrow 0$ , we have

$$e^{F(v)} - e^{F(u)} \geq \langle e^{F(u)} F'(u), \eta(v, u) \rangle + \mu\xi(\eta(v, u)).$$

This shows that  $F$  is a strongly exponentially generalized invex function.

Conversely, let  $F$  be a strongly exponentially generalized invex function on the invex set  $K_\eta$ . Then  $\forall u, v \in K_\eta, \lambda \in [0, 1], v_\lambda = u + \lambda\eta(v, u) \in K_\eta$ , using Condition C and the fact that the function  $\xi$  is even and exponentially homogeneous of degree 2, we have

$$\begin{aligned} & e^{F(v)} - e^{F(u+\lambda\eta(v,u))} \\ & \geq \langle e^{F(u+\lambda\eta(v,u))} F'(u + \lambda\eta(v, u)), \eta(v, u + \lambda\eta(v, u)) \rangle + \mu\xi(\eta(v, u + \lambda\eta(v, u))) \\ & = (1-\lambda) \langle e^{F(u+\lambda\eta(v,u))} F'(u + \lambda\eta(v, u)), \eta(v, u) \rangle + \mu(1-\lambda)^2 \xi(\eta(v, u)). \end{aligned} \quad (3.1)$$

In a similar way, we have

$$\begin{aligned} & e^{F(u)} - e^{F(u+\lambda\eta(v,u))} \\ & \geq \langle e^{F(u+\lambda\eta(v,u))} F'(u + \lambda\eta(v, u)), \eta(u, u + \lambda\eta(v, u)) \rangle + \mu\xi(\eta(u, u + \lambda\eta(v, u))) \\ & = -\lambda e^{F(u+\lambda\eta(v,u))} F'(u + \lambda\eta(v, u)), \eta(v, u) \rangle + \mu\lambda^2 \xi(\eta(v, u)). \end{aligned} \quad (3.2)$$

Multiplying (3.1) by  $\lambda$  and (3.2) by  $(1-\lambda)$  and adding the resultant, we have

$$e^{F(u+\lambda\eta(v,u))} \leq (1-\lambda)e^{F(u)} + \lambda e^{F(v)} - \lambda(1-\lambda)\mu\xi(\eta(v, u)).$$

□

**Theorem 3.2.** *Let  $F$  be differentiable on the invex set  $K_\eta$  and let Condition A and Condition C Hold. Let the function  $\xi$  be an even and exponentially homogeneous of degree 2. The function  $F$  is a strongly exponentially generalized invex function, if and only if,  $F'$  is strongly exponentially generalized  $\eta$ -monotone.*

*Proof.* Let  $F$  be a strongly exponentially generalized invex function on the invex set  $K_\eta$ . Then

$$e^{F(v)} - e^{F(u)} \geq \langle e^{F(u+\lambda\eta(v,u))} F'(u), \eta(v, u) \rangle + \mu\xi(\eta(v, u)) \quad \forall u, v \in K_\eta. \quad (3.3)$$

Changing the role of  $u$  and  $v$  in (3.3), we have

$$e^{F(u)} - e^{F(v)} \geq \langle e^{F(v)} F'(v), \eta(u, v) \rangle + \mu\xi(\eta(u, v)) \quad \forall u, v \in K_\eta. \quad (3.4)$$

Adding (3.3) and (3.4), we have

$$\langle e^{F(u)} F'(u), \eta(v, u) \rangle + \langle e^{F(v)} F'(v), \eta(u, v) \rangle \geq -\mu\{\xi(\eta(v, u)) + \xi(\eta(u, v))\}, \quad (3.5)$$

which shows that  $F'$  is strongly exponentially generalized  $\eta$ -monotone.

