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Research article

Traveling wave solutions in closed form for some nonlinear fractional evolution equations related to conformable fractional derivative

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Abstract: Fractional order nonlinear evolution equations involving conformable fractional derivative are formulated and revealed for attractive solutions to depict the physical phenomena of nonlinear mechanisms in the real world. The core aim of this article is to explore further new general exact traveling wave solutions of nonlinear fractional evolution equations, namely, the space time fractional (2+1)-dimensional dispersive long wave equations, the (3+1)-dimensional space time fractional mKdV-ZK equation and the space time fractional modified regularized long-wave equation. The mentioned equations are firstly turned into the fractional order ordinary differential equations with the aid of a suitable composite transformation and then hunted their solutions by means of recently established fractional generalized $(D_\xi^\alpha G/G)$-expansion method. This productive method successfully generates many new and general closed form traveling wave solutions in accurate, reliable and efficient way in terms of hyperbolic, trigonometric and rational. The obtained results might play important roles for describing the complex phenomena related to science and engineering and also be newly recorded in the literature for their high acceptance. The suggested method will draw the attention to the researchers to establish further new solutions to any other nonlinear evolution equations.

Keywords: The fractional generalized $(D_\xi^\alpha G/G)$-expansion method; conformable fractional derivative; composite transformation; fractional order nonlinear evolution equations; closed form solutions
Mathematics Subject Classification: 34A08, 35R11

1. Introduction

The physical phenomena of real world can effectively be modeled by making use of the theory of derivatives and integrals of fractional order. In this sense, the nonlinear fractional order evolution equations (NLFEEs) have recently become a burning topic to the researchers for searching their exact traveling wave solutions to depict the physical phenomena due to the nonlinear mechanisms arisen in various fields. The use of nonlinear equations is extensive as the nonlinearity exists everywhere in the world. NLFEEs have been attracted great interest due to their frequent appearance in many applications such as in biology, physics, chemistry, electromagnetic, polymeric materials, neutron point kinetic model, control and vibration, image and signal processing, system identifications, the finance, acoustics and fluid dynamics [1–3]. Many researchers have offered different approaches to construct analytic and numerical solutions to NLFEEs as well as NLEEs and put them forward for searching traveling wave solutions, such as the exponential decay law [4], the Ibragimov's nonlocal conservation method [5], the reproducing kernel method [6], the Jacobi elliptic function method [7], the (G'/G)-expansion method and its various modifications [8–12], the Exp-function method [13,14], the sub-equation method [15,16], the first integral method [17,18], the functional variable method [19], the modified trial equation method [20,21], the simplest equation method [22], the Lie group analysis method [23], the fractional characteristic method [24], the auxiliary equation method [25,26], the finite element method [27], the differential transform method [28], the Adomian decomposition method [29,30], the variational iteration method [31], the finite difference method [32], the various homotopy perturbation method [33–37] and the He's variational principle [38] etc. But no method is uniquely appreciable to investigate the exact solutions to all kind of NLFEEs. That is why; it is very much needed to introduce new method. In this study, we implemented recently established effectual and reliable productive method, called the fractional generalized $(D_\xi^\alpha G/G)$-expansion method to construct closed form analytic wave solutions to some NLFEEs in the sense of conformable fractional derivative [39]. The results obtained throughout the article have been compared with those existing in the literature and shown that the achieved solutions are new and much more general. We have finally concluded that the solutions might bring up their importance through the contribution and be recorded in the literature.

2. Preliminaries and methodology

2.1. Conformable fractional derivative

A new and simple definition of derivative for fractional order introduced by Khalil et al. [39] is called conformable fractional derivative. This definition is analogous to the ordinary derivative

$$\frac{d\psi}{dx} = \lim_{\varepsilon \to 0} \frac{\psi(x+\varepsilon) - \psi(x)}{\varepsilon},$$

where $\psi(x):[0,\infty] \to \mathbf{R}$ and $x > 0$. According to this classical definition, $\frac{d(x^n)}{dx} = nx^{n-1}$. According to this perception, Khalil has introduced α order fractional derivative of ψ as

$$T_\alpha \psi(x) = \lim_{\varepsilon \to 0} \frac{\psi(x + \varepsilon x^{1-\alpha}) - \psi(x)}{\varepsilon}, \; 0 < \alpha \le 1.$$

If the function ψ is α -differentiable in $(0, r)$ for $r > 0$ and $\lim_{x \to 0^+} T_\alpha \psi(x)$ exists, then the conformable derivative at $x = 0$ is defined as $T_\alpha \psi(0) = \lim_{x \to 0^+} T_\alpha \psi(x)$. The conformable integral of ψ is

$$I_\alpha^r \psi(x) = \int_r^x \frac{\psi(t)}{t^{1-\alpha}} dt, \; r \ge 0, \; 0 < \alpha \le 1.$$

This integral represents usual Riemann improper integral.

The conformable fractional derivative satisfies the following useful properties [39]:

If the functions $u(x)$ and $v(x)$ are α -differentiable at any point $x > 0$, for $\alpha \in (0,1]$, then

(a) $T_\alpha(au + bv) = aT_\alpha(u) + bT_\alpha(v) \;\; \forall a, b \in \mathbf{R}$.

(b) $T_\alpha(x^n) = n x^{n-\alpha} \;\; \forall n \in \mathbf{R}$.

(c) $T_\alpha(c) = 0$, where c is any constant.

(d) $T_\alpha(uv) = uT_\alpha(v) + vT_\alpha(u)$.

(e) $T_\alpha(u / v) = \dfrac{vT_\alpha(u) - uT_\alpha(v)}{v^2}$.

(f) if u is differentiable, then $T_\alpha(u)(x) = x^{1-\alpha} \dfrac{du}{dx}(x)$.

Many researchers used this new derivative of fractional order in physical applications due to its convenience, simplicity and usefulness [40–42].

2.2. The fractional generalized $(D_\xi^\alpha G / G)$-expansion method

Consider a nonlinear partial differential equation of fractional order in the independent variables $t, x_1, x_2, \ldots, x_n$ as

$$F(u, D_t^\alpha u, D_{x_1}^\alpha u, D_{x_2}^\alpha u, \ldots, D_{x_n}^\alpha u, D_t^{2\alpha} u, D_{x_1}^{2\alpha} u, D_{x_2}^{2\alpha} u, \ldots, D_{x_n}^{2\alpha} u, \ldots) = 0, \tag{2.2.1}$$

where F is a polynomial in $u(t, x_1, x_2, \ldots, x_n)$ and it's various conformable fractional derivatives. The main steps of the fractional generalized $(D_\xi^\alpha G / G)$-expansion method referred as follows:

Step 1: Use the wave transformation

$$\xi = \xi(t, x_1, x_2, ..., x_n), \quad u_i = u_i(t, x_1, x_2, ..., x_n) = U_i(\xi) \tag{2.2.2}$$

to reduce Eq. (2.2.1) to the following ordinary differential equation of fractional order with respect to the variable ξ:

$$P(U, D_\xi^\alpha U, D_\xi^{2\alpha} U, ...) = 0, \tag{2.2.3}$$

Take anti-derivative of Eq. (2.2.3), if possible, one or more times and the integral constant can be set to zero as soliton solutions are sought.

Step 2: Assume that Eq. (2.2.3) has solution in the form

$$u(\xi) = \sum_{i=0}^{n} a_i (d + D_\xi^\alpha G / G)^i + \sum_{i=1}^{n} b_i (d + D_\xi^\alpha G / G)^{-i}, \tag{2.2.4}$$

where $a_i (i = 0,1,2,...,n)$, $b_i (i = 1,2,3,...,n)$ and d are arbitrary constants with at least one of a_n and b_n is nonzero and $G = G(\xi)$ satisfies the auxiliary equation

$$AGD_\xi^{2\alpha} G - BGD_\xi^\alpha G - EG^2 - C(D_\xi^\alpha G)^2 = 0, \tag{2.2.5}$$

where $0 < \alpha \le 1$ and $D_\xi^\alpha G(\xi)$ stands for the conformable fractional derivative $G(\xi)$ of order α with respect to ξ; A, B, C and E are real parameters.

