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Research article

Some new inequalities of the Grüss type for conformable fractional integrals

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Abstract: In the paper, the authors establish some new inequalities of the Grüss type for conformable fractional integrals. These inequalities generalize some known results.

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1. Introduction

In [7], Grüss showed an integral inequality which connects the integral of the product of two functions and the product of integrals for these two function. This inequality reads that, if f and g are two continuous functions on $[a, b]$ satisfying $m \le f(\tau) \le M$ and $n \le g(\tau) \le N$ for all $\tau \in [a, b]$ and $m, M, n, N \in \mathbb{R}$, then

$$\left|\frac{1}{b-a}\int_a^b f(\tau)g(\tau)\,\mathrm{d}\,t - \frac{1}{(b-a)^2}\int_a^b f(\tau)\,\mathrm{d}\,t\int_a^b g(\tau)\,\mathrm{d}\,t\right| \le \frac{1}{4}(M-m)(N-n). \tag{1.1}$$

For more information on the Grüss inequality (1.1), please refer to [11, Chapter X] and closely related references therein.

In the latest decades, the fractional integral inequalities involving the Riemann–Liouville fractional integrals have been widely studied by various researchers. The interested readers can refer to the work in [1–3, 5, 17, 22, 24]. In [3], Dahmani introduced the following fractional integral inequalities for the

Riemann–Liouville fractional integrals: if f and g are two integrable functions on $[0,\infty)$ satisfying $m \le f(\tau) \le M$ and $n \le g(\tau) \le N$ for all $\tau \in [0,\infty)$ and $m, M, n, N \in \mathbb{R}$, then, for all $\tau, \alpha, \beta > 0$, we have

$$\left|\frac{\tau^{\alpha}}{\Gamma(\alpha+1)}\mathfrak{I}^{\alpha}(fg)(\tau) - \mathfrak{I}^{\alpha}f(\tau)\mathfrak{I}^{\alpha}g(\tau)\right| \le \left[\frac{\tau^{\alpha}}{\Gamma(\alpha+1)}\right]^{2}(M-m)(N-n)$$

and

$$\begin{aligned}&\left[\frac{\tau^{\alpha}}{\Gamma(\alpha+1)}\mathfrak{I}^{\beta}(fg)(\tau) - \frac{\tau^{\beta}}{\Gamma(\beta+1)}\mathfrak{I}^{\alpha}(fg)(\tau) - \mathfrak{I}^{\alpha}f(\tau)\mathfrak{I}^{\beta}g(\tau) - \mathfrak{I}^{\beta}f(\tau)\mathfrak{I}^{\alpha}g(\tau)\right]^{2}\\
&\le \left[\left(M\frac{\tau^{\alpha}}{\Gamma(\alpha+1)} - \mathfrak{I}^{\alpha}f(\tau)\right)\left(\mathfrak{I}^{\beta}f(\tau) - m\frac{\tau^{\beta}}{\Gamma(\beta+1)}\right) + \left(\mathfrak{I}^{\alpha}f(\tau) - m\frac{\tau^{\alpha}}{\Gamma(\alpha+1)}\right)\left(M\frac{\tau^{\beta}}{\Gamma(\beta+1)} - \mathfrak{I}^{\beta}f(\tau)\right)\right]\\
&\times\left[\left(N\frac{\tau^{\alpha}}{\Gamma(\alpha+1)} - \mathfrak{I}^{\alpha}g(\tau)\right)\left(\mathfrak{I}^{\beta}g(\tau) - n\frac{\tau^{\beta}}{\Gamma(\beta+1)}\right) + \left(\mathfrak{I}^{\alpha}g(\tau) - n\frac{\tau^{\alpha}}{\Gamma(\alpha+1)}\right)\left(N\frac{\tau^{\beta}}{\Gamma(\beta+1)} - \mathfrak{I}^{\beta}g(\tau)\right)\right],\end{aligned}$$

where Γ is the classical Euler gamma function [12,13,21] and the Riemann–Liouville fractional integral $\mathfrak{I}^{\mu}$ of order $\mu \ge 0$ for a function $f \in L^{1}((0,\infty),\mathbb{R})$ is defined [10] by $\mathfrak{I}^{0}f(x) = f(x)$ and

$$\mathfrak{I}^{\mu}f(x) = \frac{1}{\Gamma(\mu)}\int_{0}^{x}(x-\tau)^{\mu-1}f(\tau)\,\mathrm{d}\tau, \quad \Re(\mu) > 0.$$

The Riemann–Liouville fractional integrals $\mathfrak{I}^{\mu}_{\alpha^{+}}$ and $\mathfrak{I}^{\mu}_{\beta^{-}}$ of order $\mu > 0$ are defined respectively by

$$\mathfrak{I}^{\mu}_{\alpha^{+}}f(x) = \frac{1}{\Gamma(\mu)}\int_{\alpha}^{x}(x-\tau)^{\mu-1}f(\tau)\,\mathrm{d}\tau, \quad x > \alpha, \quad \Re(\mu) > 0 \tag{1.2}$$

and

$$\mathfrak{I}^{\mu}_{\beta^{-}}f(x) = \frac{1}{\Gamma(\mu)}\int_{x}^{\beta}(x-\tau)^{\mu-1}f(\tau)\,\mathrm{d}\tau, \quad x < \beta, \quad \Re(\mu) > 0. \tag{1.3}$$

For more details about fractional integral operators (1.2) and (1.3), please refer to [4,6,9,10,15,16,18] and closely related references.

