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Research article

Phase-field system with two temperatures and a nonlinear coupling term

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Abstract: The subject of this paper is the qualitative study of a generalization of Caginalp phase-field system involving two temperatures and a nonlinear coupling. First, we prove the well-posedness of the corresponding initial and boundary value problem, and we study the dissipativity properties of the system, in terms of bounded absorbing sets. We end by analyzing the spatial behavior of solutions in a semi-infinite cylinder, assuming the existence of such solutions.

Keywords: Caginalp phase-field system; two temperatures; well-posedness; disspativity; spatial bahavior; Phragmén-Lindelöf alternative
Mathematics Subject Classification: 35K55, 80A22

1. Introduction

The Caginalp phase-field system

$$\frac{\partial u}{\partial t} - \Delta u + f(u) = \theta, \tag{1.1}$$

$$\frac{\partial \theta}{\partial t} - \Delta \theta = -\frac{\partial u}{\partial t}, \tag{1.2}$$

has been introduced in [1] in order to describe the phase transition phenomena in certain class of material. In this context, θ denotes the relative temperature (relative to the equilibrium melting temperature), and u is the phase-field or order parameter, f is a given function (precisely, the derivaritve of a double-well potential F). This system has received much attention (see for example, [2], [3], [4], [5], [6], [7], [8], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [23], [30], [34] and [42]). These equations can be derived by introducing the (total Ginzburg-Landau) free energy:

$$\psi = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + F(u) - u\theta - \frac{1}{2}\theta^2\right) dx, \tag{1.3}$$

where Ω is the domain occupied by the system (here, we assume that it is a bounded and smooth domain of $\mathbb{R}^n$, $n = 1, 2$ or 3, with boundary $\partial\Omega$), and the enthalpy

$$H = u + \theta. \tag{1.4}$$

Then, the evolution equation for the order parameter u is given by:

$$\frac{\partial u}{\partial t} = -\delta_u \psi, \tag{1.5}$$

where δ_u stands for the variational derivative with respect to u, which yields (1.1). Then, we have the energy equation

$$\frac{\partial H}{\partial t} = -\operatorname{div} q, \tag{1.6}$$

where q is the heat flux. Assuming finally the classical Fourier law for heat conduction, which prescribes the heat flux as

$$q = -\nabla\theta, \tag{1.7}$$

we obtain (1.2). Now, a well-known side effect of the Fourier heat law is the infinite speed of propagation of thermal disturbances, deemed physically unreasonable and thus called paradox of heat conduction (see, for example, [9]). In order to account for more realistic features, several variations of (1.7), based, for example, on the Maxwell-Cattaneo law or recent laws from thermomechanics, have been proposed in the context of the Caginalp phase-field system (see, for example, [20], [21], [22], [24], [25], [26], [27], [28], [29], [31], [32], [36], [37], [38], [39], [45], [46] and [47]).

A different approach to heat conduction was proposed in the Sixties (see, [48], [49] and [50]), where it was observed that two temperatures are involved in the definition of the entropy: the conductive temperature θ, influencing the heat conduction contribution, and the thermodynamic temperature, appearing in the heat supply part. For time-independent models, it appears that these two temperatures coincide in absence of heat supply. Actually, they are different generally in the time depedent case see, for example, [20] and references therein for more discussion on the subject. In particular, this happens for non-simple materials. In that case, the two temperatures are related as follows (see [43], [44]):

$$\theta = \varphi - \Delta\varphi. \tag{1.8}$$

Our aim in this paper is to study a generalization of the Caginalp phase-field system based on this two temperatures theory and the usual Fourier law with a nonlinear coupling.

The purpose of our study is the following initial and boundary value problem

$$\frac{\partial u}{\partial t} - \Delta u + f(u) = g(u)(\varphi - \Delta\varphi), \tag{1.9}$$

$$\frac{\partial \varphi}{\partial t} - \Delta\frac{\partial \varphi}{\partial t} - \Delta\varphi = -g(u)\frac{\partial u}{\partial t}, \tag{1.10}$$

$$u = \varphi = 0 \text{ on } \partial\Omega, \tag{1.11}$$

$$u|_{t=0} = u_0, \ \varphi|_{t=0} = \varphi_0. \tag{1.12}$$

The paper is organized as follows. In Section 2, we give the derivation of the model. The Section 3 states existence, regularity and uniqueness results. In Section 4, we address the question of dissipativity properties of the system. The last section, analyzes the spatial behavior of solutions in a semi-infinite cylinder, assuming their existence.

Thoughout this paper, the same letters c, c', c'', and sometimes c''' denote constants which may change from line to line and also $\|.\|_p$ will denote the usual L^p norm and $(.,.)$ the usual L^2 scalar product. More generally, we will denote by $\|.\|_X$ the norm in the Banach space X. When there is no possible confusion, $\|.\|$ will be noted instead of $\|.\|_2$.

2. Derivation of the model

In our case, to obtain equations (1.9) and (1.10), the total free energy reads in terms of the conductive temperature θ,

$$\psi(u,\theta) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + F(u) - G(u)\theta - \frac{1}{2}\theta^2\right) dx, \tag{2.1}$$

where $f = F'$ and $g = G'$, and (1.5) yields, in view of (1.8), the evolution equation for the order parameter (1.9). Furthermore, the enthalpy now reads

$$H = G(u) + \theta = G(u) + \varphi - \Delta\varphi, \tag{2.2}$$

which yields thanks to (1.6), the energy equation,

$$\frac{\partial\varphi}{\partial t} - \Delta\frac{\partial\varphi}{\partial t} + \operatorname{div} q = -g(u)\frac{\partial u}{\partial t}. \tag{2.3}$$

Considering the usual Fourier law ($q = -\nabla\varphi$), one has (1.10).

