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Research article

Exact solutions of the generalized (2+1)-dimensional BKP equation by the G'/G-expansion method and the first integral method

Huaji Cheng and Yanxia Hu*

School of Mathematics and Physics, North China Electric Power University, Beijing, 102206, China

* **Correspondence:** yxiahu@163.com

Abstract: In this paper, the G'/G-expansion method and the first integral method are performed to the generalized (2+1)-dimensional BKP equation. Rational function solutions, periodic function solutions and hyperbolic function solutions of the equation are obtained under some parametric conditions.

Keywords: Generalized (2+1)-dimensional BKP equation; G'/G-expansion method; first integral method
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1. Introduction

During the past decades, the exact solutions of nonlinear partial differential equations have been investigated by many authors. Meanwhile, many powerful methods have been proposed by them, such as Backlund transformation method [1], multiple exp-function method [2], homogeneous balance principle [3], tanh-sech method [4], G'/G-expansion method [5–7], the first integral method [8, 9] and so on.

The G'/G-expansion method was first presented by Wang [5] which can be used to deal with all types of nonlinear evolution equations. The first integral method was first proposed by Feng [8] for obtaining the exact solutions of Burgers-KdV equation which is based on the ring theory of commutative algebra. The basic idea of the first integral method is to construct a first integral with polynomial coefficients of an explicit form to an equivalent autonomous planer system by using the division theorem. Both the G'/G-expansion method and the first integral method are powerful methods for computing the exact solutions of nonlinear partial differential equations. They are direct, elementary and effective algebraic methods.

In this paper, we consider the following generalized (2+1)-dimensional BKP equation [10]

$$\begin{cases}(w^n)_t + (w^m)_{xxx} + (w^m)_{yyy} + \alpha(uw)_x + \beta(vw)_y = 0,\\ u_y = w_x,\\ v_x = w_y,\end{cases} \tag{1.1}$$

where α, β are arbitrary constants and $\alpha + \beta \neq 0, m, n$ are integers and $m, n \geq 2$. In [10], authors studied traveling wave solutions in the parameter space of this system by bifurcation theory of dynamical systems and they obtained some exact explicit parametric representations of periodic cusp wave solutions, solitary wave solutions and compacton solutions. In this paper, we continue to consider the problem of solving system (1.1) by using the G'/G-expansion method and the first integral method and we obtain the rational function solutions, periodic function solutions and the hyperbolic function solutions of (1.1) under some parametric conditions and the values of m, n in several cases.

Specially, when $m = 1, n = 1, \alpha = \beta = 6$, (1.1) becomes

$$\begin{cases}w_t + w_{xxx} + w_{yyy} + 6(uw)_x + 6(vw)_y = 0,\\ u_y = w_x,\\ v_x = w_y.\end{cases}$$

It is the famous (2+1)-dimensional BKP equation which was introduced by Date et al. [11] and describes the processes of interaction of exponentially localized structures. It is one of a hierarchy of integrable systems emerging from a bilinear identity related to a Clifford algebra which is generated by two neutral fermion fields [12]. This equation has been studied by using many methods, such as the sine-cosine method [13], the G'/G-expansion method [6], the improved G'/G-expansion method [14] and so on.

The aim of this paper is to extract the exact solutions of the generalized (2+1)-dimensional BKP equation by using the G'/G-expansion method and the first integral method. The paper is arranged as follows: In section 2, we apply the G'/G-expansion method to this equation. In section 3, we apply the first integral method to solve this equation. In section 4, we give the conclusion of the paper.

2. Application of the G'/G-expansion method to the generalized (2+1)-dimensional BKP equation

We suppose the wave transformations

$$w(x, y, t) = w(\xi), u(x, y, t) = u(\xi), v(x, y, t) = v(\xi), \qquad \xi = k_1 x + l_1 y + \lambda_1 t \tag{2.1}$$

where k_1, l_1, λ_1 are constants. By using the wave transformations (2.1), (1.1) can be converted into ODEs

$$\begin{cases}\lambda_1(w^n)' + (k_1^3 + l_1^3)(w^m)''' + \alpha k_1(u'w + uw') + \beta l_1(v'w + vw') = 0,\\ l_1 u' = k_1 w',\\ k_1 v' = l_1 w',\end{cases} \tag{2.2}$$

where "′" is the derivative with respect to ξ. Integrating the second and third equation of (2.2) and neglecting integral constants, we obtain

$$\begin{cases} l_1 u = k_1 w, \\ k_1 v = l_1 w. \end{cases}$$

Substituting the above equations into the first equation of (2.2) and integrating it, then it becomes

$$\lambda_1 w^n + (k_1^3 + l_1^3)(w^m)'' + (\frac{\alpha k_1^2}{l_1} + \frac{\beta l_1^2}{k_1})w^2 = g, \tag{2.3}$$

where g is an integral constant. We assume that (2.3) has the following formal solutions [7, 15]:

$$w(\xi) = D\Big(\frac{G'}{G}\Big)^N, \quad D \neq 0, \tag{2.4}$$

where D is a constant to be determined later. N is determined by balancing the linear term of the highest order derivatives with the highest order nonlinear term in (2.3) and G satisfies a second order constant coefficient ODE which is

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0, \tag{2.5}$$

where $\lambda,\ \mu$ are constants and will be determined later. Next, we will obtain the exact solutions of (1.1) by considering the values of m and n in several cases.

2.1. $m \neq n, m > 2, n > 2$

Balancing $(w^m)''$ with w^n of (2.3), we have $mN + 2 = nN$, i.e., $N = 2/(n - m)$. Thus, we assume

$$w(\xi) = D_1\Big(\frac{G'}{G}\Big)^{\frac{2}{n-m}}, \quad D_1 \neq 0 \tag{2.6}$$

where D_1 is a constant to be determined later. Then, we have

$$\begin{aligned} w^n &= D_1^n\Big(\frac{G'}{G}\Big)^{\frac{2n}{n-m}}, \qquad w^2 = D_1^2\Big(\frac{G'}{G}\Big)^{\frac{4}{n-m}}, \\ (w^m)'' &= \frac{2m}{n-m}D_1^m\Big[(\frac{2m}{n-m}+1)\Big(\frac{G'}{G}\Big)^{\frac{2m}{n-m}+2} + (\frac{4m}{n-m}+1)\lambda\Big(\frac{G'}{G}\Big)^{\frac{2m}{n-m}+1} \\ &+ \frac{2m}{n-m}(2\mu+\lambda^2)\Big(\frac{G'}{G}\Big)^{\frac{2m}{n-m}} + (\frac{4m}{n-m}-1)\lambda\mu\Big(\frac{G'}{G}\Big)^{\frac{2m}{n-m}-1} \\ &+ (\frac{2m}{n-m}-1)\mu^2\Big(\frac{G'}{G}\Big)^{\frac{2m}{n-m}-2}\Big]. \end{aligned}$$

