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Research article

A semilnear singular problem for the fractional laplacian

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Abstract: We study the problem $(-\Delta)^s u = -au^{-\gamma} + \lambda h$ in Ω, $u = 0$ in $\mathbb{R}^n \setminus \Omega$, $u > 0$ in Ω, where $0 < s < 1$, Ω is a bounded domain in $\mathbb{R}^n$ with $C^{1,1}$ boundary, a and h are nonnegative bounded functions, $h \not\equiv 0$, and $\lambda > 0$. We prove that if $\gamma \in (0, s)$ then, for λ positive and large enough, there exists a weak solution such that $c_1 d_\Omega^s \leq u \leq c_2 d_\Omega^s$ in Ω for some positive constants c_1 and c_2. A somewhat more general result is also given.

Keywords: singular elliptic problems; positive solutions; fractional Laplacian; sub and supersolutions
Mathematics Subject Classification: Primary 35A15; Secondary 35S15, 47G20, 46E35

1. Introduction and statement of the main results

Elliptic problems with singular nonlinearities appear in many nonlinear phenomena, for instance, in the study of chemical catalysts process, non-Newtonian fluids, and in the study of the temperature of electrical conductors whose resistance depends on the temperature (see e.g., [3, 6, 10, 15] and the references therein). The seminal work [7] is the start point of a large literature concerning singular elliptic problems, see for instance, [1, 3, 5, 6, 8, 9, 10, 13, 15, 17, 18, 21, 22, 23, 24], and [30]. For additional references and a systematic study of singular elliptic problems see also [26].

In [10], Diaz, Morel and Oswald considered problems of the form

$$\begin{cases} -\Delta u = -u^{-\gamma} + \lambda h(x) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \\ u > 0 \text{ in } \Omega \end{cases} \tag{1.1}$$

where Ω is a bounded and regular enough domain, $0 < \gamma < 1$, $\lambda > 0$ and $h \in L^\infty(\Omega)$ is a nonnegative and nonidentically zero function. They proved (see [10], Theorem 1, Corollary 1, Lemma 2 and

Theorem 3) that there exists $\lambda_0 > 0$ such that. for $\lambda > \lambda_0$, problem (1.1) has a unique maximal solution $u \in H_0^1(\Omega)$ and has no solution when $\lambda < \lambda_0$.

Concerning nonlocal singular problems, Barrios, De Bonis, Medina, and Peral proved in [2] that if Ω is a bounded and regular enough domain in $\mathbb{R}^n$, $0 < s < 1$, $n > 2s$, f is a nonnegative function in a suitable Lebesgue space, $\lambda > 0$, $M > 0$ and $1 < p < \frac{n+2s}{n-2s}$, then the problem

$$\begin{cases} (-\Delta)^s u = \lambda f(x) u^{-\gamma} + Mu^p \text{ in } \Omega, \\ u = 0 \text{ on } \mathbb{R}^n \setminus \Omega, \\ u > 0 \text{ in } \Omega, \end{cases} \tag{1.2}$$

has a solution, in a suitable weak sense whenever $\lambda > 0$ and $M > 0$, and that, if $M = 1$ and $f = 1$, then there exists $\Lambda > 0$ such that (1.2) has at least two solutions when $\lambda < \Lambda$ and has no solution when $\lambda > \Lambda$.

A natural question is to ask if an analogous of the quoted result of [10] hold in the nonlocal case, i.e., when $-\Delta$ is replaced by the fractional laplacian $(-\Delta)^s$, $s \in (0,1)$, and with the boundary condition $u = 0$ on $\partial\Omega$ replaced by $u = 0$ on $\mathbb{R}^n \setminus \Omega$. Our aim in this paper is to obtain such a result. Note that the approach of [10] need to be modified in order to be used in the fractional case. Indeed, a step in [10] was to observe that, if φ_1 denotes a positive principal eigenfunction for $-\Delta$ on Ω, with Dirichlet boundary condition, then

$$-\Delta\varphi_1^{\frac{2}{1+\gamma}} = \frac{2}{1+\gamma}\lambda_1\varphi_1^{\frac{2}{1+\gamma}} - \frac{2(1-\gamma)}{(\gamma+1)^2}|\nabla\varphi_1|^2\varphi_1^{-\frac{2\gamma}{1+\gamma}} \text{ in } \Omega, \tag{1.3}$$

where λ_1 is the corresponding principal eigenvalue. From this fact, and using the properties of a principal eigenfunction, Diaz, Morel and Oswald proved that, for ε positive and small enough, $\varepsilon\varphi_1^{\frac{2}{1+\gamma}}$ is a subsolution of problem (1.1). Since formula (1.3), is not avalaible for the principal eigenfunction of $(-\Delta)^s$, the arguments of [10] need to be modified in order to deal with the fractional case.

Let us state the functional setting for our problem. For $s \in (0,1)$ and $n \in \mathbb{N}$, let

$$H^s(\mathbb{R}^n) := \left\{u \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}dxdy < \infty\right\},$$

and for $u \in H^s(\mathbb{R}^n)$, let $\|u\|_{H^s(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} u^2 + \int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}dxdy\right)^{\frac{1}{2}}$. Let Ω be a bounded domain in $\mathbb{R}^n$ with $C^{1,1}$ boundary and let

$$X_0^s(\Omega) := \{u \in H^s(\mathbb{R}^n) : u = 0 \; a.e. \text{ in } \mathbb{R}^n \setminus \Omega\},$$

and for $u \in X_0^s(\Omega)$, let $\|u\|_{X_0^s(\Omega)} := \left(\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}dxdy\right)^{\frac{1}{2}}$.

With these norms, $H^s(\mathbb{R}^n)$ and $X_0^s(\Omega)$ are Hilbert spaces (see e.g., [29], Lemma 7), $C_c^\infty(\Omega)$ is dense in $X_0^s(\Omega)$ (see [16], Theorem 6). Also, $X_0^s(\Omega)$ is a closed subspace of $H^s(\mathbb{R}^n)$, and from the fractional Poincaré inequality (as stated e.g., in [11], Theorem 6.5; see Remark 2.1 below), if $n > 2s$ then $\|.\|_{X_0^s(\Omega)}$ and $\|.\|_{H^s(\mathbb{R}^n)}$ are equivalent norms on $X_0^s(\Omega)$. For $f \in L^1_{loc}(\Omega)$ we say that $f \in \left(X_0^s(\Omega)\right)'$ if there exists a positive constant c such that $\left|\int_\Omega f\varphi\right| \leq c\,\|u\|_{X_0^s(\Omega)}$ for any $\varphi \in X_0^s(\Omega)$. For $f \in \left(X_0^s(\Omega)\right)'$ we will write $((-\Delta)^s)^{-1} f$ for the unique weak solution u (given by the Riesz theorem) of the problem

$$\begin{cases} (-\Delta)^s u = f \text{ in } \Omega, \\ u = 0 \text{ in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{1.4}$$

Here and below, the notion of weak solution that we use is the given in the following definition:

Definition 1.1. Let $s \in (0,1)$, let $f : \Omega \to \mathbb{R}$ be a Lebesgue measurable function such that $f\varphi \in L^1(\Omega)$ for any $\varphi \in X_0^s(\Omega)$. We say that $u : \Omega \to \mathbb{R}$ is a weak solution to the problem

$$\begin{cases} (-\Delta)^s u = f \text{ in } \Omega, \\ u = 0 \text{ in } \mathbb{R}^n \setminus \Omega \end{cases}$$

if $u \in X_0^s(\Omega)$, $u = 0$ in $\mathbb{R}^n \setminus \Omega$ and, for any $\varphi \in X_0^s(\Omega)$,

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x-y|^{n+2s}} dxdy = \int_\Omega f\varphi.$$

For $u \in X_0^s(\Omega)$ and $f \in L^1_{loc}(\Omega)$, we will write $(-\Delta)^s u \leq f$ in Ω (respectively $(-\Delta)^s u \geq f$ in Ω) to mean that, for any nonnegative $\varphi \in H_0^s(\Omega)$, it hold that $f\varphi \in L^1(\Omega)$ and

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x-y|^{n+2s}} dxdy \leq \int_\Omega f\varphi \text{ (resp. } \geq \int_\Omega f\varphi).$$

For $u, v \in X_0^s(\Omega)$, we will write $(-\Delta)^s u \leq (-\Delta)^s v$ in Ω (respectively $(-\Delta)^s u \geq (-\Delta)^s v$ in Ω), to mean that $(-\Delta)^s (u - v) \leq 0$ in Ω (resp. $(-\Delta)^s (u - v) \geq 0$ in Ω).

Let

$$\mathcal{E} := \{u \in X_0^s(\Omega) : c d_\Omega^s \leq u \leq c' d_\Omega^s \ a.e. \text{ in } \Omega, \text{ for some positive constants } c \text{ and } c'\}$$

where, for $x \in \Omega$, $d_\Omega(x) := dist(x, \partial\Omega)$. With these notations, our main results read as follows:

Theorem 1.2. *Let Ω be a bounded domain in $\mathbb{R}^n$ with $C^{1,1}$ boundary, let $s \in (0,1)$, and assume $n > 2s$. Let $h \in L^\infty(\Omega)$ be such that $0 \leq h \not\equiv 0$ in Ω (i.e., $|\{x \in \Omega : h(x) > 0\}| > 0$) and let $g : \Omega \times (0,\infty) \to [0,\infty)$ be a function satisfying the following conditions g1)-g5)*
g1) $g : \Omega \times (0,\infty) \to [0,\infty)$ is a Carathéodory function, $g(.,s) \in L^\infty(\Omega)$ for any $s > 0$ and $\lim_{\sigma\to\infty} \|(g(.,\sigma))\|_\infty = 0$.
g2) $\sigma \to g(x,\sigma)$ is non increasing on $(0,\infty)$ a.e. $x \in \Omega$.
g3) $g\left(.,\sigma d_\Omega^s\right) \in \left(X_0^s(\Omega)\right)'$ and $d_\Omega^{-s}((-\Delta)^s)^{-1}\left(d_\Omega^s g\left(.,\sigma d_\Omega^s\right)\right) \in L^\infty(\Omega)$ for all $\sigma > 0$.
g4) It hold that:
$\lim_{\sigma\to\infty} \left\|\left(\sigma d_\Omega^s\right)^{-1}((-\Delta)^s)^{-1}\left(d_\Omega^s g\left(.,\sigma d_\Omega^s\right)\right)\right\|_\infty = 0,$ *and*
$\lim_{\sigma\to\infty} \left\|d_\Omega^{-s}((-\Delta)^s)^{-1}(g(.,\sigma))\right\|_{L^\infty(\Omega)} = 0.$
g5) $d_\Omega^s g\left(.,\sigma d_\Omega^s\right) \in L^2(\Omega)$ for any $\sigma > 0$.
Consider the problem

$$\begin{cases} (-\Delta)^s u = -g(.,u) + \lambda h \ in \ \Omega, \\ u = 0 \ in \ \mathbb{R}^n \setminus \Omega, \\ u > 0 \ in \ \Omega \end{cases} \tag{1.5}$$

Then there exists $\lambda^* \geq 0$ *such that:*
i) If $\lambda > \lambda^*$ *then (1.5) has a weak solution* $u^{(\lambda)} \in \mathcal{E}$*, which is maximal in the following sense: If* $v \in \mathcal{E}$ *satisfies* $(-\Delta)^s v \leq -g(.,v) + \lambda h$ *in* Ω*, then* $u^{(\lambda)} \geq v$ *a.e. in* Ω *.*
ii)If $\lambda < \lambda^*$*, no weak solution exists in* $\mathcal{E}$*.*
iii) If, in addition, there exists $b \in L^\infty(\Omega)$ *such that* $0 \leq b \not\equiv 0$ *in* Ω *and* $g(.,s) \geq bs^{-\beta}$ *a.e. in* Ω *for any* $s \in (0,\infty)$ *, then* $\lambda^* > 0$*.*