Remark 2.1. *We can note that we still have an infinite speed of propagation here.*

3. Existence and uniqueness of solutions

Before stating the existence result, we make some assumptions on nonlinearities f and g:

$$|G(s)|^2 \le c_1 F(s) + c_2, \quad c_0, c_1, c_2 \ge 0, \tag{3.1}$$

$$|g(s)s| \le c_3(|G(s)|^2 + 1), \quad c_3 \ge 0, \tag{3.2}$$

$$c_4\, s^{k+2} - c_5 \le F(s) \le f(s)s + c_0 \le c_6\, s^{k+2} - c_7, \quad c_4, c_6 > 0,\ c_5, c_7 \ge 0, \tag{3.3}$$

$$|g(s)| \le c_8(|s| + 1), \quad |g'(s)| \le c_9 \quad c_8, c_9 \ge 0, \tag{3.4}$$

$$|f'(s)| \le c_{10}(|s|^k + 1), \quad c_{10} \ge 0, \tag{3.5}$$

where k is an integer.

Theorem 3.1. *We assume that* (3.1)–(3.4) *hold true. For all initial data* $(u_0,\ \varphi_0) \in H^1_0(\Omega) \cap L^{k+2}(\Omega) \times H^1_0(\Omega) \cap H^2(\Omega)$*, the problem* (1.9)–(1.12) *possesses at least one solution* (u, φ) *with the following regularity* $u \in L^\infty(0,T; H^1_0(\Omega)) \cap L^{k+2}(\Omega)$, $\frac{\partial u}{\partial t} \in L^2(0,T; L^2(\Omega))$, $\varphi \in L^\infty(0,T; H^1_0(\Omega) \cap H^2(\Omega))$ *and* $\frac{\partial\varphi}{\partial t} \in L^2(0,T; H^1_0(\Omega))$.

Proof. The proof is based on the Galerkin scheme. Here, we just make formally computations to get a priori estimates, having in mind that these estimates can be rigourously justified using the Galerkin scheme see, for example, [11], [12] and [41] for details.

Multiplying (1.9) by $\frac{\partial u}{\partial t}$ and integrating over Ω, we get

$$\frac{1}{2}\frac{d}{dt}\Big(\|\nabla u\|^2 + 2\int_\Omega F(u)\,dx\Big) + \Big\|\frac{\partial u}{\partial t}\Big\|^2 = \int_\Omega g(u)\frac{\partial u}{\partial t}(\varphi - \Delta\varphi)\,dx. \tag{3.6}$$

Multiplying (1.10) by $\varphi - \Delta\varphi$ and integrating over Ω, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|\varphi\|^2 + 2\|\nabla\varphi\|^2 + \|\Delta\varphi\|^2) + \|\nabla\varphi\|^2 + \|\Delta\varphi\|^2 \\ = -\int_\Omega g(u)\frac{\partial u}{\partial t}(\varphi - \Delta\varphi)\,dx.\end{aligned} \tag{3.7}$$

Now, summing (3.6) and (3.7), we are led to,

$$\begin{aligned}\frac{d}{dt}\Big(\|\nabla u\|^2 + 2\int_\Omega F(u)\,dx + \|\varphi\|^2 + 2\|\nabla\varphi\|^2 + \|\Delta\varphi\|^2\Big) \\ + 2\Big(\Big\|\frac{\partial u}{\partial t}\Big\|^2 + \|\nabla\varphi\|^2 + \|\Delta\varphi\|^2\Big) = 0.\end{aligned} \tag{3.8}$$

Multiplying (1.9) by u and integrating over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \|\nabla u\|^2 + \int_\Omega f(u)u\,dx = \int_\Omega g(u)u(\varphi - \Delta\varphi)\,dx. \tag{3.9}$$

Using (3.2)–(3.3), (3.9) becomes

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|^2 + \|\nabla u\|^2 + c\int_\Omega F(u)\,dx \\ \le c'\int_\Omega |G(u)|^2\,dx + \frac{1}{2}(\|\varphi\|^2 + \|\Delta\varphi\|^2) + c''.\end{aligned} \tag{3.10}$$

Adding (3.8) and (3.10), one has

$$\begin{aligned}\frac{dE_1}{dt} + 2\Big(\|\nabla u\|^2 + c\int_\Omega F(u)\,dx + \Big\|\frac{\partial u}{\partial t}\Big\|^2 + \|\nabla\varphi\|^2\Big) + \|\Delta\varphi\|^2 \\ \le c'\int_\Omega |G(u)|^2\,dx + \|\varphi\|^2 + c'',\end{aligned} \tag{3.11}$$

where

$$E_1 = \|u\|^2 + \|\nabla u\|^2 + 2\int_\Omega F(u)\,dx + \|\varphi\|^2 + 2\|\nabla\varphi\|^2 + \|\Delta\varphi\|^2 \tag{3.12}$$

enjoys

$$E_1 \le c\Big(\|u\|^2_{H^1(\Omega)} + \|u\|^{k+2}_{k+2} + \|\varphi\|^2_{H^2(\Omega)}\Big) - c' \tag{3.13}$$

and

$$E_1 \le c''\Big(\|u\|^2_{H^1(\Omega)} + \|u\|^{k+2}_{k+2} + \|\varphi\|^2_{H^2(\Omega)}\Big) - c'''. \tag{3.14}$$

Multiplying now (1.10) by $\frac{\partial\varphi}{\partial t}$ and integrating over Ω, we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla\varphi\|^2 + \left\|\frac{\partial\varphi}{\partial t}\right\|^2 + \left\|\nabla\frac{\partial\varphi}{\partial t}\right\|^2 = -\int_\Omega g(u)\frac{\partial u}{\partial t}\frac{\partial\varphi}{\partial t}\,dx. \tag{3.15}$$