The left and right sided fractional conformable integral operators are respectively defined [8] by

$${}^{\lambda}\mathfrak{I}^{\mu}_{a^{+}}f(x) = \frac{1}{\Gamma(\lambda)}\int_{a}^{x}\left[\frac{(x-a)^{\mu} - (\tau-a)^{\mu}}{\mu}\right]^{\lambda-1}\frac{f(\tau)}{(\tau-a)^{1-\mu}}\,\mathrm{d}\tau \tag{1.4}$$

and

$${}^{\lambda}\mathfrak{I}^{\mu}_{b^{-}}f(x) = \frac{1}{\Gamma(\lambda)}\int_{x}^{b}\left[\frac{(b-x)^{\mu} - (b-\tau)^{\mu}}{\mu}\right]^{\lambda-1}\frac{f(\tau)}{(b-\tau)^{1-\mu}}\,\mathrm{d}\tau, \tag{1.5}$$

where $\Re(\lambda) > 0$. Obviously, if taking $a = 0$ and $\mu = 1$, then (1.4) reduces to the Riemann–Liouville fractional integral (1.2). Similarly, if setting $b = 0$ and $\mu = 1$, then (1.5) becomes the Riemann–Liouville fractional integral (1.3).

In [19] the conformable fractional integral

$${}^{\beta}\mathfrak{I}^{\mu}f(x) = \frac{1}{\Gamma(\beta)}\int_{0}^{x}\left(\frac{x^{\mu} - \tau^{\mu}}{\mu}\right)^{\beta-1}\frac{f(\tau)}{\tau^{1-\mu}}\,\mathrm{d}\tau. \tag{1.6}$$

was defined. From (1.6), one can obtain easily that ${}^{\alpha}\mathfrak{I}^{\mu\beta}\mathfrak{I}^{\mu}f(x) = {}^{\beta+\alpha}\mathfrak{I}^{\mu}f(x)$ and ${}^{\alpha}\mathfrak{I}^{\mu\beta}\mathfrak{I}^{\mu}f(x) = {}^{\beta}\mathfrak{I}^{\mu\alpha}\mathfrak{I}^{\mu}f(x)$.

In this paper, we will empoy the conformable fractional integral (1.6) to establish some new inequalities of the Grüss type for conformable fractional integrals.

2. Main results

We are now in a position to state and prove our main results.

Theorem 2.1. *Let f be an integrable function on $[0,\infty)$. Assume that there exist two integrable functions ϕ_1, ϕ_2 on $[0,\infty)$ such that*

$$\phi_1(x) \le f(x) \le \phi_2(x), \quad x \in [0,\infty). \tag{2.1}$$

Then, for $x, \alpha, \beta > 0$, we have

$$^{\beta}\mathfrak{I}^{\mu}\phi_1(x)^{\alpha}\mathfrak{I}^{\mu}f(x) + {}^{\alpha}\mathfrak{I}^{\mu}\phi_2(x)^{\beta}\mathfrak{I}^{\mu}f(x) \ge {}^{\alpha}\mathfrak{I}^{\mu}\phi_2(x)^{\beta}\mathfrak{I}^{\mu}\phi_1(x) + {}^{\alpha}\mathfrak{I}^{\mu}f(x)^{\beta}\mathfrak{I}^{\mu}f(x). \tag{2.2}$$

Proof. From (2.1), for all $\tau, \rho \ge 0$, it follows that $[\phi_2(\tau) - f(\tau)][f(\rho) - \phi_1(\rho)] \ge 0$. Therefore, we have

$$\phi_2(\tau)f(\rho) + \phi_1(\rho)f(\tau) \ge \phi_1(\rho)\phi_2(\tau) + f(\tau)f(\rho). \tag{2.3}$$

Multiplying both sides of (2.3) by $\frac{1}{\Gamma(\alpha)}(\frac{x^\mu-\tau^\mu}{\mu})^{\alpha-1}\tau^{\mu-1}$ and integrating over $\tau \in (0,x)$ lead to

$$\begin{aligned} f(\rho)\int_0^x \frac{1}{\Gamma(\alpha)}\Big(\frac{x^\mu-\tau^\mu}{\mu}\Big)^{\alpha-1}\tau^{\mu-1}\phi_2(\tau)\,\mathrm{d}\tau + \phi_1(\rho)\int_0^x \frac{1}{\Gamma(\alpha)}\Big(\frac{x^\mu-\tau^\mu}{\mu}\Big)^{\alpha-1}\tau^{\mu-1}f(\tau)\,\mathrm{d}\tau \\ \ge \phi_1(\rho)\int_0^x \frac{1}{\Gamma(\alpha)}\Big(\frac{x^\mu-\tau^\mu}{\mu}\Big)^{\alpha-1}\tau^{\mu-1}\phi_2(\tau)\,\mathrm{d}\tau + f(\rho)\int_0^x \frac{1}{\Gamma(\alpha)}\Big(\frac{x^\mu-\tau^\mu}{\mu}\Big)^{\alpha-1}\tau^{\mu-1}f(\tau)\,\mathrm{d}\tau \end{aligned}$$

which gives

$$f(\rho)^{\alpha}\mathfrak{I}^{\mu}\phi_2(x) + \phi_1(\rho)^{\alpha}\mathfrak{I}^{\mu}f(x) \ge \phi_1(\rho)^{\alpha}\mathfrak{I}^{\mu}\phi_2(x) + f(\rho)^{\alpha}\mathfrak{I}^{\mu}f(x). \tag{2.4}$$