Conversely, let  $F'$  be strongly exponentially generalized  $\eta$ -monotone. From (3.5), we have

$$\langle e^{F(v)} F'(v), \eta(u, v) \rangle \geq \langle e^{F(u)} F'(u), \eta(v, u) \rangle - \{\xi(\eta(v, u)) + \xi(\eta(u, v))\}, \quad (3.6)$$

Since  $K_\eta$  is an invex set,  $\forall u, v \in K_\eta, t \in [0, 1] v_\lambda = u + \lambda\eta(v, u) \in K_\eta$ . Taking  $v = v_\lambda$  in (3.6), using Condition C and the fact that the function  $h$  is even and exponentially homogeneous of degree 2, we have

$$\begin{aligned} \langle e^{F(v_\lambda)} F'(v_\lambda), \eta(u, u + t\eta(v, u)) \rangle &\leq \langle e^{F(u)} F'(u), \eta(u + \lambda\eta(v, u), u) \rangle \\ &\quad - \mu\{\xi(\eta(u + \lambda\eta(v, u), u)) + \xi(\eta(u, u + \lambda\eta(v, u)))\} \\ &= -\lambda\langle e^{F(u)} F'(u), \eta(v, u) \rangle - 2\lambda^2\mu\xi(\eta(v, u)), \end{aligned}$$

which implies that

$$\langle e^{F(v_\lambda)} F'(v_\lambda), \eta(v, u) \rangle \geq \langle e^{F(u)} F'(u), \eta(v, u) \rangle + 2\mu\lambda\xi(\eta(v, u)). \quad (3.7)$$

Let

$$\varphi(\lambda) = e^{F(u+\lambda\eta(v,u))}. \quad (3.8)$$

Then

$$\varphi(0) = e^{F(u)}, \quad \varphi(1) = e^{F(u+\lambda\eta(v,u))}. \quad (3.9)$$

From (3.7), we have

$$\begin{aligned} \varphi'(t) &= \langle e^{F(u+\lambda\eta(v,u))} F'(u + \lambda\eta(v, u)), \eta(v, u) \rangle \\ &\geq \langle e^{F(u)} F'(u), \eta(v, u) \rangle + 2\mu\lambda\xi(\eta(v, u)). \end{aligned} \quad (3.10)$$

Integrating (3.10) between 0 and 1, we have

$$\varphi(1) - \varphi(0) \geq \langle e^{F(u)} F'(u), \eta(v, u) \rangle + \mu\xi(\eta(v, u)).$$

that is,

$$e^{F(u+\xi\eta(v,u))} - e^{F(u)} \geq \langle e^{F(u)} F'(u), \eta(v, u) \rangle + \mu\xi(\eta(v, u)).$$

By using Condition A, we have

$$e^{F(v)} - e^{F(u)} \geq \langle e^{F(u)} F'(u), \eta(v, u) \rangle + \mu\xi(\eta(v, u)).$$

which shows that  $F$  is strongly exponentially generalized invex function on the invex set  $K_\eta$ .  $\square$

**Theorem 3.3.** Let  $F$  be a differentiable strongly exponentially generalized preinvex function with modulus  $\mu > 0$ . If  $u \in K_\eta$  is the minimum of the function  $F$ , then

$$e^{F(v)} - e^{F(u)} \geq \mu \xi(\eta(v, u)), \quad \forall u, v \in K_\eta. \quad (3.11)$$

*Proof.* Let  $u \in K_\eta$  be a minimum of the function  $F$ . Then

$$F(u) \leq F(v), \quad \forall v \in K_\eta$$

from which, we have

$$e^{F(u)} \leq e^{F(v)}, \quad \forall v \in K_\eta. \quad (3.12)$$

Since  $K + \eta$  is an invex set, so,  $\forall u, v \in K_\eta, \lambda \in [0, 1]$ ,

$$v_\lambda = u + \lambda \eta(v, u) \in K_\eta.$$

Taking  $v = v_\lambda$  in (3.12), we have

$$0 \leq \lim_{\lambda \rightarrow 0} \left\{ \frac{e^{F(u+\lambda\eta(v,u))} - e^{F(u)}}{\lambda} \right\} = \langle e^{F(u)} F'(u), \eta(v, u) \rangle. \quad (3.13)$$