Substituting the above formulas into (2.3) and collecting all terms with the same order of G'/G together, we can convert the left-hand side of (2.3) into a polynomial in G'/G. Then, setting each coefficient of each polynomial to zero, we can derive a set of algebraic equation for λ, μ and D_1:

$\Big(\frac{G'}{G}\Big)^{\frac{2m}{n-m}+2}$ coeff:

$$(k_1^3 + l_1^3)(\frac{2m}{n-m}+1)\frac{2m}{n-m}D_1^m + \lambda_1 D_1^n = 0, \tag{2.7}$$

$\left(\frac{G'}{G}\right)^{\frac{2m}{n-m}+1}$ coeff:

$$(k_1^3+l_1^3)(\frac{4m}{n-m}+1)\frac{2m}{n-m}\lambda D_1^m=0. \tag{2.8}$$

Here, we need to consider the value of $4/(n-m)$ in the following cases:

Case 1. $\frac{4}{n-m}=\frac{2m}{n-m}-1$

$\left(\frac{G'}{G}\right)^{\frac{2m}{n-m}}$ coeff:

$$(k_1^3+l_1^3)(\frac{2m}{n-m})^2(2\mu+\lambda^2)D_1^m=0, \tag{2.9}$$

$\left(\frac{G'}{G}\right)^{\frac{2m}{n-m}-1}$ coeff:

$$(k_1^3+l_1^3)(\frac{4m}{n-m}-1)\frac{2m}{n-m}\lambda\mu D_1^m+(\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1})D_1^2=0, \tag{2.10}$$

$\left(\frac{G'}{G}\right)^{\frac{2m}{n-m}-2}$ coeff:

$$(k_1^3+l_1^3)(\frac{2m}{n-m}-1)\frac{2m}{n-m}\mu^2 D_1^m=0. \tag{2.11}$$

Solving the set of (2.7)-(2.11), we obtain

$$\lambda=\mu=0,\quad g=0,\quad \frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1}=0,\quad D_1=\left(-\frac{(k_1^3+l_1^3)(\frac{2m}{n-m}+1)\frac{2m}{n-m}}{\lambda_1}\right)^{1/(n-m)}. \tag{2.12}$$

Case 2. $\frac{4}{n-m}=\frac{2m}{n-m}-2$

$\left(\frac{G'}{G}\right)^{\frac{2m}{n-m}}$ coeff:

$$(k_1^3+l_1^3)(\frac{2m}{n-m})^2(2\mu+\lambda^2)D_1^m=0, \tag{2.13}$$

$\left(\frac{G'}{G}\right)^{\frac{2m}{n-m}-1}$ coeff:

$$(k_1^3+l_1^3)(\frac{4m}{n-m}-1)\frac{2m}{n-m}\lambda\mu D_1^m=0, \tag{2.14}$$

$\left(\frac{G'}{G}\right)^{\frac{2m}{n-m}-2}$ coeff:

$$(k_1^3+l_1^3)(\frac{2m}{n-m}-1)\frac{2m}{n-m}\mu^2 D_1^m+(\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1})D_1^2=0. \tag{2.15}$$

Solving the set of (2.7)-(2.8) and (2.13)-(2.15), we get the same results as those of Case 1.
Case 3. $\frac{4}{n-m} \neq \frac{2m}{n-m} - 1$ and $\frac{4}{n-m} \neq \frac{2m}{n-m} - 2$

$\left(\frac{G'}{G}\right)^{\frac{2m}{n-m}}$ coeff:

$$(k_1^3 + l_1^3)(\frac{2m}{n-m})^2(2\mu + \lambda^2)D_1^m = 0, \tag{2.16}$$

$\left(\frac{G'}{G}\right)^{\frac{2m}{n-m}-1}$ coeff:

$$(k_1^3 + l_1^3)(\frac{4m}{n-m} - 1)\frac{2m}{n-m}\lambda\mu D_1^m = 0, \tag{2.17}$$

$\left(\frac{G'}{G}\right)^{\frac{2m}{n-m}-2}$ coeff:

$$(k_1^3 + l_1^3)(\frac{2m}{n-m} - 1)\frac{2m}{n-m}\mu^2 D_1^m = 0, \tag{2.18}$$

$\left(\frac{G'}{G}\right)^{\frac{4}{n-m}}$ coeff:

$$(\frac{\alpha k_1^2}{l_1} + \frac{\beta l_1^2}{k_1})D_1^2 = 0. \tag{2.19}$$

Solving the set of (2.7)-(2.8) and (2.16)-(2.19), we obtain the same results as those of former cases. Substituting (2.12) into (2.5) and (2.6), then, we can get the rational function solutions

$$\begin{aligned} w(x,y,t) &= \left(-\frac{(k_1^3 + l_1^3)(\frac{2m}{n-m} + 1)\frac{2m}{n-m}}{\lambda_1}\right)^{\frac{1}{n-m}}\left(\frac{C_1}{C_1(k_1x + l_1y + \lambda_1 t) + C_2}\right)^{\frac{2}{n-m}}, \\ v(x,y,t) &= \frac{l_1}{k_1}\left(-\frac{(k_1^3 + l_1^3)(\frac{2m}{n-m} + 1)\frac{2m}{n-m}}{\lambda_1}\right)^{\frac{1}{n-m}}\left(\frac{C_1}{C_1(k_1x + l_1y + \lambda_1 t) + C_2}\right)^{\frac{2}{n-m}}, \\ u(x,y,t) &= \frac{k_1}{l_1}\left(-\frac{(k_1^3 + l_1^3)(\frac{2m}{n-m} + 1)\frac{2m}{n-m}}{\lambda_1}\right)^{\frac{1}{n-m}}\left(\frac{C_1}{C_1(k_1x + l_1y + \lambda_1 t) + C_2}\right)^{\frac{2}{n-m}}, \end{aligned}$$

where C_1, C_2 are arbitrary constants and $\alpha k_1^3 + \beta l_1^3 = 0$.