Multiplying both sides of (2.4) by $\frac{1}{\Gamma(\beta)}(\frac{x^\mu-\rho^\mu}{\mu})^{\beta-1}\rho^{\mu-1}$ and integrating over $\rho \in (0,x)$ result in

$$\begin{aligned} {}^{\alpha}\mathfrak{I}^{\mu}\phi_2(x)\int_0^x \frac{1}{\Gamma(\beta)}\Big(\frac{x^\mu-\rho^\mu}{\mu}\Big)^{\beta-1}\rho^{\mu-1}f(\rho)\,\mathrm{d}\rho + {}^{\alpha}\mathfrak{I}^{\mu}f(x)\int_0^x \frac{1}{\Gamma(\beta)}\Big(\frac{x^\mu-\rho^\mu}{\mu}\Big)^{\beta-1}\rho^{\mu-1}\phi_1(\rho)\,\mathrm{d}\rho \\ \ge {}^{\alpha}\mathfrak{I}^{\mu}\phi_2(x)\int_0^x \frac{1}{\Gamma(\beta)}\Big(\frac{x^\mu-\rho^\mu}{\mu}\Big)^{\beta-1}\rho^{\mu-1}\phi_1(\rho)\,\mathrm{d}\rho + {}^{\alpha}\mathfrak{I}^{\mu}f(x)\int_0^x \frac{1}{\Gamma(\beta)}\Big(\frac{x^\mu-\rho^\mu}{\mu}\Big)^{\beta-1}\rho^{\mu-1}f(\rho)\,\mathrm{d}\rho. \end{aligned}$$

which gives the required inequality (2.2). □

From Theorem 2.1, we can derive the following two corollaries.

Corollary 2.1. *Let f be an integrable function on $[0,\infty)$ satisfying $m \le f(x) \le M$ for all $x \in [0,\infty)$ and $m, M \in \mathbb{R}$. Then, for $x, \alpha, \beta > 0$, we have*

$$m\frac{x^{\mu\beta}}{\mu^{\beta}\Gamma(\beta+1)}{}^{\alpha}\mathfrak{I}^{\mu}f(x) + M\frac{x^{\mu\alpha}}{\mu^{\alpha}\Gamma(\alpha+1)}{}^{\beta}\mathfrak{I}^{\mu}f(x) \ge mM\frac{x^{\mu(\alpha+\beta)}}{\mu^{\alpha+\beta}\Gamma(\alpha+1)\Gamma(\beta+1)} + {}^{\alpha}\mathfrak{I}^{\mu}f(x)^{\beta}\mathfrak{I}^{\mu}f(x).$$

Corollary 2.2. *Let f be an integrable function on $[0,\infty)$ satisfying $x^\mu \le f(x) \le x^\mu + 1$ for all $x \in [0,\infty)$. Then, for $x, \alpha > 0$, we have*

$$\Big[\frac{2x^{\mu(\alpha+1)}}{\mu^{\alpha}\Gamma(\alpha+2)} + \frac{x^{\mu\alpha}}{\mu^{\alpha}\Gamma(\alpha+1)}\Big]^{\alpha}\mathfrak{I}^{\mu}f(x) \ge \Big[\frac{x^{\mu(\alpha+1)}}{\mu^{\alpha}\Gamma(\alpha+2)} + \frac{x^{\mu\alpha}}{\mu^{\alpha}\Gamma(\alpha+1)}\Big]\Big[\frac{x^{\mu(\alpha+1)}}{\mu^{\alpha}\Gamma(\alpha+2)}\Big] + [{}^{\alpha}\mathfrak{I}^{\mu}f(x)]^2.$$

Theorem 2.2. *Let f and g be two integrable function on $[0,\infty)$. Suppose that the inequality (2.1) holds and that there exist two integrable functions ψ_1 and ψ_2 on $[0,\infty)$ such that*

$$\psi_1(x) \le g(x) \le \psi_2(x), \quad x \in [0,\infty). \tag{2.5}$$

Then, for $x,\alpha,\beta > 0$, the following four inequalities hold:

$$^{\beta}\mathfrak{I}^{\mu}\psi_1(x)^{\alpha}\mathfrak{I}^{\mu}f(x) + {}^{\alpha}\mathfrak{I}^{\mu}\phi_2(x)^{\beta}\mathfrak{I}^{\mu}g(x) \ge {}^{\alpha}\mathfrak{I}^{\mu}\phi_2(x)^{\beta}\mathfrak{I}^{\mu}\psi_1(x) + {}^{\alpha}\mathfrak{I}^{\mu}f(x)^{\beta}\mathfrak{I}^{\mu}g(x), \tag{2.6}$$

$$^{\beta}\mathfrak{I}^{\mu}\phi_1(x)^{\alpha}\mathfrak{I}^{\mu}g(x) + {}^{\alpha}\mathfrak{I}^{\mu}\psi_2(x)^{\beta}\mathfrak{I}^{\mu}f(x) \ge {}^{\alpha}\mathfrak{I}^{\mu}\phi_1(x)^{\beta}\mathfrak{I}^{\mu}\psi_2(x) + {}^{\alpha}\mathfrak{I}^{\mu}f(x)^{\beta}\mathfrak{I}^{\mu}g(x), \tag{2.7}$$