Making use of the transformation $G(\xi) = H(\eta)$, $\eta = \xi^\alpha / \alpha$, Eq. (2.2.5) turns into the equation

$$AHH'' - BHH' - EH^2 - C(H')^2 = 0, \tag{2.2.6}$$

whose solutions are well-known. Utilizing the solutions in Ref. [10] of Eq. (2.2.6) together with the transformation $D_\xi^\alpha G(\xi) = D_\xi^\alpha H(\eta) = H'(\eta) D_\xi^\alpha \eta = H'(\eta)$, which can be derived by using conformable fractional derivative, we can easily obtain the following solutions to Eq. (2.2.5):

Family 1: When $B \ne 0$, $\psi = A - C$ and $\Omega = B^2 + 4E\psi > 0$,

$$(D_\xi^\alpha G / G) = \frac{B}{2\psi} + \frac{\sqrt{\Omega}}{2\psi} \frac{C_1 \sinh\left(\frac{\sqrt{\Omega}\,\xi^\alpha}{2A\alpha}\right) + C_2 \cosh\left(\frac{\sqrt{\Omega}\,\xi^\alpha}{2A\alpha}\right)}{C_1 \cosh\left(\frac{\sqrt{\Omega}\,\xi^\alpha}{2A\alpha}\right) + C_2 \sinh\left(\frac{\sqrt{\Omega}\,\xi^\alpha}{2A\alpha}\right)} \tag{2.2.7}$$

Family 2: When $B \neq 0$, $\psi = A - C$ and $\Omega = B^2 + 4E\psi < 0$,

$$(D_\xi^\alpha G/G) = \frac{B}{2\psi} + \frac{\sqrt{-\Omega}}{2\psi} \frac{-C_1 \sin\left(\frac{\sqrt{-\Omega}\,\xi^\alpha}{2A\alpha}\right) + C_2 \cos\left(\frac{\sqrt{-\Omega}\,\xi^\alpha}{2A\alpha}\right)}{C_1 \cos\left(\frac{\sqrt{-\Omega}\,\xi^\alpha}{2A\alpha}\right) + C_2 \sin\left(\frac{\sqrt{-\Omega}\,\xi^\alpha}{2A\alpha}\right)} \tag{2.2.8}$$

Family 3: When $B \neq 0$, $\psi = A - C$ and $\Omega = B^2 + 4E\psi = 0$,

$$(D_\xi^\alpha G/G) = \frac{B}{2\psi} + \frac{\alpha C_2}{C_1\alpha + C_2\xi^\alpha} \tag{2.2.9}$$

Family 4: When $B = 0$, $\psi = A - C$ and $\Delta = \psi E > 0$,

$$(D_\xi^\alpha G/G) = \frac{\sqrt{\Delta}}{\psi} \frac{C_1 \sinh\left(\frac{\sqrt{\Delta}\,\xi^\alpha}{A\alpha}\right) + C_2 \cosh\left(\frac{\sqrt{\Delta}\,\xi^\alpha}{A\alpha}\right)}{C_1 \cosh\left(\frac{\sqrt{\Delta}\,\xi^\alpha}{A\alpha}\right) + C_2 \sinh\left(\frac{\sqrt{\Delta}\,\xi^\alpha}{A\alpha}\right)} \tag{2.2.10}$$

Family 5: When $B = 0$, $\psi = A - C$ and $\Delta = \psi E < 0$,

$$(D_\xi^\alpha G/G) = \frac{\sqrt{-\Delta}}{\psi} \frac{-C_1 \sin\left(\frac{\sqrt{-\Delta}\,\xi^\alpha}{A\alpha}\right) + C_2 \cos\left(\frac{\sqrt{-\Delta}\,\xi^\alpha}{A\alpha}\right)}{C_1 \cos\left(\frac{\sqrt{-\Delta}\,\xi^\alpha}{A\alpha}\right) + C_2 \sin\left(\frac{\sqrt{-\Delta}\,\xi^\alpha}{A\alpha}\right)} \tag{2.2.11}$$

Step 3: Take the homogeneous balance between the highest order linear and nonlinear terms appearing in Eq. (2.2.3) to determine the positive constant n.

Step 4: Using the value of n obtained in step 3, Eq. (2.2.3) along with Eqs. (2.2.4), (2.2.5) makes available polynomials in $(d + D_\xi^\alpha G/G)^n$ $(n = 0,1,2,...)$ and $(d + D_\xi^\alpha G/G)^{-n}$ $(n = 1,2,3,...)$. Equalize each coefficient of these polynomials to zero yield a system of algebraic equations for a_i $(i = 0,1,2,...)$, b_i $(i = 1,2,3,...)$ and d. Solve this system by Maple to obtain the values of a_i $(i = 0,1,2,...)$, b_i $(i = 1,2,3,...)$ and d.

Step 5: Eq. (2.2.4) together with Eqs. (2.2.7)–(2.2.11) and the values appeared in step 4 provides traveling wave solutions of the nonlinear evolution equation (2.2.1) in closed form.

3. Formulation of the solutions

In this section, the closed form traveling wave solutions to the suggested equations are examined.

3.1. The space time fractional (2+1)-dimensional dispersive long wave equations

Consider the space time fractional (2+1)-dimensional dispersive long wave equations

$$\left.\begin{aligned} &\frac{\partial^{2\alpha}u}{\partial y^{\alpha}\partial t^{\alpha}}+\frac{\partial^{2\alpha}v}{\partial x^{2\alpha}}+\frac{\partial^{\alpha}(u(\partial^{\alpha}u/\partial x^{\alpha}))}{\partial y^{\alpha}}=0\\ &\frac{\partial^{\alpha}v}{\partial t^{\alpha}}+\frac{\partial^{\alpha}u}{\partial x^{\alpha}}+\frac{\partial^{\alpha}(uv)}{\partial x^{\alpha}}+\frac{\partial^{3\alpha}u}{\partial x^{2\alpha}\partial y^{\alpha}}=0 \end{aligned}\right\}, \tag{3.1.1}$$

where $0<\alpha\le 1$; the notation $\frac{\partial^{\alpha}u}{\partial x^{\alpha}}$ denotes the α-order partial derivative of u with respect to x and the other notations are so. This system of equations was first obtained by Boiti et al. [43] as compatibility condition for a weak Lax pair.

The composite transformation

$$u(t,x,y)=U(\xi),\ v(t,x,y)=V(\xi),\ \xi=k^{1/\alpha}x+l^{1/\alpha}y+w^{1/\alpha}t, \tag{3.1.2}$$

where k, l and w are constants, with the aid of chain rule

$$D_x^{\alpha}U=\frac{\partial^{\alpha}U}{\partial x^{\alpha}}=\frac{\partial^{\alpha}U}{\partial\xi^{\alpha}}.\frac{\partial\xi^{\alpha}}{\partial x^{\alpha}}=\frac{\partial^{\alpha}U}{\partial\xi^{\alpha}}.\left(\frac{\partial\xi}{\partial x}\right)^{\alpha}=k\,D_{\xi}^{\alpha}U,$$

forces Eq. (3.1.1) to take the form

$$\left.\begin{aligned} &lwD_{\xi}^{2\alpha}U+k^2D_{\xi}^{2\alpha}V+kl(UD_{\xi}^{2\alpha}U+(D_{\xi}^{\alpha}U)^2)=0\\ &wD_{\xi}^{\alpha}V+kD_{\xi}^{\alpha}U+kD_{\xi}^{\alpha}(UV)+k^2lD_{\xi}^{3\alpha}U=0 \end{aligned}\right\} \tag{3.1.3}$$

The positive constant n under the homogeneous balance from Eq. (3.1.3) reduces Eq. (2.2.4) into the form

$$\left.\begin{aligned} &U(\xi)=a_0+a_1\phi+b_1\phi^{-1}\\ &V(\xi)=c_0+c_1\phi+c_2\phi^2+d_1\phi^{-1}+d_2\phi^{-2} \end{aligned}\right\}, \tag{3.1.4}$$

where $\phi=(d+D_{\xi}^{\alpha}G/G)$. Eq. (3.1.3) with the aid of Eq. (3.1.4) and Eq. (2.2.5) produces a polynomial in ϕ. Setting each coefficient of this polynomial to zero gives a set of algebraic equations for the constants in Eq. (3.1.4). Solving these equations by the computational software Maple present the following results:

Set-1: $a_0=\pm\frac{1}{Ak^{3/2}}\{k^3(2d\psi+B)\mp Aw\sqrt{k}\}$, $a_1=\mp\frac{2}{A}k^{3/2}\psi$, $b_1=\mp\frac{2k^{3/2}}{A}(d^2\psi+Bd-E)$,

