Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 3(4): 514–523
DOI:10.3934/Math.2018.4.514
Received: 19 August 2018
Accepted: 11 October 2018
Published: 01 November 2018

Research article

Lower bounds for the blow-up time to a nonlinear viscoelastic wave equation with strong damping

Xiaoming Peng[1], **Xiaoxiao Zheng**[2] **and Yadong Shang**[3,*]

[1] School of Statistics and Mathematics, Guangdong University of Finance and Economics, Guangzhou 510320, China

[2] School of Mathematical Sciences, Qufu Normal University, Qufu 273155, Shandong, China

[3] School of Mathematics and Information Science, Guangzhou University, Guangzhou 510006, China

* **Correspondence:** Email: gzydshang@126.com.

Abstract: This paper deals with a nonlinear viscoelastic wave equation with strong damping. By the means of the interpolation inequalities and differential inequality technique, we obtain a lower bound for blow-up time of the solution. This result extends our earlier work Peng et al. [Appl. Math. Lett., 76, 2018].

Keywords: lower bound; blow up; viscoelastic; strong damping; memory
Mathematics Subject Classification: 35B44, 35L20

1. Introduction

In this paper, we study the following initial boundary problem

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds - \Delta u_t = |u|^{p-2}u, (x,t) \in \Omega \times [0,T), \\ u(x,t) = 0, \ \ x \in \partial\Omega, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with a smooth boundary $\partial\Omega$.

It is well known that viscoelastic materials present a natural damping, which is due to some properties of these materials to keep memory of their past trace. This type of equations with viscoelastic term describe a variety of important physical processes [1] and the reference therein. There is a vast literature on the existence or nonexistence of global solutions, blow up results in finite time, and the asymptotic behavior of the solutions for the viscoelastic equations, we refer the interested readers to [2–11] and the references therein. In particular, Song and Zhong [5] studied problem (1.1). They established a blow-up result for solutions with positive initial energy. Later, Song and Xue [6] extended this blow up result to solutions whose initial data have arbitrarily high initial energy.

Since Payne et al. [12, 13] applied a differential inequality technique to obtain a lower bound on blow-up time for solutions of the semilinear heat equation. Many authors have given attention to this problem and obtained many profound results [14–18] and the references therein. However, there seems to have been little work devoted to obtaining lower bounds on blow-up time to solutions of viscoelastic problems. To our best knowledge, only few articles dealt with this questions, see [19–21]. Yang et al. [19] established a lower bound for the blow-up time of the following equation

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds - |u_t|^{m-2}u_t = |u|^{p-2}u.$$

Tian [20] considered a semilinear parabolic equation with viscoelastic term

$$u_t - \Delta u + \int_0^t g(t-s)\Delta u(s)ds = |u|^{p-2}u.$$

By the means of differential inequality technique, they obtained a lower bound for blow-up time of the solution. Recently, Peng et al. [21] obtained a lower bound for the blow-up time to problem (1.1) by establishing a differential inequality. But they can only derived a lower bounds for blow up time t^* when $2 < p \leqslant \frac{2(n^2-2)}{n(n-2)}$. Compared with the condition of blow up result for p in [6], there exists a gap for p between $\frac{2(n^2-2)}{n(n-2)}$ and $\frac{2n}{n-2}$. It is still open whether a lower bound estimate can be obtained if p lies in this gap. Inspired by [18, 20], the goal of this paper is to gives an answer to the problem unsolved in our earlier work Peng et. al [21]. By introducing a new auxiliary functional and using interpolation inequalities, we obtain lower bounds for the blow-up time for the problem (1.1).

2. Main results

Throughout the paper, we use $\|\cdot\|$ to denote the L^p- norm for $1 \leqslant p \leqslant \infty$. Before stating our main results, let us recall the results on the local existence, uniqueness and blow-up in finite time of solutions to (1.1).

Theorem 2.1 ([6]). *Let $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ be given. Let g be a C^1 function satisfying*

$$1 - \int_0^\infty g(s)ds = l > 0. \tag{2.1}$$

Let p be such that

$$\begin{cases} 2 < p < \infty & n = 1, 2, \\ 2 < p \leqslant \dfrac{2n}{n-2} & n \geqslant 3. \end{cases} \tag{2.2}$$

Then problem (1.1) *has a unique local solution*

$$u \in C([0, T_m); H_0^1(\Omega)),\ u_t \in C([0, T_m); L^2(\Omega)) \cap L^2([0, T_m); H_0^1(\Omega)),$$

for some $T_m > 0$.