$$^{\alpha}\mathfrak{I}^{\mu}\phi_2(x)^{\beta}\mathfrak{I}^{\mu}\psi_2(x) + {}^{\alpha}\mathfrak{I}^{\mu}f(x)^{\beta}\mathfrak{I}^{\mu}g(x) \ge {}^{\alpha}\mathfrak{I}^{\mu}\phi_2(x)^{\beta}\mathfrak{I}^{\mu}g(x) + {}^{\beta}\mathfrak{I}^{\mu}\psi_2(x)^{\alpha}\mathfrak{I}^{\mu}f(x), \tag{2.8}$$

$$^{\alpha}\mathfrak{I}^{\mu}\phi_1(x)^{\beta}\mathfrak{I}^{\mu}\psi_1(x) + {}^{\alpha}\mathfrak{I}^{\mu}f(x)^{\beta}\mathfrak{I}^{\mu}g(x) \ge {}^{\alpha}\mathfrak{I}^{\mu}\phi_1(x)^{\beta}\mathfrak{I}^{\mu}g(x) + {}^{\alpha}\mathfrak{I}^{\mu}f(x)^{\beta}\mathfrak{I}^{\mu}\psi_1(x). \tag{2.9}$$

Proof. From (2.1) and (2.5) and for $x \in [0,\infty)$, we have $[\phi_2(\tau) - f(\tau)][g(\rho) - \psi_1(\rho)] \ge 0$. Therefore, it follows that

$$\phi_2(\tau)g(\rho) + \psi_1(\rho)f(\tau) \ge \psi_1(\rho)\phi_2(\tau) + f(\tau)g(\rho). \tag{2.10}$$

Multiplying both sides of (2.10) by $\frac{1}{\Gamma(\alpha)}(\frac{x^{\mu}-\tau^{\mu}}{\mu})^{\alpha-1}\tau^{\mu-1}$ and integrating over $\tau \in (0,x)$ arrive at

$$\begin{aligned} g(\rho)\int_0^x \frac{1}{\Gamma(\alpha)}\Big(\frac{x^{\mu}-\tau^{\mu}}{\mu}\Big)^{\alpha-1}\tau^{\mu-1}\phi_2(\tau)\,\mathrm{d}\tau + \psi_1(\rho)\int_0^x \frac{1}{\Gamma(\alpha)}\Big(\frac{x^{\mu}-\tau^{\mu}}{\mu}\Big)^{\alpha-1}\tau^{\mu-1}f(\tau)\,\mathrm{d}\tau \\ \ge \psi_1(\rho)\int_0^x \frac{1}{\Gamma(\alpha)}\Big(\frac{x^{\mu}-\tau^{\mu}}{\mu}\Big)^{\alpha-1}\tau^{\mu-1}\phi_2(\tau)\,\mathrm{d}\tau + g(\rho)\int_0^x \frac{1}{\Gamma(\alpha)}\Big(\frac{x^{\mu}-\tau^{\mu}}{\mu}\Big)^{\alpha-1}\tau^{\mu-1}f(\tau)\,\mathrm{d}\tau \end{aligned}$$

which gives

$$g(\rho)^{\alpha}\mathfrak{I}^{\mu}\phi_2(x) + \psi_1(\rho)^{\alpha}\mathfrak{I}^{\mu}f(x) \ge \psi_1(\rho)^{\alpha}\mathfrak{I}^{\mu}\phi_2(x) + g(\rho)^{\alpha}\mathfrak{I}^{\mu}f(x). \tag{2.11}$$

Multiplying both sides of (2.11) by $\frac{1}{\Gamma(\beta)}(\frac{x^{\mu}-\rho^{\mu}}{\mu})^{\beta-1}\rho^{\mu-1}$ and integrating over $\rho \in (0,x)$ reveal

$$\begin{aligned} {}^{\alpha}\mathfrak{I}^{\mu}\phi_2(x)\int_0^x \frac{1}{\Gamma(\beta)}\Big(\frac{x^{\mu}-\rho^{\mu}}{\mu}\Big)^{\beta-1}\rho^{\mu-1}g(\rho)\,\mathrm{d}\rho + {}^{\alpha}\mathfrak{I}^{\mu}f(x)\int_0^x \frac{1}{\Gamma(\beta)}\Big(\frac{x^{\mu}-\rho^{\mu}}{\mu}\Big)^{\beta-1}\rho^{\mu-1}\psi_1(\rho)\,\mathrm{d}\rho \\ \ge {}^{\alpha}\mathfrak{I}^{\mu}\phi_2(x)\int_0^x \frac{1}{\Gamma(\beta)}\Big(\frac{x^{\mu}-\rho^{\mu}}{\mu}\Big)^{\beta-1}\rho^{\mu-1}\psi_1(\rho)\,\mathrm{d}\rho + {}^{\alpha}\mathfrak{I}^{\mu}f(x)\int_0^x \frac{1}{\Gamma(\beta)}\Big(\frac{x^{\mu}-\rho^{\mu}}{\mu}\Big)^{\beta-1}\rho^{\mu-1}g(\rho)\,\mathrm{d}\rho. \end{aligned}$$

which gives the required inequality (2.6).