$c_1=\frac{2k^2l\psi}{A^2}(2d\psi+B)$, $c_2=-\frac{2}{A^2}k^2l\psi^2$, $d_1=\frac{2k^2l}{A^2}(2d\psi+B)(d^2\psi+Bd-E)$,

$$c_0=-1,\ d_2=-\frac{2k^2l}{A^2}(d^2\psi+Bd-E)^2 \tag{3.1.5}$$

Set-2: $a_0 = \pm\frac{1}{Ak^{3/2}}\{k^3(2d\psi + B) \mp Aw\sqrt{k}\}$, $a_1 = 0$, $b_1 = \mp\frac{2k^{3/2}}{A}(d^2\psi + Bd - E)$, $c_1 = 0$,

$c_0 = -\frac{1}{A^2}\{2k^2 l\psi(d^2\psi + Bd - E) + A^2\}$, $d_1 = \frac{2k^2 l}{A^2}\{(2d^2\psi + 3Bd - 2E)d\psi + B^2 d - BE\}$,

$$c_2 = 0,\ d_2 = -\frac{2k^2 l}{A^2}\{d^2\psi(d^2\psi + 2Bd - 2E) + E^2 + B^2 d^2 - 2BdE\} \quad (3.1.6)$$

Set-3: $a_0 = \pm\frac{1}{Ak^{3/2}}\{k^3(2d\psi + B) \mp Aw\sqrt{k}\}$, $a_1 = \mp\frac{2}{A}k^{3/2}\psi$, $b_1 = 0$, $c_1 = \frac{2k^2 l\psi}{A^2}(2d\psi + B)$,

$$c_0 = -\frac{1}{A^2}\{k^2 l\psi(2d^2\psi + 2Bd - 2E) + A^2\},\ c_2 = -\frac{2}{A^2}k^2 l\psi^2,\ d_1 = 0,\ d_2 = 0 \quad (3.1.7)$$

Putting Eqs. (3.1.5)–(3.1.6) into Eq. (3.1.4) yields

$$U_1(\xi) = \pm\frac{1}{Ak^{3/2}}\{k^3(2d\psi + B) \mp Aw\sqrt{k}\} \mp \frac{2}{A}k^{3/2}\psi(d + D_\xi^\alpha G/G) \mp \frac{2k^{3/2}}{A}(d^2\psi + Bd - E)(d + D_\xi^\alpha G/G)^{-1} \quad (3.1.8)$$

$$V_1(\xi) = -1 + \frac{2k^2 l\psi}{A^2}(2d\psi + B)(d + D_\xi^\alpha G/G) - \frac{2}{A^2}k^2 l\psi^2(d + D_\xi^\alpha G/G)^2 + \frac{2k^2 l}{A^2}(2d\psi + B)(d^2\psi + Bd - E)(d + D_\xi^\alpha G/G)^{-1} - \frac{2k^2 l}{A^2}(d^2\psi + Bd - E)^2(d + D_\xi^\alpha G/G)^{-2} \quad (3.1.9)$$

$$U_2(\xi) = \pm\frac{1}{Ak^{3/2}}\{k^3(2d\psi + B) \mp Aw\sqrt{k}\} \mp \frac{2k^{3/2}}{A}(d^2\psi + Bd - E)(d + D_\xi^\alpha G/G) \quad (3.1.10)$$

$$V_2(\xi) = -\frac{1}{A^2}\{2k^2 l\psi(d^2\psi + Bd - E) + A^2\} + \frac{2k^2 l}{A^2}\{(2d^2\psi + 3Bd - 2E)d\psi + B^2 d - BE\}(d + D_\xi^\alpha G/G)^{-1} - \frac{2k^2 l}{A^2}\{d^2\psi(d^2\psi + 2Bd - 2E) + E^2 + B^2 d^2 - 2BdE\}(d + D_\xi^\alpha G/G)^{-2} \quad (3.1.11)$$

$$U_3(\xi) = \pm\frac{1}{Ak^{3/2}}\{k^3(2d\psi + B) \mp Aw\sqrt{k}\} \mp \frac{2}{A}k^{3/2}\psi(d + D_\xi^\alpha G/G) \quad (3.1.12)$$

$$V_3(\xi) = -\frac{1}{A^2}\{k^2 l\psi(2d^2\psi + 2Bd - 2E) + A^2\} + \frac{2k^2 l\psi}{A^2}(2d\psi + B)(d + D_\xi^\alpha G/G) - \frac{2}{A^2}k^2 l\psi^2(d + D_\xi^\alpha G/G)^2 \quad (3.1.13)$$

where $\xi = k^{1/\alpha}x + l^{1/\alpha}y + w^{1/\alpha}t$.

Eqs. (3.1.8), (3.1.9) together with Eqs. (2.2.7)–(2.2.11) make available the following three types solutions in terms hyperbolic function, trigonometric function and rational function as:

When $B \neq 0$, $\psi = A - C$ and $\Omega = B^2 + 4E\psi > 0$,

$$U_1^1(\xi) = \pm\frac{1}{Ak^{3/2}}\{k^3(2d\psi + B) \mp Aw\sqrt{k}\} \mp \frac{2}{A}k^{3/2}\psi\left\{d + \frac{B}{2\psi} + \frac{\sqrt{\Omega}}{2\psi}\frac{C_1\sinh\Phi + C_2\cosh\Phi}{C_1\cosh\Phi + C_2\sinh\Phi}\right\} \mp \frac{2k^{3/2}}{A}(d^2\psi + Bd - E)\left\{d + \frac{B}{2\psi} + \frac{\sqrt{\Omega}}{2\psi}\frac{C_1\sinh\Phi + C_2\cosh\Phi}{C_1\cosh\Phi + C_2\sinh\Phi}\right\}^{-1} \tag{3.1.14}$$

$$\begin{aligned} V_1^1(\xi) = &-1 + \frac{2k^2 l\psi}{A^2}(2d\psi + B)\left\{d + \frac{B}{2\psi} + \frac{\sqrt{\Omega}}{2\psi}\frac{C_1\sinh\Phi + C_2\cosh\Phi}{C_1\cosh\Phi + C_2\sinh\Phi}\right\} \\ &- \frac{2}{A^2}k^2 l\psi^2\left\{d + \frac{B}{2\psi} + \frac{\sqrt{\Omega}}{2\psi}\frac{C_1\sinh\Phi + C_2\cosh\Phi}{C_1\cosh\Phi + C_2\sinh\Phi}\right\}^2 + \frac{2k^2 l}{A^2}(2d\psi + B)(d^2\psi + Bd - E) \\ &\times\left\{d + \frac{B}{2\psi} + \frac{\sqrt{\Omega}}{2\psi}\frac{C_1\sinh\Phi + C_2\cosh\Phi}{C_1\cosh\Phi + C_2\sinh\Phi}\right\}^{-1} - \frac{2k^2 l}{A^2}(d^2\psi + Bd - E)^2\left\{d + \frac{B}{2\psi} + \frac{\sqrt{\Omega}}{2\psi}\frac{C_1\sinh\Phi + C_2\cosh\Phi}{C_1\cosh\Phi + C_2\sinh\Phi}\right\}^{-2} \end{aligned} \tag{3.1.15}$$

where $\Phi = \dfrac{\sqrt{\Omega}\,\xi^\alpha}{2A\alpha}$.

Since C_1 and C_2 are arbitrary constants, if $C_1 \neq 0$ and $C_2 = 0$, then Eqs. (3.1.14), (3.1.15) become

$$U_1^1(\xi) = \pm\frac{1}{Ak^{3/2}}\{k^3(2d\psi + B) \mp Aw\sqrt{k}\} \mp \frac{k^{3/2}}{A}\left\{2d\psi + B + \sqrt{\Omega}\tanh\frac{\sqrt{\Omega}\,\xi^\alpha}{2A\alpha}\right\} \mp \frac{4\psi k^{3/2}}{A}(d^2\psi + Bd - E)\left\{2d\psi + B + \sqrt{\Omega}\tanh\frac{\sqrt{\Omega}\,\xi^\alpha}{2A\alpha}\right\}^{-1} \tag{3.1.16}$$

$$\begin{aligned} V_1^1(\xi) = &-1 + \frac{k^2 l}{A^2}(2d\psi + B)\left\{2d\psi + B + \sqrt{\Omega}\tanh\frac{\sqrt{\Omega}\,\xi^\alpha}{2A\alpha}\right\} - \frac{k^2 l}{2A^2}\left\{2d\psi + B + \sqrt{\Omega}\tanh\frac{\sqrt{\Omega}\,\xi^\alpha}{2A\alpha}\right\}^2 \\ &+ \frac{4\psi k^2 l}{A^2}(2d\psi + B)(d^2\psi + Bd - E)\times\left\{2d\psi + B + \sqrt{\Omega}\tanh\frac{\sqrt{\Omega}\,\xi^\alpha}{2A\alpha}\right\}^{-1} \\ &- \frac{8\psi^2 k^2 l}{A^2}(d^2\psi + Bd - E)^2\left\{2d\psi + B + \sqrt{\Omega}\tanh\frac{\sqrt{\Omega}\,\xi^\alpha}{2A\alpha}\right\}^{-2} \end{aligned} \tag{3.1.17}$$

where $\xi = k^{1/\alpha}x + l^{1/\alpha}y + w^{1/\alpha}t$.