Theorem 1.2 allows $g(x,s)$ to be singular at $s = 0$. In fact, in Lemma 3.2, using some estimates from [4] for the Green function of $(-\Delta)^s$ in Ω (with homogeneous Dirichlet boundary condition on $\mathbb{R}^n \setminus \Omega$), we show that if $g(x,s) = as^{-\beta}$ with a a nonnegative function in $L^\infty(\Omega)$ and $\beta \in [0,s)$, then g satisfies the assumptions of Theorem 1.2. Thus, as a consequence of Theorem 1.2, we obtain the following:

Theorem 1.3. *Let* Ω*,* s*, and* h *be as in the statement of Theorem 1.2, and let* $g : \Omega \times (0,\infty) \to [0,\infty)$ *. Then the assertions i)-iii) of Theorem 1.2 remain true if we assume, instead of the conditions g1)-g5), that the following conditions g6) and g7) hold:*
g6) $g : \Omega \times (0,\infty) \to [0,\infty)$ *is a Carathéodory function and* $s \to g(x,s)$ *is nonincreasing for a.e.* $x \in \Omega$*.*
g7) There exist positive constants a *and* $\beta \in [0,s)$ *such that* $g(.,s) \leq as^{-\beta}$ *a.e. in* Ω *for any* $s \in (0,\infty)$ *.*

Let us sketch our approach: In Section 2 we consider, for $\varepsilon > 0$, the following approximated problem

$$\begin{cases} (-\Delta)^s u = -g(.,u+\varepsilon) + \lambda h \text{ in } \Omega, \\ u = 0 \text{ in } \mathbb{R}^n \setminus \Omega, \\ u > 0 \text{ in } \Omega. \end{cases} \tag{1.6}$$

Let us mention that, in order to deal with problems involving the $(p;q)$-Laplacian and a convection term, this type of approximation was considered in [14] (see problem P_ε therein).
Lemma 2.5 gives a positive number λ_0, independent of ε and such that, for $\lambda = \lambda_0$, problem (1.6) has a weak solution w_ε. From this result, and from some properties of the function w_ε, in Lemma 2.11 we show that, for $\lambda \geq \lambda_0$ and for any $\varepsilon > 0$, there exists a weak solution u_ε of problem (1.6), with the following properties:
a) $cd_\Omega^s \leq u_\varepsilon \leq c' d_\Omega^s$ for some positive constants c and c' independent of ε,
b) $u_\varepsilon \leq \overline{u}$, where $\overline{u}$ is the solution of the problem $(-\Delta)^s \overline{u} = \lambda h$ in Ω, $\overline{u} = 0$ in $\mathbb{R}^n \setminus \Omega$,
c) $u_\varepsilon \geq \psi$ for any $\psi \in X_0^s(\Omega)$ such that $(-\Delta)^s \psi = -g(.,\psi+\varepsilon) + \lambda h$ in Ω.
In section 3 we prove Theorems 1.2 and 1.3. To prove Theorem 1.2, we consider a decreasing sequence $\left\{\varepsilon_j\right\}_{j\in\mathbb{N}}$ such that $\lim_{j\to\infty} \varepsilon_j = 0$, and we show that, for $\lambda \geq \lambda_0$, the sequence of functions $\left\{u_{\varepsilon_j}\right\}_{j\in\mathbb{N}}$ given by Lemma 2.11 converges, in $X_0^s(\Omega)$, to a weak solution u of problem (1.5) which has the properties required by the theorem. An adaptation of some of the arguments of [10] gives that, if problem (1.5) has a weak solution in $\mathcal{E}$, then it has a maximal (in the sense stated in the theorem) weak solution in $\mathcal{E}$ and that if for some $\lambda = \lambda'$ (1.5) has a weak solution in $\mathcal{E}$, then it has a weak solution in $\mathcal{E}$ for any $\lambda \geq \lambda'$. Finally, the assertion *iii)* of Theorem 1.2 is proved with the same argument given in [10].

2. Preliminaries and auxiliary results

We fix, from now on, $h \in L^\infty(\Omega)$ such that $0 \leq h \not\equiv 0$ in Ω. We assume also from now on (except in Lemma 3.2) that $g : \Omega \times (0,\infty) \to [0,\infty)$ satisfies the assumptions g1)-g5 of Theorem 1.2.

In the next remark we collect some general facts concerning the operator $(-\Delta)^s$.

Remark 2.1. i) (see e.g., [27], Proposition 4.1 and Corollary 4.2) The following comparison principle holds: If $u, v \in X_0^s(\Omega)$ and $(-\Delta)^s u \geq (-\Delta)^s v$ in Ω then $u \geq v$ in Ω. In particular, the following maximum principle holds: If $v \in X_0^s(\Omega)$, $(-\Delta)^s v \geq 0$ in Ω and $v \geq 0$ in $\mathbb{R}^n \setminus \Omega$, then $v \geq 0$ in Ω.
ii) (see e.g., [27], Lemma 7.3) If $f : \Omega \to \mathbb{R}$ is a nonnegative and not identically zero measurable function in $f \in \left(X_0^s(\Omega)\right)'$, then the weak solution u of problem (1.4) satisfies, for some positive constant c,

$$u \geq c d_\Omega^s \text{ in } \Omega. \tag{2.1}$$

iii) (see e.g., [28], Proposition 1.1) If $f \in L^\infty(\Omega)$ then the weak solution u of problem (1.4) belongs to $C^s(\mathbb{R}^n)$. In particular, there exists a positive constant c such that

$$|u| \leq c d_\Omega^s \text{ in } \Omega. \tag{2.2}$$

iv) (Poincaré inequality, see [11], Theorem 6.5) Let $s \in (0,1)$ and let $2_s^* := \frac{2n}{n-2s}$. Then there exists a positive constant $C = C(n,s)$ such that, for any measurable and compactly supported function $f : \mathbb{R}^n \to \mathbb{R}$,

$$\|f\|_{L^{2_s^*}(\mathbb{R}^n)} \leq C \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(f(x) - f(y))^2}{|x-y|^{n+sp}} dxdy.$$

v) If $v \in L^{(2_s^*)'}(\Omega)$ then $v \in \left(X_0^s(\Omega)\right)'$, and $\|v\|_{(X_0^s(\Omega))'} \leq C \|v\|_{(2_s^*)'}$, with C as in i). Indeed, for $\varphi \in X_0^s(\Omega)$, from the Hölder inequality and iii), $\int_\Omega |v\varphi| \leq \|v\|_{(2_s^*)'} \|\varphi\|_{2_s^*} \leq C \|v\|_{(2_s^*)'} \|\varphi\|_{X_0^s(\Omega)}$.
vi) (Hardy inequality, see [25], Theorem 2.1) There exists a positive constant c such that, for any $\varphi \in X_0^s(\Omega)$,

$$\left\| d_\Omega^{-s} \varphi \right\|_2 \leq c \|\varphi\|_{X_0^s(\Omega)}. \tag{2.3}$$

Remark 2.2. Let $z^* \in H^s(\mathbb{R}^n)$ be the solution of the problem

$$\begin{cases} (-\Delta)^s z^* = \tau_1 h \text{ in } \Omega \\ z^* = 0 \text{ in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{2.4}$$

with τ_1 chosen such that $\|z^*\|_{L^\infty(\mathbb{R}^n)} = 1$. Since $h \in L^\infty(\mathbb{R}^n)$, Remark 2.1 iii) gives $z^* \in C(\mathbb{R}^n)$ (see also [12], Theorem 1.2). Thus, since $supp(z^*) \subset \overline{\Omega}$ and $z^* \in C\left(\overline{\Omega}\right)$, we have $z^* \in L^\infty(\mathbb{R}^n)$. Moreover, by Remark 2.1 ii), there exists a positive constant c^* such that

$$z^* \geq c^* d_\Omega^s \text{ in } \Omega. \tag{2.5}$$

Remark 2.3. There exist positive numbers M_0 and M_1 such that

$$\begin{aligned} &\frac{1}{2} c^* M_1 \geq \left\| d_\Omega^{-s} \left((-\Delta)^s\right)^{-1} \left(g\left(., \frac{1}{2} c^* M_1 d_\Omega^s\right)\right)\right\|_\infty, \\ &M_1 < M_0, \\ &\frac{1}{2} c^* M_1 \geq \left\| d_\Omega^{-s} \left((-\Delta)^s\right)^{-1} \left(g(., M_0)\right)\right\|_{L^\infty(\Omega)}. \end{aligned} \tag{2.6}$$

Indeed, by g4), $\lim_{\sigma\to\infty}\left\|\left(\sigma d_\Omega^s\right)^{-1}((-\Delta)^s)^{-1}\left(d_\Omega^s g\left(.,\sigma d_\Omega^s\right)\right)\right\|_\infty = 0$ and so the first one of the above inequalities hold for M_1 large enough. Fix such a M_1. Since, from g4), $\lim_{\sigma\to\infty}\left\|d_\Omega^{-s}((-\Delta)^s)^{-1}(g(.,\sigma))\right\|_{L^\infty(\Omega)} = 0$, the remaining inequalities of (2.6) hold for M_0 large enough.

Lemma 2.4. *Let $\varepsilon > 0$ and let z^*, τ_1 and c^* be as in Remark 2.2. Let M_0 and M_1 be as in Remark 2.3. Let $z := M_1 z^*$ and let $w^{0,\varepsilon} : \mathbb{R}^n \to \mathbb{R}$ be the constant function $w^{0,\varepsilon} = M_0$. Then there exist sequences $\left\{w^{j,\varepsilon}\right\}_{j\in\mathbb{N}}$ and $\left\{\zeta^{j,\varepsilon}\right\}_{j\in\mathbb{N}}$ in $X_0^s(\Omega)\cap L^\infty(\Omega)$ such that, for all $j\in\mathbb{N}$:*
i) $w^{j-1,\varepsilon} \ge w^{j,\varepsilon} \ge 0$ in $\mathbb{R}^n$,
ii) $w^{j,\varepsilon} \ge \frac{1}{2}c^* M_1 d_\Omega^s$ in Ω,
iii) $w^{j,\varepsilon}$ is a weak solution of the problem

$$\begin{cases} (-\Delta)^s w^{j,\varepsilon} = -g\left(.,w^{j-1,\varepsilon}+\varepsilon\right) + \tau_1 M_1 h \text{ in } \Omega, \\ w^{j,\varepsilon} = 0 \text{ in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{2.7}$$

iv) $w^{j,\varepsilon} = z - \zeta^{j,\varepsilon}$ in $\mathbb{R}^n$ and $\zeta^{j,\varepsilon}$ is a weak solution of the problem

$$\begin{cases} (-\Delta)^s \zeta^{j,\varepsilon} = g\left(.,w^{j-1,\varepsilon}+\varepsilon\right) \text{ in } \Omega, \\ \zeta^{j,\varepsilon} = 0 \text{ in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{2.8}$$

v) $\left\|w^{j,\varepsilon}\right\|_{X_0^s(\Omega)} \le c$ for some positive constant c independent of j and ε.