Taking into account (3.4) and using Hölder's inequality, we get

$$\frac{1}{2}\frac{d}{dt}\|\nabla\varphi\|^2 + \frac{1}{2}\left\|\frac{\partial\varphi}{\partial t}\right\|^2 + \left\|\nabla\frac{\partial\varphi}{\partial t}\right\|^2 \le c(\|\nabla u\|^2 + 1)\left\|\frac{\partial u}{\partial t}\right\|^2 \tag{3.16}$$

and then, summing (3.11) and (3.16), we have

$$\begin{aligned}&\frac{dE_2}{dt} + 2\Big(\|\nabla u\|^2 + c\int_\Omega F(u)\,dx + \left\|\frac{\partial u}{\partial t}\right\|^2\\ &\quad + \|\nabla\varphi\|^2 + \frac{1}{2}\|\Delta\varphi\|^2 + \frac{1}{2}\left\|\frac{\partial\varphi}{\partial t}\right\|^2 + \left\|\nabla\frac{\partial\varphi}{\partial t}\right\|^2\Big)\\ &\quad \le c\int_\Omega |G(u)|^2\,dx + \|\varphi\|^2 + c''(\|\nabla u\|^2+1)\left\|\frac{\partial u}{\partial t}\right\|^2 + c''',\end{aligned} \tag{3.17}$$

where

$$E_2 = E_1 + \|\nabla\varphi\|^2 \tag{3.18}$$

satisfies similar estimates as E_1. We deduce from (3.1) and (3.17)

$$\frac{dE_2}{dt} + c\Big(\left\|\frac{\partial\varphi}{\partial t}\right\|^2 + \left\|\nabla\frac{\partial\varphi}{\partial t}\right\|^2\Big) \le c'E_2 + c'', \tag{3.19}$$

which achieve the proof.

□

For more regularity on solutions, we make following additional assumptions:

$$f(0) = 0 \text{ and } f'(s) \ge -c,\ c \ge 0. \tag{3.20}$$

We have:

Theorem 3.2. *Under assumptions of Theorem 3.1 and assuming that* (3.20) *is satisfied. For every initial data* $(u_0, \varphi_0) \in H_0^1(\Omega)\cap L^{k+2}(\Omega)\times H_0^1(\Omega)\cap H^2(\Omega)$*, the problem* (1.9)–(1.12) *admits at least one solution* (u, φ) *such that* $u \in L^\infty(0,T;H_0^1(\Omega))\cap L^{k+2}(\Omega)$, $\frac{\partial u}{\partial t} \in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega))$, $\varphi \in L^\infty(0,T;H_0^1(\Omega)\cap H^2(\Omega))$ *and* $\frac{\partial\varphi}{\partial t} \in L^2(0,T;H_0^1(\Omega)\cap H^2(\Omega))$.

Proof. As above proof, we focus on a priori estimates.

We multiply (1.10) by $-\Delta\frac{\partial\varphi}{\partial t}$ and have, integrating over Ω,

$$\frac{1}{2}\frac{d}{dt}\|\nabla\varphi\|^2 + \left\|\nabla\frac{\partial\varphi}{\partial t}\right\|^2 + \left\|\Delta\frac{\partial\varphi}{\partial t}\right\|^2 = \int_\Omega g(u)\frac{\partial u}{\partial t}\Delta\frac{\partial\varphi}{\partial t}\,dx. \tag{3.21}$$

Thanks to (3.4) and Hölder's inequality:

$$\begin{aligned}\int_\Omega g(u)\frac{\partial u}{\partial t}\Delta\frac{\partial\varphi}{\partial t}\,dx &\le c\int_\Omega(|u|+1)\left|\frac{\partial u}{\partial t}\right|\left|\Delta\frac{\partial\varphi}{\partial t}\right|dx\\ &\le c(\|\nabla u\|^2+1)\left\|\frac{\partial u}{\partial t}\right\|^2+\frac{1}{2}\left\|\Delta\frac{\partial\varphi}{\partial t}\right\|^2\end{aligned}\tag{3.22}$$

and then,

$$\frac{1}{2}\frac{d}{dt}\|\nabla\varphi\|^2+\left\|\nabla\frac{\partial\varphi}{\partial t}\right\|^2+\frac{1}{2}\left\|\Delta\frac{\partial\varphi}{\partial t}\right\|^2\le c(\|\nabla u\|^2+1)\left\|\frac{\partial u}{\partial t}\right\|^2.\tag{3.23}$$

Differentiating (1.9) with respect to time, we get

$$\frac{\partial^2 u}{\partial t^2}-\Delta\frac{\partial u}{\partial t}+f'(u)\frac{\partial u}{\partial t}=g'(u)\frac{\partial u}{\partial t}(\varphi-\Delta\varphi)+g(u)\Big(\frac{\partial\varphi}{\partial t}-\Delta\frac{\partial\varphi}{\partial t}\Big).\tag{3.24}$$

Multiplying (3.24) by $\frac{\partial u}{\partial t}$ and integrating over Ω, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|\frac{\partial u}{\partial t}\right\|^2+\left\|\nabla\frac{\partial u}{\partial t}\right\|^2+\int_\Omega f'(u)\left|\frac{\partial u}{\partial t}\right|^2dx\\ &=\int_\Omega g'(u)\left|\frac{\partial u}{\partial t}\right|^2(\varphi-\Delta\varphi)\,dx+\int_\Omega g(u)\frac{\partial u}{\partial t}\Big(\frac{\partial\varphi}{\partial t}-\Delta\frac{\partial\varphi}{\partial t}\Big)dx.\end{aligned}\tag{3.25}$$

Using (1.10), we write,

$$\begin{aligned}\int_\Omega g(u)\frac{\partial u}{\partial t}\Big(\frac{\partial\varphi}{\partial t}-\Delta\frac{\partial\varphi}{\partial t}\Big)dx&=\int_\Omega g(u)\frac{\partial u}{\partial t}\Big(-g(u)\frac{\partial u}{\partial t}+\Delta\varphi\Big)dx\\ &=-\int_\Omega\left|g(u)\frac{\partial u}{\partial t}\right|^2dx+\int_\Omega g(u)\frac{\partial u}{\partial t}\Delta\varphi\,dx.\end{aligned}\tag{3.26}$$