When $B \neq 0$, $\psi = A - C$ and $\Omega = B^2 + 4E\psi < 0$,

$$U_1^{\,2}(\xi)=\pm\frac{1}{Ak^{3/2}}\{k^3(2d\psi+B)\mp Aw\sqrt{k}\}\mp\frac{k^{3/2}}{A}\left\{2d\psi+B+\sqrt{-\Omega}\,\frac{-C_1\sin\Phi+C_2\cos\Phi}{C_1\cos\Phi+C_2\sin\Phi}\right\}$$
$$\mp\frac{4\psi k^{3/2}}{A}(d^2\psi+Bd-E)\left\{2d\psi+B+\sqrt{-\Omega}\,\frac{-C_1\sin\Phi+C_2\cos\Phi}{C_1\cos\Phi+C_2\sin\Phi}\right\}^{-1} \quad (3.1.18)$$

$$V_1^{\,2}(\xi)=-1+\frac{k^2l}{A^2}(2d\psi+B)\left\{2d\psi+B+\sqrt{-\Omega}\,\frac{-C_1\sin\Phi+C_2\cos\Phi}{C_1\cos\Phi+C_2\sin\Phi}\right\}-\frac{k^2l}{2A^2}\left\{2d\psi+B+\sqrt{-\Omega}\,\frac{-C_1\sin\Phi+C_2\cos\Phi}{C_1\cos\Phi+C_2\sin\Phi}\right\}^2$$
$$+\frac{4k^2l\psi}{A^2}(2d\psi+B)(d^2\psi+Bd-E)\left\{2d\psi+B+\sqrt{-\Omega}\,\frac{-C_1\sin\Phi+C_2\cos\Phi}{C_1\cos\Phi+C_2\sin\Phi}\right\}^{-1} \quad (3.1.19)$$
$$-\frac{8k^2l\psi^2}{A^2}(d^2\psi+Bd-E)^2\left\{2d\psi+B+\sqrt{-\Omega}\,\frac{-C_1\sin\Phi+C_2\cos\Phi}{C_1\cos\Phi+C_2\sin\Phi}\right\}^{-2}$$

where $\Phi=\dfrac{\sqrt{-\Omega}\,\xi^\alpha}{2A\alpha}$.

In particular, if $C_1\neq 0$ and $C_2=0$, then Eqs. (3.1.18), (3.1.19) are simplified as

$$U_1^{\,2}(\xi)=\pm\frac{1}{Ak^{3/2}}\{k^3(2d\psi+B)\mp Aw\sqrt{k}\}\mp\frac{k^{3/2}}{A}(2d\psi+B-\sqrt{-\Omega}\tan\Phi)$$
$$\mp\frac{4\psi k^{3/2}}{A}(d^2\psi+Bd-E)(2d\psi+B-\sqrt{-\Omega}\tan\Phi)^{-1} \quad (3.1.20)$$

$$V_1^{\,2}(\xi)=-1+\frac{k^2l}{A^2}(2d\psi+B)(2d\psi+B-\sqrt{-\Omega}\tan\Phi)-\frac{k^2l}{2A^2}(2d\psi+B-\sqrt{-\Omega}\tan\Phi)^2+\frac{4k^2l\psi}{A^2}(2d\psi+B)$$
$$(d^2\psi+Bd-E)(2d\psi+B-\sqrt{-\Omega}\tan\Phi)^{-1}-\frac{8k^2l\psi^2}{A^2}(d^2\psi+Bd-E)^2(2d\psi+B-\sqrt{-\Omega}\tan\Phi)^{-2} \quad (3.1.21)$$

where $\Phi=\dfrac{\sqrt{-\Omega}\,\xi^\alpha}{2A\alpha}$ and $\xi=k^{1/\alpha}x+l^{1/\alpha}y+w^{1/\alpha}t$.

When $B\neq 0$, $\psi=A-C$ and $\Omega=B^2+4E\psi=0$,

$$U_1^{\,3}(\xi)=\pm\frac{1}{Ak^{3/2}}\{k^3(2d\psi+B)\mp Aw\sqrt{k}\}\mp\frac{k^{3/2}}{A}\left\{2d\psi+B+\frac{2\psi\alpha C_2}{C_1\alpha+C_2\xi^\alpha}\right\}$$
$$\mp\frac{4\psi k^{3/2}}{A}(d^2\psi+Bd-E)\left\{2d\psi+B+\frac{2\psi\alpha C_2}{C_1\alpha+C_2\xi^\alpha}\right\}^{-1} \quad (3.1.22)$$

$$V_1^{\,3}(\xi)=-1+\frac{k^2l}{A^2}(2d\psi+B)\left\{2d\psi+B+\frac{2\psi\alpha C_2}{C_1\alpha+C_2\xi^\alpha}\right\}-\frac{k^2l}{2A^2}\left\{2d\psi+B+\frac{2\psi\alpha C_2}{C_1\alpha+C_2\xi^\alpha}\right\}^2$$
$$+\frac{4k^2l\psi}{A^2}(2d\psi+B)(d^2\psi+Bd-E)\left\{2d\psi+B+\frac{2\psi\alpha C_2}{C_1\alpha+C_2\xi^\alpha}\right\}^{-1} \quad (3.1.23)$$
$$-\frac{8\psi^2k^2l}{A^2}(d^2\psi+Bd-E)^2\left\{2d\psi+B+\frac{2\psi\alpha C_2}{C_1\alpha+C_2\xi^\alpha}\right\}^{-2}$$

If $C_1 = 0$ and $C_2 \neq 0$, then

$$U_1^{3}(\xi) = \pm\frac{1}{Ak^{3/2}}\{k^3(2d\psi + B) \mp Aw\sqrt{k}\} \mp \frac{k^{3/2}}{A}(2d\psi + B + 2\psi\alpha/\xi^{\alpha}) \mp \frac{4\psi k^{3/2}}{A}(d^2\psi + Bd - E)(2d\psi + B + 2\psi\alpha/\xi^{\alpha})^{-1} \tag{3.1.24}$$

$$V_1^{3}(\xi) = -1 + \frac{k^2 l}{A^2}(2d\psi + B)(2d\psi + B + 2\psi\alpha/\xi^{\alpha}) - \frac{k^2 l}{2A^2}(2d\psi + B + 2\psi\alpha/\xi^{\alpha})^2 + \frac{4k^2 l\psi}{A^2}(2d\psi + B)(d^2\psi + Bd - E)(2d\psi + B + 2\psi\alpha/\xi^{\alpha})^{-1} - \frac{8\psi^2 k^2 l}{A^2}(d^2\psi + Bd - E)^2(2d\psi + B + 2\psi\alpha/\xi^{\alpha})^{-2} \tag{3.1.25}$$

where $\xi = k^{1/\alpha}x + l^{1/\alpha}y + w^{1/\alpha}t$.
When $B = 0$, $\psi = A - C$ and $\Delta = \psi E > 0$,

$$U_1^{4}(\xi) = \pm\frac{1}{Ak^{3/2}}\{2d\psi k^3 \mp Aw\sqrt{k}\} \mp \frac{2}{A}k^{3/2}\left\{d\psi + \sqrt{\Delta}\frac{C_1\sinh L + C_2\cosh L}{C_1\cosh L + C_2\sinh L}\right\} \mp \frac{2\psi k^{3/2}}{A}(d^2\psi - E)\left\{d\psi + \sqrt{\Delta}\frac{C_1\sinh L + C_2\cosh L}{C_1\cosh L + C_2\sinh L}\right\}^{-1} \tag{3.1.26}$$

$$V_1^{4}(\xi) = -1 + \frac{4d\psi k^2 l}{A^2}\left\{d\psi + \sqrt{\Delta}\frac{C_1\sinh L + C_2\cosh L}{C_1\cosh L + C_2\sinh L}\right\} - \frac{2}{A^2}k^2 l\left\{d\psi + \sqrt{\Delta}\frac{C_1\sinh L + C_2\cosh L}{C_1\cosh L + C_2\sinh L}\right\}^2 + \frac{4dk^2 l\psi^2}{A^2}(d^2\psi - E)\left\{d\psi + \sqrt{\Delta}\frac{C_1\sinh L + C_2\cosh L}{C_1\cosh L + C_2\sinh L}\right\}^{-1} - \frac{2\psi^2 k^2 l}{A^2}(d^2\psi - E)^2\left\{d\psi + \sqrt{\Delta}\frac{C_1\sinh L + C_2\cosh L}{C_1\cosh L + C_2\sinh L}\right\}^{-2} \tag{3.1.27}$$