Since  $F$  is differentiable strongly exponentially generalized preinvex function, so

$$e^{F(u+\lambda\eta(v,u))} \leq e^{F(u)} + \lambda(e^{F(v)} - e^{F(u)}) - \mu\lambda(1-\lambda)h(\eta(v, u)), \quad u, v \in K_\eta, \lambda \in [0, 1],$$

from which, using (3.13), we have

$$\begin{aligned} e^{F(v)} - e^{F(u)} &\geq \lim_{\lambda \rightarrow 0} \left\{ \frac{e^{F(u+\lambda\eta(v,u))} - e^{F(u)}}{\lambda} \right\} + \mu \xi(\eta(v, u)). \\ &= \langle e^{F(u)} F'(u), v - u \rangle + \mu \xi(\eta(v, u)) \\ &\geq \mu h(\eta(v, u)), \end{aligned}$$

the required result (3.11). □

**Remark 3.1.** If

$$\langle e^{F(u)} F'(u), v - u \rangle + \mu \xi(\eta(v, u)) \geq 0, \quad \forall u, v \in K_\eta,$$

then  $u \in K_\eta$  is the minimum of the function  $F$ .

We would like to emphasize that the minimum  $u \in K_\eta$  of the strongly exponentially generalized preinvex functions can be characterized by inequality

$$\langle e^{F(u)} F'(u), \eta(v, u) \rangle \geq 0, \quad \forall v \in K_\eta, \quad (3.14)$$

which is called the exponential variational-like inequality and appears to be a new one. It is an interesting problem to study the existence of a unique solution of the exponentially variational-like inequality (3.14) and its applications.



**Theorem 3.4.** *Let  $F'$  be exponentially strongly generalized relaxed  $\eta$ -pseudomonotone and Condition A and C hold. If the function  $\xi$  is even and exponentially homogeneous of degree 2, then  $F$  is a strongly exponentially generalized  $\eta$ -pseudo-invex function.*

*Proof.* Let  $F'$  be strongly exponentially relaxed generalized  $\eta$ -pseudomonotone. Then

$$\langle e^{F(u)} F'(u), \eta(v, u) \rangle \geq 0, \quad \forall u, v \in K_\eta,$$

implies that

$$-\langle e^{F(v)} F'(v), \eta(u, v) \rangle \geq \alpha \xi(\eta(u, v)). \quad (3.15)$$

Since  $K_\eta$  is an invex set,  $\forall u, v \in K_\eta, \lambda \in [0, 1], v_\lambda = u + \lambda\eta(v, u) \in K_\eta$ . Taking  $v = v_\lambda$  in (3.15), using Condition C and the fact that the function  $\xi$  is even and exponentially homogeneous of degree 2, we have

$$-\langle e^{F(u+\lambda\eta(v,u))} F'(u + \lambda\eta(v, u)), \eta(u, v) \rangle \geq \lambda \alpha \xi(\eta(v, u)). \quad (3.16)$$

Let

$$\varphi(\lambda) = e^{F(u+\lambda\eta(v,u))}, \quad \forall u, v \in K_\eta, \lambda \in [0, 1].$$

Then, using (3.16), we have

$$\varphi'(\lambda) = \langle e^{F(u+\lambda\eta(v,u))} F'(u + \lambda\eta(v, u)), \eta(u, v) \rangle \geq \lambda \alpha \xi(\eta(v, u)).$$

Integrating the above relation between 0 to 1, we have

$$\varphi(1) - \varphi(0) \geq \frac{\alpha}{2} \xi(\eta(v, u)),$$

that is,

$$e^{F(u+\lambda\eta(v,u))} - e^{F(u)} \geq \frac{\alpha}{2} \xi(\eta(v, u)),$$

which implies, using Condition A, that

$$e^{F(v)} - e^{F(u)} \geq \frac{\alpha}{2} \xi(\eta(v, u)),$$

showing that  $F$  is a strongly exponentially generalized  $\eta$ -pseudo-invex function.  $\square$

As special cases of Theorem 3.4, we have the following:

**Theorem 3.5.** *Let the differential  $F'(u)$  of a function  $F(u)$  on the invex set  $K_\eta$  be exponentially  $\eta$ -pseudomonotone and let Conditions A and C hold. If the function  $\xi$  is even and exponentially homogeneous of degree 2, then  $F$  is exponentially pseudo  $\eta$ -invex function.*