Owing to (3.26), (3.25) reads

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|\frac{\partial u}{\partial t}\right\|^2+\left\|\nabla\frac{\partial u}{\partial t}\right\|^2+\int_\Omega f'(u)\left|\frac{\partial u}{\partial t}\right|^2dx\\ &=\int_\Omega g'(u)\left|\frac{\partial u}{\partial t}\right|^2(\varphi-\Delta\varphi)\,dx+\int_\Omega g(u)\frac{\partial u}{\partial t}\Delta\varphi\,dx-\int_\Omega\left|g(u)\frac{\partial u}{\partial t}\right|^2dx,\end{aligned}\tag{3.27}$$

since

$$\begin{aligned}\int_\Omega g'(u)\left|\frac{\partial u}{\partial t}\right|^2(\varphi-\Delta\varphi)\,dx&\le c\int_\Omega\left|\frac{\partial u}{\partial t}\right|^2(|\varphi|+|\Delta\varphi|)\,dx\\ &\le\frac{1}{2}\left\|\nabla\frac{\partial u}{\partial t}\right\|^2+c(\|\varphi\|^2+\|\Delta\varphi\|^2),\end{aligned}\tag{3.28}$$

$$\int_\Omega g(u)\frac{\partial u}{\partial t}\Delta\varphi\,dx=-\int_\Omega g'(u)\nabla u\frac{\partial u}{\partial t}\nabla\varphi\,dx-\int_\Omega g(u)\nabla\frac{\partial u}{\partial t}\nabla\varphi\,dx\tag{3.29}$$

and then,

$$\begin{aligned}\left|\int_\Omega g'(u)\nabla u\frac{\partial u}{\partial t}\nabla\varphi\,dx\right| &\le c\int_\Omega|\nabla u|\left|\frac{\partial u}{\partial t}\right||\nabla\varphi|\,dx\\ &\le\frac{1}{6}\left\|\nabla\frac{\partial u}{\partial t}\right\|^2+c\|\nabla u\|^2\|\Delta\varphi\|^2\end{aligned}\tag{3.30}$$

and

$$\begin{aligned}\left|\int_\Omega g(u)\nabla\frac{\partial u}{\partial t}\nabla\varphi\,dx\right| &\le c\int_\Omega(|u|+1)\left|\nabla\frac{\partial u}{\partial t}\right||\nabla\varphi|\,dx\\ &\le\frac{1}{6}\left\|\nabla\frac{\partial u}{\partial t}\right\|^2+c(\|\nabla u\|^2+1)\|\nabla\varphi\|^2.\end{aligned}\tag{3.31}$$

Furthemore,

$$\begin{aligned}\int_\Omega\left|g(u)\frac{\partial u}{\partial t}\right|^2\,dx &\le c\int_\Omega(|u|+1)^2\left|\frac{\partial u}{\partial t}\right|^2\,dx\\ &\le c(\|\nabla u\|^2+\|u\|^2+1)\left\|\frac{\partial u}{\partial t}\right\|^2.\end{aligned}\tag{3.32}$$

Now, collecting (3.27)–(3.32) and owing to (3.20), we are led to

$$\frac{d}{dt}\left\|\frac{\partial u}{\partial t}\right\|^2+c\left\|\nabla\frac{\partial u}{\partial t}\right\|^2\le c'(\|u\|^2_{H^1(\Omega)}+1)\Big(\left\|\frac{\partial u}{\partial t}\right\|^2+\|\varphi\|^2_{H^2(\Omega)}\Big).\tag{3.33}$$

Adding (3.19), ε_1(3.23) and ε_2(3.33), with $\varepsilon_i>0$, $i=1,2$, small enough, we obtain

$$\frac{d\,E_3}{dt}+c\Big(\left\|\frac{\partial u}{\partial t}\right\|^2_{H^1(\Omega)}+\left\|\frac{\partial\varphi}{\partial t}\right\|^2_{H^2(\Omega)}\Big)\le c'E_3+c'',\tag{3.34}$$

where

$$E_3=E_2+\varepsilon_1\|\nabla\varphi\|^2+\varepsilon_2\left\|\frac{\partial u}{\partial t}\right\|^2\tag{3.35}$$

enjoys

$$E_3\ge c(\|u\|^2_{H^(\Omega)}+\|u\|^{k+2}_{k+2}+\|\varphi\|^2_{H^2(\Omega)})-c'\tag{3.36}$$

and

$$E_3\le c''(\|u\|^2_{H^(\Omega)}+\|u\|^{k+2}_{k+2}+\|\varphi\|^2_{H^2(\Omega)})-c'''.\tag{3.37}$$

We complete the proof applying Gronwall's lemma.

□

We now give a uniqueness result

Theorem 3.3. *Under assumptions of Theorem 3.2 and assuming that* (3.5) *holds true. The problem* (1.9)–(1.12) *has a unique solution* (u,φ)*, with the above regularity.*

Proof. We suppose the existence of two solutions (u_1, φ_1) and (u_2, φ_2) to problem (1.9)–(1.11) associated to initial conditions (u_{01}, φ_{01}) and (u_{02}, φ_{02}), respectively. We then have

$$\frac{\partial u}{\partial t} - \Delta u + f(u_1) - f(u_2) = g(u_1)\Big(\varphi - \Delta\varphi\Big) + (g(u_1) - g(u_2))\Big(\varphi_2 - \Delta\varphi_2\Big), \tag{3.38}$$

$$\frac{\partial \varphi}{\partial t} - \Delta\frac{\partial \varphi}{\partial t} - \Delta\varphi = -g(u_1)\frac{\partial u}{\partial t} - (g(u_1) - g(u_2))\frac{\partial u_2}{\partial t}, \tag{3.39}$$

$$u|_{\partial\Omega} = \varphi|_{\partial\Omega} = 0, \tag{3.40}$$

$$u|_{t=0} = u_{01} - u_{02},\ \varphi|_{t=0} = \varphi_{01} - \varphi_{02}, \tag{3.41}$$

with $u = u_1 - u_2$, $\varphi = \varphi_1 - \varphi_2$, $u_0 = u_{01} - u_{02}$ and $\varphi_0 = \varphi_{01} - \varphi_{02}$.