Proof. The sequences $\left\{w^{j,\varepsilon}\right\}_{j\in\mathbb{N}}$ and $\left\{\zeta^{j,\varepsilon}\right\}_{j\in\mathbb{N}}$ with the properties *i)-v)* will be constructed inductively. Let $\zeta^{1,\varepsilon}\in X_0^1(\Omega)$ be the solution of the problem

$$\begin{cases} (-\Delta)^s \zeta^{1,\varepsilon} = g\left(.,w^{0,\varepsilon}+\varepsilon\right) \text{ in } \Omega \\ \zeta^{1,\varepsilon} = 0 \text{ in } \mathbb{R}^n \setminus \Omega \end{cases}$$

(thus *iv)* holds for $j = 1$). From *g1)* and *g2)* we have $0 \le g\left(.,w^{0,\varepsilon}+\varepsilon\right) \le g(.,\varepsilon) \in L^\infty(\Omega)$. Thus $g\left(.,w^{0,\varepsilon}+\varepsilon\right)\in L^\infty(\Omega)$. Then, by Remark 2.1 *iii)*, $\zeta^{1,\varepsilon}\in C(\mathbb{R}^n)$. Therefore, since $supp\left(\zeta^{1,\varepsilon}\right)\subset\overline{\Omega}$, we have $\zeta^{1,\varepsilon}\in L^\infty(\Omega)$. By *g1)*, $g(.,M_0)\in L^\infty(\Omega)$ and so $g(.,M_0)\in\left(X_0^1(\Omega)\right)'$. Let $u_0 := ((-\Delta)^s)^{-1}(g(.,M_0))$. Then, by *g1)* and *g3)*, $d_\Omega^{-s}u_0\in L^\infty(\Omega)$. We have, in weak sense,

$$\begin{cases} (-\Delta)^s\left(\zeta^{1,\varepsilon}-u_0\right) = g\left(.,w^{0,\varepsilon}+\varepsilon\right) - g(.,M_0) \le 0 \text{ in } \Omega \\ \zeta^{1,\varepsilon}-u_0 = 0 \text{ in } \mathbb{R}^n\setminus\Omega. \end{cases}$$

Then, by the maximum principle of Remark 2.1 *i)*,

$$0 \le \zeta^{1,\varepsilon} \le u_0 \le \left\|d_\Omega^{-s}u_0\right\|_{L^\infty(\Omega)} d_\Omega^s \text{ in } \Omega. \tag{2.9}$$

Let $z := M_1 z^*$. By Remark 2.2, $z\in H^s(\mathbb{R}^n)\cap C\left(\overline{\Omega}\right)$ and

$$z \ge c^* M_1 d_\Omega^s \text{ in } \Omega. \tag{2.10}$$

Also, $z \le M_1$ in Ω, and z is a weak solution of the problem

$$\begin{cases} (-\Delta)^s z = \tau_1 M_1 h \text{ in } \Omega, \\ z = 0 \text{ in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

Let $w^{1,\varepsilon} := z - \zeta^{1,\varepsilon}$. Then $w^{1,\varepsilon} \in H^s(\mathbb{R}^n)$ and $w^{1,\varepsilon} = 0$ in $\mathbb{R}^n \setminus \Omega$. Thus $w^{1,\varepsilon} \in X_0^s(\Omega)$. Also $w^{1,\varepsilon} \in L^\infty(\Omega)$. Since $\zeta^{1,\varepsilon} \ge 0$ in Ω, we have

$$w^{0,\varepsilon} - w^{1,\varepsilon} = M_0 - z + \zeta^{1,\varepsilon} \ge M_0 - z \ge M_0 - M_1 > 0 \text{ in } \Omega.$$

Then $w^{1,\varepsilon} \le w^{0,\varepsilon}$ in Ω. Thus *i)* holds for $j = 1$. Now, in weak sense,

$$\begin{cases} (-\Delta)^s w^{1,\varepsilon} = (-\Delta)^s \left(z - \zeta^{1,\varepsilon}\right) = \tau_1 M_1 h - (-\Delta)^s \left(\zeta^{1,\varepsilon}\right) \\ = -g\left(., w^{0,\varepsilon} + \varepsilon\right) + \tau_1 M_1 h \text{ in } \Omega, \\ w^{1,\varepsilon} = 0 \text{ in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

and so *iii)* holds for $j = 1$. Also, from (2.9), (2.10), and taking into account that (2.6),

$$\begin{aligned} w^{1,\varepsilon} &= z - \zeta^{1,\varepsilon} \ge c^* M_1 d_\Omega^s - \left\| d_\Omega^{-s} \left((-\Delta)^s\right)^{-1} \left(g\left(., M_0\right)\right) \right\|_{L^\infty(\Omega)} d_\Omega^s \\ &\ge \frac{1}{2} c^* M_1 d_\Omega^s \text{ in } \Omega. \end{aligned}$$

and then $w^{1,\varepsilon} \ge \frac{1}{2} c^* M_1 d_\Omega^s$ in Ω. Thus *ii)* holds for $j = 1$.
Suppose constructed, for $k \ge 1$, functions $w^{1,\varepsilon}, ..., w^{k,\varepsilon}$ and $\zeta^{1,\varepsilon}, ..., \zeta^{k,\varepsilon}$, belonging to $X_0^s(\Omega) \cap L^\infty(\Omega)$, and with the properties *i)-iv)*. Let $\zeta^{k+1,\varepsilon} \in X_0^s(\Omega)$ be the solution of the problem

$$\begin{cases} (-\Delta)^s \zeta^{k+1,\varepsilon} = g\left(., w^{k,\varepsilon} + \varepsilon\right) \text{ in } \Omega, \\ \zeta^{k+1,\varepsilon} = 0 \text{ on } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{2.11}$$

(and so *iv)* holds for $j = k + 1$) and let $w^{k+1,\varepsilon} := z - \zeta^{k+1,\varepsilon}$. Then $w^{k+1,\varepsilon} \in H^s(\mathbb{R}^n)$ and $w^{k+1,\varepsilon} = 0$ in $\mathbb{R}^n \setminus \Omega$. Thus $w^{k+1,\varepsilon} \in X_0^s(\Omega)$. Also,

$$w^{k,\varepsilon} - w^{k+1,\varepsilon} = \zeta^{k+1,\varepsilon} - \zeta^{k,\varepsilon} \text{ in } \mathbb{R}^n \tag{2.12}$$

and

$$\begin{cases} (-\Delta)^s \left(\zeta^{k+1,\varepsilon} - \zeta^{k,\varepsilon}\right) = g\left(., w^{k,\varepsilon} + \varepsilon\right) - g\left(., w^{k-1,\varepsilon} + \varepsilon\right) \ge 0 \text{ in } \Omega, \\ \zeta^{k+1,\varepsilon} - \zeta^{k,\varepsilon} = 0 \text{ in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

the last inequality because, by *g1)*, $s \to g(., s)$ is nonincreasing and (by our inductive hypothesis) $w^{k,\varepsilon} \le w^{k-1,\varepsilon}$ in Ω. Then, by the maximum principle, $\zeta^{k+1,\varepsilon} - \zeta^{k,\varepsilon} \ge 0$ in $\mathbb{R}^n$. Therefore, by (2.12), $w^{k,\varepsilon} \ge w^{k+1,\varepsilon}$ in $\mathbb{R}^n$,and then *i)* holds for $j = k + 1$. Also,

$$\begin{cases} (-\Delta)^s w^{k+1,\varepsilon} = (-\Delta)^s z - (-\Delta)^s \zeta^{k+1,\varepsilon} = -g\left(., w^{k,\varepsilon} + \varepsilon\right) + \tau_1 M_1 h \text{ in } \Omega, \\ w^{k+1,\varepsilon} = 0 \text{ in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

Then *iii)* holds for $j = k + 1$. By *g4)*, $g\left(., \frac{1}{2} c^* M_1 d_\Omega^s\right) \in \left(X_0^s(\Omega)\right)'$. Let $u_1 := \left((-\Delta)^s\right)^{-1} \left(g\left(., \frac{1}{2} c^* M_1 d_\Omega^s\right)\right) \in X_0^s(\Omega)$. By the inductive hypothesis we have $w^{k,\varepsilon} \ge \frac{1}{2} c^* M_1 d_\Omega^s$ in Ω. Now,

$$\begin{cases} (-\Delta)^s \left(\zeta^{k+1,\varepsilon} - u_1\right) \\ = g\left(., w^{k,\varepsilon} + \varepsilon\right) - g\left(., \frac{1}{2} c^* M_1 d_\Omega^s\right) \le 0 \text{ in } \Omega, \\ \zeta^{k+1,\varepsilon} - u_1 = 0 \text{ on } \mathbb{R}^n \setminus \Omega, \end{cases}$$

then the comparison principle gives $\zeta^{k+1,\varepsilon} \leq u_1$. Thus, in Ω,

$$\begin{aligned} w^{k+1,\varepsilon} &= z - \zeta^{k+1,\varepsilon} \geq c^* M_1 d_\Omega^s - u_1 \\ &= c^* M_1 d_\Omega^s - ((-\Delta)^s)^{-1} \left(g\left(., \frac{1}{2} c^* M_1 d_\Omega^s\right)\right) \\ &\geq c^* M_1 d_\Omega^s - \left\| d_\Omega^{-s} ((-\Delta)^s)^{-1} \left(g\left(., \frac{1}{2} c^* M_1 d_\Omega^s\right)\right)\right\|_\infty d_\Omega^s \geq \frac{1}{2} c^* M_1 d_\Omega^s, \end{aligned}$$

the last inequality by (2.6). Thus *ii)* holds for $j = k+1$. This complete the inductive construction of the sequences $\left\{w^{j,\varepsilon}\right\}_{j\in\mathbb{N}}$ and $\left\{\zeta^{j,\varepsilon}\right\}_{j\in\mathbb{N}}$ with the properties *i)-iv)*.
To see *v)*, we take $\zeta^{j,\varepsilon}$ as a test function in (2.8). Using *ii),* the Hölder inequality, the Poincaré inequality of Remark 2.1 *iv)*, we get, for any $j \in \mathbb{N}$,

$$\begin{aligned} \left\|\zeta^{j,\varepsilon}\right\|^2_{X_0^s(\Omega)} &= \int_\Omega g\left(., w^{j-1,\varepsilon} + \varepsilon\right)\zeta^{j,\varepsilon} \leq \int_\Omega g\left(., \frac{1}{2} c^* M_1 d_\Omega^s\right)\zeta^{j,\varepsilon} \\ &= \int_\Omega d_\Omega^s g\left(., \frac{1}{2} c^* M_1 d_\Omega^s\right) d_\Omega^{-s}\zeta^{j,\varepsilon} \\ &\leq \left\| d_\Omega^s g\left(., \frac{1}{2} c^* M_1 d_\Omega^s\right)\right\|_2 \left\| d_\Omega^{-s}\zeta^{j,\varepsilon}\right\|_2 \leq c \left\|\zeta^{j,\varepsilon}\right\|_{X_0^s(\Omega)}. \end{aligned}$$

where c is a positive constant c independent of j and ε, and where, in the last inequality, we have used *g5).* Then $\left\|\zeta^{j,\varepsilon}\right\|_{X_0^s(\Omega)}$ has an upper bound independent of j and ε. Since $w^{j,\varepsilon} = z - \zeta^{j,\varepsilon}$, the same assertion holds for $w^{j,\varepsilon}$. □

Lemma 2.5. *Let $\varepsilon > 0$ and let τ_1 and c^* be as in Remark 2.2. Let M_0 and M_1 be as in Remark 2.3 and let $\left\{w^{j,\varepsilon}\right\}_{j\in\mathbb{N}}$ and $\left\{\zeta^{j,\varepsilon}\right\}_{j\in\mathbb{N}}$ be as in Lemma 2.4. Let $w_\varepsilon := \lim_{j\to\infty} w^{j,\varepsilon}$ and let $\zeta_\varepsilon := \lim_{j\to\infty}\zeta^{j,\varepsilon}$. Then*
i) w_ε and ζ_ε belong to $H^s(\mathbb{R}^n) \cap C\left(\overline{\Omega}\right)$,
ii) $\frac{1}{2} c^* M_1 d_\Omega^s \leq w_\varepsilon \leq M_0$ in Ω, and there exists a positive constant c independent of ε such that $w^\varepsilon \leq c d_\Omega^s$ in Ω.
iii) w_ε satisfies, in weak sense,

$$\begin{cases} -\Delta w_\varepsilon = -g\left(., w_\varepsilon + \varepsilon\right) + \tau_1 M_1 h \text{ in } \Omega, \\ w_\varepsilon = 0 \text{ in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{2.13}$$

iv) ζ_ε satisfies, in weak sense,

$$\begin{cases} (-\Delta)^s \zeta_\varepsilon = g\left(., w_\varepsilon + \varepsilon\right) \text{ in } \Omega, \\ \zeta_\varepsilon = 0 \text{ in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{2.14}$$