2.2. $m = 2, n > 2$

(2.3) becomes

$$\lambda_1 w^n + (k_1^3 + l_1^3)(w^2)'' + (\frac{\alpha k_1^2}{l_1} + \frac{\beta l_1^2}{k_1})w^2 = g. \tag{2.20}$$

Balancing $(w^2)''$ with w^n, we have $N = 2/(n-2)$. Thus, (2.20) has the following formal solutions

$$w(\xi) = D_2\left(\frac{G'}{G}\right)^{\frac{2}{n-2}}, \quad D_2 \neq 0, \tag{2.21}$$

where D_2 is a constant to be determined later and G satisfies (2.5). Similarly, we can get a set of algebraic equations:
$\left(\frac{G'}{G}\right)^{\frac{4}{n-2}+2}$ coeff:

$$(k_1^3 + l_1^3)(\frac{4}{n-2} + 1)\frac{4}{n-2}D_2^2 + \lambda_1 D_2^n = 0, \tag{2.22}$$

$\left(\frac{G'}{G}\right)^{\frac{4}{n-2}+1}$ coeff:

$$(k_1^3 + l_1^3)(\frac{8}{n-2} + 1)\frac{4}{n-2}\lambda D_2^2 = 0, \tag{2.23}$$

$\left(\frac{G'}{G}\right)^{\frac{4}{n-2}}$ coeff:

$$(k_1^3 + l_1^3)(\frac{4}{n-2})^2(2\mu + \lambda^2)D_2^2 + (\frac{\alpha k_1^2}{l_1} + \frac{\beta l_1^2}{k_1})D_2^2 = 0, \tag{2.24}$$

I. The case $g = 0$

$\left(\frac{G'}{G}\right)^{\frac{4}{n-2}-1}$ coeff:

$$(k_1^3 + l_1^3)(\frac{8}{n-2} - 1)\frac{4}{n-2}\lambda\mu D_2^2 = 0, \tag{2.25}$$

$\left(\frac{G'}{G}\right)^{\frac{4}{n-2}-2}$ coeff:

$$(k_1^3 + l_1^3)(\frac{4}{n-2} - 1)\frac{4}{n-2}\mu^2 D_2^2 = 0. \tag{2.26}$$

Solving that set of (2.22)-(2.26), we obtain

$$\lambda = \mu = 0, \quad \frac{\alpha k_1^2}{l_1} + \frac{\beta l_1^2}{k_1} = 0, \quad D_2 = \left(-\frac{(k_1^3 + l_1^3)(\frac{4}{n-2} + 1)\frac{4}{n-2}}{\lambda_1}\right)^{1/(n-2)}. \tag{2.27}$$

Specially, when $\frac{4}{n-2} - 1 = 0$, i.e., $n = 6$, we obtain

$$\lambda = 0, \quad \mu = -\frac{\frac{\alpha k_1^2}{l_1} + \frac{\beta l_1^2}{k_1}}{2(k_1^3 + l_1^3)}, \quad D_2 = \left(\frac{-2(k_1^3 + l_1^3)}{\lambda_1}\right)^{1/4}. \tag{2.28}$$

Substituting (2.27) into (2.5) and (2.21), then, we can get the rational function solutions

$$w(x, y, t) = \left(-\frac{(k_1^3 + l_1^3)(\frac{4}{n-2} + 1)\frac{4}{n-2}}{\lambda_1}\right)^{\frac{1}{n-2}}\left(\frac{C_3}{C_3(k_1 x + l_1 y + \lambda_1 t) + C_4}\right)^{\frac{2}{n-2}},$$

$$
\begin{aligned}
v(x,y,t) &= \frac{l_1}{k_1}\Big(-\frac{(k_1^3+l_1^3)(\frac{4}{n-2}+1)\frac{4}{n-2}}{\lambda_1}\Big)^{\frac{1}{n-2}}\Big(\frac{C_3}{C_3(k_1x+l_1y+\lambda_1t)+C_4}\Big)^{\frac{2}{n-2}},\\
u(x,y,t) &= \frac{k_1}{l_1}\Big(-\frac{(k_1^3+l_1^3)(\frac{4}{n-2}+1)\frac{4}{n-2}}{\lambda_1}\Big)^{\frac{1}{n-2}}\Big(\frac{C_3}{C_3(k_1x+l_1y+\lambda_1t)+C_4}\Big)^{\frac{2}{n-2}},
\end{aligned}
$$

where C_3, C_4 are arbitrary constants and $\alpha k_1^3+\beta l_1^3=0$. Substituting (2.28) into (2.5) and (2.21), then, we have

$$
G''+\Big(-\frac{\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1}}{2(k_1^3+l_1^3)}\Big)G=0.
$$

Case 1. $\frac{\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1}}{2(k_1^3+l_1^3)}>0$

We obtain the hyperbolic function solutions

$$
\begin{aligned}
w(x,y,t) = {} & \Big(\frac{-(\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1})^2}{2\lambda_1(k_1^3+l_1^3)}\Big)^{1/4}\\
& \left\{\frac{C_5\sinh\Big(\frac{\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1}}{2(k_1^3+l_1^3)}\Big)^{1/2}(k_1x+l_1y+\lambda_1t)+C_6\cosh\Big(\frac{\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1}}{2(k_1^3+l_1^3)}\Big)^{1/2}(k_1x+l_1y+\lambda_1t)}{C_5\cosh\Big(\frac{\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1}}{2(k_1^3+l_1^3)}\Big)^{1/2}(k_1x+l_1y+\lambda_1t)+C_6\sinh\Big(\frac{\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1}}{2(k_1^3+l_1^3)}\Big)^{1/2}(k_1x+l_1y+\lambda_1t)}\right\}^{1/2},\\
v(x,y,t) = {} & \frac{l_1}{k_1}\Big(\frac{-(\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1})^2}{2\lambda_1(k_1^3+l_1^3)}\Big)^{1/4}\\
& \left\{\frac{C_5\sinh\Big(\frac{\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1}}{2(k_1^3+l_1^3)}\Big)^{1/2}(k_1x+l_1y+\lambda_1t)+C_6\cosh\Big(\frac{\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1}}{2(k_1^3+l_1^3)}\Big)^{1/2}(k_1x+l_1y+\lambda_1t)}{C_5\cosh\Big(\frac{\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1}}{2(k_1^3+l_1^3)}\Big)^{1/2}(k_1x+l_1y+\lambda_1t)+C_6\sinh\Big(\frac{\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1}}{2(k_1^3+l_1^3)}\Big)^{1/2}(k_1x+l_1y+\lambda_1t)}\right\}^{1/2},\\
u(x,y,t) = {} & \frac{k_1}{l_1}\Big(\frac{-(\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1})^2}{2\lambda_1(k_1^3+l_1^3)}\Big)^{1/4}\\
& \left\{\frac{C_5\sinh\Big(\frac{\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1}}{2(k_1^3+l_1^3)}\Big)^{1/2}(k_1x+l_1y+\lambda_1t)+C_6\cosh\Big(\frac{\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1}}{2(k_1^3+l_1^3)}\Big)^{1/2}(k_1x+l_1y+\lambda_1t)}{C_5\cosh\Big(\frac{\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1}}{2(k_1^3+l_1^3)}\Big)^{1/2}(k_1x+l_1y+\lambda_1t)+C_6\sinh\Big(\frac{\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1}}{2(k_1^3+l_1^3)}\Big)^{1/2}(k_1x+l_1y+\lambda_1t)}\right\}^{1/2},
\end{aligned}
$$