Multiplying (3.38) by $\frac{\partial u}{\partial t}$ and integrating over Ω, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2 + \int_\Omega (f(u_1 - f(u_2)))\frac{\partial u}{\partial t}\,dx \\ = \int_\Omega g(u_1)(\varphi - \Delta\varphi)\frac{\partial u}{\partial t}\,dx + \int_\Omega (g(u_1) - g(u_2))(\varphi_2 - \Delta\varphi_2)\frac{\partial u}{\partial t}\,dx.\end{aligned} \tag{3.42}$$

Multiplying (3.39) by φ and integrating over Ω, one has

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|\varphi\|^2 + \|\nabla\varphi\|^2) + \|\nabla\varphi\|^2 = &- \int_\Omega g(u_1)\frac{\partial u}{\partial t}\varphi\,dx \\ &- \int_\Omega (g(u_1) - g(u_2))\frac{\partial u_2}{\partial t}\varphi\,dx.\end{aligned} \tag{3.43}$$

Multiplying (3.39) by $-\Delta\varphi$ and integrating over Ω, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|\nabla\varphi\|^2 + \|\Delta\varphi\|^2) + \|\Delta\varphi\|^2 = &\int_\Omega g(u_1)\frac{\partial u}{\partial t}\Delta\varphi\,dx \\ &+ \int_\Omega (g(u_1) - g(u_2))\frac{\partial u_2}{\partial t}\Delta\varphi\,dx.\end{aligned} \tag{3.44}$$

Finally, adding (3.42), (3.43) and (3.44), we get

$$\begin{aligned}\frac{dE_4}{dt} + \left\|\frac{\partial u}{\partial t}\right\|^2 + \|\nabla\varphi\|^2 + \|\Delta\varphi\|^2 + \int_\Omega (f(u_1) - f(u_2))\frac{\partial u}{\partial t}\,dx \\ = \int_\Omega (g(u_1) - g(u_2))(\varphi_2 - \Delta\varphi_2)\frac{\partial u}{\partial t}\,dx \\ - \int_\Omega (g(u_1) - g(u_2))(\varphi - \Delta\varphi)\frac{\partial u_2}{\partial t}\,dx,\end{aligned} \tag{3.45}$$

where

$$E_4 = \|\nabla u\|^2 + \|\varphi\|^2 + 2\|\nabla\varphi\|^2 + \|\Delta\varphi\|^2. \tag{3.46}$$

Now, owing to (3.5), and applying Hölder's inequality for $k = 2$, when $n = 3$, we can write

$$\begin{aligned}\int_\Omega (f(u_1) - f(u_2))\frac{\partial u}{\partial t}\,dx \le c\int_\Omega (|u_2|^k + 1)|u|\left|\frac{\partial u}{\partial t}\right|\,dx \\ \le c(\|\nabla u_2\|^{2k} + 1)\|\nabla u\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2,\end{aligned} \tag{3.47}$$

we also get, thanks to (3.4), and applying Hölder's inequality,

$$\begin{aligned}\int_\Omega (g(u_1)-g(u_2))(\varphi_2-\Delta\varphi_2)\frac{\partial u}{\partial t}\,dx &\le c\int_\Omega |u||\varphi_2-\Delta\varphi_2|\left|\frac{\partial u}{\partial t}\right|\,dx\\ &\le c\|\nabla u\|^2(\|\varphi_2\|^2+\|\Delta\varphi_2\|^2)+\left\|\frac{\partial u}{\partial t}\right\|^2\end{aligned} \tag{3.48}$$

and

$$\begin{aligned}\int_\Omega (g(u_1)-g(u_2))(\varphi-\Delta\varphi)\frac{\partial u_2}{\partial t}\,dx &\le c\int_\Omega |u|\left|\frac{\partial u}{\partial t}\right||\varphi-\Delta\varphi|\,dx\\ &\le c\left\|\frac{\partial u_2}{\partial t}\right\|^2(\|\varphi\|^2+\|\Delta\varphi\|^2)+\|\nabla u\|^2.\end{aligned} \tag{3.49}$$

From (3.45)–(3.49), we deduce a differential inequality of the type

$$\frac{dE_4}{dt}+c\left\|\frac{\partial u}{\partial t}\right\|^2 \le c(\|\nabla u_2\|^{2k}+\left\|\frac{\partial u_2}{\partial t}\right\|^2+\|\varphi_2\|^2+\|\Delta\varphi_2\|^2+1)E_4. \tag{3.50}$$

In particular,

$$\frac{dE_4}{dt}\le cE_4 \tag{3.51}$$

and then applying the Gronwall's lemma to (3.51), we end the proof.

□

4. Dissipativity properties of the system

This section is devoted to the existence of bounded absorbing sets for the semigroup $S(t)$, $t \ge 0$. To this end, we consider a more restrictive assumption on G, namely,

$$\forall\,\epsilon>0,\ |G(u)|^2\le \epsilon F(s)+c_\epsilon,\ s\in\mathbb{R}. \tag{4.1}$$

We then have

Theorem 4.1. *Under the assumptions of the Theorem 3.3 and assuming that* (4.1) *holds true. Then,* $u \in L^\infty(\mathbb{R}^+; H_0^1(\Omega)) \cap L^{k+2}(\Omega)$, $\varphi \in L^\infty(\mathbb{R}^+; H_0^1(\Omega)\cap H^2(\Omega))$.

Proof. Going from (3.8) and (3.10), we get, summing (3.8) and δ(3.10), with $\delta > 0$, as small as we need,

$$\begin{aligned}&\frac{dE_5}{dt}+2\Big(c\|\nabla u\|^2+\delta\int_\Omega F(u)\,dx+\left\|\frac{\partial u}{\partial t}\right\|^2+\|\nabla\varphi\|^2+\|\Delta\varphi\|^2\Big)\\ &\le 2c'\delta\int_\Omega |G(u)|^2\,dx+\delta(\|\varphi\|^2+\|\Delta\varphi\|^2)+c''\\ &\le 2c'\delta\int_\Omega |G(u)|^2\,dx+\delta(c\|\nabla\varphi\|^2+\|\Delta\varphi\|^2)+c'',\end{aligned} \tag{4.2}$$

where

$$E_5=\delta\|u\|^2+\|\nabla u\|^2+2\int_\Omega F(u)\,dx+\|\varphi\|^2+2\|\nabla\varphi\|^2+\|\Delta\varphi\|^2 \tag{4.3}$$

satisfies

$$E_5 \geq c\Big(||u||^2_{H^1(\Omega)} + ||u||^{k+2}_{k+2} + ||\varphi||^2_{H^2(\Omega)}\Big) - c' \tag{4.4}$$

and

$$E_5 \leq c''\Big(||u||^2_{H^1(\Omega)} + ||u||^{k+2}_{k+2} + ||\varphi||^2_{H^2(\Omega)}\Big) - c'''. \tag{4.5}$$