Proof. Let z^* be as in Remark 2.2, and let $z := M_1 z^*$. Let M_0 and M_1 be as in Remark 2.3. By Lemma 2.4, $\left\{w^{j,\varepsilon}\right\}_{j\in\mathbb{N}}$ is a nonincreasing sequence of nonnegative functions in $\mathbb{R}^n$, and so there exists $w_\varepsilon = \lim_{j\to\infty} w^{j,\varepsilon}$. Since $\zeta^{j,\varepsilon} = z - w^{j-1,\varepsilon}$, there exists also $\zeta_\varepsilon = \lim_{j\to\infty}\zeta^{j,\varepsilon}$. Again by Lemma 2.4 we have, for any $j \in \mathbb{N}$, $0 \leq w^{j,\varepsilon} = z - \zeta^{j,\varepsilon} \leq z \in L^\infty(\Omega)$. Thus, by the Lebesgue dominated convergence theorem,

$$\left\{w^{j,\varepsilon}\right\}_{j\in\mathbb{N}} \text{ converges to } w_\varepsilon \text{ in } L^p(\Omega) \text{ for any } p \in [1,\infty), \tag{2.15}$$

and so $\left\{g\left(.,w^{j,\varepsilon}+\varepsilon\right)\right\}_{j\in\mathbb{N}}$ converges to $g\left(.,w_{\varepsilon}+\varepsilon\right)$ in $L^{p}\left(\Omega\right)$ for any $p\in[1,\infty)$. We claim that

$$\zeta_{\varepsilon}\in X_{0}^{s}\left(\Omega\right)\text{ and }\left\{\zeta^{j,\varepsilon}\right\}_{j\in\mathbb{N}}\text{ converges in }X_{0}^{s}\left(\Omega\right)\text{ to }\zeta_{\varepsilon}. \tag{2.16}$$

Indeed, for $j,k\in\mathbb{N}$, from (2.8),

$$\begin{cases}(-\Delta)^{s}\left(\zeta^{j,\varepsilon}-\zeta^{k,\varepsilon}\right)=g\left(.,w^{j-1,\varepsilon}+\varepsilon\right)-g\left(.,w^{k-1,\varepsilon}+\varepsilon\right)\text{ in }\Omega,\\ \zeta^{j,\varepsilon}-\zeta^{k,\varepsilon}=0\text{ in }\mathbb{R}^{n}\setminus\Omega.\end{cases} \tag{2.17}$$

We take $\zeta^{j,\varepsilon}-\zeta^{k,\varepsilon}$ as a test function in (2.17) to obtain

$$\begin{aligned}\left\|\zeta^{j,\varepsilon}-\zeta^{k,\varepsilon}\right\|_{X_{0}^{s}(\Omega)}^{2}&=\int_{\Omega}\left(\zeta^{j,\varepsilon}-\zeta^{k,\varepsilon}\right)\left(g\left(.,w^{j-1,\varepsilon}+\varepsilon\right)-g\left(.,w^{k-1,\varepsilon}+\varepsilon\right)\right)\\&\leq\left\|\zeta^{j,\varepsilon}-\zeta^{k,\varepsilon}\right\|_{2_{s}^{*}}\left\|g\left(.,w^{j-1,\varepsilon}+\varepsilon\right)-g\left(.,w^{k-1,\varepsilon}+\varepsilon\right)\right\|_{(2_{s}^{*})'}\end{aligned}$$

where $2_{s}^{*}:=\frac{2n}{n-2s}$. Then

$$\left\|\zeta^{j,\varepsilon}-\zeta^{k,\varepsilon}\right\|_{X_{0}^{s}(\Omega)}\leq c\left\|g\left(.,w^{j-1,\varepsilon}+\varepsilon\right)-g\left(.,w^{k-1,\varepsilon}+\varepsilon\right)\right\|_{(2_{s}^{*})'}.$$

where c is a constant independent of j and k. Since $\left\{g\left(.,w^{j-1,\varepsilon}+\varepsilon\right)\right\}_{j\in\mathbb{N}}$ converges to $g\left(.,w_{\varepsilon}+\varepsilon\right)$ in $L^{(2_{s}^{*})'}\left(\Omega\right)$, we get

$$\lim_{j,k\to\infty}\left\|\zeta^{j,\varepsilon}-\zeta^{k,\varepsilon}\right\|_{X_{0}^{s}(\Omega)}=0,$$

and thus $\left\{\zeta^{j,\varepsilon}\right\}_{j\in\mathbb{N}}$ converges in $X_{0}^{s}\left(\Omega\right)$. Since $\left\{\zeta^{j,\varepsilon}\right\}_{j\in\mathbb{N}}$ converges to ζ_{ε} in pointwise sense, (2.16) follows. Also, $w^{j,\varepsilon}=z-\zeta^{j,\varepsilon}$, and then $\left\{w^{j,\varepsilon}\right\}_{j\in\mathbb{N}}$ converges to $w_{\varepsilon,\rho}$ in $X_{0}^{s}\left(\Omega\right)$. Thus

$$w_{\varepsilon}\in X_{0}^{s}\left(\Omega\right)\text{ and }\left\{w^{j,\varepsilon}\right\}_{j\in\mathbb{N}}\text{ converges in }X_{0}^{s}\left(\Omega\right)\text{ to }w_{\varepsilon}. \tag{2.18}$$

To prove (2.14) observe that, from (2.8), we have, for any $\varphi\in X_{0}^{s}\left(\Omega\right)$ and $j\in\mathbb{N}$,

$$\int_{\mathbb{R}^{n}\times\mathbb{R}^{n}}\frac{\left(\zeta^{\varepsilon,j}\left(x\right)-\zeta^{\varepsilon,j}\left(y\right)\right)\left(\varphi\left(x\right)-\varphi\left(y\right)\right)}{|x-y|^{n+2s}}dxdy=\int_{\Omega}g\left(.,w^{\varepsilon,j-1}+\varepsilon\right)\varphi. \tag{2.19}$$

Taking $\lim_{j\to\infty}$ in (2.19) and using (2.16) and (2.15), we obtain (2.14). From (2.14) and since, by *g1)* and *g2)*, $g\left(.,w_{\varepsilon}+\varepsilon\right)\in L^{\infty}\left(\Omega\right)$, Remark 2.1 *iii)* gives that, in addition, $\zeta_{\varepsilon}\in C\left(\overline{\Omega}\right)$ (and so, since $w_{\varepsilon}=z-\zeta_{\varepsilon}$, then also $w_{\varepsilon}\in C\left(\overline{\Omega}\right)$).
Let us see that (2.13) holds. Let $\varphi\in X_{0}^{s}\left(\Omega\right)$. From (2.7), we have, for any $j\in\mathbb{N}$,

$$\begin{aligned}&\int_{\mathbb{R}^{n}\times\mathbb{R}^{n}}\frac{\left(w^{j,\varepsilon}\left(x\right)-w^{j,\varepsilon}\left(y\right)\right)\left(\varphi\left(x\right)-\varphi\left(y\right)\right)}{|x-y|^{n+2s}}dxdy\\&=\int_{\Omega\setminus\overline{B_{\rho}(y)}}\left(-g\left(.,w^{j-1,\varepsilon}+\varepsilon\right)+\tau_{1}M_{1}h\right)\varphi.\end{aligned} \tag{2.20}$$

Since $\varphi \in X_0^s(\Omega)$ and $\left\{w^{j,\varepsilon}\right\}_{j\in\mathbb{N}}$ converges to w_ε in $X_0^s(\Omega)$ we have

$$\lim_{j\to\infty}\int_{\mathbb{R}^n\times\mathbb{R}^n}\frac{\left(w^{j,\varepsilon}(x)-w^{j,\varepsilon}(y)\right)(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}dxdy \tag{2.21}$$
$$=\int_{\mathbb{R}^n\times\mathbb{R}^n}\frac{(w_\varepsilon(x)-w_\varepsilon(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}dxdy.$$

Also, $w_\varepsilon(x)=\lim_{j\to\infty}w^{j,\varepsilon}(x)$ for any $x\in\Omega$, and

$$\left|g\left(.,w^{j-1,\varepsilon}+\varepsilon\right)\varphi\right|\le g(.,\varepsilon)|\varphi|\in L^1(\Omega),$$

and clearly $|\tau_1 M_1 h\varphi|\in L^1(\Omega)$. Then, by the Lebesgue dominated convergence theorem,

$$\lim_{j\to\infty}\int_\Omega\left(-g\left(.,w^{j-1,\varepsilon}+\varepsilon\right)+\tau_1 M_1 h\right)\varphi=\int_\Omega(-g(.,w_\varepsilon+\varepsilon)+\tau_1 M_1 h)\varphi. \tag{2.22}$$

Now (2.13) follows from (2.20), (2.21) and (2.22). Finally, by Lemma 2.4 we have, for all $j\in\mathbb{N}$, $\frac{1}{2}c^*M_1d_\Omega^s\le w^{j,\varepsilon}$ in Ω and so the same inequality hold with $w^{j,\varepsilon}$ replaced by w_ε. Also, since $w^{j,\varepsilon}\le z_0$ in Ω we have $w^{j,\varepsilon}\le cd_\Omega^s$ with c independent of j and ε. □

Remark 2.6. Let $G:\Omega\times\Omega\to\mathbb{R}\cup\{\infty\}$ be the Green function for $(-\Delta)^s$ in Ω, with homogeneous Dirichlet boundary condition on $\mathbb{R}^n\setminus\Omega$. Then, for $f\in C\left(\overline{\Omega}\right)$ the solution u of problem (1.4) is given by $u(x)=\int_\Omega G(x,y)f(y)dy$ for $x\in\Omega$ and by $u(x)=0$ for $x\in\mathbb{R}^n\setminus\Omega$. Let us recall the following estimates from [4]:

i) (see [4], Theorems 1.1 and 1.2) There exist positive constants c and c', depending only on Ω and s, such that for $x;y\in\Omega$,

$$G(x,y)\le c\frac{d_\Omega(x)^s}{|x-y|^{n-s}}, \tag{2.23}$$
$$G(x,y)\le c\frac{d_\Omega(x)^s}{d_\Omega(y)^s}\frac{1}{|x-y|^{n-2s}}, \tag{2.24}$$
$$G(x,y)\le c\frac{d_\Omega(x)^s d_\Omega(y)^s}{|x-y|^n} \tag{2.25}$$
$$G(x,y)\ge c'\frac{1}{|x-y|^{n-2s}}\text{ if }|x-y|\le\max\left\{\frac{d_\Omega(x)}{2},\frac{d_\Omega(y)}{2}\right\} \tag{2.26}$$
$$G(x,y)\ge c'\frac{d_\Omega(x)^s d_\Omega(y)^s}{|x-y|^n}\text{ if }|x-y|>\max\left\{\frac{d_\Omega(x)}{2},\frac{d_\Omega(y)}{2}\right\} \tag{2.27}$$

ii) From i) it follows that there exists a positive constant c'', depending only on Ω and s, such that for $x;y\in\Omega$,

$$G(x;y)\ge c''d_\Omega(x)^s d_\Omega(y)^s. \tag{2.28}$$

Indeed:
If $|x-y|>\max\left\{\frac{d_\Omega(x)}{2},\frac{d_\Omega(y)}{2}\right\}$ then, from (2.27), $G(x;y)\ge c'\frac{d_\Omega(x)^s d_\Omega(y)^s}{|x-y|^n}$ and so $G(x;y)\ge c'\frac{d_\Omega(x)^s d_\Omega(y)^s}{(diam(\Omega))^n}$.
If $|x-y|\le\max\left\{\frac{d_\Omega(x)}{2},\frac{d_\Omega(y)}{2}\right\}$ then either $|x-y|\le\frac{d_\Omega(x)}{2}$ or $|x-y|\le\frac{d_\Omega(y)}{2}$. If $|x-y|\le\frac{d_\Omega(x)}{2}$ consider $z\in\partial\Omega$

such that $d_\Omega(y) = |z-y|$. then $d_\Omega(y) = |z-y| \geq |x-z| - |x-y| \geq d_\Omega(x) - |x-y| \geq \frac{1}{2}d_\Omega(x)$. Then $d_\Omega(y) \geq \frac{1}{2}d_\Omega(x) \geq |x-y|$. Thus, since also $|x-y| \leq \frac{d_\Omega(x)}{2}$, we have $|x-y| \leq \frac{1}{\sqrt{2}}(d_\Omega(x)\,d_\Omega(y))^{\frac{1}{2}}$, and so, from (2.26), $G(x,y) \geq c'\frac{1}{|x-y|^{n-2s}} \geq c'\frac{1}{\left(\frac{1}{\sqrt{2}}(d_\Omega(x)d_\Omega(y))^{\frac{1}{2}}\right)^{n-2s}} = 2^{\frac{n}{2}-s}c'\frac{d_\Omega^s(x)d_\Omega^s(y)}{(d_\Omega(x)d_\Omega(y))^{\frac{n}{2}}} \geq \frac{2^{\frac{n}{2}-s}c'}{(diam(\Omega))^n}d_\Omega^s(x)\,d_\Omega^s(y)$. If $|x-y| \leq \frac{d_\Omega(y)}{2}$, by interchanging the roles of x and y in the above argument, the same conclusion is reached.