Consider the arbitrary constants as $C_1 \neq 0$ and $C_2 = 0$, then

$$U_1^{4}(\xi) = \pm\frac{1}{Ak^{3/2}}\{2d\psi k^3 \mp Aw\sqrt{k}\} \mp \frac{2}{A}k^{3/2}(d\psi + \sqrt{\Delta}\tanh L) \mp \frac{2\psi k^{3/2}}{A}(d^2\psi - E)(d\psi + \sqrt{\Delta}\tanh L)^{-1} \tag{3.1.28}$$

$$V_1^{4}(\xi) = -1 + \frac{4d\psi k^2 l}{A^2}(d\psi + \sqrt{\Delta}\tanh L) - \frac{2}{A^2}k^2 l(d\psi + \sqrt{\Delta}\tanh L)^2 + \frac{4d\psi^2 k^2 l}{A^2} (d^2\psi - E)(d\psi + \sqrt{\Delta}\tanh L)^{-1} - \frac{2\psi^2 k^2 l}{A^2}(d^2\psi - E)^2(d\psi + \sqrt{\Delta}\tanh L)^{-2} \tag{3.1.29}$$

where $L=\dfrac{\sqrt{\Delta}\,\xi^{\alpha}}{A\alpha}$ and $\xi=k^{1/\alpha}x+l^{1/\alpha}y+w^{1/\alpha}t$.

When $B=0$, $\psi=A-C$ and $\Delta=\psi E<0$,

$$U_1^{5}(\xi)=\pm\frac{1}{Ak^{3/2}}\{2d\psi k^{3}\mp Aw\sqrt{k}\}\mp\frac{2}{A}k^{3/2}\left\{d\psi+\sqrt{-\Delta}\,\frac{-C_1\sin M+C_2\cos M}{C_1\cos M+C_2\sin M}\right\}$$
$$\mp\frac{2k^{3/2}\psi}{A}(d^2\psi-E)\left\{d\psi+\sqrt{-\Delta}\,\frac{-C_1\sin M+C_2\cos M}{C_1\cos M+C_2\sin M}\right\}^{-1} \tag{3.1.30}$$

$$V_1^{5}(\xi)=-1+\frac{4d\psi k^2 l}{A^2}\left\{d\psi+\sqrt{-\Delta}\,\frac{-C_1\sin M+C_2\cos M}{C_1\cos M+C_2\sin M}\right\}-\frac{2}{A^2}k^2 l\left\{d\psi+\sqrt{-\Delta}\,\frac{-C_1\sin M+C_2\cos M}{C_1\cos M+C_2\sin M}\right\}^{2}$$
$$+\frac{4d\psi^2k^2l}{A^2}(d^2\psi-E)\left\{d\psi+\sqrt{-\Delta}\,\frac{-C_1\sin M+C_2\cos M}{C_1\cos M+C_2\sin M}\right\}^{-1}$$
$$-\frac{2k^2l\psi^2}{A^2}(d^2\psi-E)^2\left\{d\psi+\sqrt{-\Delta}\,\frac{-C_1\sin M+C_2\cos M}{C_1\cos M+C_2\sin M}\right\}^{-2} \tag{3.1.31}$$

For particular case $C_1\neq 0$ and $C_2=0$,

$$U_1^{5}(\xi)=\pm\frac{1}{Ak^{3/2}}\{2d\psi k^{3}\mp Aw\sqrt{k}\}\mp\frac{2}{A}k^{3/2}(d\psi-\sqrt{-\Delta}\tan M)$$
$$\mp\frac{2k^{3/2}\psi}{A}(d^2\psi-E)(d\psi-\sqrt{-\Delta}\tan M)^{-1} \tag{3.1.32}$$

$$V_1^{5}(\xi)=-1+\frac{4d\psi k^2l}{A^2}(d\psi-\sqrt{-\Delta}\tan M)-\frac{2}{A^2}k^2l(d\psi-\sqrt{-\Delta}\tan M)^2+\frac{4d\psi^2k^2l}{A^2}$$
$$(d^2\psi-E)(d\psi-\sqrt{-\Delta}\tan M)^{-1}-\frac{2k^2l\psi^2}{A^2}(d^2\psi-E)^2(d\psi-\sqrt{-\Delta}\tan M)^{-2} \tag{3.1.33}$$

where $M=\dfrac{\sqrt{-\Delta}\,\xi^{\alpha}}{A\alpha}$ and $\xi=k^{1/\alpha}x+l^{1/\alpha}y+w^{1/\alpha}t$.

Following the same procedure as above for Eqs. (3.1.10)–(3.1.13) together with Eqs. (2.2.7)–(2.2.11), we might obtain more general closed form traveling wave solutions to the space time fractional (2+1)-dimensional dispersive long wave equations in terms of hyperbolic function, trigonometric function and rational function. To avoid the disturbance of readers the results have not been recorded here.

3.2. The (3+1)-dimensional space time fractional mKdV-ZK equation

Let us consider the (3+1)-dimensional space time fractional mKdV-ZK equation in the form

$$D_t^{\alpha}u+\delta u^2D_x^{\alpha}u+D_x^{3\alpha}u+D_x^{\alpha}(D_y^{2\alpha}u)+D_x^{\alpha}(D_z^{2\alpha}u)=0, \tag{3.2.1}$$

where $0<\alpha\le 1$ and δ is an arbitrary constant. This equation is derived for plasma comprised of cool and hot electrons and a species of fluid ions [44].

Using the fractional composite transformation

$$u(x,y,z,t)=U(\xi),\ \xi=l^{1/\alpha}x+m^{1/\alpha}y+n^{1/\alpha}z-\omega^{1/\alpha}t, \tag{3.2.2}$$

where l, m, n and ω are non-zero parameters, Eq. (3.2.1) is turned into the ordinary differential equation

$$-\omega D_\xi^\alpha U+\delta l U^2 D_\xi^\alpha U+(l^3+lm^2+ln^2)D_\xi^{3\alpha}U=0. \tag{3.2.3}$$

The anti-derivative of Eq. (3.2.3) with integral constant zero possesses

$$-\omega U+\frac{\delta l}{3}U^3+(l^3+lm^2+ln^2)D_\xi^{2\alpha}U=0. \tag{3.2.4}$$

Considering the value of n obtained by homogeneous balance to Eq. (3.2.4), Eq. (2.2.4) takes the form

$$U(\xi)=a_0+a_1(d+D_\xi^\alpha G/G)+b_1(d+D_\xi^\alpha G/G)^{-1} \tag{3.2.5}$$

where at least one of a_1 and b_1 is non-zero.

Substitute Eq. (3.2.5) with the help of Eq. (2.2.5) into Eq. (3.2.4), we obtain a polynomial in $(d+D_\xi^\alpha G/G)$. Equating each term of this polynomial to zero gives a set of algebraic equations for $a_0,\ a_1,\ b_0,\ b_1,$ and ω. Solving these equations by Maple gives the following set of solutions:

Set-1: $a_0=\pm\dfrac{(2d\psi+B)}{A}\sqrt{\dfrac{3(l^2+m^2+n^2)}{-2\delta}}$, $a_1=\pm\dfrac{\psi}{\delta A}\sqrt{-6\delta(l^2+m^2+n^2)}$, $b_1=0$,

$$\omega=-\frac{l(l^2+m^2+n^2)(4E\psi+B)}{2A^2} \tag{3.2.6}$$

Set-2: $a_0=\pm\dfrac{(2d\psi+B)}{2\delta A}\sqrt{-6\delta(l^2+m^2+n^2)}$, $\omega=-\dfrac{l(l^2+m^2+n^2)(4E\psi+B)}{2A^2}$,

$$a_1=0,\ b_1=\mp\frac{(d^2\psi+Bd-E)}{\delta A}\sqrt{-6\delta(l^2+m^2+n^2)} \tag{3.2.7}$$

Utilizing Eqs. (3.2.6), (3.2.7) into Eq. (3.2.5) yields the following general expressions for solutions:

$$U_1(\xi)=\mp\frac{\sqrt{-6\delta(l^2+m^2+n^2)}}{2A\delta}(B-2\psi D_\xi^\alpha G/G) \tag{3.2.8}$$

$$U_2(\xi)=\pm\frac{\sqrt{-6\delta(l^2+m^2+n^2)}}{2A\delta}\{2d\psi+B-2(d^2\psi+Bd-E)(d+D_\xi^\alpha G/G)^{-1}\}, \tag{3.2.9}$$

where $\xi=l^{1/\alpha}x+m^{1/\alpha}y+n^{1/\alpha}z-\left\{-\dfrac{l(l^2+m^2+n^2)(4E\psi+B)}{2A^2}\right\}^{1/\alpha}t$.