From (4.2) and owing to (4.1), we obtain

$$\begin{aligned}
&\frac{dE_5}{dt} + 2\Big(c||\nabla u||^2 + \delta \int_\Omega F(u)\,dx + \left\|\frac{\partial u}{\partial t}\right\|^2 + ||\nabla\varphi||^2 + ||\Delta\varphi||^2\Big) \\
&\leq C_\epsilon \int_\Omega F(u)\,dx + \delta(c||\nabla\varphi||^2 + ||\Delta\varphi||^2) + C'_\epsilon,
\end{aligned} \tag{4.6}$$

where C_ϵ and C'_ϵ are positive constants which depend on ϵ. Now, choosing ϵ and δ such that:

$$2\delta \geq C_\epsilon \text{ and } 2 > c\delta, \tag{4.7}$$

we then deduce from (4.6),

$$\frac{dE_5}{dt} + c\Big(E_5 + \left\|\frac{\partial u}{\partial t}\right\|^2\Big) \leq c', \tag{4.8}$$

we complete the proof applying the Gronwall's lemma.

□

Remark 4.2. *It follows from theorems 3.1, 3.2 and 4.1 that we can define the family solving operators:*

$$\begin{aligned}
S(t) :&\Phi \longrightarrow \Phi, \\
&(u_0, \varphi_0) \mapsto (u(t), \varphi(t)),\ \forall\, t \geq 0,
\end{aligned} \tag{4.9}$$

where $\Phi = H^1_0(\Omega) \times H^1_0(\Omega) \cap H^2(\Omega)$*, and* (u, φ) *is the unique solution to the problem* (1.9)–(1.12). *Moreover, this family of solving operators forms a continuous semigroup i.e.,* $S(0) = Id$ *and* $S(t+\tau) = S(t) \circ S(\tau)$, $\forall\, t,\ \tau \geq 0$. *And then, it follows from* (4.8) *that* $S(t)$ *is dissipative in* Φ*, it means that it possesses a bounded absorbing set* $\mathbb{B}_0 \subset \Phi$ *i.e.,* $\forall\, B \subset \Phi$*(bounded),* $\exists\, t_0 = t_0(B)$ *such that* $t \geq t_0$ *implies* $S(t)B \subset \mathbb{B}_0$*. (see, e.g., [33], [35] for details).*

5. Spatial behavior of solutions

The aim of this section is to study the spatial behavior of solutions in a semi-infinite cylinder, assuming that such solutions exist. This study is motivated by the possibility of extending results obtained above to the case of unbounded domains like semi-infinite cylinders. To do so, we will study the behavior of solutions in a semi-infinite cylinder denoted $R = (0, +\infty) \times D$, where D is a smooth bounded domain of $\mathbb{R}^{n-1}$, n being the space dimension. We then consider the problem defined by the system (1.9)–(1.10) in the semi-infinite R, with $n = 3$. Furthermore, we endow to this system following boundary conditions:

$$u = \varphi = 0 \text{ on } (0,\ +\infty) \times \partial D \times (0, T) \tag{5.1}$$

and

$$u(0, x_2, x_3; t) = h(x_2, x_3; t),\ \varphi(0, x_2, x_3; t) = l(x_2, x_3; t) \text{ on } \{0\} \times D \times (0, T), \tag{5.2}$$

where $T > 0$ is a given final time.

We also consider following initial data

$$u|_{t=0} = \varphi|_{t=0} = 0 \text{ on } R. \tag{5.3}$$

Let us suppose that such solutions exist. We consider the function

$$F_w(z,t) = \int_0^t \int_{D(z)} e^{-ws}\Big(u_s u_{,1} + \varphi(\varphi_{,1} + \varphi_{,1s}) + \varphi_s \varphi_{,1}\Big)\, dads, \tag{5.4}$$

where $D(z) = \{x \in R : x_1 = z\}$, $u_{,1} = \frac{\partial u}{\partial x_1}$, $u_s = \frac{\partial u}{\partial s}$ and w is a positive constant. Using the divergence theorem and owing to (5.1), we have

$$\begin{aligned} F_w(z+h,t) - F_w(z,t) &= \frac{e^{-wt}}{2}\int_{R(z,z+h)}\Big(|\nabla u|^2 + 2F(u) + |\varphi|^2 + 2|\nabla\varphi|^2 + |\Delta\varphi|^2\Big)\,dx \\ &+ \int_0^t \int_{R(z,z+h)} e^{-ws}\Big(|u_s|^2 + |\nabla\varphi|^2 + |\Delta\varphi|^2\Big)\,dxds \\ &+ \frac{w}{2}\int_0^t \int_{R(z,z+h)} e^{-ws}\Big(|\nabla u|^2 + 2F(u) + |\varphi|^2 + 2|\nabla\varphi|^2 + |\Delta\varphi|^2\Big)\,dxds, \end{aligned} \tag{5.5}$$

where $R(z, z+h) = \{x \in R : z < x_1 < z+h\}$.

Hence,

$$\begin{aligned} \frac{\partial F_w}{\partial t}(z,t) &= \frac{e^{-wt}}{2}\int_{D(z)}\Big(|\nabla u|^2 + 2F(u) + |\varphi|^2 + 2|\nabla\varphi|^2 + |\Delta\varphi|^2\Big)\,da \\ &+ \int_0^t \int_{D(z)} e^{-ws}\Big(|u_s|^2 + |\nabla\varphi|^2 + |\Delta\varphi|^2\Big)\,dads \\ &+ \frac{w}{2}\int_0^t \int_{D(z)} e^{-ws}\Big(|\nabla u|^2 + 2F(u) + |\varphi|^2 + 2|\nabla\varphi|^2 + |\Delta\varphi|^2\Big)\,dads. \end{aligned} \tag{5.6}$$

We consider a second function, namely,

$$G_w(z,t) = \int_0^t \int_{D(z)} e^{-ws}\Big(u_s u_{,1} + \varphi(\theta_{,1} + \varphi_{,1s})\Big)\,dads, \tag{5.7}$$

where $\theta = \int_0^t \varphi(s)\,ds$.