iii) If $0<\beta<s$, then

$$G(x,y) \leq c\frac{d_\Omega(x)^s\,d_\Omega(y)^\beta}{|x-y|^{n-s+\beta}}. \tag{2.29}$$

Indeed, If $d_\Omega(y) \geq |x-y|$ then, from (2.23),

$$G(x,y) \leq c\frac{d_\Omega(x)^s}{|x-y|^{n-s}} = c\frac{d_\Omega(x)^s\,d_\Omega(y)^\beta}{|x-y|^{n-s}\,d_\Omega(y)^\beta} \leq c\frac{d_\Omega(x)^s\,d_\Omega(y)^\beta}{|x-y|^{n-s+\beta}},$$

and if $d_\Omega(y) \leq |x-y|$ then, from (2.27),

$$G(x,y) \leq c\frac{d_\Omega(x)^s\,d_\Omega(y)^s}{|x-y|^n} = c\frac{d_\Omega(x)^s\,d_\Omega(y)^\beta\,d_\Omega(y)^{s-\beta}}{|x-y|^{n-s+\beta}\,|x-y|^{s-\beta}} \leq c\frac{d_\Omega(x)^s\,d_\Omega(y)^\beta}{|x-y|^{n-s+\beta}},$$

iv) If $f \in C\left(\overline{\Omega}\right)$ then the unique solution $u \in X_0^s(\Omega)$ of problem (1.4) is given by $u(x) := \int_\Omega G(x,y)\,f(y)\,dy$ for $x \in \Omega$, and $u(x) := 0$ for $x \in \mathbb{R}^n \setminus \Omega$.

Lemma 2.7. *Let $a \in L^\infty(\Omega)$ and let $\beta \in [0,s)$. Then $ad_\Omega^{-\beta} \in \left(X_0^s(\Omega)\right)'$ and the weak solution $u \in X_0^s(\Omega)$ of the problem*

$$\begin{cases} (-\Delta)^s u = ad_\Omega^{-\beta} \text{ in } \Omega, \\ u = 0 \text{ in } \mathbb{R}^n \setminus \Omega \end{cases} \tag{2.30}$$

satisfies $d_\Omega^{-s}u \in L^\infty(\Omega)$.

Proof. Let $\varphi \in X_0^s(\Omega)$. By the Hölder and Hardy inequalities we have $\int_\Omega \left|ad_\Omega^{-\beta}\varphi\right| = \int_\Omega \left|ad_\Omega^{s-\beta}d_\Omega^{-s}\varphi\right| \leq \|a\|_\infty \left\|d_\Omega^{s-\beta}\right\|_2 \left\|d_\Omega^{-s}\varphi\right\|_2 \leq c\,\|\varphi\|_{X_0^s(\Omega)}$ with c a positive constant independent of φ. Thus $ad_\Omega^{-\beta} \in \left(X_0^s(\Omega)\right)'$. Let $u \in X_0^s(\Omega)$ be the unique weak solution (given by the Riesz Theorem) of problem (2.30) and consider a decreasing sequence $\left\{\varepsilon_j\right\}_{j\in N}$ in $(0,1)$ such that $\lim_{j\to\infty}\varepsilon_j = 0$. For $j \in \mathbb{N}$, let $u_{\varepsilon_j} \in X_0^s(\Omega)$ be the weak solution of the problem

$$\begin{cases} (-\Delta)^s u_{\varepsilon_j} = a\left(d_\Omega + \varepsilon_j\right)^{-\beta} \text{ in } \Omega, \\ u_{\varepsilon_j} = 0 \text{ in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{2.31}$$

Thus $u_{\varepsilon_j} = \int_\Omega G(.,y)\,a(y)\left(d_\Omega(y) + \varepsilon_j\right)^{-\beta} dy$ in Ω, where G is the Green function for $(-\Delta)^s$ in Ω, with homogeneous Dirichlet boundary condition on $\mathbb{R}^n \setminus \Omega$. Since $\beta < s$ we have $\int_\Omega \frac{1}{|x-y|^{n-s+\beta}}dy < \infty$. Thus, recalling (2.29), there exists a positive constant c such that, for any $j \in \mathbb{N}$ and $(x,y) \in \Omega \times \Omega$,

$$0 \leq G(x,y)\,a(y)\left(d_\Omega(y) + \varepsilon_j\right)^{-\beta} \leq c\frac{d_\Omega^s(x)\,d_\Omega^\beta(y)}{|x-y|^{n-s+\beta}}\left(d_\Omega(y) + \varepsilon_j\right)^{-\beta}$$

$$\leq c d_\Omega^s (x) \frac{1}{|x-y|^{n-s+\beta}} \in L^1 (\Omega, dy).$$

Since also $\lim_{j\to\infty} G(x,y) a(y) \left(d_\Omega (y) + \varepsilon_j\right)^{-\beta} = G(x,y) a(y) d_\Omega^{-\beta} (y)$ for *a.e.* $y \in \Omega$, by the Lebesgue dominated convergence theorem, $\left\{u_{\varepsilon_j} (x)\right\}_{j\in\mathbb{N}}$ converges to $\int_\Omega G(x,y) a(y) d_\Omega^{-\beta} (y)\, dy$ for any $x \in \Omega$. Let $u(x) := \lim_{j\to\infty} u_{\varepsilon_j} (x)$. Thus $u(x) = \int_\Omega G(x,y) a(y) d_\Omega^{-\beta} (y)\, dy$. Again from (2.29), $u \leq c d_\Omega^s$ *a.e.* in Ω, with c constant c independent of x. Now we take u_{ε_j} as a test function in (2.31) to obtain that

$$\begin{aligned}
\int_{\Omega\times\Omega} \frac{\left(u_{\varepsilon_j} (x) - u_{\varepsilon_j} (y)\right)^2}{|x-y|^{n+2s}} &= \int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\left(u_{\varepsilon_j} (x) - u_{\varepsilon_j} (y)\right)^2}{|x-y|^{n+2s}} \\
&= \int_\Omega u_{\varepsilon_j} (y) \left(d_\Omega (y) + \varepsilon_j\right)^{-\beta} dy \\
&\leq c \int_\Omega d_\Omega^s (y) \left(d_\Omega (y) + \varepsilon\right) j^{-\beta} dy \leq c' \int_\Omega d_\Omega^{s-\beta} (y)\, dy = c'',
\end{aligned}$$

with c and c' constants independent of j. For $j \in \mathbb{N}$, let U_{ε_j} and U be the functions, defined on $\mathbb{R}^n \times \mathbb{R}^n$, by

$$U_{\varepsilon_j} (x,y) := u_{\varepsilon_j} (x) - u_{\varepsilon_j} (y),\ U(x,y) := u(x) - u(y).$$

Then $\left\{U_{\varepsilon_j}\right\}_{j\in\mathbb{N}}$ is bounded in $\mathcal{H} = L^2\left(\mathbb{R}^n \times \mathbb{R}^n, \frac{1}{|x-y|^{n+2s}} dxdy\right)$. Thus, after pass to a subsequence if necessary, we can assume that $\left\{U_{\varepsilon_j}\right\}_{j\in\mathbb{N}}$ is weakly convergent in $\mathcal{H}$ to some $V \in \mathcal{H}$. Since $\left\{U_{\varepsilon_j}\right\}_{j\in\mathbb{N}}$ converges pointwise to U on $\mathbb{R}^n \times \mathbb{R}^n$, we conclude that $U \in \mathcal{H}$ and that $\left\{U_{\varepsilon_j}\right\}_{j\in\mathbb{N}}$ converges weakly to U in $\mathcal{H}$. Thus $u \in X_0^s(\Omega)$ and, for any $\varphi \in X_0^s(\Omega)$,

$$\begin{aligned}
&\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x-y|^{n+2s}} dxdy \\
&= \lim_{j\to\infty} \int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\left(u_{\varepsilon_j} (x) - u_{\varepsilon_j} (y)\right)(\varphi(x) - \varphi(y))}{|x-y|^{n+2s}} dxdy \\
&= \lim_{j\to\infty} \int_\Omega a \left(d_\Omega + \varepsilon_j\right)^{-\beta} \varphi = \int_\Omega a d_\Omega^{-\beta} \varphi,
\end{aligned}$$

Then u is the weak solution of (2.30). Finally, since for all j, $u_{\varepsilon_j} \leq c' d_\Omega^s$ *a.e.* in Ω, we have $u \leq c' d_\Omega^s$ *a.e.* in Ω. □

Lemma 2.8. *Let $\lambda > 0$ and let $\varepsilon \geq 0$. Suppose that $\left\{u_j\right\}_{j\in\mathbb{N}} \subset X_0^s(\Omega)$ is a nonincreasing sequence with the following properties i) and ii):*
i) There exist positive constants c and c' such that $c d_\Omega^s \leq u_j \leq c' d_\Omega^s$ a.e. in Ω for any $j \in \mathbb{N}$.
ii) for any $j \in \mathbb{N}$, u_j is a weak solution of the problem

$$\begin{cases} (-\Delta)^s u_j = -g\left(., u_j + \varepsilon\right) + \lambda h \text{ in } \Omega, \\ u_j = 0 \text{ in } \mathbb{R}^n \setminus \Omega, \\ u_j > .0 \text{ in } \Omega \end{cases} \tag{2.32}$$

Then $\left\{u_j\right\}_{j\in\mathbb{N}}$ *converges in* $X_0^s(\Omega)$ *to a weak solution* u *of the problem*

$$\begin{cases} (-\Delta)^s u = -g(.,u+\varepsilon) + \lambda h \text{ in } \Omega, \\ u_j = 0 \text{ in } \mathbb{R}^n \setminus \Omega, \\ u > 0 \text{ in } \Omega, \end{cases} \tag{2.33}$$

which satisfies $cd_\Omega^s \le u \le c' d_\Omega^s$ *a.e. in* Ω*. Moreover, the same conclusions holds if, instead of ii), we assume the following ii'):*
ii') for any $j \ge 2$, u_j *is a weak solution of the problem*

$$\begin{cases} (-\Delta)^s u_j = -g\left(.,u_{j-1}+\varepsilon\right) + \lambda h \text{ in } \Omega, \\ u_j = 0 \text{ in } \mathbb{R}^n \setminus \Omega, \\ u_j > 0 \text{ in } \Omega. \end{cases}$$