where C_5, C_6 are arbitrary constants.

Case 2. $\frac{\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1}}{2(k_1^3+l_1^3)}<0$

We obtain the hyperbolic function solutions

$$
w(x,y,t) = \Big(\frac{-(\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1})^2}{2\lambda_1(k_1^3+l_1^3)}\Big)^{1/4}
$$

$$\left\{\frac{-C_7\sin\left(\frac{-(\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1})}{2(k_1^3+l_1^3)}\right)^{1/2}(k_1x+l_1y+\lambda_1t)+C_8\cos\left(\frac{-(\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1})}{2(k_1^3+l_1^3)}\right)^{1/2}(k_1x+l_1y+\lambda_1t)}{C_7\cos\left(\frac{-(\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1})}{2(k_1^3+l_1^3)}\right)^{1/2}(k_1x+l_1y+\lambda_1t)+C_8\sin\left(\frac{-(\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1})}{2(k_1^3+l_1^3)}\right)^{1/2}(k_1x+l_1y+\lambda_1t)}\right\}^{1/2},$$

$$v(x,y,t) = \frac{l_1}{k_1}\left(\frac{-(\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1})^2}{2\lambda_1(k_1^3+l_1^3)}\right)^{1/4}$$
$$\left\{\frac{-C_7\sin\left(\frac{-(\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1})}{2(k_1^3+l_1^3)}\right)^{1/2}(k_1x+l_1y+\lambda_1t)+C_8\cos\left(\frac{-(\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1})}{2(k_1^3+l_1^3)}\right)^{1/2}(k_1x+l_1y+\lambda_1t)}{C_7\cos\left(\frac{-(\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1})}{2(k_1^3+l_1^3)}\right)^{1/2}(k_1x+l_1y+\lambda_1t)+C_8\sin\left(\frac{-(\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1})}{2(k_1^3+l_1^3)}\right)^{1/2}(k_1x+l_1y+\lambda_1t)}\right\}^{1/2},$$

$$u(x,y,t) = \frac{k_1}{l_1}\left(\frac{-(\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1})^2}{2\lambda_1(k_1^3+l_1^3)}\right)^{1/4}$$
$$\left\{\frac{-C_7\sin\left(\frac{-(\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1})}{2(k_1^3+l_1^3)}\right)^{1/2}(k_1x+l_1y+\lambda_1t)+C_8\cos\left(\frac{-(\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1})}{2(k_1^3+l_1^3)}\right)^{1/2}(k_1x+l_1y+\lambda_1t)}{C_7\cos\left(\frac{-(\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1})}{2(k_1^3+l_1^3)}\right)^{1/2}(k_1x+l_1y+\lambda_1t)+C_8\sin\left(\frac{-(\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1})}{2(k_1^3+l_1^3)}\right)^{1/2}(k_1x+l_1y+\lambda_1t)}\right\}^{1/2},$$

where C_7, C_8 are arbitrary constants.

Case 3. $\frac{\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1}}{2(k_1^3+l_1^3)} = 0$

We obtain the rational function solutions

$$w(x,y,t) = \left(\frac{-2(k_1^3+l_1^3)}{\lambda_1}\right)^{1/4}\left\{\frac{C_9}{C_9(k_1x+l_1y+\lambda_1t)+C_{10}}\right\}^{1/2},$$
$$v(x,y,t) = \frac{l_1}{k_1}\left(\frac{-2(k_1^3+l_1^3)}{\lambda_1}\right)^{1/4}\left\{\frac{C_9}{C_9(k_1x+l_1y+\lambda_1t)+C_{10}}\right\}^{1/2},$$
$$u(x,y,t) = \frac{k_1}{l_1}\left(\frac{-2(k_1^3+l_1^3)}{\lambda_1}\right)^{1/4}\left\{\frac{C_9}{C_9(k_1x+l_1y+\lambda_1t)+C_{10}}\right\}^{1/2},$$

where C_9, C_{10} are arbitrary constants.

II. The case $g \neq 0$

When $\frac{4}{n-2} - 2 = 0$, i.e, $n = 4$.

$\left(\frac{G'}{G}\right)$ coeff:

$$6(k_1^3+l_1^3)\lambda\mu D_2^2 = 0, \tag{2.29}$$

$\left(\frac{G'}{G}\right)^0$ coeff:

$$2(k_1^3+l_1^3)\mu^2 D_2^2 = g. \tag{2.30}$$

Solving the set of (2.22)-(2.24) and (2.29)-(2.30), we obtain

$$\lambda = 0, \quad \mu = -\frac{\frac{\alpha k_1^2}{l_1} + \frac{\beta l_1^2}{k_1}}{2(k_1^3 + l_1^3)}, \quad D_2 = \Big(\frac{-6(k_1^3 + l_1^3)}{\lambda_1}\Big)^{1/2}, \quad \frac{\alpha k_1^2}{l_1} + \frac{\beta l_1^2}{k_1} \neq 0, \quad \frac{-3(\frac{\alpha k_1^2}{l_1} + \frac{\beta l_1^2}{k_1})^2}{\lambda_1} = g. \quad (2.31)$$

Similarly, we can obtain the hyperbolic function solutions and trigonometric function solutions