Define the energy functional $E(t)$ associated to the problem (1.1)

$$E(t)=\frac{1}{2}\|u_t\|_2^2+\frac{1}{2}\Big(1-\int_0^t g(s)ds\Big)\|\nabla u\|_2^2+\frac{1}{2}(g\circ\nabla u)(t)-\frac{1}{p}\|u\|_p^p,$$

where

$$(g\circ v)(t)=\int_0^t g(t-s)\|v(t)-v(s)\|_2^2ds.$$

Theorem 2.2 ([6]). *Assume that $p>2$ satisfies (2.2) and let g be a C^1 function satisfying*

$$g(s)\geqslant 0,\quad g'(s)\leqslant 0,\quad \int_0^\infty g(s)ds<1-\frac{1}{(p-1)^2}. \tag{2.3}$$

Let $u(t)$ be a solution of problem (1.1) satisfying

$$\Big(2\int_\Omega uu_tdx+\|\nabla u(t)\|_2^2\Big)|_{t=0}>\frac{2p}{\kappa}E(0), \tag{2.4}$$

then $u(t)$ blow up in finite time, where

$$\kappa=\max_{\eta_1\in(0,1)}\kappa(\eta_1)=\kappa(\eta^*),$$
$$\kappa(\eta_1)=\min(\sqrt{(p+2)\delta\eta_1\lambda_1},\delta(1-\eta_1)),$$

λ_1 being the first eigenvalue of $-\Delta$, $\delta=(p-2)l-\frac{1}{p}(1-l)$, η^* is the root of the equation $\sqrt{(p+2)\delta\eta_1\lambda_1}=\delta(1-\eta_1)$.

Let us introduce an auxiliary function

$$\varphi(t)=\frac{1}{2}\|u_t\|_2^2+\frac{1}{2}\Big(1-\int_0^t g(s)ds\Big)\|\nabla u\|_2^2+\frac{1}{2}(g\circ\nabla u)(t)+\frac{1}{p}\|u\|_p^p, \tag{2.5}$$

with

$$\varphi(0)=\frac{1}{2}\|u_1(x)\|_2^2+\frac{1}{2}\|\nabla u_0(x)\|_2^2+\frac{1}{p}\|u_0(x)\|_p^p. \tag{2.6}$$

Theorem 2.3. *Under the conditions (2.3) and (2.4), assume p satisfy*

$$\begin{cases}2<p<\infty, & n=1,2,\\ 2<p<\dfrac{2n}{n-2}, & n\geqslant 3,\end{cases}$$

then the solution $u(x,t)$ of problem (1.1) blows up in finite time t^*.

(1) *If $n\geqslant 3$, then t^* is bounded below by*

$$t^*\geqslant\frac{2n-np+4p}{2K_1(p-2)}[\varphi(0)]^{\frac{4-2p}{2n-np+4p}},$$

where K_1 is given in (2.22).

(2) *If* $n = 1$, *then* t^* *is bounded below by*

$$t^* \geqslant \frac{2(p-1)}{K_2(p-2)}[\varphi(0)]^{\frac{2-p}{2(p-1)}},$$

where K_2 *is given in* (2.29).

(3) *If* $n = 2$, *then* t^* *is bounded below by*

$$t^* \geqslant \frac{p-2}{K_3(p+2)}[\varphi(0)]^{\frac{2-p}{p+2}}.$$

where K_3 *is given in* (2.34).

Proof. According to Theorem 2.2, the solution $u(x,t)$ of (1.1) blows up in a finite time t^*. Besides, Song and Xue [6] proved that

$$\lim_{t\to t^*}\left[\|u_t\|_2^2 + \left(1+\frac{1}{\lambda_1}\right)\|\nabla u\|_2^2\right] = +\infty,$$

which implies that

$$\lim_{t\to t^*}\varphi(t) = +\infty. \tag{2.7}$$

Multiplying Eq. (1.1) by u_t and integrating over Ω yields

$$\frac{1}{2}\frac{d}{dt}\left\{\int_\Omega |u_t|^2dx + \int_\Omega |\nabla u|^2dx\right\} = \int_0^t g(t-s)\int_\Omega \nabla u_t\cdot\nabla u dxds - \int_\Omega |\nabla u_t|^2dx + \int_\Omega |u|^{p-2}uu_tdx. \tag{2.8}$$