Proof. Assume *i)* and *ii)*. For $x \in \mathbb{R}^n$, let $u(x) := \lim_{j\to\infty} u_j(x)$. For $j,k \in \mathbb{N}$ we have, in weak sense,

$$\begin{cases} (-\Delta)^s \left(u_j - u_k\right) = g(.,u_k+\varepsilon) - g\left(u_j+\varepsilon\right) \text{ in } \Omega, \\ u_j - u_k = 0 \text{ in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{2.34}$$

We take $u_j - u_k$ as a test function in (2.34) to get

$$\begin{aligned} \left\|u_j-u_k\right\|_{X_0^s(\Omega)}^2 &= \int_\Omega \left(g(.,u_k+\varepsilon) - g\left(.,u_j+\varepsilon\right)\right)\left(u_j-u_k\right) \\ &= \int_\Omega d_\Omega^s \left(g(.,u_k+\varepsilon) - g\left(.,u_j+\varepsilon\right)\right) d_\Omega^{-s}\left(u_j-u_k\right) \\ &\le \left\|d_\Omega^{-s}\left(\overline{u}_j-\overline{u}_k\right)\right\|_2 \left\|d_\Omega^s\left(g(.,\overline{u}_k+\varepsilon) - g\left(.,\overline{u}_j+\varepsilon\right)\right)\right\|_2. \end{aligned} \tag{2.35}$$

By the Hardy inequality, $\left\|d_\Omega^{-s}\left(u_j-u_k\right)\right\|_2 \le c'' \left\|u_j-u_k\right\|_{X_0^s(\Omega)}$ where c'' is a constant independent of j and k. Thus, from (2.35),

$$\left\|u_j-u_k\right\|_{X_0^s(\Omega)} \le c'' \left\|d_\Omega^s\left(g(.,u_k+\varepsilon) - g\left(.,u_j+\varepsilon\right)\right)\right\|_2. \tag{2.36}$$

Now, $\lim_{j,k\to\infty} \left|d_\Omega^s\left(g(.,u_k+\varepsilon) - g\left(.,u_j+\varepsilon\right)\right)\right|^2 = 0$ *a.e.* in Ω. Also, since $u_l \ge cd_\Omega^s$ *a.e.* in Ω for any $l \in \mathbb{N}$, and taking into account *g5)* and *g2),*

$$\left|d_\Omega^s\left(g(.,u_k+\varepsilon) - g\left(.,u_j+\varepsilon\right)\right)\right|^2 \le c' \left(d_\Omega^s g(.,cd_\Omega^s)\right)^2 \in L^1(\Omega),$$

where c' is a constant independent of j and k. Then, by the Lebesgue dominated convergence theorem $\lim_{j,k\to\infty} \left\|d_\Omega^s\left(g(.,u_k+\varepsilon) - g\left(.,u_j+\varepsilon\right)\right)\right\|_2 = 0$. Therefore, from (2.36), $\lim_{j,k\to\infty} \left\|u_j-u_k\right\|_{X_0^s(\Omega)} = 0$ and so $\left\{u_j\right\}_{j\in\mathbb{N}}$ converges in $X_0^s(\Omega)$ to some $u^* \in X_0^s(\Omega)$. Then, by the Poincaré inequality of Remark 2.1 *iv),* $\left\{u_j\right\}_{j\in\mathbb{N}}$ converges to u^* in $L^{2_s^*}(\Omega)$, and thus there exists a subsequence $\left\{u_{j_k}\right\}_{k\in\mathbb{N}}$ that converges to u^*

$a.e.$ in Ω. Since $\left\{u_{j_k}\right\}_{k\in\mathbb{N}}$ converges pointwise to u_ε, we conclude that $u^* = u$. Then $\left\{u_j\right\}_{j\in\mathbb{N}}$ converges to u_ε in $X_0^s(\Omega)$. Now, for $\varphi \in X_0^s(\Omega)$ and $j \in \mathbb{N}$,

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\left(u_j(x) - u_j(y)\right)(\varphi(x) - \varphi(y))}{|x-y|^{n+2s}} dxdy = \int_\Omega \left(-g\left(., u_j + \varepsilon\right) + \lambda h\right)\varphi. \tag{2.37}$$

Since $\left\{u_j\right\}_{j\in\mathbb{N}}$ converges to u in $X_0^s(\Omega)$, we have

$$\begin{aligned}&\lim_{j\to\infty} \int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\left(u_j(x) - u_j(y)\right)(\varphi(x) - \varphi(y))}{|x-y|^{n+2s}} dxdy \\ &= \int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x-y|^{n+2s}} dxdy.\end{aligned} \tag{2.38}$$

On the other hand, $\left|\left(-g\left(., u_j + \varepsilon\right) + \lambda h\right)\varphi\right| \le (g(., cd_\Omega) + \lambda \|h\|_\infty)|\varphi| \in L^1(\Omega)$ (with c as in *i)*). Also, $\left\{\left(-g\left(u_j + \varepsilon\right) + \lambda h\right)\varphi\right\}_{j\in\mathbb{N}}$ converges to $(-g(u+\varepsilon) + \lambda h)\varphi$ $a.e.$ in Ω. Then, by the Lebesgue dominated convergence theorem,

$$\lim_{j\to\infty} \int_\Omega \left(-g\left(., u_j + \varepsilon\right) + \lambda h\right)\varphi = \int_\Omega (-g(., u+\varepsilon) + \lambda h)\varphi. \tag{2.39}$$

From (2.37), (2.38) and (2.39) we get, for any $\varphi \in X_0^s(\Omega)$,

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x-y|^{n+2s}} dxdy = \int_\Omega (-g(., u+\varepsilon) + \lambda h)\varphi.$$

and so u is a weak solution of problem (2.33) which clearly satisfies $cd_\Omega^s \le u \le c'd_\Omega^s$ $a.e.$ in Ω. If instead of *ii)* we assume *ii')*, the proof is the same. Only replace, for $j \ge 2$, $k \ge 2$ and in each appearance, $g\left(., u_j\right)$ and $g(., u_k)$ by $g\left(., u_{j-1}\right)$ and $g(., u_{k-1})$ respectively. □

Lemma 2.9. *Let $\lambda > 0$, and let $\overline{u}$ be the weak solution of*

$$\begin{cases} (-\Delta)^s \overline{u} = \lambda h \text{ in } \Omega, \\ \overline{u} = 0 \text{ in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{2.40}$$

Assume that, for each $\varepsilon > 0$, we have a function $\widetilde{v}_\varepsilon \in X_0^s(\Omega)$ satisfying, in weak sense,

$$\begin{cases} (-\Delta)^s \widetilde{v}_\varepsilon \le -g(., \widetilde{v}_\varepsilon + \varepsilon) + \lambda h \text{ in } \Omega, \\ \widetilde{v}_\varepsilon = 0 \text{ in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{2.41}$$

and such that $\widetilde{v}_\varepsilon \ge cd_\Omega^s$ a.e. in Ω, where c is a positive constant independent of ε. Then for any $\varepsilon > 0$ there exists a sequence $\left\{u_j\right\}_{j\in\mathbb{N}} \subset X_0^s(\Omega)$ such that:

i) $u_1 = \overline{u}$ and $u_j \le u_{j-1}$ for any $j \ge 2$.

ii) $\widetilde{v}_\varepsilon \le u_j \le \overline{u}$ for all $j \in N$.

iii) For any $j \ge 2$, u_j satisfies, in weak sense,

$$\begin{cases} (-\Delta)^s u_j = -g\left(., u_{j-1} + \varepsilon\right) + \lambda h \text{ in } \Omega, \\ u_j = 0 \text{ in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

iv) There exist positive constants c and c′ independent of ε and j such that, for all j, $cd_\Omega^s \le u_j \le c'd_\Omega^s$ $a.e.$ in Ω.

Proof. By Remark 2.1 *iii)*, there exists a positive constant c' such that $\overline{u} \leq c'd_{\Omega}^{s}$ in Ω. We construct inductively a sequence $\left\{u_j\right\}_{j\in\mathbb{N}}$ satisfying the assertions *i)-iii)* of the lemma: Let $u_1 := \overline{u}$. Thus, in weak sense, $(-\Delta)^s u_1 = \lambda h \geq -g\left(.,\widetilde{v}_\varepsilon + \varepsilon\right) + \lambda h$ in Ω. Thus $(-\Delta)^s \left(u_1 - \widetilde{v}_\varepsilon\right) \geq 0$ in Ω. Then, by the maximum principle in Remark 2.1 *i)*, $u_1 \geq \widetilde{v}_\varepsilon$ in Ω, and so $u_1 \geq cd_{\Omega}^{s}$ in Ω. Then, for some positive constant c'', $\left|-g\left(.,u_1 + \varepsilon\right) + \lambda h\right| \leq c''\left(1 + g\left(.,cd_{\Omega}^{s}\right)\right)$ in Ω and, by *g1)*, $g\left(.,cd_{\Omega}^{s}\right) \in L^{\infty}(\Omega)$. Thus there exists a weak solution $u_2 \in X_0^s(\Omega)$ to the problem

$$\begin{cases} (-\Delta)^s u_2 = -g\left(.,u_1 + \varepsilon\right) + \lambda h \text{ in } \Omega, \\ u_2 = 0 \text{ in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

Since, in weak sense, $(-\Delta)^s u_2 \leq \lambda h = (-\Delta)^s u_1$ in Ω, the maximum principle in Remark 2.1 *i)* gives $u_2 \leq u_1$ in Ω. Since $u_1 \geq \widetilde{v}_\varepsilon$ in Ω we have, in weak sense, $(-\Delta)^s u_2 = -g\left(.,u_1 + \varepsilon\right) + \lambda h \geq -g\left(.,\widetilde{v}_\varepsilon + \varepsilon\right) + \lambda h$ in Ω. Also, $(-\Delta)^s \widetilde{v}_\varepsilon \leq -g\left(.,\widetilde{v}_\varepsilon + \varepsilon\right) + \lambda h$ in Ω and so, by the maximum principle, $u_2 \geq \widetilde{v}_\varepsilon$ in Ω. Then *i)-iii)* hold for $j = 1$.
Supposed constructed $u_1, ..., u_k$ such that *i)-iii)* hold for $1 \leq j \leq k$. Then, for some positive constant c''', $\left|-g\left(.,u_k + \varepsilon\right) + \lambda h\right| \leq c''\left(1 + g\left(.,cd_{\Omega}^{s}\right)\right)$ in Ω and so, as before, there exists a weak solution $u_{k+1} \in X_0^s(\Omega)$ to the problem

$$\begin{cases} (-\Delta)^s u_{k+1} = -g\left(.,u_k + \varepsilon\right) + \lambda h \text{ in } \Omega, \\ u_{k+1} = 0 \text{ in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

By our inductive hypothesis, $u_k \geq \widetilde{v}_\varepsilon$ in Ω. Then, in weak sense, $(-\Delta)^s u_{k+1} = -g\left(.,u_k + \varepsilon\right) + \lambda h \geq -g\left(.,\widetilde{v}_\varepsilon + \varepsilon\right) + \lambda h \geq (-\Delta)^s \widetilde{v}_\varepsilon$ in Ω and thus, by the maximum principle, $u_{k+1} \geq \widetilde{v}_\varepsilon$ in Ω. If $k = 2$ we have $u_k \leq u_{k-1}$ in Ω. If $k > 2$, by the inductive hypothesis we have, in weak sense, $(-\Delta)^s u_k = -g\left(.,u_{k-1} + \varepsilon\right) + \lambda h \leq -g\left(.,u_{k-2} + \varepsilon\right) + \lambda h$ in Ω. Also, $(-\Delta)^s u_k = -g\left(.,u_{k-1} + \varepsilon\right) + \lambda h$ in Ω. Thus, by the maximum principle, $u_{k+1} \leq u_k$ in Ω. Again by the inductive hypothesis $u_k \leq \overline{u}$ in Ω and then, since $u_{k+1} \leq u_k$ in Ω, we get $u_{k+1} \leq \overline{u}$ in Ω.
Since for all j, $v_\varepsilon \leq u_j \leq \overline{u}$ in Ω, *iv)* follows from the facts that $\overline{u} \leq c'd_{\Omega}^{s}$ in Ω, and that $\widetilde{v}_\varepsilon \geq cd_{\Omega}^{s}$ in Ω, with c and c' positive constants independent of ε and j. □