Eq. (3.2.8) along with Eqs. (2.2.7)–(2.2.11) provides the following solutions in terms of hyperbolic, trigonometric and rational:

When $B \neq 0$, $\psi = A - C$ and $\Omega = B^2 + 4E\psi > 0$,

$$U_1^{\,1}(\xi) = \pm \frac{\sqrt{-6\delta(l^2+m^2+n^2)}}{2A\delta} \sqrt{\Omega}\, \frac{C_1 \sinh\left(\dfrac{\sqrt{\Omega}\,\xi^\alpha}{2A\alpha}\right) + C_2 \cosh\left(\dfrac{\sqrt{\Omega}\,\xi^\alpha}{2A\alpha}\right)}{C_1 \cosh\left(\dfrac{\sqrt{\Omega}\,\xi^\alpha}{2A\alpha}\right) + C_2 \sinh\left(\dfrac{\sqrt{\Omega}\,\xi^\alpha}{2A\alpha}\right)} \tag{3.2.10}$$

The choices for arbitrary constants as $C_1 \neq 0$, $C_2 = 0$, reduces Eq. (3.2.10) to

$$U_1^{\,1}(\xi) = \pm \frac{\sqrt{-6\delta(l^2+m^2+n^2)}}{2A\delta} \sqrt{\Omega} \tanh\left(\frac{\sqrt{\Omega}\,\xi^\alpha}{2A\alpha}\right) \tag{3.2.11}$$

where $\xi = l^{1/\alpha} x + m^{1/\alpha} y + n^{1/\alpha} z - \left\{ -\dfrac{l(l^2+m^2+n^2)(4E\psi + B)}{2A^2} \right\}^{1/\alpha} t$.

When $B \neq 0$, $\psi = A - C$ and $\Omega = B^2 + 4E\psi < 0$,

$$U_1^{\,2}(\xi) = \pm \frac{\sqrt{-6\delta(l^2+m^2+n^2)}}{2A\delta} \sqrt{-\Omega}\, \frac{-C_1 \sin\left(\dfrac{\sqrt{-\Omega}\,\xi^\alpha}{2A\alpha}\right) + C_2 \cos\left(\dfrac{\sqrt{-\Omega}\,\xi^\alpha}{2A\alpha}\right)}{C_1 \cos\left(\dfrac{\sqrt{-\Omega}\,\xi^\alpha}{2A\alpha}\right) + C_2 \sin\left(\dfrac{\sqrt{-\Omega}\,\xi^\alpha}{2A\alpha}\right)} \tag{3.2.12}$$

The choices for arbitrary constants as $C_1 \neq 0$, $C_2 = 0$, reduces Eq. (3.2.12) to

$$U_1^{\,2}(\xi) = \mp \frac{\sqrt{-6\delta(l^2+m^2+n^2)}}{2A\delta} \sqrt{-\Omega} \tan\left(\frac{\sqrt{-\Omega}\,\xi^\alpha}{2A\alpha}\right), \tag{3.2.13}$$

where $\xi = l^{1/\alpha} x + m^{1/\alpha} y + n^{1/\alpha} z - \left\{ -\dfrac{l(l^2+m^2+n^2)(4E\psi + B)}{2A^2} \right\}^{1/\alpha} t$.

When $B \neq 0$, $\psi = A - C$ and $\Omega = B^2 + 4E\psi = 0$,

$$U_1^{\,3}(\xi) = \pm \frac{\psi\alpha\, C_2 \sqrt{-6\delta(l^2+m^2+n^2)}}{A\delta(C_1\alpha + C_2\xi^\alpha)} \tag{3.2.14}$$

In particular, if $C_1 = 0$, $C_2 \neq 0$, then

$$U_1^{\,3}(\xi) = \pm \frac{\psi\alpha}{A\delta\xi^\alpha} \sqrt{-6\delta(l^2+m^2+n^2)} \tag{3.2.15}$$

where $\xi = l^{1/\alpha} x + m^{1/\alpha} y + n^{1/\alpha} z - \left\{ -\dfrac{l(l^2+m^2+n^2)(4E\psi + B)}{2A^2} \right\}^{1/\alpha} t$.

When $B=0$, $\psi = A-C$ and $\Delta = \psi E > 0$,

$$U_1^{\,4}(\xi) = \pm \frac{\sqrt{-6\Delta\delta(l^2+m^2+n^2)}}{A\delta}\left\{\frac{C_1\sinh\left(\dfrac{\sqrt{\Delta}\,\xi^\alpha}{A\alpha}\right)+C_2\cosh\left(\dfrac{\sqrt{\Delta}\,\xi^\alpha}{A\alpha}\right)}{C_1\cosh\left(\dfrac{\sqrt{\Delta}\,\xi^\alpha}{A\alpha}\right)+C_2\sinh\left(\dfrac{\sqrt{\Delta}\,\xi^\alpha}{A\alpha}\right)}\right\} \tag{3.2.16}$$

For $C_1 \neq 0$, $C_2 = 0$, Eq. (3.2.16) becomes

$$U_1^{\,4}(\xi) = \pm \frac{\sqrt{-6\Delta\delta(l^2+m^2+n^2)}}{A\delta}\tanh\left(\frac{\sqrt{\Delta}\,\xi^\alpha}{A\alpha}\right) \tag{3.2.17}$$

where $\xi = l^{1/\alpha}\,x + m^{1/\alpha}\,y + n^{1/\alpha}\,z - \left\{-\dfrac{4E\psi\, l(l^2+m^2+n^2)}{2A^2}\right\}^{1/\alpha} t$.

When $B=0$, $\psi = A-C$ and $\Delta = \psi E < 0$,

$$U_1^{\,5}(\xi) = \pm \frac{\sqrt{6\Delta\delta(l^2+m^2+n^2)}}{A\delta}\left\{\frac{-C_1\sin\left(\dfrac{\sqrt{-\Delta}\,\xi^\alpha}{A\alpha}\right)+C_2\cos\left(\dfrac{\sqrt{-\Delta}\,\xi^\alpha}{A\alpha}\right)}{C_1\cos\left(\dfrac{\sqrt{-\Delta}\,\xi^\alpha}{A\alpha}\right)+C_2\sin\left(\dfrac{\sqrt{-\Delta}\,\xi^\alpha}{A\alpha}\right)}\right\} \tag{3.2.18}$$

If $C_1 \neq 0$, $C_2 = 0$, Eq. (3.2.18) becomes

$$U_1^{\,5}(\xi) = \mp \frac{\sqrt{6\Delta\delta(l^2+m^2+n^2)}}{A\delta}\tan\left(\frac{\sqrt{-\Delta}\,\xi^\alpha}{A\alpha}\right) \tag{3.2.19}$$

where $\xi = l^{1/\alpha}\,x + m^{1/\alpha}\,y + n^{1/\alpha}\,z - \left\{-\dfrac{4E\psi\, l(l^2+m^2+n^2)}{2A^2}\right\}^{1/\alpha} t$.

Making use of Eq. (3.2.9) as above will also provide further new and general exact traveling wave solutions in terms of hyperbolic, trigonometric and rational. For convenience of readers we have not record these all solutions in this study. Guner et al. [45] obtained only four solutions by (G'/G)-expansion method where as our applied fractional generalized $(D_\xi^\alpha G/G)$-expansion method has ensured many solutions which are further new and general. To the best of our knowledge, these solutions have not been visible in any earlier study.