Similarly, we have

$$\begin{aligned} G_w(z+h,t) - G_w(z,t) &= \frac{e^{-wt}}{2}\int_{R(z,z+h)}(|u|^2 + |\nabla\theta|^2)\,dx \\ &+ \int_0^t \int_{R(z,z+h)} e^{-ws}\Big(|\nabla u|^2 + f(u)u + u\Delta\varphi + |\varphi|^2 + |\nabla\varphi|^2\Big)\,dxds \\ &+ \frac{w}{2}\int_0^t \int_{R(z,z+h)} e^{-ws}(|u|^2 + |\nabla\theta|^2)\,dxds \\ &+ \int_0^t \int_{R(z,z+h)} e^{-ws}(G(u) - g(u)u)\varphi\,dxds \end{aligned} \tag{5.8}$$

and then

$$\begin{aligned}\frac{\partial G_w}{\partial t}(z,t) &= \frac{e^{-wt}}{2}\int_{D(z)}(|u|^2+|\nabla\theta|^2)\,da\\&+\int_0^t\int_{D(z)}e^{-ws}\Big(|\nabla u|^2+f(u)u+u\Delta\varphi+|\varphi|^2+|\nabla\varphi|^2\Big)\,dads\\&+\frac{w}{2}\int_0^t\int_{D(z)}e^{-ws}(|u|^2+|\nabla\theta|^2)\,dads\\&+\int_0^t\int_{D(z)}e^{-ws}(G(u)-g(u)u)\varphi\,dads.\end{aligned}\tag{5.9}$$

We choose τ large enough such as

$$2F(u)+\tau u^2\geq C_1u^2, C_1>0.\tag{5.10}$$

Now, we focus on the nonliear part i.e.,

$$w(F(u)+\frac{\tau}{2}|u|^2)+\tau f(u)u+\tau(G(u)-g(u)u)\varphi+\frac{w}{2}|\varphi|^2.\tag{5.11}$$

We assume that $G(s)-g(s)s\leq c(|s|^{k+2}+s^2)$.

For τ large enough, we have $F(u)+\frac{\tau}{2}|u|^2\geq C_2(|u|^{k+2}+|u|^2)$, $C_2>0$. Thus, for $w\gg\tau$, we deduce that

$$\begin{aligned}w(F(u)+\frac{\tau}{2}|u|^2)+\tau f(u)u+\tau(G(u)-g(u)u)\varphi&\\+\frac{w}{2}|\varphi|^2\geq C_3(|u|^2+|\varphi|^2+|\Delta\varphi|^2).&\end{aligned}\tag{5.12}$$

Taking into account previous choices, it clearly appears that the following function

$$H_w=F_w+\tau G_w\tag{5.13}$$

satisfies

$$\begin{aligned}\frac{\partial H_w}{\partial t}(z,t)\geq C_4\int_0^t\int_{D(z)}e^{-ws}\Big(&|u|^2+|\nabla u|^2+|u_s|^2\\&+|\varphi|^2+|\nabla\varphi|^2+|\Delta\varphi|^2+|\nabla\theta|^2\Big)\,dads.\end{aligned}\tag{5.14}$$

We give now an estimate of $|H_w|$ in terms of $\frac{\partial H_w}{\partial t}$. Applying Cauchy-Schwarz's inequality, one has

$$
\begin{aligned}
|F_w| &\leq \Big(\int_0^t \int_{D(z)} e^{-ws} u_s^2 \, dads\Big)^{1/2} \Big(e^{-ws} u_{,1}^2\Big)^{1/2} \\
&+ \Big(\int_0^t \int_{D(z)} e^{-ws} \varphi^2 \, dads\Big)^{1/2} \Big(e^{-ws} \varphi_{,1}^2\Big)^{1/2} \\
&+ \Big(\int_0^t \int_{D(z)} e^{-ws} \varphi^2 \, dads\Big)^{1/2} \Big(e^{-ws} \varphi_{,1s}^2\Big)^{1/2} \\
&+ \Big(\int_0^t \int_{D(z)} e^{-ws} \varphi_s^2 \, dads\Big)^{1/2} \Big(e^{-ws} \varphi_{,1}^2\Big)^{1/2} \\
&\leq C_5 \int_0^t \int_{D(z)} e^{-ws} \Big(|\nabla u|^2 + |u_s|^2 + |\varphi|^2 + |\nabla \varphi|^2 \\
&+ |\varphi_s|^2 + |\nabla \varphi_s|^2\Big) dads, \ C_5 > 0.
\end{aligned}
\tag{5.15}
$$

Similarly,

$$
\begin{aligned}
|G_w| &\leq \Big(\int_0^t \int_{D(z)} e^{-ws} u^2 \, dads\Big)^{1/2} \Big(\int_0^t \int_{D(z)} e^{-ws} u_{,1}^2 \, dads\Big)^{1/2} \\
&+ \Big(\int_0^t \int_{D(z)} e^{-ws} \varphi^2 \, dads\Big)^{1/2} \Big(\int_0^t \int_{D(z)} e^{-ws} \theta_{,1}^2 \, dads\Big)^{1/2} \\
&+ \Big(\int_0^t \int_{D(z)} e^{-ws} \varphi_s^2 \, dads\Big)^{1/2} \Big(\int_0^t \int_{D(z)} e^{-ws} \varphi_{,1}^2 \, dads\Big)^{1/2} \\
&\leq C_6 \int_0^t \int_{D(z)} e^{-ws} \Big(|u|^2 + |\nabla u|^2 + |\varphi|^2 \\
&+ |\nabla \varphi|^2 + |\nabla \theta|^2\Big) dads, \ C_6 > 0.
\end{aligned}
\tag{5.16}
$$