Making use of the inequalities $[\psi_2(\tau) - g(\tau)][f(\rho) - \phi_1(\rho)] \ge 0$, $[\phi_2(\tau) - f(\tau)][g(\rho) - \psi_2(\rho)] \le 0$, and $[\phi_1(\tau) - f(\tau)][g(\rho) - \psi_1(\rho)] \le 0$, we can prove (2.7) to (2.9). □

Corollary 2.3. *Let f and g be two integrable functions on $[0,\infty)$. Assume that there exist $m, M, n, N \in \mathbb{R}$ such that $m \le f(x) \le M$ and $n \le g(x) \le N$ for $x \in [0,\infty)$. Then we have the following four inequalities*

$$\begin{aligned} n\frac{x^{\mu\beta}}{\mu^{\beta}\Gamma(\beta+1)}{}^{\alpha}\mathfrak{I}^{\mu}f(x) + M\frac{x^{\mu\alpha}}{\mu^{\alpha}\Gamma(\alpha+1)}{}^{\beta}\mathfrak{I}^{\mu}g(x) &\ge nM\frac{x^{\mu(\alpha+\beta)}}{\mu^{\alpha+\beta}\Gamma(\alpha+1)\Gamma(\beta+1)} + {}^{\alpha}\mathfrak{I}^{\mu}f(x)^{\beta}\mathfrak{I}^{\mu}g(x), \\ m\frac{x^{\mu\beta}}{\mu^{\beta}\Gamma(\beta+1)}{}^{\alpha}\mathfrak{I}^{\mu}g(x) + N\frac{x^{\mu\alpha}}{\mu^{\alpha}\Gamma(\alpha+1)}{}^{\beta}\mathfrak{I}^{\mu}f(x) &\ge mN\frac{x^{\mu(\alpha+\beta)}}{\mu^{\alpha+\beta}\Gamma(\alpha+1)\Gamma(\beta+1)} + {}^{\beta}\mathfrak{I}^{\mu}f(x)^{\alpha}\mathfrak{I}^{\mu}g(x), \end{aligned}$$

$$MN\frac{x^{\mu(\alpha+\beta)}}{\mu^{\alpha+\beta}\Gamma(\alpha+1)\Gamma(\beta+1)}+{}^{\beta}\Im^{\mu}f(x)^{\alpha}\Im^{\mu}g(x)\geq M\frac{x^{\mu\alpha}}{\mu^{\alpha}\Gamma(\alpha+1)}{}^{\beta}\Im^{\mu}g(x)+N\frac{x^{\mu\beta}}{\mu^{\beta}\Gamma(\beta+1)}{}^{\alpha}\Im^{\mu}f(x),$$

and

$$mn\frac{x^{\mu(\alpha+\beta)}}{\mu^{\alpha+\beta}\Gamma(\alpha+1)\Gamma(\beta+1)}+{}^{\beta}\Im^{\mu}f(x)^{\alpha}\Im^{\mu}g(x)\geq m\frac{x^{\mu\alpha}}{\mu^{\alpha}\Gamma(\alpha+1)}{}^{\beta}\Im^{\mu}g(x)+n\frac{x^{\mu\beta}}{\mu^{\beta}\Gamma(\beta+1)}{}^{\alpha}\Im^{\mu}f(x).$$

Theorem 2.3. *Let f be an integrable function on $[0,\infty)$ and let ϕ_1 and ϕ_2 be two integrable functions on $[0,\infty)$. Assume that the inequality (2.1) holds. Then, for $x,\alpha>0$, we have*

$$\begin{aligned}\frac{x^{\mu\alpha}}{\mu^{\alpha}\Gamma(\alpha+1)}{}^{\alpha}\Im^{\mu}f^2(x)-[{}^{\alpha}\Im^{\mu}f(x)]^2&=[{}^{\alpha}\Im^{\mu}\phi_2(x)-{}^{\alpha}\Im^{\mu}f(x)][{}^{\alpha}\Im^{\mu}f(x)-{}^{\alpha}\Im^{\mu}\phi_1(x)]\\&-\frac{x^{\mu\alpha}}{\mu^{\alpha}\Gamma(\alpha+1)}{}^{\alpha}\Im^{\mu}[\phi_2(x)-f(x)][f(x)-\phi_1(x)]+\frac{x^{\mu\alpha}}{\mu^{\alpha}\Gamma(\alpha+1)}{}^{\alpha}\Im^{\mu}\phi_1f(x)-{}^{\alpha}\Im^{\mu}\phi_1(x)^{\alpha}\Im^{\mu}f(x)\\&+\frac{x^{\mu\alpha}}{\mu^{\alpha}\Gamma(\alpha+1)}{}^{\alpha}\Im^{\mu}\phi_2f(x)-{}^{\alpha}\Im^{\mu}\phi_2(x)^{\alpha}\Im^{\mu}f(x)+{}^{\alpha}\Im^{\mu}\phi_1(x)^{\alpha}\Im^{\mu}\phi_2(x)-\frac{x^{\mu\alpha}}{\mu^{\alpha}\Gamma(\alpha+1)}{}^{\alpha}\Im^{\mu}\phi_1\phi_2(x).\end{aligned}\tag{2.12}$$

Proof. For $\tau,\rho>0$, we have

$$\begin{aligned}&[\phi_2(\rho)-f(\rho)][f(\tau)-\phi_1(\tau)]+[\phi_2(\tau)-f(\tau)][f(\rho)-\phi_1(\rho)]\\&\qquad-[\phi_2(\tau)-f(\tau)][f(\tau)-\phi_1(\tau)]-[\phi_2(\rho)-f(\rho)][f(\rho)-\phi_1(\rho)]\\&=f^2(\tau)+f^2(\rho)-2f(\tau)f(\rho)+\phi_2(\rho)f(\tau)+\phi_1(\tau)f(\rho)-\phi_1(\tau)\phi_2(\rho)+\phi_2(\tau)f(\rho)+\phi_1(\rho)f(\tau)\\&-\phi_1(\rho)\phi_2(\tau)-\phi_2(\tau)f(\tau)+\phi_1(\tau)\phi_2(\tau)-\phi_1(\tau)f(\tau)-\phi_2(\rho)f(\rho)+\phi_1(\rho)\phi_2(\rho)-\phi_1(\rho)f(\rho).\end{aligned}\tag{2.13}$$