3.3. The space time fractional modified regularized long-wave equation

The following nonlinear space-time fractional modified regularized long-wave equation is considered to be examined for further exact traveling wave solutions:

$$D_t^\alpha u + \delta\, D_x^\alpha u + \tau\, u^2 D_x^\alpha u - \eta\, D_t^\alpha D_x^{2\alpha} u = 0,\ 0<\alpha\le 1 \tag{3.3.1}$$

where δ, τ and η are constants. This equation proposed by Benjamin et al. to describe approximately the unidirectional propagation of long waves in certain dispersive systems is supposed to be alternative to the modified KdV equation. Eq. (3.3.1) has been modeled to demonstrate some physical phenomena like transverse waves in shallow water and magneto hydrodynamic waves in plasma and photon packets in nonlinear crystals [46–48].

The fractional complex transformation

$$u(x,t)=U(\xi),\ \ \xi=x-v^{1/\alpha}t\,, \tag{3.3.2}$$

reduces Eq. (3.3.1) to the ODE

$$(\delta-v)D_{\xi}^{\alpha}U+\tau U^{2}D_{\xi}^{\alpha}U+v\eta D_{\xi}^{3\alpha}U=0 \tag{3.3.3}$$

Integrating Eq. (3.3.3) and setting integral constant to zero gives

$$(\delta-v)U+\frac{\tau}{3}U^{3}+\eta vD_{\xi}^{2\alpha}U=0 \tag{3.3.4}$$

Taking homogeneous balance between highest order linear term and highest nonlinear term from Eq. (3.3.4) yields $n=1$ and the solution Eq. (2.2.4) is reduced to

$$U(\xi)=a_{0}+a_{1}(d+D_{\xi}^{\alpha}G/G)+b_{1}(d+D_{\xi}^{\alpha}G/G)^{-1} \tag{3.3.5}$$

where at least one of a_1 and b_1 is nonzero.

Eq. (3.3.4) with the help of Eq. (2.2.5) and Eq. (3.3.5) makes a polynomial in $(d+D_{\xi}^{\alpha}G/G)$. Set each coefficient of this polynomial to zero and obtain a system of equations for $a_0,\ a_1,\ b_1$ and v. Calculating these equations by Maple gives the following solutions:

Set-1: $a_{0}=\pm\frac{3l\eta\delta(2d\psi+B)}{\sqrt{-3\tau\eta\delta(2A^{2}+4l^{2}\eta E\psi+l^{2}\eta B^{2})}},\ a_{1}=\mp\frac{6l\eta\delta\psi}{\sqrt{-3\tau\eta\delta(2A^{2}+4l^{2}\eta E\psi+l^{2}\eta B^{2})}},$

$$b_{1}=0,\ v=\frac{2l\delta A^{2}}{2A^{2}+4l^{2}\eta E\psi+l^{2}\eta B^{2}} \tag{3.3.6}$$

Set-2: $a_{0}=\pm\frac{3l\eta\delta(2d\psi+B)}{\sqrt{-3\tau\eta\delta(2A^{2}+4l^{2}\eta E\psi+l^{2}\eta B^{2})}},\ b_{1}=\mp\frac{6l\eta\delta(d^{2}\psi+Bd-E)}{\sqrt{-3\tau\eta\delta(2A^{2}+4l^{2}\eta E\psi+l^{2}\eta B^{2})}},$

$$a_{1}=0,\ v=\frac{2l\delta A^{2}}{2A^{2}+4l^{2}\eta E\psi+l^{2}\eta B^{2}} \tag{3.3.7}$$

Inserting the values appearing in Eq. (3.3.6) and Eq. (3.3.7) into Eq. (3.3.5) possesses the following expressions for solutions:

$$U_{1}(\xi)=\pm\frac{3l\eta\delta}{\sqrt{-3\tau\eta\delta(2A^{2}+4l^{2}\eta E\psi+l^{2}\eta B^{2})}}(B-2\psi D_{\xi}^{\alpha}G/G) \tag{3.3.8}$$

$$U_{2}(\xi)=\pm3l\eta\delta\frac{(2d\psi+B)-2(d^{2}\psi+Bd-E)(d+D_{\xi}^{\alpha}G/G)^{-1}}{\sqrt{-3\tau\eta\delta(2A^{2}+4l^{2}\eta E\psi+l^{2}\eta B^{2})}} \tag{3.3.9}$$

where $\xi=x-\left(\frac{2l\delta A^{2}}{2A^{2}+4l^{2}\eta E\psi+l^{2}\eta B^{2}}\right)^{1/\alpha}t$.

Eq. (3.3.8) with the help of Eqs. (2.2.7)–(2.2.11) provides the following solutions:

Case 1: When $B \neq 0$, $\psi = A - C$ and $\Omega = B^2 + 4E\psi > 0$,

$$U_1^1(\xi) = \mp \frac{3l\eta\delta\sqrt{\Omega}}{\sqrt{-3\tau\eta\delta(2A^2 + 4l^2\eta E\psi + l^2\eta B^2)}} \times \frac{C_1 \sinh\left(\frac{\sqrt{\Omega}\,\xi^\alpha}{2A\alpha}\right) + C_2 \cosh\left(\frac{\sqrt{\Omega}\,\xi^\alpha}{2A\alpha}\right)}{C_1 \cosh\left(\frac{\sqrt{\Omega}\,\xi^\alpha}{2A\alpha}\right) + C_2 \sinh\left(\frac{\sqrt{\Omega}\,\xi^\alpha}{2A\alpha}\right)} \quad (3.3.10)$$

Since C_1 and C_2 are arbitrary constants, one may choose $C_1 \neq 0$, $C_2 = 0$ and under simplification Eq. (3.3.10) becomes

$$U_1^1(\xi) = \mp \frac{3l\eta\delta\sqrt{\Omega}}{\sqrt{-3\tau\eta\delta(2A^2 + 4l^2\eta E\psi + l^2\eta B^2)}} \times \tanh\left(\frac{\sqrt{\Omega}\,\xi^\alpha}{2A\alpha}\right) \quad (3.3.11)$$

where $\xi = x - \left(\frac{2l\delta A^2}{2A^2 + 4l^2\eta E\psi + l^2\eta B^2}\right)^{1/\alpha} t$.

Case 2: When $B \neq 0$, $\psi = A - C$ and $\Omega = B^2 + 4E\psi < 0$,

$$U_1^2(\xi) = \mp \frac{3l\eta\delta\sqrt{-\Omega}}{\sqrt{-3\tau\eta\delta(2A^2 + 4l^2\eta E\psi + l^2\eta B^2)}} \times \frac{-C_1 \sin\left(\frac{\sqrt{-\Omega}\,\xi^\alpha}{2A\alpha}\right) + C_2 \cos\left(\frac{\sqrt{-\Omega}\,\xi^\alpha}{2A\alpha}\right)}{C_1 \cos\left(\frac{\sqrt{-\Omega}\,\xi^\alpha}{2A\alpha}\right) + C_2 \sin\left(\frac{\sqrt{-\Omega}\,\xi^\alpha}{2A\alpha}\right)} \quad (3.3.12)$$

In particular, if we choose $C_1 \neq 0$, $C_2 = 0$, then under simplification Eq. (3.3.12) reduces to

$$U_1^2(\xi) = \pm \frac{3l\eta\delta\sqrt{-\Omega}}{\sqrt{-3\tau\eta\delta(2A^2 + 4l^2\eta E\psi + l^2\eta B^2)}} \times \tan\left(\frac{\sqrt{-\Omega}\,\xi^\alpha}{2A\alpha}\right) \quad (3.3.13)$$

where $\xi = x - \left(\frac{2l\delta A^2}{2A^2 + 4l^2\eta E\psi + l^2\eta B^2}\right)^{1/\alpha} t$.

Case 3: When $B \neq 0$, $\psi = A - C$ and $\Omega = B^2 + 4E\psi = 0$,

$$U_1^3(\xi) = \mp \frac{3l\eta\delta}{\sqrt{-3\tau\eta\delta(2A^2 + 4l^2\eta E\psi + l^2\eta B^2)}} \times \frac{2\psi\alpha C_2}{C_1\alpha + C_2\xi^\alpha} \quad (3.3.14)$$

If $C_1 = 0$, $C_2 \neq 0$, Eq. (3.3.14) is simplified to

$$U_1^3(\xi) = \mp \frac{6\psi\alpha l\eta\delta}{\xi^\alpha\sqrt{-3\tau\eta\delta(2A^2 + 4l^2\eta E\psi + l^2\eta B^2)}} \quad (3.3.15)$$

where $\xi = x - \left(\dfrac{2l\delta A^2}{2A^2 + 4l^2\eta E\psi + l^2\eta B^2} \right)^{1/\alpha} t$.