We then deduce the existence of a positive constant $C_7 = \frac{C_5 + \tau C_6}{C_4}$ such that

$$
|H_w| \leq C_7 \frac{\partial H_w}{\partial z}. \tag{5.17}
$$

Remark 5.1. *The inequality* (5.17) *is well known in the study of spatial estimates and leads to the Phragmén-Lindelöf alternative (see, e.g., [10], [40]).*

In particular, if there exist $z_0 \geq 0$ such that $F_w(z_0, t) > 0$, then the solution satisfies

$$
H_w(z, t) \geq H_w(z_0, t) e^{C_7^{-1}(z - z_0)}, \ z \geq z_0. \tag{5.18}
$$

The estimate (5.18) gives information in terms of measure defined in the cylinder. Actually, from

(5.18), we deduce that

$$\begin{aligned}
&\frac{e^{-wt}}{2}\int_{R(0,z)}\Big(|\nabla u|^2+2F(u)+|\varphi|^2+2|\nabla\varphi|^2+|\Delta\varphi|^2\Big)\,dx\\
&+\tau\frac{e^{-wt}}{2}\int_{R(0,z)}(|u|^2+|\nabla\theta|^2)\,dx\\
&+\int_0^t\int_{R(0,z)}e^{-ws}\Big(|u_s|^2+|\nabla\varphi|^2+|\Delta\varphi|^2\Big)\,dxds\\
&+\tau\int_0^t\int_{R(0,z)}e^{-ws}\Big(|\nabla u|^2+f(u)u+g(u)u\Delta\varphi+|\varphi|^2+2|\nabla\varphi|^2\Big)\,dxds \qquad (5.19)\\
&+\frac{w}{2}\int_0^t\int_{R(0,z)}e^{-ws}\Big(|\nabla u|^2+2F(u)+|\varphi|^2+2|\nabla\varphi|^2+|\Delta\varphi|^2\Big)\,dxds\\
&+\tau\frac{w}{2}\int_0^t\int_{R(0,z)}e^{-ws}(|u|^2+|\nabla\theta|^2)\,dx\\
&+\tau\int_0^t\int_{R(0,z)}e^{-ws}(G(u)-g(u)u)\varphi\,dxds
\end{aligned}$$

tends to infinity exponentially fast. On the other hand, if $H_w(z,t)\le 0$, for every $z\ge 0$, we deduce that the solution decreases and we get an inequality of the type

$$-H_w(z,t)\le -H_w(0,t)e^{C_7^{-1}z},\ z\ge 0, \qquad (5.20)$$

where

$$\begin{aligned}
E_w(z,t)=&\frac{e^{-wt}}{2}\int_{R(z)}\Big(|\nabla u|^2+2F(u)+|\varphi|^2+2|\nabla\varphi|^2+|\Delta\varphi|^2\Big)\,dx\\
&+\tau\frac{e^{-wt}}{2}\int_{R(z)}(|u|^2+|\nabla\theta|^2)\,dx\\
&+\int_0^t\int_{R(z)}e^{-ws}\Big(|u_s|^2+|\nabla\varphi|^2+|\Delta\varphi|^2\Big)\,dxds\\
&+\tau\int_0^t\int_{R(z)}e^{-ws}\Big(|\nabla u|^2+f(u)u+g(u)u\Delta\varphi+|\varphi|^2+2|\nabla\varphi|^2\Big)\,dxds \qquad (5.21)\\
&+\frac{w}{2}\int_0^t\int_{R(z)}e^{-ws}\Big(|\nabla u|^2+2F(u)+|\varphi|^2+2|\nabla\varphi|^2+|\Delta\varphi|^2\Big)\,dxds\\
&+\tau\frac{w}{2}\int_0^t\int_{R(z)}e^{-ws}(|u|^2+|\nabla\theta|^2)\,dx\\
&+\tau\int_0^t\int_{R(z)}e^{-ws}(G(u)-g(u)u)\varphi\,dxds
\end{aligned}$$

and $R(z)=\{x\in R: x_1>z\}$.

Finally, setting

$$
\begin{aligned}
&\mathcal{E}_w(z,t) = \frac{1}{2}\int_{R(z)}\Big(|\nabla u|^2 + 2F(u) + |\varphi|^2 + 2|\nabla\varphi|^2 + |\Delta\varphi|^2\Big)\,dx \\
&+ \tau\frac{1}{2}\int_{R(z)}(|u|^2 + |\nabla\theta|^2)\,dx \\
&+ \int_0^t\int_{R(z)}\Big(|u_s|^2 + |\nabla\varphi|^2 + |\Delta\varphi|^2\Big)\,dxds \\
&+ \tau\int_0^t\int_{R(z)}\Big(|\nabla u|^2 + f(u)u + g(u)u\Delta\varphi + |\varphi|^2 + 2|\nabla\varphi|^2\Big)\,dxds \\
&+ \frac{w}{2}\int_0^t\int_{R(z)}\Big(|\nabla u|^2 + 2F(u) + |\varphi|^2 + 2|\nabla\varphi|^2 + |\Delta\varphi|^2\Big)\,dxds \\
&+ \tau\frac{w}{2}\int_0^t\int_{R(z)}(|u|^2 + |\nabla\theta|^2)\,dx \\
&+ \tau\int_0^t\int_{R(z)}(G(u) - g(u)u)\varphi\,dxds.
\end{aligned}
\tag{5.22}
$$

We have the following result

Theorem 5.2. *Let* (u,φ) *be a solution to the problem given by* (1.9)–(1.10)*, boundary conditions* (5.1)–(5.2) *and initial data* (5.3)*. Then, either this solution satisfies* (5.18)*, or it satisfies*

$$\mathcal{E}_w(z,t) \le E_w(0,t)e^{wt - C_7^{-1}z},\ z \ge 0, \tag{5.23}$$

where the energy $\mathcal{E}_w$ *is given by* (5.22)*.*

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Conflict of interest

The author declares no conflicts of interest in this paper.

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