Case 1. $\frac{\frac{\alpha k_1^2}{l_1} + \frac{\beta l_1^2}{k_1}}{2(k_1^3 + l_1^3)} > 0$

$$\begin{aligned} w(x, y, t) = & \Big(\frac{-3(\frac{\alpha k_1^2}{l_1} + \frac{\beta l_1^2}{k_1})}{\lambda_1}\Big)^{1/2} \\ & \left\{\frac{C_{11} \sinh\Big(\frac{\frac{\alpha k_1^2}{l_1} + \frac{\beta l_1^2}{k_1}}{2(k_1^3 + l_1^3)}\Big)^{1/2}(k_1 x + l_1 y + \lambda_1 t) + C_{12} \cosh\Big(\frac{\frac{\alpha k_1^2}{l_1} + \frac{\beta l_1^2}{k_1}}{2(k_1^3 + l_1^3)}\Big)^{1/2}(k_1 x + l_1 y + \lambda_1 t)}{C_{11} \cosh\Big(\frac{\frac{\alpha k_1^2}{l_1} + \frac{\beta l_1^2}{k_1}}{2(k_1^3 + l_1^3)}\Big)^{1/2}(k_1 x + l_1 y + \lambda_1 t) + C_{12} \sinh\Big(\frac{\frac{\alpha k_1^2}{l_1} + \frac{\beta l_1^2}{k_1}}{2(k_1^3 + l_1^3)}\Big)^{1/2}(k_1 x + l_1 y + \lambda_1 t)}\right\}, \quad (2.32) \\ v(x, y, t) = & \frac{l_1}{k_1}\Big(\frac{-3(\frac{\alpha k_1^2}{l_1} + \frac{\beta l_1^2}{k_1})}{\lambda_1}\Big)^{1/2} \\ & \left\{\frac{C_{11} \sinh\Big(\frac{\frac{\alpha k_1^2}{l_1} + \frac{\beta l_1^2}{k_1}}{2(k_1^3 + l_1^3)}\Big)^{1/2}(k_1 x + l_1 y + \lambda_1 t) + C_{12} \cosh\Big(\frac{\frac{\alpha k_1^2}{l_1} + \frac{\beta l_1^2}{k_1}}{2(k_1^3 + l_1^3)}\Big)^{1/2}(k_1 x + l_1 y + \lambda_1 t)}{C_{11} \cosh\Big(\frac{\frac{\alpha k_1^2}{l_1} + \frac{\beta l_1^2}{k_1}}{2(k_1^3 + l_1^3)}\Big)^{1/2}(k_1 x + l_1 y + \lambda_1 t) + C_{12} \sinh\Big(\frac{\frac{\alpha k_1^2}{l_1} + \frac{\beta l_1^2}{k_1}}{2(k_1^3 + l_1^3)}\Big)^{1/2}(k_1 x + l_1 y + \lambda_1 t)}\right\}, \\ u(x, y, t) = & \frac{k_1}{l_1}\Big(\frac{-3(\frac{\alpha k_1^2}{l_1} + \frac{\beta l_1^2}{k_1})}{\lambda_1}\Big)^{1/2} \\ & \left\{\frac{C_{11} \sinh\Big(\frac{\frac{\alpha k_1^2}{l_1} + \frac{\beta l_1^2}{k_1}}{2(k_1^3 + l_1^3)}\Big)^{1/2}(k_1 x + l_1 y + \lambda_1 t) + C_{12} \cosh\Big(\frac{\frac{\alpha k_1^2}{l_1} + \frac{\beta l_1^2}{k_1}}{2(k_1^3 + l_1^3)}\Big)^{1/2}(k_1 x + l_1 y + \lambda_1 t)}{C_{11} \cosh\Big(\frac{\frac{\alpha k_1^2}{l_1} + \frac{\beta l_1^2}{k_1}}{2(k_1^3 + l_1^3)}\Big)^{1/2}(k_1 x + l_1 y + \lambda_1 t) + C_{12} \sinh\Big(\frac{\frac{\alpha k_1^2}{l_1} + \frac{\beta l_1^2}{k_1}}{2(k_1^3 + l_1^3)}\Big)^{1/2}(k_1 x + l_1 y + \lambda_1 t)}\right\}, \end{aligned}$$

where C_{11}, C_{12} are arbitrary constants and $\lambda_1(k_1^3 + l_1^3) < 0$.

Case 2. $\frac{\frac{\alpha k_1^2}{l_1} + \frac{\beta l_1^2}{k_1}}{2(k_1^3 + l_1^3)} < 0$

$$\begin{aligned} w(x, y, t) = & \Big(\frac{-3(\frac{\alpha k_1^2}{l_1} + \frac{\beta l_1^2}{k_1})}{\lambda_1}\Big)^{1/2} \\ & \left\{\frac{-C_{13} \sin\Big(\frac{-(\frac{\alpha k_1^2}{l_1} + \frac{\beta l_1^2}{k_1})}{2(k_1^3 + l_1^3)}\Big)^{1/2}(k_1 x + l_1 y + \lambda_1 t) + C_{14} \cos\Big(\frac{-(\frac{\alpha k_1^2}{l_1} + \frac{\beta l_1^2}{k_1})}{2(k_1^3 + l_1^3)}\Big)^{1/2}(k_1 x + l_1 y + \lambda_1 t)}{C_{13} \cos\Big(\frac{-(\frac{\alpha k_1^2}{l_1} + \frac{\beta l_1^2}{k_1})}{2(k_1^3 + l_1^3)}\Big)^{1/2}(k_1 x + l_1 y + \lambda_1 t) + C_{14} \sin\Big(\frac{-(\frac{\alpha k_1^2}{l_1} + \frac{\beta l_1^2}{k_1})}{2(k_1^3 + l_1^3)}\Big)^{1/2}(k_1 x + l_1 y + \lambda_1 t)}\right\}, \quad (2.33) \\ v(x, y, t) = & \frac{l_1}{k_1}\Big(\frac{-3(\frac{\alpha k_1^2}{l_1} + \frac{\beta l_1^2}{k_1})}{\lambda_1}\Big)^{1/2} \end{aligned}$$

$$\left\{\frac{-C_{13}\sin\left(\frac{-(\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1})}{2(k_1^3+l_1^3)}\right)^{1/2}(k_1x+l_1y+\lambda_1t)+C_{14}\cos\left(\frac{-(\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1})}{2(k_1^3+l_1^3)}\right)^{1/2}(k_1x+l_1y+\lambda_1t)}{C_{13}\cos\left(\frac{-(\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1})}{2(k_1^3+l_1^3)}\right)^{1/2}(k_1x+l_1y+\lambda_1t)+C_{14}\sin\left(\frac{-(\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1})}{2(k_1^3+l_1^3)}\right)^{1/2}(k_1x+l_1y+\lambda_1t)}\right\},$$