For the first term on the right-hand side of (2.8), we have

$$\begin{aligned}\int_0^t g(t-s)\int_\Omega \nabla u_t\cdot\nabla u dxds =&\frac{1}{2}\frac{d}{dt}\left\{\int_0^t g(s)ds\int_\Omega |\nabla u(t)|^2dx - \int_0^t g(t-s)ds\int_\Omega |\nabla u(s)-\nabla u(t)|^2dx\right\}\\ &-\frac{1}{2}g(t)\int_\Omega |\nabla u(t)|^2dx + \frac{1}{2}\int_0^t g'(t-s)ds\int_\Omega |\nabla u(s)-\nabla u(t)|^2dx.\end{aligned} \tag{2.9}$$

Inserting (2.9) into (2.8) gives

$$\begin{aligned}&\frac{d}{dt}\left\{\frac{1}{2}\int_\Omega |u_t|^2dx + \frac{1}{2}\left(1-\int_0^t g(s)ds\right)\int_\Omega |\nabla u|^2dx + \frac{1}{2}(g\circ\nabla u)(t) + \frac{1}{p}\int_\Omega |u|^pdx\right\} = -\int_\Omega |\nabla u_t(t)|^2dx\\ &\quad+\frac{1}{2}\int_0^t g'(t-s)\int_\Omega |\nabla u(s)-\nabla u(t)|^2dxds - \frac{1}{2}g(t)\int_\Omega |\nabla u(t)|^2dx + 2\int_\Omega |u|^{p-2}uu_tdx.\end{aligned}$$

From (2.5), the above identity can be rewritten as

$$\varphi'(t) = -\|\nabla u_t(t)\|^2 - \frac{1}{2}g(t)\|\nabla u(t)\|^2 + \frac{1}{2}(g'\circ\nabla u)(t) + 2\int_\Omega |u|^{p-2}uu_tdx. \tag{2.10}$$

Since $g'(s)\leqslant 0$ and $g(s)\geqslant 0$, it follows from (2.10) that

$$\varphi'(t) \leqslant -\|\nabla u_t(t)\|^2 + 2\int_\Omega |u|^{p-2}uu_tdx.$$

Using Hölder inequality, we have

$$\varphi'(t) \leqslant -\|\nabla u_t(t)\|^2 + 2\|u\|_p^{p-1}\|u_t\|_p. \tag{2.11}$$

Next, we are going to estimate the second term on the right-hand side of (2.11).

Firstly, we consider the case $n \geqslant 3$. Using interpolation inequality yields

$$\|u_t\|_p \leqslant \|u_t\|_2^{\frac{2n-p(n-2)}{2p}} \|u_t\|_{\frac{2n}{n-2}}^{\frac{n(p-2)}{2p}}. \tag{2.12}$$

For any $\varepsilon > 0, r, s, \theta > 1$, we have the following Young inequality

$$abc \leqslant \frac{\varepsilon}{r}a^r + \frac{\varepsilon^{-\frac{s}{2r}}}{s}b^s + \frac{\varepsilon^{-\frac{\theta}{2r}}}{\theta}c^\theta,\ \frac{1}{r} + \frac{1}{s} + \frac{1}{\theta} = 1. \tag{2.13}$$

Combing (2.12) with (2.13) gives

$$\begin{aligned} 2\|u\|_p^{p-1}\|u_t\|_p &\leqslant 2\|u\|_p^{p-1}\|u_t\|_2^{\frac{2n-p(n-2)}{2p}} \|u_t\|_{\frac{2n}{n-2}}^{\frac{n(p-2)}{2p}} \\ &\leqslant \frac{\varepsilon}{r}\|u_t\|_{\frac{2n}{n-2}}^2 + \frac{\varepsilon^{-\frac{s}{2r}}}{s}\|u_t\|_2^{\frac{2n-p(n-2)}{2p}s} + \frac{\varepsilon^{-\frac{\theta}{2r}}2^\theta}{\theta}\|u\|_p^{\theta(p-1)}, \end{aligned} \tag{2.14}$$

with

$$\begin{aligned} r &= \frac{4p}{n(p-2)} > 1, \\ s &= \frac{4p(2n-np+6p-4)}{(2n-np+2p)(2n-np+4p)} > \frac{4p}{2n-np+2p} > \frac{4p}{2p} = 2, \\ \theta &= \frac{2n-np+2p}{4(p-1)}s = \frac{p(2n-np+6p-4)}{(p-1)(2n-np+4p)} > \frac{2n-np+6p-4}{2n-np+4p} > 1. \end{aligned}$$