**Theorem 3.6.** *Let the differential  $F'(u)$  of a function  $F(u)$  on the invex set  $K_\eta$  be strongly exponentially generalized  $\eta$ -pseudomonotone and Conditions A and C hold. If the function  $h$  is even and exponentially homogeneous of degree 2, then  $F$  is strongly exponentially generalized pseudo  $\eta$ -invex function.*

**Theorem 3.7.** Let the differential  $F'(u)$  of a function  $F(u)$  on the invex set  $K_\eta$  be strongly exponentially generalized  $\eta$ -pseudomonotone and let Conditions A and C hold. If the function  $h$  is even and exponentially homogeneous of degree 2, then  $F$  is strongly exponentially generalized pseudo  $\eta$ -invex function.

**Theorem 3.8.** Let the differential  $F'(u)$  of a function  $F(u)$  on the invex set be exponentially  $\eta$ -pseudomonotone and Conditions A and C hold. If the function  $h$  is even and exponentially homogeneous of degree 2, then  $F$  is exponentially pseudo invex function.

**Theorem 3.9.** Let the differential  $e^{F(u)}F'(u)$  of a differentiable preinvex function  $F(u)$  be Lipschitz continuous on the invex set  $K_\eta$  with a constant  $\beta > 0$ . Then

$$e^{F(u+\eta(v,u))} - e^{F(u)} \leq \langle e^{F(u)}F'(u), \eta(v, u) \rangle + \frac{\beta}{2} \|\eta(v, u)\|^2, \quad u, v \in K_\eta.$$

*Proof.* Its proof follows from Noor and Noor [30]. □

**Definition 3.1.** The function  $F$  is said to be sharply strongly generalized pseudo preinvex, if there exists a constant  $\mu > 0$  such that

$$\langle F'(u), \eta(v, u) \rangle \geq 0 \Rightarrow F(v) \geq F(v + \lambda\eta(v, u)) + \mu\lambda(1 - \lambda)\xi(\eta(v, u)), \quad \forall u, v \in K_\eta, \lambda \in [0, 1].$$

**Theorem 3.10.** Let  $F$  be a sharply strongly generalized pseudo preinvex function on  $K_\eta$  with a constant  $\mu > 0$ . Then

$$-\langle F'(v), \eta(v, u) \rangle \geq \mu\xi(\eta(v, u)), \quad \forall u, v \in K_\eta.$$

*Proof.* Let  $F$  be a sharply strongly generalized pseudo preinvex function on  $K_\eta$ . Then

$$F(v) \geq F(v + \lambda\eta(v, u)) + \mu\lambda(1 - \lambda)\xi(\eta(v, u)), \quad \forall u, v \in K_\eta, \lambda \in [0, 1].$$

from which we have

$$\left\{ \frac{F(v + \lambda\eta(v, u)) - F(v)}{\lambda} \right\} + \mu(1 - \lambda)\xi(\eta(v, u)) \leq 0.$$

Taking limit in the above inequality, as  $t \rightarrow 0$ , we have

$$-\langle F'(v), \eta(v, u) \rangle \geq \mu\xi(\eta(v, u)),$$

the required result. □

**Definition 3.2.** A function  $F$  is said to be a exponentially generalized pseudo preinvex function, if there exists a strictly positive bifunction  $B(., .)$ , such that

$$\begin{aligned} e^{F(v)} &< e^{F(u)} \\ \Rightarrow \\ e^{F(u+\lambda\eta(v,u))} &< e^{F(u)} + \lambda(\lambda - 1)B(v, u), \forall u, v \in K_\eta, \lambda \in [0, 1]. \end{aligned}$$

**Theorem 3.11.** *If the function  $F$  is strongly exponentially generalized preinvex function such that  $e^{F(v)} < e^{F(u)}$ , then the function  $F$  is strongly exponentially generalized pseudo preinvex.*