$$u(x,y,t) = \frac{k_1}{l_1}\left(\frac{-3(\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1})}{\lambda_1}\right)^{1/2}$$

$$\left\{\frac{-C_{13}\sin\left(\frac{-(\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1})}{2(k_1^3+l_1^3)}\right)^{1/2}(k_1x+l_1y+\lambda_1t)+C_{14}\cos\left(\frac{-(\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1})}{2(k_1^3+l_1^3)}\right)^{1/2}(k_1x+l_1y+\lambda_1t)}{C_{13}\cos\left(\frac{-(\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1})}{2(k_1^3+l_1^3)}\right)^{1/2}(k_1x+l_1y+\lambda_1t)+C_{14}\sin\left(\frac{-(\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1})}{2(k_1^3+l_1^3)}\right)^{1/2}(k_1x+l_1y+\lambda_1t)}\right\},$$

where C_{13}, C_{14} are arbitrary constants and $\lambda_1(k_1^3 + l_1^3) < 0$.

2.3. $m > 2, n = 2$

(2.3) becomes

$$\left(\lambda_1 + \frac{\alpha k_1^2}{l_1} + \frac{\beta l_1^2}{k_1}\right)w^2 + (k_1^3 + l_1^3)(w^m)'' = g. \tag{2.34}$$

Balancing $(w^m)''$ with w^2, we have $N = 2/(2-m)$. Thus, (2.34) has the following formal solution

$$w(\xi) = D_3\left(\frac{G'}{G}\right)^{\frac{2}{2-m}}, \quad D_3 \neq 0, \tag{2.35}$$

where D_3 is a constant to be determined later and G satisfies (2.5). Similarly, we can get a set of algebraic equations:

$\left(\frac{G'}{G}\right)^{\frac{2m}{2-m}+2}$ coeff:

$$(k_1^3 + l_1^3)(\frac{2m}{2-m} + 1)(\frac{2m}{2-m})D_3^m + \left(\lambda_1 + \frac{\alpha k_1^2}{l_1} + \frac{\beta l_1^2}{k_1}\right)D_3^2 = 0,$$

$\left(\frac{G'}{G}\right)^{\frac{2m}{2-m}+1}$ coeff:

$$(k_1^3 + l_1^3)(\frac{4m}{2-m} + 1)\frac{2m}{2-m}\lambda D_3^m = 0,$$

$\left(\frac{G'}{G}\right)^{\frac{2m}{2-m}}$ coeff:

$$(k_1^3 + l_1^3)(\frac{2m}{2-m})^2(2\mu + \lambda^2)D_3^m = 0,$$

$\left(\frac{G'}{G}\right)^{\frac{2m}{2-m}-1}$ coeff:

$$(k_1^3 + l_1^3)(\frac{4m}{2-m} - 1)\frac{2m}{2-m}\lambda\mu D_3^m = 0,$$

$\left(\frac{G'}{G}\right)^{\frac{2m}{2-m}-2}$ coeff:

$$(k_1^3+l_1^3)(\frac{2m}{2-m}-1)\frac{2m}{2-m}\mu^2 D_3^m=0.$$

Solving the above algebraic equations, we obtain

$$\lambda=\mu=0,\quad g=0,\quad D_3=\Big(-\frac{(k_1^3+l_1^3)(\frac{2m}{2-m}+1)\frac{2m}{2-m}}{\lambda_1+\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1}}\Big)^{1/(2-m)}. \tag{2.36}$$

Substituting (2.36) into (2.5) and (2.35), then, when $m\neq n=2$, we have the rational function solutions

$$\begin{aligned}
w(x,y,t) &= \Big(-\frac{(k_1^3+l_1^3)(\frac{2m}{2-m}+1)\frac{2m}{2-m}}{\lambda_1+\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1}}\Big)^{\frac{1}{2-m}}\Big(\frac{C_{15}}{C_{15}(k_1x+l_1y+\lambda_1t)+C_{16}}\Big)^{\frac{2}{2-m}},\\
v(x,y,t) &= \frac{l_1}{k_1}\Big(-\frac{(k_1^3+l_1^3)(\frac{2m}{2-m}+1)\frac{2m}{2-m}}{\lambda_1+\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1}}\Big)^{\frac{1}{2-m}}\Big(\frac{C_{15}}{C_{15}(k_1x+l_1y+\lambda_1t)+C_{16}}\Big)^{\frac{2}{2-m}},\\
u(x,y,t) &= \frac{k_1}{l_1}\Big(-\frac{(k_1^3+l_1^3)(\frac{2m}{2-m}+1)\frac{2m}{2-m}}{\lambda_1+\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1}}\Big)^{\frac{1}{2-m}}\Big(\frac{C_{15}}{C_{15}(k_1x+l_1y+\lambda_1t)+C_{16}}\Big)^{\frac{2}{2-m}},
\end{aligned}$$

where C_{15},C_{16} are arbitrary constants and $\lambda_1+\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1}\neq 0$.

2.4. $m=n=2$

Now, (2.3) can be converted into a second order ODE

$$\Big(\lambda_1+\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1}\Big)w^2+(k_1^3+l_1^3)(w^2)''=g. \tag{2.37}$$

Obviously, the characteristic equation of (2.37) is $r^2+\left(\frac{\lambda_1+\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1}}{k_1^3+l_1^3}\right)=0$, r is the characteristic value.