Applying Sobolev inequality to the first term on the right-hand side of (2.13), we have

$$\|u_t\|_{\frac{2n}{n-2}}^2 \leqslant C_1^2\|\nabla u_t\|_2^2, \tag{2.15}$$

where C_1 is the best constant of the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{\frac{2n}{n-2}}(\Omega)$.

Recalling (2.5), we have

$$\frac{1}{p}\|u\|_p^p \leqslant \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\Big(1 - \int_0^t g(s)ds\Big)\|\nabla u\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t) + \frac{1}{p}\|u\|_p^p = \varphi(t), \tag{2.16}$$

$$\frac{1}{2}\|u_t\|_2^2 \leqslant \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\Big(1 - \int_0^t g(s)ds\Big)\|\nabla u\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t) + \frac{1}{p}\|u\|_p^p = \varphi(t). \tag{2.17}$$

Plugging (2.15)–(2.17) into (2.14), it follows that

$$\begin{aligned} 2\|u\|_p^{p-1}\|u_t\|_p &\leqslant \frac{\varepsilon C_1^2}{r}\|\nabla u_t\|_2^2 + \frac{\varepsilon^{-\frac{s}{2r}}}{s}\|u_t\|_2^{\frac{2n-p(n-2)}{2p}s} + \frac{\varepsilon^{-\frac{\theta}{2r}}2^\theta}{\theta}\|u\|_p^{\theta(p-1)} \\ &\leqslant \frac{\varepsilon C_1^2}{r}\|\nabla u_t\|_2^2 + \frac{\varepsilon^{-\frac{s}{2r}}}{s}2^{\frac{(2n-np+2p)s}{4p}}[\varphi(t)]^{\frac{(2n-np+2p)s}{4p}} + \frac{\varepsilon^{-\frac{\theta}{2r}}2^\theta}{\theta}p^{\frac{\theta(p-1)}{p}}[\varphi(t)]^{\frac{\theta(p-1)}{p}}. \end{aligned} \tag{2.18}$$

Noting that

$$\frac{(2n-np+2p)s}{4p}=\frac{\theta(p-1)}{p}=\frac{2n-np+6p-4}{2n-np+4p}>1,$$

(2.18) can be rewritten as

$$2\|u\|_p^{p-1}\|u_t\|_p \leqslant \frac{\varepsilon C_1^2}{r}\|\nabla u_t\|_2^2+\left[\frac{2^\gamma}{s}\varepsilon^{-\frac{s}{2r}}+\frac{p^\gamma 2^\theta}{\theta}\varepsilon^{-\frac{\theta}{2r}}\right][\varphi(t)]^\gamma. \tag{2.19}$$

where

$$\gamma=\frac{2n-np+6p-4}{2n-np+4p}>1. \tag{2.20}$$

Inserting (2.19) into (2.11), we obtain

$$\varphi'(t)\leqslant\left(\frac{\varepsilon C_1^2}{r}-1\right)\|\nabla u_t\|_2^2+\left[\frac{2^\gamma}{s}\varepsilon^{-\frac{s}{2r}}+\frac{p^\gamma 2^\theta}{\theta}\varepsilon^{-\frac{\theta}{2r}}\right][\varphi(t)]^\gamma. \tag{2.21}$$

Taking $\varepsilon=\frac{4p}{n(p-2)C_1^2}$ in (2.21) leads to

$$\varphi'(t)\leqslant K_1[\varphi(t)]^\gamma. \tag{2.22}$$

where

$$K_1=\frac{2n-np+2p}{p\gamma 2^{2-\gamma}}\left[\frac{n(p-2)C_1^2}{4p}\right]^{\frac{n\gamma(p-2)}{2n-np+2p}}+2^{\frac{p\gamma}{p-1}}\frac{p-1}{\gamma p^{1-\gamma}}\left[\frac{n(p-2)C_1^2}{4p}\right]^{\frac{n\gamma(p-2)}{8(p-1)}}. \tag{2.23}$$