Multiplying both sides of (2.13) by $\frac{1}{\Gamma(\alpha)}(\frac{x^{\mu}-\tau^{\mu}}{\mu})^{\alpha-1}\tau^{\mu-1}$ and integrating over $\tau\in(0,x)$ yield

$$\begin{aligned}&[\phi_2(\rho)-f(\rho)][{}^{\alpha}\Im^{\mu}f(x)-{}^{\alpha}\Im^{\mu}\phi_1(x)]+[{}^{\alpha}\Im^{\mu}\phi_2(x)-{}^{\alpha}\Im^{\mu}f(x)][f(\rho)-\phi_1(\rho)]\\&\qquad-{}^{\alpha}\Im^{\mu}[\phi_2(x)-f(x)][f(x)-\phi_1(x)]-[\phi_2(\rho)-f(\rho)][f(\rho)-\phi_1(\rho)]\frac{x^{\mu\alpha}}{\mu^{\alpha}\Gamma(\alpha+1)}\\&={}^{\alpha}\Im^{\mu}f^2(x)+f^2(\rho)\frac{x^{\mu\alpha}}{\mu^{\alpha}\Gamma(\alpha+1)}-2f(\rho)^{\alpha}\Im^{\mu}f(x)+\phi_2(\rho)^{\alpha}\Im^{\mu}f(x)\\&+f(\rho)^{\alpha}\Im^{\mu}\phi_1(x)-\phi_2(\rho)^{\alpha}\Im^{\mu}\phi_1(x)+f(\rho)^{\alpha}\Im^{\mu}\phi_2(x)+\phi_1(\rho)^{\alpha}\Im^{\mu}f(x)\\&-\phi_1(\rho)^{\alpha}\Im^{\mu}\phi_2(x)-{}^{\alpha}\Im^{\mu}\phi_2f(x)+{}^{\alpha}\Im^{\mu}\phi_1\phi_2(x)-{}^{\alpha}\Im^{\mu}\phi_1f(x)\\&-\phi_2(\rho)f(\rho)\frac{x^{\mu\alpha}}{\mu^{\alpha}\Gamma(\alpha+1)}+\phi_1(\rho)\phi_2(\rho)\frac{x^{\mu\alpha}}{\mu^{\alpha}\Gamma(\alpha+1)}-\phi_1(\rho)f(\rho)\frac{x^{\mu\alpha}}{\mu^{\alpha}\Gamma(\alpha+1)}.\end{aligned}\tag{2.14}$$

Multiplying both sides of (2.14) by $\frac{1}{\Gamma(\alpha)}(\frac{x^{\mu}-\rho^{\mu}}{\mu})^{\alpha-1}\rho^{\mu-1}$ and integrating the resultant identity with respect to ρ from 0 to x bring out

$$\begin{aligned}&[{}^{\alpha}\Im^{\mu}\phi_2(x)-{}^{\alpha}\Im^{\mu}f(x)][{}^{\alpha}\Im^{\mu}f(x)-{}^{\alpha}\Im^{\mu}\phi_1(x)]+[{}^{\alpha}\Im^{\mu}\phi_2(x)-{}^{\alpha}\Im^{\mu}f(x)][{}^{\alpha}\Im^{\mu}f(x)-{}^{\alpha}\Im^{\mu}\phi_1(x)]\\&-{}^{\alpha}\Im^{\mu}[\phi_2(x)-f(x)][f(x)-\phi_1(x)]\frac{x^{\mu\alpha}}{\mu^{\alpha}\Gamma(\alpha+1)}-{}^{\alpha}\Im^{\mu}[\phi_2(\rho)-f(\rho)][f(\rho)-\phi_1(\rho)]\frac{x^{\mu\alpha}}{\mu^{\alpha}\Gamma(\alpha+1)}\\&=\frac{x^{\mu\alpha}}{\mu^{\alpha}\Gamma(\alpha+1)}{}^{\alpha}\Im^{\mu}f^2(x)+\frac{x^{\mu\alpha}}{\mu^{\alpha}\Gamma(\alpha+1)}{}^{\alpha}\Im^{\mu}f^2(x)-2{}^{\alpha}\Im^{\mu}f(x)^{\alpha}\Im^{\mu}f(x)+{}^{\alpha}\Im^{\mu}\phi_2(x)^{\alpha}\Im^{\mu}f(x)\end{aligned}$$