Case 1. $\frac{\lambda_1+\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1}}{k_1^3+l_1^3}<0$

We can obtain the exact solution

$$\begin{aligned}
w(x,y,t) &= \Big(C_{17}e^{\left(-\frac{\lambda_1+\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1}}{k_1^3+l_1^3}\right)^{1/2}(k_1x+l_1y+\lambda_1t)}+C_{18}e^{-\left(-\frac{\lambda_1+\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1}}{k_1^3+l_1^3}\right)^{1/2}(k_1x+l_1y+\lambda_1t)}+\frac{g}{\lambda_1+\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1}}\Big)^{1/2}, \qquad (2.38)\\
v(x,y,t) &= \frac{l_1}{k_1}\Big(C_{17}e^{\left(-\frac{\lambda_1+\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1}}{k_1^3+l_1^3}\right)^{1/2}(k_1x+l_1y+\lambda_1t)}+C_{18}e^{-\left(-\frac{\lambda_1+\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1}}{k_1^3+l_1^3}\right)^{1/2}(k_1x+l_1y+\lambda_1t)}+\frac{g}{\lambda_1+\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1}}\Big)^{1/2},\\
u(x,y,t) &= \frac{k_1}{l_1}\Big(C_{17}e^{\left(-\frac{\lambda_1+\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1}}{k_1^3+l_1^3}\right)^{1/2}(k_1x+l_1y+\lambda_1t)}+C_{18}e^{-\left(-\frac{\lambda_1+\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1}}{k_1^3+l_1^3}\right)^{1/2}(k_1x+l_1y+\lambda_1t)}+\frac{g}{\lambda_1+\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1}}\Big)^{1/2},
\end{aligned}$$

where C_{17}, C_{18} are arbitrary constants.

Case 2. $\frac{\lambda_1+\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1}}{k_1^3+l_1^3} > 0$

We can obtain the periodic function solutions

$$\begin{aligned} w(x,y,t) &= \Bigg(C_{19}\cos\Big[\Big(\frac{\lambda_1+\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1}}{k_1^3+l_1^3}\Big)^{1/2}(k_1x+l_1y+\lambda_1 t)\Big] \\ &\quad +C_{20}\sin\Big[\Big(\frac{\lambda_1+\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1}}{k_1^3+l_1^3}\Big)^{1/2}(k_1x+l_1y+\lambda_1 t)\Big]+\frac{g}{\lambda_1+\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1}}\Bigg)^{1/2}, \qquad (2.39) \\ v(x,y,t) &= \frac{l_1}{k_1}\Bigg(C_{19}\cos\Big[\Big(\frac{\lambda_1+\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1}}{k_1^3+l_1^3}\Big)^{1/2}(k_1x+l_1y+\lambda_1 t)\Big] \\ &\quad +C_{20}\sin\Big[\Big(\frac{\lambda_1+\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1}}{k_1^3+l_1^3}\Big)^{1/2}(k_1x+l_1y+\lambda_1 t)\Big]+\frac{g}{\lambda_1+\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1}}\Bigg)^{1/2}, \\ u(x,y,t) &= \frac{k_1}{l_1}\Bigg(C_{19}\cos\Big[\Big(\frac{\lambda_1+\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1}}{k_1^3+l_1^3}\Big)^{1/2}(k_1x+l_1y+\lambda_1 t)\Big] \\ &\quad +C_{20}\sin\Big[\Big(\frac{\lambda_1+\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1}}{k_1^3+l_1^3}\Big)^{1/2}(k_1x+l_1y+\lambda_1 t)\Big]+\frac{g}{\lambda_1+\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1}}\Bigg)^{1/2}, \end{aligned}$$

where C_{19}, C_{20} are arbitrary constants.

Case 3. $\frac{\lambda_1+\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1}}{k_1^3+l_1^3} = 0$

We can obtain the rational function solutions

$$\begin{aligned} w(x,y,t) &= \Big(\frac{g}{2(k_1^3+l_1^3)}(k_1x+l_1y+\lambda_1 t)^2+C_{21}(k_1x+l_1y+\lambda_1 t)+C_{22}\Big)^{1/2}, \qquad (2.40) \\ v(x,y,t) &= \frac{l_1}{k_1}\Big(\frac{g}{2(k_1^3+l_1^3)}(k_1x+l_1y+\lambda_1 t)^2+C_{21}(k_1x+l_1y+\lambda_1 t)+C_{22}\Big)^{1/2}, \\ u(x,y,t) &= \frac{k_1}{l_1}\Big(\frac{g}{2(k_1^3+l_1^3)}(k_1x+l_1y+\lambda_1 t)^2+C_{21}(k_1x+l_1y+\lambda_1 t)+C_{22}\Big)^{1/2}, \end{aligned}$$

where C_{21}, C_{22} are arbitrary constants.

3. Application of the first integral method to the generalized (2+1)-dimensional BKP equation

3.1. $m \neq n$

For simplicity, we let $g = 0$ and propose a transformation $w = \varphi^{\frac{2}{n-m}}$. Then, (2.3) is converted to

$$\lambda_1\varphi^4+(k_1^3+l_1^3)(\frac{2m}{n-m}-1)\frac{2m}{n-m}(\varphi')^2+(k_1^3+l_1^3)\frac{2m}{n-m}\varphi\varphi''+\Big(\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1}\Big)\varphi^{2-\frac{2m-4}{n-m}}=0. \qquad (3.1)$$

Let $x = \varphi, y = \frac{\mathrm{d}\varphi}{\mathrm{d}\xi}$, thus (3.1) is equivalent to the two dimensional autonomous system

$$\begin{cases} x' = y, \\ y' = -\left(\frac{\lambda_1 x^4 + (\frac{\alpha k_1^2}{l_1} + \frac{\beta l_1^2}{k_1}) x^{2-\frac{2m-4}{n-m}} + (k_1^3 + l_1^3)(\frac{2m}{n-m} - 1)\frac{2m}{n-m} y^2}{(k_1^3 + l_1^3)\frac{2m}{n-m} x}\right). \end{cases} \quad (3.2)$$

Making the transformation $d\eta = \frac{d\xi}{(k_1^3+l_1^3)\frac{2m}{n-m}x}$, then, (3.2) becomes

$$\begin{cases} \frac{dx}{d\eta} = (k_1^3 + l_1^3)\frac{2m}{n-m} xy, \\ \frac{dy}{d\eta} = -\left(\lambda_1 x^4 + (\frac{\alpha k_1^2}{l_1} + \frac{\beta l_1^2}{k_1}) x^{2-\frac{2m-4}{n-m}} + (k_1^3 + l_1^3)(\frac{2m}{n-m} - 1)\frac{2m}{n-m} y^2\right). \end{cases} \quad (3.3)$$