Integrating (2.22) from 0 to t results in

$$\frac{1}{1-\gamma}\left\{[\varphi(t)]^{1-\gamma}-[\varphi(0)]^{1-\gamma}\right\}\leqslant K_1 t. \tag{2.24}$$

Thus, letting $t\to t^*$ and taking into account (2.7), we have the lower bound for t^*

$$t^*\geqslant\frac{1}{K_1(\gamma-1)}[\varphi(0)]^{1-\gamma}=\frac{2n-np+4p}{2K_1(p-2)}[\varphi(0)]^{\frac{2p-4}{2n-np+4p}}.$$

Next, we continue to estimate (2.11) for the case $n=1$. Using Hölder inequality and Sobolev inequality, we have

$$\|u_t\|_p^p\leqslant\|u_t\|_2^{p-2}\|u_t\|_\infty^2\leqslant\|u_t\|_2^{p-2}(C_2\|\nabla u_t\|_2)^2, \tag{2.25}$$

where C_2 is the best constant of the Sobolev embedding $H_0^1(\Omega)\hookrightarrow L^\infty(\Omega)$.

Using again (2.13), we arrive at

$$\begin{aligned}2\|u\|_p^{p-1}\|u_t\|_p&\leqslant 2C_2^{\frac{2}{p}}\|u\|_p^{p-1}\|\nabla u_t\|_2^{\frac{p-2}{p}}\|u_t\|_2^{\frac{2p-np+2n}{2p}}\\&\leqslant\frac{\varepsilon}{p}\|\nabla u_t\|_2^2+\frac{\varepsilon^{-\frac{s}{2r}}}{s}\|u_t\|_2^{\frac{(2p-np+2n)s}{2p}}+\frac{\varepsilon^{-\frac{\theta}{2r}}}{\theta}2^\theta C_2^\theta\|u\|_p^{\theta(p-1)},\end{aligned} \tag{2.26}$$

with

$$r=p>2,$$

$$s = \frac{p(3p-4)}{(p-1)(p-2)} = \frac{2p(p-2)+p^2}{(p-1)(p-2)} > \frac{2p}{p-1} = 2,$$
$$\theta = \frac{p(3p-4)}{2(p-1)^2}s = \frac{3(p-1)^2+2p-3}{2(p-1)^2} > \frac{3}{2}.$$

Combining (2.11), (2.16), (2.17) with (2.26) yields

$$\begin{aligned}\varphi'(t) &\leqslant (\frac{\varepsilon}{p}-1)\|\nabla u_t\|_2^2 + \frac{\varepsilon^{-\frac{s}{2r}}}{s}\|u_t\|_2^{\frac{(p-2)s}{p}} + \frac{\varepsilon^{-\frac{\theta}{2r}}}{\theta}2^{\theta}C_2^{\frac{2\theta}{p}}\|u\|_p^{\theta(p-1)} \\ &\leqslant (\frac{\varepsilon}{p}-1)\|\nabla u_t\|_2^2 + \left[\frac{\varepsilon^{-\frac{s}{2r}}}{s}2^{\frac{(p-2)s}{2p}} + \frac{\varepsilon^{-\frac{\theta}{2r}}}{\theta}2^{\theta}C_2^{\frac{2\theta}{p}}p^{\frac{\theta(p-1)}{p}}\right][\varphi(t)]^{\frac{3p-4}{2(p-1)}}.\end{aligned} \tag{2.27}$$

Taking $\varepsilon = p$ in (2.27), we have

$$\varphi'(t) \leqslant K_2[\varphi(t)]^{\frac{3p-4}{2(p-1)}}, \tag{2.28}$$

where

$$K_2 = \frac{(p-1)(p-2)}{p(3p-4)}(p2^{p-2})^{\frac{3p-4}{2(p-1)(p-2)}}\left[1+\frac{2(p-1)}{p-2}p^{\frac{2p^2-9p+8}{2(p-1)(p-2)}}(2C_2)^{\frac{3p-4}{2(p-1)^2}}\right]. \tag{2.29}$$

Noting that $\frac{3p-4}{2(p-1)} > 1$ and integrating (2.28) from 0 to t^* results in

$$\frac{2(p-1)}{p-2}[\varphi(0)]^{\frac{2-p}{2(p-1)}} \leqslant K_2 t^*,$$

which implies that

$$t^* \geqslant \frac{2(p-1)}{K_2(p-2)}[\varphi(0)]^{\frac{2-p}{2(p-1)}}.$$

Finally, we estimate (2.11) for the case $n = 2$. Using interpolation theorem [22], we have

$$\|u_t\|_p \leqslant N\|\nabla u_t\|_2^{\frac{p-2}{p}}\|u_t\|_2^{\frac{2}{p}}, \tag{2.30}$$

where K is an embedding constant.