Lemma 2.10. *Let $\lambda > 0$. Assume that we have, for each $\varepsilon > 0$, a function $\widetilde{v}_\varepsilon \in X_0^s(\Omega)$ satisfying, in weak sense, (2.41), and such that $\widetilde{v}_\varepsilon \geq cd_{\Omega}^{s}$ a.e. in Ω, with c a positive constant independent of ε. Then for any $\varepsilon > 0$ there exists a weak solution u_ε of the problem*

$$\begin{cases} (-\Delta)^s u_\varepsilon = -g\left(.,u_\varepsilon + \varepsilon\right) + \lambda h \textit{ in } \Omega, \\ u_\varepsilon = 0 \textit{ in } \mathbb{R}^n \setminus \Omega, \\ u_\varepsilon > 0 \textit{ in } \Omega. \end{cases} \tag{2.42}$$

such that:
i) $u_\varepsilon \geq \widetilde{v}_\varepsilon$ and there exist positive constants c and c' independent of ε such that $cd_{\Omega}^{s} \leq u_\varepsilon \leq c'd_{\Omega}^{s}$ in Ω,
ii) If $\underline{u}_\varepsilon \in X_0^s(\Omega)$ and $(-\Delta)^s \underline{u}_\varepsilon \leq -g\left(.,\underline{u}_\varepsilon + \varepsilon\right) + \lambda h$ in Ω, then $\underline{u}_\varepsilon \leq u_\varepsilon$ in Ω,
iii) If $0 < \varepsilon < \eta$ then $u_\varepsilon \leq u_\eta$.

Proof. Let $\left\{u_j\right\}_{j\in\mathbb{N}}$ be as given by Lemma 2.9. For $x \in \Omega$, let $u_\varepsilon(x) := \lim_{j\to\infty} u_j(x)$. By Lemma 2.8, $\left\{u_j\right\}_{j\in\mathbb{N}}$ converges to u_ε in $X_0^s(\Omega)$ and u_ε is a weak solution to (2.42). From Lemma 2.9 *iv)* we have $u_\varepsilon \geq \widetilde{v}_\varepsilon$ in Ω and $cd_{\Omega}^{s} \leq u_\varepsilon \leq c'd_{\Omega}^{s}$ in Ω, for some positive constants c and c' independent of ε. Then *i)* holds. If $\underline{u}_\varepsilon \in X_0^s(\Omega)$ and $(-\Delta)^s \underline{u}_\varepsilon \leq -g\left(.,\underline{u}_\varepsilon + \varepsilon\right) + \lambda h$ in Ω, then $(-\Delta)^s \underline{u}_\varepsilon \leq \lambda h = (-\Delta)^s u_1$ in Ω, and so $\underline{u}_\varepsilon \leq u_1$. Thus $(-\Delta)^s \underline{u}_\varepsilon \leq -g\left(.,\underline{u}_\varepsilon + \varepsilon\right) + \lambda h \leq -g\left(.,u_1 + \varepsilon\right) + \lambda h = (-\Delta)^s u_2$, then $\underline{u}_\varepsilon \leq u_2$ and, iterating this procedure, we obtain that $\underline{u}_\varepsilon \leq u_j$ for all j. Then $\underline{u}_\varepsilon \leq u_\varepsilon$. Thus *ii)* holds. Finally, *iii)* is immediate from *ii)*. □

Lemma 2.11. *Let $\varepsilon > 0$ and let τ_1 and M_1 be as in Remarks 2.2, and 2.3 respectively. Let w_ε be as in Lemma 2.5. Then, for $\lambda \geq \tau_1 M_1$, there exists a weak solution $u_\varepsilon \in X_0^s(\Omega)$ of problem (2.42) such that*
i) $u_\varepsilon \geq w_\varepsilon$ and there exist positive constants c and c', both independent of ε, such that $cd_\Omega^s \leq u_\varepsilon \leq c'd_\Omega^s$ in Ω,
ii) If $\underline{u}_\varepsilon \in X_0^s(\Omega)$ and $(-\Delta)^s \underline{u}_\varepsilon \leq -g\left(.,\underline{u}_\varepsilon + \varepsilon\right) + \lambda h$ in Ω, then $\underline{u}_\varepsilon \leq u_\varepsilon$ in Ω,
iii) If $0 < \varepsilon_1 < \varepsilon_2$ then $u_{\varepsilon_1} \leq u_{\varepsilon_2}$.

Proof. Let $\lambda \geq \tau_1 M_1$ and let w_ε be as in Lemma 2.5. We have, in weak sense,

$$\begin{cases} -\Delta w_\varepsilon = -g\left(.,w_\varepsilon + \varepsilon\right) + \tau_1 M_1 h \text{ in } \Omega, \\ w_\varepsilon = 0 \text{ on } \mathbb{R}^n \setminus \Omega, \end{cases}$$

Also, $-g\left(.,w_\varepsilon + \varepsilon\right) + \tau_1 M_1 h \leq -g\left(.,w_\varepsilon + \varepsilon\right) + \lambda h$ in Ω, and $cd_\Omega^s \leq w_\varepsilon \leq c'd_\Omega^s$ in Ω, with c and c' positive constants independent of ε. Then the lemma follows from Lemma 2.10. □

3. Proof of the main results

Lemma 3.1. *Let $\lambda > 0$. If problem (1.5) has a weak solution in $\mathcal{E}$, then it has a weak solution $u \in \mathcal{E}$ satisfying $u \geq \psi$ a.e. in Ω for any $\psi \in \mathcal{E}$ such that $-\Delta\psi \leq -g\left(.,\psi\right) + \lambda h$ in Ω.*

Proof. Let $u^* \in \mathcal{E}$ be a weak solution of (1.5), and let $\overline{u}$ be as in (2.40). By the comparison principle $u^* \leq \overline{u}$ in Ω. We construct inductively a sequence $\left\{u_j\right\}_{j\in\mathbb{N}} \subset \mathcal{E}$ with the following properties: $u_1 = \overline{u}$ and
i) $u^* \leq u_j \leq \overline{u}$ for all $j \in \mathbb{N}$
ii) $g\left(.,u_j\right) \in \left(X_0^s(\Omega)\right)'$ for all $j \in \mathbb{N}$.
iii) $u_j \leq u_{j-1}$ for all $j \geq 2$.
iv) For all $j \geq 2$

$$\begin{cases} (-\Delta)^s u_j = -g\left(.,u_{j-1}\right) + \lambda h \text{ in } \Omega, \\ u_j = 0 \text{ on } \mathbb{R}^n \setminus \Omega. \end{cases}$$

To do it, define $u_1 =: \overline{u}$. Thus $u_1 \in \mathcal{E}$. By the comparison principle, $u^* \leq \overline{u}$, i.e., $u^* \leq u_1$. By Remark 2.1 there exist positive constants c and c' such that $cd_\Omega^s \leq \overline{u} \leq c'd_\Omega^s$ in Ω. Thus $\left|-g\left(.,\overline{u}\right) + \lambda h\right| \leq g\left(.,cd_\Omega^s\right) + \lambda \|h\|_\infty$ and so $-g\left(.,u_1\right) + \lambda h \in \left(X_0^s(\Omega)\right)'$. Thus *i)* and *ii)* hold for $j = 1$. Define u_2 as the weak solution of

$$\begin{cases} (-\Delta)^s u_2 = -g\left(.,u_1\right) + \lambda h \text{ in } \Omega, \\ u_2 = 0 \text{ on } \mathbb{R}^n \setminus \Omega. \end{cases}$$

Then, in weak sense,

$$\begin{cases} (-\Delta)^s u_2 \leq (-\Delta)^s u_1 \text{ in } \Omega, \\ u_2 = 0 = u_1 \text{ on } \mathbb{R}^n \setminus \Omega. \end{cases}$$

and so $u_2 \leq u_1 = \overline{u}$ *a.e.* in Ω. Since $u_1 \geq u^*$, we have, in weak sense,

$$\begin{cases} (-\Delta)^s u_2 = -g(., u_1) + \lambda h \geq -g(., u^*) + \lambda h = (-\Delta)^s u^* \text{ in } \Omega, \\ u_2 = 0 = u^* \text{ on } \mathbb{R}^n \setminus \Omega. \end{cases}$$

and then $u_2 \geq u^*$ *a.e.* in Ω. Thus $u^* \leq u_2 \leq \overline{u}$. In particular this gives $u_2 \in \mathcal{E}$. Let $c'' > 0$ such that $u^* \geq c'' d_\Omega$ in Ω. Now, $|-g(., u_2) + \lambda h| \leq g(., u_2) + \lambda h \leq g(., u^*) + \lambda h \leq g\left(., c'' d_\Omega^s\right) + \lambda \|h\|_\infty$ and so $-g(., u_2) + \lambda h \in \left(X_0^s(\Omega)\right)'$. Thus *i)-iv)* hold for $j = 2$. Suppose constructed, for $2 \leq j \leq k$, functions $u_j \in \mathcal{E}$ with the properties *i)-iv)*. Define u_{k+1} by

$$\begin{cases} (-\Delta)^s u_{k+1} = -g(., u_k) + \lambda h \text{ in } \Omega, \\ u_{k+1} = 0 \text{ on } \mathbb{R}^n \setminus \Omega. \end{cases}$$

Thus, by the comparison principle, $u_{k+1} \leq \overline{u}$. Also, by the inductive hypothesis, $u_k \geq u^*$, then

$$\begin{cases} (-\Delta)^s u_{k+1} = -g(., u_k) + \lambda h \geq -g(., u^*) + \lambda h \text{ in } \Omega, \\ u_{k+1} = 0 = u^* \text{ on } \mathbb{R}^n \setminus \Omega. \end{cases}$$

and so $u_{k+1} \geq u^*$. Then $u^* \leq u_{k+1} \leq \overline{u}$. In particular $u_{k+1} \in \mathcal{E}$. Again by the inductive hypothesis, $u_k \leq u_{k-1}$. Then

$$\begin{cases} (-\Delta)^s u_{k+1} = -g(., u_k) + \lambda h \leq -g(., u_{k-1}) + \lambda h = (-\Delta)^s u_k \text{ in } \Omega, \\ u_{k+1} = 0 = u_k \text{ on } \mathbb{R}^n \setminus \Omega. \end{cases}$$

and so $u_{k+1} \leq u_k$. Also, $|-g(., u_{k+1}) + \lambda h| \leq g(., u_{k+1}) + \lambda h \leq g(., u^*) + \lambda h \leq g\left(., c'' d_\Omega^s\right) + \lambda \|h\|_\infty$ and so $-g(., u_2) + \lambda h \in \left(X_0^s(\Omega)\right)'$. Thus *i)-iv)* hold for $j = k + 1$, which completes the inductive construction of the sequence $\left\{u_j\right\}_{j \in \mathbb{N}}$. For $x \in \mathbb{R}^n$ let $u(x) := \lim_{j \to \infty} u_j(x)$. By *i)* we have $c'' d_\Omega^s \leq u_j \leq c' d_\Omega^s$ in Ω for all $j \in \mathbb{N}$, and so $c'' d_\Omega^s \leq u \leq c' d_\Omega^s$ in Ω. By Lemma 2.8 $\left\{u_j\right\}_{j \in \mathbb{N}}$ converges in $X_0^s(\Omega)$ to some weak solution $u^{**} \in X_0^s(\Omega)$ of problem (1.5). Thus, by the Poincaré inequality, $\left\{u_j\right\}_{j \in \mathbb{N}}$ converges to u^{**} in $L^{2_s^*}(\Omega)$, which implies $u = u^{**}$. Thus $u \in X_0^s(\Omega)$ and u is a weak solution of problem (1.5). Since $c'' d_\Omega^s \leq u_j \leq c' d_\Omega^s$ in Ω for all j, we have $c'' d_\Omega^s \leq u \leq c' d_\Omega^s$ in Ω. Thus $u \in \mathcal{E}$. Let $\psi \in \mathcal{E}$ such that $-\Delta \psi \leq -g(., \psi) + \lambda h$ in Ω. By the comparison principle, $\psi \leq u$ *a.e.* in Ω. An easy induction shows that $\psi \leq u_j$ for all j. Indeed, by the comparison principle, $\psi \leq \overline{u} = u_1$. Then

$$\begin{cases} (-\Delta)^s \psi \leq -g(., \psi) + \lambda h \leq -g(., u_1) + \lambda h = (-\Delta)^s u_2 \text{ in } \Omega, \\ \psi = 0 = u_2 \text{ on } \mathbb{R}^n \setminus \Omega. \end{cases}$$