$$+{}^{\alpha}\Im^{\mu} f(x){}^{\alpha}\Im^{\mu}\phi_1(x) - {}^{\alpha}\Im^{\mu}\phi_2(x){}^{\alpha}\Im^{\mu}\phi_1(x) + {}^{\alpha}\Im^{\mu} f(x){}^{\alpha}\Im^{\mu}\phi_2(x) + {}^{\alpha}\Im^{\mu}\phi_1(x){}^{\alpha}\Im^{\mu} f(x)$$
$$-{}^{\alpha}\Im^{\mu}\phi_1(x){}^{\alpha}\Im^{\mu}\phi_2(x) - \frac{x^{\mu\alpha}}{\mu^{\alpha}\Gamma(\alpha+1)}{}^{\alpha}\Im^{\mu}\phi_2 f(x) + \frac{x^{\mu\alpha}}{\mu^{\alpha}\Gamma(\alpha+1)}{}^{\alpha}\Im^{\mu}\phi_1\phi_2(x) - \frac{x^{\mu\alpha}}{\mu^{\alpha}\Gamma(\alpha+1)}{}^{\alpha}\Im^{\mu}\phi_1 f(x)$$
$$-\frac{x^{\mu\alpha}}{\mu^{\alpha}\Gamma(\alpha+1)}{}^{\alpha}\Im^{\mu}\phi_2 f(x) + \frac{x^{\mu\alpha}}{\mu^{\alpha}\Gamma(\alpha+1)}{}^{\alpha}\Im^{\mu}\phi_1\phi_2(x) - \frac{x^{\mu\alpha}}{\mu^{\alpha}\Gamma(\alpha+1)}{}^{\alpha}\Im^{\mu}\phi_1 f(x)$$

which yields the required inequality (2.12). □

Corollary 2.4. *Let f be an integrable function on $[0,\infty)$ satisfying $m \le f(x) \le M$ for all $x \in [0,\infty)$. Then, for all $x, \alpha > 0$, we have*

$$\frac{x^{\mu\alpha}}{\mu^{\alpha}\Gamma(\alpha+1)}{}^{\alpha}\Im^{\mu} f^2(x) - [{}^{\alpha}\Im^{\mu} f(x)]^2$$
$$= \left[M\frac{x^{\mu\alpha}}{\mu^{\alpha}\Gamma(\alpha+1)} - {}^{\alpha}\Im^{\mu} f(x)\right]\left[{}^{\alpha}\Im^{\mu} f(x) - m\frac{x^{\mu\alpha}}{\mu^{\alpha}\Gamma(\alpha+1)}\right] - \frac{x^{\mu\alpha}}{\mu^{\alpha}\Gamma(\alpha+1)}{}^{\alpha}\Im^{\mu}[M - f(x)][f(x) - m].$$

Theorem 2.4. *Let $f, g, \phi_1, \phi_2, \psi_1$, and ψ_2 be integrable functions on $[0,\infty)$ satisfying (2.1) and (2.5) on $[0,\infty)$. Then, for all $x, \alpha > 0$, we have*

$$\left|\frac{x^{\mu\alpha}}{\mu^{\alpha}\Gamma(\alpha+1)}{}^{\alpha}\Im^{\mu} fg(x) - {}^{\alpha}\Im^{\mu} f(x){}^{\alpha}\Im^{\mu} g(x)\right| \le \sqrt{T(f,\phi_1,\phi_2)T(g,\psi_1,\psi_2)}, \tag{2.15}$$

where

$$T(u,v,w) = [{}^{\alpha}\Im^{\mu} w(x) - {}^{\alpha}\Im^{\mu} u(x)][{}^{\alpha}\Im^{\mu} u(x) - {}^{\alpha}\Im^{\mu} v(x)] + \frac{x^{\mu\alpha}}{\mu^{\alpha}\Gamma(\alpha+1)}{}^{\alpha}\Im^{\mu} vu(x) - {}^{\alpha}\Im^{\mu} v(x){}^{\alpha}\Im^{\mu} u(x)$$
$$+ \frac{x^{\mu\alpha}}{\mu^{\alpha}\Gamma(\alpha+1)}{}^{\alpha}\Im^{\mu} wu(x) - {}^{\alpha}\Im^{\mu} w(x){}^{\alpha}\Im^{\mu} u(x) + {}^{\alpha}\Im^{\mu} v(x){}^{\alpha}\Im^{\mu} w(x) - \frac{x^{\mu\alpha}}{\mu^{\alpha}\Gamma(\alpha+1)}{}^{\alpha}\Im^{\mu} vw(x).$$

Proof. Define

$$H(\tau,\rho) = [f(\tau) - f(\rho)][g(\tau) - g(\rho)], \quad \tau,\rho \in (0,x), \quad x > 0. \tag{2.16}$$

Multiplying both sides of (2.16) by $\frac{1}{\Gamma^2(\alpha)}(\frac{x^{\mu}-\tau^{\mu}}{\mu})^{\alpha-1}(\frac{x^{\mu}-\rho^{\mu}}{\mu})^{\alpha-1}\tau^{\mu-1}\rho^{\mu-1}$ for $\tau,\rho \in (0,x)$ and integrating the desired inequality with respect to τ,ρ from 0 to x yield

$$\frac{1}{2\Gamma^2(\alpha)}\int_0^t\int_0^t\left(\frac{x^{\mu}-\tau^{\mu}}{\mu}\right)^{\alpha-1}\left(\frac{x^{\mu}-\rho^{\mu}}{\mu}\right)^{\alpha-1}\tau^{\mu-1}\rho^{\mu-1}H(\tau,\rho)\,\mathrm{d}\tau\,\mathrm{d}\rho$$
$$= \frac{x^{\mu\alpha}}{\mu^{\alpha}\Gamma(\alpha+1)}{}^{\alpha}\Im^{\mu} fg(x) + {}^{\alpha}\Im^{\mu} f(x){}^{\alpha}\Im^{\mu} g(x). \tag{2.17}$$