Case 4: When $B = 0$, $\psi = A - C$ and $\Delta = \psi E > 0$,

$$U_1^{\,4}(\xi) = \mp \frac{6l\eta\delta\sqrt{\Delta}}{\sqrt{-3\tau\eta\delta(2A^2 + 4l^2\eta E\psi)}} \times \frac{C_1 \sinh\left(\dfrac{\sqrt{\Delta}\,\xi^\alpha}{A\alpha}\right) + C_2 \cosh\left(\dfrac{\sqrt{\Delta}\,\xi^\alpha}{A\alpha}\right)}{C_1 \cosh\left(\dfrac{\sqrt{\Delta}\,\xi^\alpha}{A\alpha}\right) + C_2 \sinh\left(\dfrac{\sqrt{\Delta}\,\xi^\alpha}{A\alpha}\right)} \tag{3.3.16}$$

For particular values of the arbitrary constants as $C_1 \neq 0$, $C_2 = 0$, Eq. (3.3.16) possesses

$$U_1^{\,4}(\xi) = \mp \frac{6l\eta\delta\sqrt{\Delta}}{\sqrt{-3\tau\eta\delta(2A^2 + 4l^2\eta E\psi)}} \times \tanh\left(\frac{\sqrt{\Delta}\,\xi^\alpha}{A\alpha}\right) \tag{3.3.17}$$

where $\xi = x - \left(\dfrac{2l\delta A^2}{2A^2 + 4l^2\eta E\psi} \right)^{1/\alpha} t$.

Case 5: When $B = 0$, $\psi = A - C$ and $\Delta = \psi E < 0$,

$$U_1^{\,5}(\xi) = \pm \frac{6l\eta\delta\sqrt{-\Delta}}{\sqrt{-3\tau\eta\delta(2A^2 + 4l^2\eta E\psi)}} \times \frac{C_1 \sin\left(\dfrac{\sqrt{-\Delta}\,\xi^\alpha}{A\alpha}\right) + C_2 \cos\left(\dfrac{\sqrt{-\Delta}\,\xi^\alpha}{A\alpha}\right)}{C_1 \cos\left(\dfrac{\sqrt{-\Delta}\,\xi^\alpha}{A\alpha}\right) + C_2 \sin\left(\dfrac{\sqrt{-\Delta}\,\xi^\alpha}{A\alpha}\right)} \tag{3.3.18}$$

The choice of the arbitrary constants as $C_1 \neq 0$, $C_2 = 0$ forces Eq. (3.3.18) to turn into

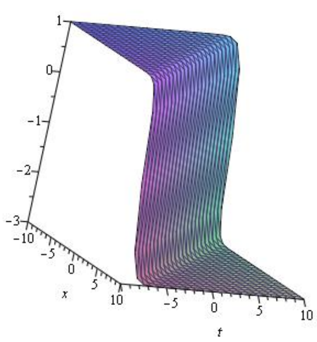
$$U_1^{\,5}(\xi) = \pm \frac{6l\eta\delta\sqrt{-\Delta}}{\sqrt{-3\tau\eta\delta(2A^2 + 4l^2\eta E\psi + l^2\eta B^2)}} \times \tan\left(\frac{\sqrt{-\Delta}\,\xi^\alpha}{A\alpha}\right) \tag{3.3.19}$$

where $\xi = x - \left(\dfrac{2l\delta A^2}{2A^2 + 4l^2\eta E\psi} \right)^{1/\alpha} t$.

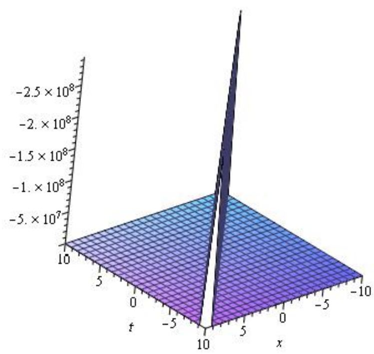
The obtained solutions in terms of hyperbolic function, trigonometric function and rational function are new and more general. In similar way, much more new and general solutions of the closed form can be constructed by using Eq. (3.3.9) along with Eqs. (2.2.7)–(2.2.11). The solutions obtained by modified simple equation method [49] and the improved fractional Riccati expansion method [50] are only in terms of hyperbolic, where as we achieved those in terms of hyperbolic function, trigonometric function and rational function in explicitly general form. We have not recorded these results to avoid the annoyance of the readers. On comparison, our solutions are general and much more in number than those of [49,50].

4. Graphical representations of the solutions

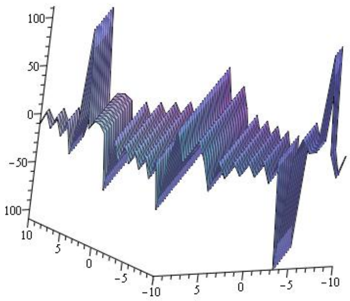
The complex physical mechanism of real world can be illustrated by means of graphical representations. The graphs (Figures 1–3) drown for the exact solutions obtained in this study has been appeared in different shape like kink type soliton, bell shape soliton, singular bell shape soliton, anti bell shape soliton, periodic solution, singular periodic solution etc. We have recorded here only few graphs rather than all for making it easily readable.



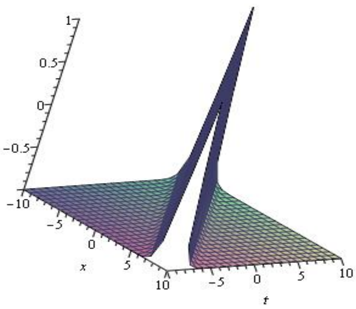
Sketch of the kink solution U_1^1 shown in (3.1.16) for $\alpha=\eta=l=C=k=w=d=1$, $y=0$, $\tau=-1$, $E=\delta=3$, $A=B=2$ within $-10\le x,t\le 0$.



Sketch of the singular bell shape soliton U_1^2 shown in equation (3.1.20) for $B=C=2$, $y=0$, $\alpha=\eta=l=A=k=w=d=1$, $E=\delta=3$, $\tau=-1$ within $-10\le x,t\le 0$.



Sketch of the traveling wave solution U_1^5 shown in (3.1.30) for $B=C=E=2$, $\tau=-1$, $\alpha=\eta=l=A=k=w=d=1$, $\delta=3$, $y=0$ within $-10\le x,t\le 0$.



Sketch of the Bell shape solution V_1^1 shown in (3.1.17) for $\alpha=\eta=l=C=k=w=d=1$, $y=0$, $\tau=-1$, $E=\delta=3$, $A=B=2$ within $-10\le x,t\le 0$.

Figure 1. The above are graphs for the solutions to the space time fractional (2+1)-dimensional dispersive long wave equations.

The kink type solution (3.2.11) for $y = z = 0$, $\alpha = l = m = n = B = C = E = 1$, $\delta = -1$, $A = 3$ within $-10 \le x, t \le 0$.

The periodic solution (3.2.19) for $x = t = 0$, $\alpha = l = m = n = A = E = 1$, $\delta = -1$, $C = 2$ within $-10 \le x, t \le 0$.

Figure 2. The above are the graphs for the solutions to the (3+1)-dimensional space time fractional mKdV-ZK equation.

The kink type solution (3.3.11) for $\delta = E = 3$, $\alpha = l = \eta = C = 1$, $\tau = -1$, $A = B = 2$ within $-10 \le x, t \le 0$.

The singular periodic solution (3.3.19) for $\delta = 3$, $\alpha = l = \eta = \tau = A = E = 1$, $C = 2$ within $-10 \le x, t \le 0$.

Figure 3. The above are the graphs for the solutions to the space time fractional modified regularized long-wave equation.

5. Conclusion

This article has been put in writing further new and general traveling wave solutions in closed form to the space time fractional (2+1)-dimensional dispersive long wave equations, the (3+1)-dimensional space time fractional mKdV-ZK equation and the space time fractional modified regularized long-wave equation. The solutions have successfully constructed in terms of hyperbolic function, trigonometric function and rational function by the newly established fractional generalized $(D_\xi^\alpha G/G)$-expansion method. To the best of our knowledge, these results are not available in the literature. The obtained solutions might play important roles to analyze the mechanisms of complex physical phenomena of the real world. The performance of the suggested method is highly appreciable for its easiest productive behavior and worthy for revealing rare solutions to more fractional order nonlinear evolution equations. Since each nonlinear equation has its own anomalous characteristic, the future research might be how the suggested method is compatible for revealing the solutions to other fractional nonlinear evolution equations.

Conflict of interest

All authors have contributed to the manuscript equally. We all discussed the outcomes of the proposed method and approved the final manuscript.

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