Thus, again by the comparison principle, $\psi \leq u_2$. Suppose that $k \geq 2$ and $\psi \leq u_k$. Then, in weak sense,

$$\begin{cases} (-\Delta)^s \psi \leq -g(., \psi) + \lambda h \leq -g(., u_k) + \lambda h = (-\Delta)^s u_{k+1} \text{ in } \Omega, \\ \psi = 0 = u_{k+1} \text{ on } \mathbb{R}^n \setminus \Omega, \end{cases}$$

which gives $\psi \leq u_{k+1}$. Thus $\psi \leq u_j$ for all j, and so $\psi \leq u$. □

Proof of Theorem 1. Let $\left\{\varepsilon_j\right\}_{j\in\mathbb{N}} \subset (0,\infty)$ be a decreasing sequence such that $\lim_{j\to\infty}\varepsilon_j = 0$. For $\lambda \geq \tau_1 M_1$ and $j \in \mathbb{N}$, let u_{ε_j} be the weak solution of problem (2.42), given by Lemma 2.11, taking there $\varepsilon = \varepsilon_j$. Then $\left\{u_{\varepsilon_j}\right\}_{j\in\mathbb{N}}$ is a nonincreasing sequence in $X_0^s(\Omega)$ and there exist positive constants c and c' such that $cd_\Omega^s \leq u_j \leq c'd_\Omega^s$ in Ω for all $j \in \mathbb{N}$. Therefore, by Lemma 2.8, $\left\{u_{\varepsilon_j}\right\}_{j\in\mathbb{N}}$ converges in $X_0^s(\Omega)$ to some weak solution $u \in X_0^s(\Omega)$ of problem (1.5). Let

$$\mathcal{T} := \{\lambda > 0 : \text{ problem (1.5) has a weak solution } u \in \mathcal{E}\}.$$

Thus $\lambda \in \mathcal{T}$ whenever $\lambda \geq \tau_1 M_1$. Consider now an arbitrary $\lambda \in \mathcal{T}$, and let $\lambda' > \lambda$. Let $u \in \mathcal{E}$ be a weak solution of the problem

$$\begin{cases} (-\Delta)^s u = -g(.,u) + \lambda h \text{ in } \Omega, \\ u = 0 \text{ on } \mathbb{R}^n \setminus \Omega. \end{cases}$$

Let $\left\{\varepsilon_j\right\}_{j\in\mathbb{N}} \subset (0,\infty)$ be a decreasing sequence such that $\lim_{j\to\infty}\varepsilon_j = 0$. We have, in weak sense,

$$\begin{cases} (-\Delta)^s u = -g(.,u) + \lambda h \leq -g\left(.,u+\varepsilon_j\right) + \lambda' h \text{ in } \Omega, \\ u = 0 \text{ on } \mathbb{R}^n \setminus \Omega. \end{cases}$$

Then, by Lemma 2.9, used with $\varepsilon = \varepsilon_j$, $\widetilde{v}_{\varepsilon_j} = u$, and with λ replaced by λ', there exists a nonincreasing sequence $\left\{\widetilde{u}_{\varepsilon_j}\right\}_{j\in\mathbb{N}} \subset X_0^s(\Omega)$ such that

$$\begin{cases} (-\Delta)^s \widetilde{u}_{\varepsilon_j} = -g\left(.,\widetilde{u}_{\varepsilon_j} + \varepsilon_j\right) + \lambda' h \text{ in } \Omega, \\ \widetilde{u}_{\varepsilon_j} = 0 \text{ in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

satisfying that $\widetilde{u}_{\varepsilon_j} \geq u$ for all $j \in \mathbb{N}$, and $cd_\Omega^s \leq \widetilde{u}_{\varepsilon_j} \leq c'd_\Omega^s$ in Ω for some positive constants c and c' independent of j. Let $\widetilde{u} := \lim_{j\to\infty}\widetilde{u}_{\varepsilon_j}$. Proceeding as in the first part of the proof, we get that $\widetilde{u} \in \mathcal{E}$ and that $\widetilde{u}$ is a weak solution of problem (1.5). Then $\lambda' \in \mathcal{T}$ whenever $\lambda' > \lambda$ for some $\lambda \in \mathcal{T}$. Thus there exists $\lambda^* \geq 0$ such that $(\lambda^*,\infty) \subset \mathcal{T} \subset [\lambda^*,\infty)$.
By Lemma 3.1, for any $\lambda \in \mathcal{T}$ there exists a weak solution $u \in \mathcal{E}$ of problem (1.5) such that $u \geq \psi$ *a.e.* in Ω for any $\psi \in \mathcal{E}$ such that $(-\Delta)^s \psi \leq -g(.,\psi) + \lambda h$ in Ω.
Suppose now that $g(.,s) \geq bs^{-\beta}$ *a.e.* in Ω for any $s \in (0,\infty)$ for some $b \in L^\infty(\Omega)$ such that $0 \leq b \not\equiv 0$ in Ω. Then there exist a constant $\eta_0 > 0$ and a measurable set $\Omega_0 \subset \Omega$ such that $|\Omega_0| > 0$ and $b \geq \eta_0$ in Ω_0. Let λ_1 be the principal eigenvalue for $(-\Delta)^s$ in Ω with Dirichlet boundary condition $\varphi_1 = 0$ on $\mathbb{R}^n \setminus \Omega$, and let $\varphi_1 \in X_0^s(\Omega)$ be an associated positive principal eigenfunction. Then

$$\lambda_1 \int_\Omega \varphi\varphi_1 = \int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(\varphi(x)-\varphi(y))(\varphi_1(x)-\varphi_1(y))}{|x-y|^{n+2s}}dxdy$$

and $\varphi_1 > 0$ *a.e.* in Ω (see e.g., [25], Theorem 3.1). Let $\lambda \in \mathcal{T}$ and let $u \in \mathcal{E}$ be a weak solution of (1.5). Thus

$$\lambda_1 \int_\Omega u\varphi_1 = \int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(u(x)-u(y))(\varphi_1(x)-\varphi_1(y))}{|x-y|^{n+2s}}dxdy$$

$$= \int_{\Omega} (-\varphi_1 g(.,u) + \lambda h\varphi_1) \leq \int_{\Omega} \left(-bu^{-\beta}\varphi_1 + \lambda h\varphi_1\right)$$

and so

$$\lambda \int_{\Omega} h\varphi_1 \geq \int_{\Omega_0} \left(\lambda_1 u + bu^{-\beta}\right)\varphi_1 \geq \inf_{s>0} \left(\lambda_1 s + \eta_0 s^{-\beta}\right) \int_{\Omega_0} \varphi_1$$

thus $\lambda \geq \inf_{s>0} \left(\lambda_1 s + \eta_0 s^{-\beta}\right)\left(\int_{\Omega} h\varphi_1\right)^{-1} \int_{\Omega_0} \varphi_1$ for any $\lambda \in \mathcal{T}$. Then $\lambda^* > 0$. □

Lemma 3.2. *Let $g : \Omega \times (0,\infty) \to [0,\infty)$ be a Carathéodory function. Assume that $s \to g(x,s)$ is nonincreasing for a.e. $x \in \Omega$, and that,for some $a \in L^\infty(\Omega)$ and $\beta \in [0,s)$, $g(.,s) \leq as^{-\beta}$ a.e. in Ω for any $s \in (0,\infty)$. Then g satisfies the conditions g1)-g5) of Theorem 1.2.*

Proof. Clearly g satisfies *g1)* and *g2).* Let $\sigma > 0$. By Lemma 2.7, $0 \leq g\left(.,\sigma d_{\Omega}^{s}\right) \leq a\sigma^{-\beta} d_{\Omega}^{-s\beta} \in \left(X_0^s(\Omega)\right)'$ and so $g\left(.,\sigma d_{\Omega}^{s}\right) \in \left(X_0^s(\Omega)\right)'$. Also, from the comparison principle, $0 \leq ((-\Delta)^s)^{-1}\left(d_{\Omega}^{s} g\left(.,\sigma a d_{\Omega}^{s}\right)\right) \leq ((-\Delta)^s)^{-1}\left(\sigma^{-\beta} a d_{\Omega}^{s-\beta}\right)$ in Ω, and, since $a d_{\Omega}^{s-\beta} \in L^\infty(\Omega)$, by Remark 2.1 *iii),* there exists a positive constant c such that $((-\Delta)^s)^{-1}\left(\sigma a d_{\Omega}^{s-\beta}\right) \leq c d_{\Omega}^{s}$ in Ω. Thus $d_{\Omega}^{-s}((-\Delta)^s)^{-1}\left(d_{\Omega}^{s} g\left(.,\sigma d_{\Omega}^{s}\right)\right) \in L^\infty(\Omega)$. Then g satisfies *g3).* In particular, $d_{\Omega}^{-s}((-\Delta)^s)^{-1}\left(d_{\Omega}^{s} g\left(.,d_{\Omega}^{s}\right)\right) \in L^\infty(\Omega)$. Since,for $\sigma \geq 1$,

$$0 \leq \left(\sigma d_{\Omega}^{s}\right)^{-1}((-\Delta)^s)^{-1}\left(d_{\Omega}^{s} g\left(.,\sigma d_{\Omega}^{s}\right)\right) \leq \sigma^{-1} d_{\Omega}^{-s}((-\Delta)^s)^{-1}\left(d_{\Omega}^{s} g\left(.,d_{\Omega}^{s}\right)\right),$$

we get $\lim_{\sigma\to\infty} \left\|\left(\sigma d_{\Omega}^{s}\right)^{-1}((-\Delta)^s)^{-1}\left(d_{\Omega}^{s} g\left(.,\sigma d_{\Omega}^{s}\right)\right)\right\|_\infty = 0$. Also, by the comparison principle,

$$0 \leq d_{\Omega}^{-s}((-\Delta)^s)^{-1}(g(.,\sigma)) \leq \sigma^{-\beta} d_{\Omega}^{-s}((-\Delta)^s)^{-1}(a),$$

and, by Remark 2.1 *iii)*, $d_{\Omega}^{-s}((-\Delta)^s)^{-1}(a) \in L^\infty(\Omega)$. Thus

$$\lim_{\sigma\to\infty} \left\|d_{\Omega}^{-s}((-\Delta)^s)^{-1}(g(.,\sigma))\right\|_{L^\infty(\Omega)} = 0.$$

Then *g4)* holds. Finally, $0 \leq d_{\Omega}^{s} g\left(.,\sigma d_{\Omega}^{s}\right) \leq \sigma^{-\beta} d_{\Omega}^{s-\beta} a$ and so *g5)* holds. □

Proof of Theorem 1.3. The theorem follows from Lemma 3.2 and Theorem 1.2. □

Conflict of interest

The author declare no conflicts of interest in this paper.

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