Applying the Cauchy–Schwartz inequality to (2.17) leads to

$$\left[\frac{x^{\mu\alpha}}{\mu^{\alpha}\Gamma(\alpha+1)}{}^{\alpha}\Im^{\mu} fg(x) + {}^{\alpha}\Im^{\mu} f(x){}^{\alpha}\Im^{\mu} g(x)\right]^2$$
$$\le \left[\frac{x^{\mu\alpha}}{\mu^{\alpha}\Gamma(\alpha+1)}{}^{\alpha}\Im^{\mu} f^2(x) - [{}^{\alpha}\Im^{\mu} f(x)]^2\right]\left[\frac{x^{\mu\alpha}}{\mu^{\alpha}\Gamma(\alpha+1)}{}^{\alpha}\Im^{\mu} g^2(x) - [{}^{\alpha}\Im^{\mu} g(x)]^2\right]. \tag{2.18}$$

Since $[\phi_2(x) - f(x)][f(x) - \phi_1(x)] \geq 0$ and $[\psi_2(x) - g(x)][g(x) - \psi_1(x)] \geq 0$ for $x \in [0, \infty)$, we have

$$\frac{x^{\mu\alpha}}{\mu^\alpha\Gamma(\alpha+1)}[\phi_2(x) - f(x)][f(x) - \phi_1(x)] \geq 0 \quad \text{and} \quad \frac{x^{\mu\alpha}}{\mu^\alpha\Gamma(\alpha+1)}[\psi_2(x) - g(x)][g(x) - \psi_1(x)] \geq 0.$$

Thus, from Theorem 2.3, we obtain

$$\begin{aligned}\frac{x^{\mu\alpha}}{\mu^\alpha\Gamma(\alpha+1)}{}^\alpha\Im^\mu f^2(x) - [{}^\alpha\Im^\mu f(x)]^2 &\leq [{}^\alpha\Im^\mu \phi_2(x) - {}^\alpha\Im^\mu f(x)][{}^\alpha\Im^\mu f(x) - {}^\alpha\Im^\mu \phi_1(x)] \\ &+ \frac{x^{\mu\alpha}}{\mu^\alpha\Gamma(\alpha+1)}{}^\alpha\Im^\mu \phi_1 f(x) - {}^\alpha\Im^\mu \phi_1(x){}^\alpha\Im^\mu f(x) + \frac{x^{\mu\alpha}}{\mu^\alpha\Gamma(\alpha+1)}{}^\alpha\Im^\mu \phi_2 f(x) - {}^\alpha\Im^\mu \phi_2(x){}^\alpha\Im^\mu f(x) \\ &+ {}^\alpha\Im^\mu \phi_1(x){}^\alpha\Im^\mu \phi_2(x) - \frac{x^{\mu\alpha}}{\mu^\alpha\Gamma(\alpha+1)}{}^\alpha\Im^\mu \phi_1\phi_2(x) = T(f, \phi_1, \phi_2). \quad (2.19)\end{aligned}$$

Similarly, we have

$$\begin{aligned}\frac{x^{\mu\alpha}}{\mu^\alpha\Gamma(\alpha+1)}{}^\alpha\Im^\mu g^2(x) - [{}^\alpha\Im^\mu g(x)]^2 &\leq [{}^\alpha\Im^\mu \psi_2(x) - {}^\alpha\Im^\mu g(x)][{}^\alpha\Im^\mu g(x) - {}^\alpha\Im^\mu \psi_1(x)] \\ &+ \frac{x^{\mu\alpha}}{\mu^\alpha\Gamma(\alpha+1)}{}^\alpha\Im^\mu \psi_1 g(x) - {}^\alpha\Im^\mu \psi_1(x){}^\alpha\Im^\mu g(x) + \frac{x^{\mu\alpha}}{\mu^\alpha\Gamma(\alpha+1)}{}^\alpha\Im^\mu \psi_2 g(x) \\ &- {}^\alpha\Im^\mu \psi_2(x){}^\alpha\Im^\mu g(x) + {}^\alpha\Im^\mu \psi_1(x){}^\alpha\Im^\mu \psi_2(x) - \frac{x^{\mu\alpha}}{\mu^\alpha\Gamma(\alpha+1)}{}^\alpha\Im^\mu \psi_1\psi_2(x) = T(g, \psi_1, \psi_2). \quad (2.20)\end{aligned}$$

Combining (2.18), (2.19), and (2.20), we obtain the desired inequality (2.15). □

Remark 2.1. For $m, M, n, N \in \mathbb{R}$, if $T(f, \phi_1, \phi_2) = T(f, m, M)$ and $T(g, \psi_1, \psi_2) = T(g, n, N)$, then the inequality (2.15) reduces to

$$\left|\frac{x^{\mu\alpha}}{\mu^\alpha\Gamma(\alpha+1)}{}^\alpha\Im^\mu fg(x) - {}^\alpha\Im^\mu f(x){}^\alpha\Im^\mu g(x)\right| \leq \left[\frac{x^{\mu\alpha}}{2\mu^\alpha\Gamma(\alpha+1)}\right]^2 (M - m)(N - n).$$

in [20, Theorem 1].

Remark 2.2. In this paper, we presented some new conformable fractional integral inequalities which generalize those corresponding ones in [23].

Remark 2.3. This paper is a slightly revised version of the preprint [14].

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Conflict of interest

The authors declare that they have no conflict of interest.

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