Using again (2.13), we arrive at

$$\begin{aligned}2\|u\|_p^{p-1}\|u_t\|_p &\leqslant 2N\|u\|_p^{p-1}\|\nabla u_t\|_2^{\frac{p-2}{p}}\|u_t\|_2^{\frac{2}{p}} \\ &\leqslant \frac{\varepsilon}{r}\|\nabla u_t\|_2^{\frac{(p-2)r}{p}} + \frac{\varepsilon^{-\frac{s}{2r}}}{s}\|u_t\|_2^{\frac{2s}{p}} + \frac{\varepsilon^{-\frac{\theta}{2r}}}{\theta}2^{\theta}N^{\theta}\|u\|_p^{\theta(p-1)}, \\ &= \frac{(p-2)\varepsilon}{2p}\|\nabla u_t\|_2^2 + \frac{\varepsilon^{-\frac{s}{2r}}}{s}\|u_t\|_2^{\frac{4p}{p+2}} + \frac{\varepsilon^{-\frac{\theta}{2r}}}{\theta}2^{\theta}N^{\theta}\|u\|_p^{\frac{2p^2}{p+2}},\end{aligned} \tag{2.31}$$

with

$$r = \frac{2p}{p-2} > 2,\ s = \frac{2p^2}{p+2} > 2,\ \theta = \frac{2p^2}{(p-1)(p+2)} > \frac{2p}{p+2} > 1.$$

Combining (2.11), (2.16), (2.17) with (2.31) yields

$$\varphi'(t) \leqslant \left[\frac{(p-2)\varepsilon}{2p}-1\right]\|\nabla u_t\|_2^2 + \frac{\varepsilon^{-\frac{s}{2r}}}{s}\|u_t\|_2^{\frac{4p}{p+2}} + \frac{\varepsilon^{-\frac{\theta}{2r}}}{\theta}2^{\theta}K^{\theta}\|u\|_p^{\frac{2p^2}{p+2}}$$

$$\leqslant \left[\frac{(p-2)\varepsilon}{2p}-1\right]\|\nabla u_t\|_2^2+\left[\frac{\varepsilon^{-\frac{s}{2r}}}{s}2^{\frac{2p}{p+2}}+\frac{\varepsilon^{-\frac{\theta}{2r}}}{\theta}2^{\theta}N^{\theta}p^{\frac{2p}{p+2}}\right][\varphi(t)]^{\frac{2p}{p+2}}. \tag{2.32}$$

Taking $\varepsilon=\frac{2p}{p-2}$ in (2.32), we have

$$\varphi'(t)\leqslant K_3[\varphi(t)]^{\frac{2p}{p+2}}, \tag{2.33}$$

where

$$K_3=\frac{p+2}{p^2}\left(\frac{2p}{p-2}\right)^{\frac{-p(p-2)}{2(p+2)}}2^{\frac{p-2}{p+2}}\left[1+(p-1)\left(\frac{2p}{p-2}\right)^{\frac{p(p-2)^2}{2(p-1)(p+2)}}2^{\frac{2p}{(p-1)(p+2)}}N^{\frac{2p^2}{(p-1)(p+2)}}p^{\frac{2p}{p+2}}\right]. \tag{2.34}$$

Noting that $\frac{2p}{p+2}>1$ and integrating (2.33) from 0 to t^* results in

$$\frac{p-2}{p+2}[\varphi(0)]^{\frac{2-p}{p+2}}\leqslant K_3t^*,$$

which implies that

$$t^*\geqslant\frac{p-2}{K_3(p+2)}[\varphi(0)]^{\frac{2-p}{p+2}}.$$

The proof is complete. □

Remark 1. From the proof of (2.14), we observe that it is clear that $2n-np+2p=0$ when $p=\frac{2n}{n-2}$ for $n\geqslant 3$. In this case, the inequality (2.14) doesn't hold. Thus we need to develop new ideas to restructure this inequality.