*Proof.* Since  $e^{F(v)} < e^{F(u)}$  and  $F$  is strongly exponentially generalized preinvex function, then  $\forall u, v \in K_\eta$ ,  $\lambda \in [0, 1]$ , we have

$$\begin{aligned} e^{F(u+\lambda\eta(v,u))} &\leq e^{F(u)} + \lambda(e^{F(v)} - e^{F(u)} - \mu\lambda(1-\lambda)\xi(\eta(v,u))) \\ &< e^{F(u)} + \lambda(1-\lambda)(e^{F(v)} - e^{F(u)} - \mu\lambda(1-\lambda)\xi(\eta(v,u))) \\ &= e^{F(u)} + \lambda(1-\lambda)(e^{F(v)} - e^{F(u)} - \mu\lambda(1-\lambda)\xi(\eta(v,u))) \\ &< e^{F(u)} + \lambda(1-\lambda)B(u,v) - \mu\lambda(1-\lambda)\xi(\eta(v,u)), \end{aligned}$$

where  $B(u, v) = e^{F(u)} - e^{F(v)} > 0$ . This shows that  $F$  is a strongly exponentially generalized preinvex function. □

It is well known that each strongly convex functions is of the form  $- \pm \|\cdot\|^2$ , where  $f$  is a convex function. We now establish a similar result for the strongly exponentially generalized preinvex functions.

**Theorem 3.12.** *Let  $f$  be a strongly exponentially affine generalized preinvex function. Then  $F$  is a strongly exponentially generalized preinvex function, if and only if,  $\zeta = F - f$  is a exponentially preinvex function.*

*Proof.* Let  $f$  be strongly exponentially affine generalized preinvex function. Then

$$e^{f(u+\lambda\eta(v,u))} = (1-\lambda)e^{f(u)} + \lambda e^{f(v)} - \mu\lambda(1-\lambda)\xi(\eta(v,u)). \quad (3.17)$$

From the strongly exponentially generalized preinvexity of  $F$ , we have

$$e^{F(u+\lambda\eta(v,u))} \leq (1-\lambda)e^{F(u)} + \lambda e^{F(v)} - \mu\lambda(1-\lambda)\xi(\eta(v,u)). \quad (3.18)$$

From (3.17) and (3.18), we have

$$e^{F(u+\lambda\eta(v,u))} - e^{f(u+\lambda\eta(v,u))} \leq (1-\lambda)(e^{F(u)} - e^{f(u)}) + \lambda(e^{F(v)} - e^{f(v)}), \quad (3.19)$$

from which it follows that

$$\begin{aligned} e^{\zeta(u+\lambda\eta(v,u))} &= e^{F(u+\lambda\eta(v,u))} - e^{f(u+\lambda\eta(v,u))} \\ &\leq (1-\lambda)e^{F(u)} + \lambda e^{F(v)} - (1-\lambda)e^{f(u)} - \lambda e^{f(v)} \\ &= (1-\lambda)(e^{F(u)} - e^{f(u)}) + \lambda(e^{F(v)} - e^{f(v)}), \end{aligned}$$

which show that  $\zeta = F - f$  is an exponentially preinvex function.

The inverse implication is obvious. □

We would like to remark that one can show that  $F$  is a strongly exponentially generalized preinvex function, if and only if,  $F$  is strongly exponentially affine preinvex function essentially using the technique of Adamek [1] and Noor et al. [17].

It is worth mentioning that the strongly exponentially generalized preinvex is also Wright strongly exponentially generalized preinvex functions. From the definition 2.2, we have

$$e^{F(u+\lambda\eta(v,u))} + e^{F(v+\lambda\eta(u,v))} \leq e^F(u) + e^F(v) - 2\mu\lambda(1-\lambda)\xi(\eta(v,u)), \forall u, v \in K_\eta, \lambda \in [0, 1],$$

which is called Wright strongly exponentially generalized preinvex function. One can studies the properties and applications of the Wright strongly exponentially generalized preinvex functions in optimization operations research.

#### 4. Conclusion

In this paper, we have introduced and studied a new class of preinvex functions with respect to any arbitrary function and bifunction. which is called strongly exponentially generalized function. It is shown that several new classes of strongly preinvex and convex functions can be obtained as special cases of these relative strongly generalized preinvex functions. Some basic properties of these functions are explored. The ideas and techniques of this paper may motivate further research.

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#### Conflict of interest

The authors declare no conflict of interest.

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