Theorem 2.4. *Let $\varphi(t)$ and $\varphi(0)$ be defined in* (2.5) *and* (2.6). *Suppose that the conditions of Theorem 2.2 hold. Then the solution of* (1.1) *blows up in finite time t^*, which is bounded below by*

$$t^*\geqslant\int_{\varphi(0)}^{\infty}\frac{d\eta}{M_1\eta^{\frac{\alpha(p-1)}{(\alpha-1)p}}+M_2},$$

where

$$1<\alpha<2,\quad M_1=\frac{\alpha}{\alpha-1}2^{\frac{\alpha}{\alpha-1}}\alpha^{-\frac{1}{\alpha-1}}p^{\frac{\alpha(p-1)}{p(\alpha-1)}},\quad M_2=\frac{2}{2-\alpha}(\frac{\alpha}{2})^{\frac{\alpha}{2-\alpha}}B_s^{\frac{2\alpha}{2-\alpha}}$$

and B_s is the best constant of the Sobolev embedding $H_0^1(\Omega)\hookrightarrow L^p(\Omega)$.

Proof. As already mentioned, going back to (2.11), we need to estimate the second term on the right hand side of (2.11). In what follows, we are going to estimate it in a different way.

For any $\varepsilon>0, r>1, s>1$, we have the following known Young inequality

$$ab\leqslant\frac{\varepsilon}{r}a^r+\frac{\varepsilon^{-\frac{s}{r}}}{s}b^s,\quad \frac{1}{r}+\frac{1}{s}=1. \tag{2.35}$$

By means of the inequality (2.35) with $r=\alpha; s=\frac{\alpha}{\alpha-1}, \varepsilon=\alpha$, it follows that

$$2\|u\|_p^{p-1}\|u_t\|_p\leqslant C_3(\|u\|_p^p)^{\frac{\alpha(p-1)}{p(\alpha-1)}}+\|u_t\|_p^{\alpha}, \tag{2.36}$$

where $C_3 = \frac{\alpha}{\alpha-1} 2^{\frac{\alpha}{\alpha-1}} \alpha^{-\frac{1}{\alpha-1}}$.

We now focus our attention on the second term on the right in (2.36). Since $1 < \alpha < 2$, using Sobolev inequality and (2.35) with $r = \frac{2}{\alpha}$, $s = \frac{2}{2-\alpha}$, $\varepsilon = \frac{2}{\alpha}$, we arrive at

$$\|u_t\|_p^\alpha \leqslant B_s^\alpha \|\nabla u_t\|_2^\alpha \leqslant \|\nabla u_t\|_2^2 + M_2, \tag{2.37}$$

where $M_2 = \frac{2}{2-\alpha}(\frac{\alpha}{2})^{\frac{\alpha}{2-\alpha}} B_s^{\frac{2\alpha}{2-\alpha}}$, B_s is the best constant of the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$.

Inserting (2.37) into (2.36) yields

$$2\|u\|_p^{p-1}\|u_t\|_p \leqslant C_3(\|u\|_p^p)^{\frac{\alpha(p-1)}{p(\alpha-1)}} + \|\nabla u_t\|_2^2 + M_2. \tag{2.38}$$

Combining (2.11), (2.16) with (2.38), we get

$$\varphi'(t) \leqslant M_1[\varphi(t)]^{\frac{\alpha(p-1)}{p(\alpha-1)}} + M_2, \tag{2.39}$$

where $M_1 = C_3 p^{\frac{\alpha(p-1)}{p(\alpha-1)}}$.

Integrating (2.39) from 0 to t yields

$$\int_{\varphi(0)}^{\varphi(t)} \frac{d\eta}{M_1 \eta^{\frac{\alpha(p-1)}{p(\alpha-1)}} + M_2} \leqslant t,$$

from which we deduce a lower bound for t^*, namely,

$$\int_{\varphi(0)}^{\infty} \frac{d\eta}{M_1 \eta^{\frac{\alpha(p-1)}{p(\alpha-1)}} + M_2} \leqslant t^*.$$

The proof is complete. □

Conflict of interest

The authors declare that there are no conflicts of interest in this paper.

References

1. A. B. Al'shin, M. O. Korpusov, A. G. Siveshnikov, *Blow up in nonlinear Sobolev type equations*, Series in Nonlinear Analysis and Applications, Vol. 15, Berlin: De Gruyter, 2011.
2. S. A. Messaoudi, *Blow up and global existence in a nonlinear viscoelastic wave equation*, Math. Nachr., **260** (2003), 58–66.
3. S. A. Messaoudi, *Blow-up of positive-initial-energy solutions of a nonlinear viscoelastic hyperbolic equation*, J. Math. Anal. Appl., **320** (2006), 902–915.
4. S. T. Wu, *Blow-up of solutions for an integro-differential equation with a nonlinear source*, Electron. J. Differ. Eq., **45** (2006), 1–9.
5. H. T. Song, C. K. Zhong, *Blow-up of solutions of a nonlinear viscoelastic wave equation*, Nonlinear Analysis: Real World Applications, **11** (2010), 3877–3883.

6. H. T. Song, D. X. Xue, *Blow up in a nonlinear viscoelastic wave equation with strong damping*, Nonlinear Analysis: Theory, Methods & Applications, **109** (2014), 245–251.
7. F. S. Li, C. L. Zhao, *Uniform energy decay rates for nonlinear viscoelastic wave equation with nonlocal boundary damping*, Nonlinear Analysis: Theory, Methods & Applications, **74** (2011), 3468–3477.
8. F. S. Li, Z. Q. Zhao, Y. F. Chen, *Global existence uniqueness and decay estimates for nonlinear viscoelastic wave equation with boundary dissipation*, Nonlinear Analysis: Real World Applications, **12** (2011), 1759–1773.
9. W. Liu, Y. Sun, G. Li, *On decay and blow-up of solutions for a singular nonlocal viscoelastic problem with a nonlinear source term*, Topol. Method Nonl. An., **49** (2017), 299–323.
10. W. Liu, D. Wang, D. Chen, *General decay of solution for a transmission problem in infinite memory-type thermoelasticity with second sound*, J. Therm. Stresses, **41** (2018), 758–775.
11. S. H. Park, M. J. Lee, J. R. Kang, *Blow-up results for viscoelastic wave equations with weak damping*, Appl. Math. Lett., **80** (2018), 20–26.
12. L. E. Payne, P. W. Schaefer, *Lower bounds for blow-up time in parabolic problems under Dirichlet conditions*, J. Math. Anal. Appl., **328** (2007), 1196–1205.
13. L. E. Payne, P. W. Schaefer, *Lower bounds for blow-up time in parabolic problems under Neumann conditions*, J. Math. Appl. Anal., **85** (2006), 1301–1311.
14. L. L. Sun, B. Guo, W. J. Gao, *A lower bound for the blow-up time to a damped semilinear wave equation*, Appl. Math. Lett., **37** (2014), 22–25.
15. G. A. Philippin, *Lower bounds for blow-up time in a class of nonlinear wave equations*, Z. Angew. Math. Phys., **66** (2014), 129–134.
16. G. A. Philippin, S.Vernier Piro, *Lower bound for the lifespan of solutions for a class of fourth order wave equations*, Appl. Math. Lett., **50** (2015), 141–145.
17. B. Guo, F. Liu, *A lower bound for the blow-up time to a viscoelastic hyperbolic equation with nonlinear sources*, Appl. Math. Lett., **60** (2016), 115–119.
18. K. Baghaei, *Lower bounds for the blow-up time in a superlinear hyperbolic equation with linear damping term*, Comput. Math. Appl., **73** (2017), 560–564.
19. L. Yang, F. Liang, Z. H. Guo, *Lower bounds for blow-up time of a nonlinear viscoelastic wave equation*, Bound. Value Probl., **2015** (2015), 219.
20. S. Y. Tian, *Bounds for blow-up time in a semilinear parabolic problem with viscoelastic term*, Comput. Math. Appl., **74** (2017), 736–743.
21. X. M. Peng, Y. D. Shang, X. X. Zheng, *Lower bounds for the blow-up time to a nonlinear viscoelastic wave equation with strong damping*, Appl. Math. Lett., **76** (2018), 66–73.
22. R. A. Adams, J. J. F. Fournier, *Sobolev Spaces*, 2Eds., New York: Academic Press, 2003.

AIMS Press