Then, we will apply the Division Theorem to seek the first integral of system (3.3). Suppose that $x = x(\eta), y = y(\eta)$ are the nontrivial solutions to (3.3), and $p(x, y) = \sum_{i=0}^{M} a_i(x) y^i$ is an irreducible polynomial in $C[x, y]$, where $a_i(x)(i = 0, 1..., M)$ are polynomials of x and $a_i(x) \neq 0$. Let $p(x(\eta), y(\eta)) = 0$ be the first integral to system (3.3). $\frac{dp}{d\eta}$ is a polynomial in x, y and $\frac{dp}{d\eta}\Big|_{(3.3)} = 0$. According to the Division Theorem, there exists a polynomial $g(x) + h(x)y$ in $C[x, y]$, such that

$$\begin{aligned} \frac{dp}{d\eta}\Big|_{(3.3)} &= \left(\frac{\partial p}{\partial x}\frac{dx}{d\eta} + \frac{\partial p}{\partial y}\frac{dy}{d\eta}\right)\Big|_{(3.3)} \\ &= \sum_{i=0}^{M} [a_i'(x) y^i \cdot (k_1^3 + l_1^3)\frac{2m}{n-m} xy] \\ &\quad - \sum_{i=0}^{2} \left[i a_i(x) y^{i-1} (\lambda_1 x^4 + (\frac{\alpha k_1^2}{l_1} + \frac{\beta l_1^2}{k_1}) x^{2-\frac{2m-4}{n-m}} + (k_1^3 + l_1^3)(\frac{2m}{n-m} - 1)\frac{2m}{n-m} y^2) \right] \\ &= [g(x) + h(x)y]\Big[\sum_{i=0}^{M} a_i(x) y^i\Big]. \end{aligned} \quad (3.4)$$

Here, let $M = 1$, thus, $p(x, y) = a_0(x) + a_1(x)y$. By comparing with the coefficients of y^i of both sides of (3.4), we have

$$(k_1^3 + l_1^3)\frac{2m}{n-m} x a_1'(x) = h(x) a_1(x) + (k_1^3 + l_1^3)(\frac{2m}{n-m} - 1)\frac{2m}{n-m} a_1(x), \quad (3.5)$$

$$(k_1^3 + l_1^3)\frac{2m}{n-m} x a_0'(x) = g(x) a_1(x) + h(x) a_0(x), \quad (3.6)$$

$$g(x) a_0(x) = -\left(\lambda_1 x^4 + (\frac{\alpha k_1^2}{l_1} + \frac{\beta l_1^2}{k_1}) x^{2-\frac{2m-4}{n-m}}\right) a_1(x). \quad (3.7)$$

Since $a_i(x)(i = 0, 1)$ are polynomials, then from (3.5), we deduce that $h(x) = -(k_1^3 + l_1^3)(\frac{2m}{n-m} - 1)\frac{2m}{n-m}$ and $a_1(x)$ is a constant. For simplicity, take $a_1(x) = 1$. Balancing the degrees of $g(x)$ and $a_0(x)$, we conclude that $\deg(g(x)) = \deg(a_0(x))$. Then, we derive $\deg(g(x)) = \deg(a_0(x)) = j, (j \in Z^+, j \geq 2)$.

When $2 - \frac{2m-4}{n-m} = 4$ $(n = 2)$ and $\deg(g(x)) = \deg(a_0(x)) = 2$, we suppose that

$$g(x) = A_0 + A_1 x + A_2 x^2,$$

$$a_0(x) = B_0 + B_1 x + B_2 x^2, \quad (A_2 \neq 0, B_2 \neq 0), \tag{3.8}$$

where $A_i, B_i, (i = 0, 1, 2)$ are all constants to be determined. Substituting (3.8) into (3.6), we obtain

$$g(x) = (k_1^3 + l_1^3)\frac{2m}{2-m}\left[(\frac{2m}{2-m} - 1)B_0 + \frac{2m}{2-m}B_1 x + (\frac{2m}{2-m} + 1)B_2 x^2\right].$$

Substituting $a_0(x), a_1(x)$ and $g(x)$ into (3.7), and setting all the coefficients of powers x to be zero, we can get a system of nonlinear algebraic equations. After solving it, we can get the following solutions

$$B_0 = B_1 = 0, \qquad B_2 = \pm\Big(-\frac{\lambda_1 + \frac{\alpha k_1^2}{l_1} + \frac{\beta l_1^2}{k_1}}{(k_1^3 + l_1^3)(\frac{2m}{2-m} + 1)\frac{2m}{2-m}}\Big)^{1/2}. \tag{3.9}$$

Using the conditions (3.9) in $p(x, y) = a_0(x) + a_1(x)y = 0$, we obtain

$$y \pm \Big(-\frac{\lambda_1 + \frac{\alpha k_1^2}{l_1} + \frac{\beta l_1^2}{k_1}}{(k_1^3 + l_1^3)(\frac{2m}{2-m} + 1)\frac{2m}{2-m}}\Big)^{1/2} x^2 = 0. \tag{3.10}$$

Combining (3.3) with (3.10), we find

$$\frac{dx}{d\eta} = \pm(k_1^3 + l_1^3)\frac{2m}{n-m}\Big(-\frac{\lambda_1 + \frac{\alpha k_1^2}{l_1} + \frac{\beta l_1^2}{k_1}}{(k_1^3 + l_1^3)(\frac{2m}{2-m} + 1)\frac{2m}{2-m}}\Big)^{1/2} x^3.$$

Thus, (3.10) can be reduced to

$$\frac{d\varphi}{d\xi} = \pm\Big(-\frac{\lambda_1 + \frac{\alpha k_1^2}{l_1} + \frac{\beta l_1^2}{k_1}}{(k_1^3 + l_1^3)(\frac{2m}{2-m} + 1)\frac{2m}{2-m}}\Big)^{1/2} \varphi^2.$$

Then, we have

$$\varphi(\xi) = \Big[\pm\Big(-\frac{\lambda_1 + \frac{\alpha k_1^2}{l_1} + \frac{\beta l_1^2}{k_1}}{(k_1^3 + l_1^3)(\frac{2m}{2-m} + 1)\frac{2m}{2-m}}\Big)^{1/2} \xi + C_{23}\Big]^{-1}.$$

Thus, we can have the rational function solutions

$$\begin{aligned}
w(x, y, t) &= \Big[\pm\Big(-\frac{\lambda_1 + \frac{\alpha k_1^2}{l_1} + \frac{\beta l_1^2}{k_1}}{(k_1^3 + l_1^3)(\frac{2m}{2-m} + 1)\frac{2m}{2-m}}\Big)^{1/2}(k_1 x + l_1 y + \lambda_1 t) + C_{23}\Big]^{2/(m-2)}, \\
v(x, y, t) &= \frac{l_1}{k_1}\Big[\pm\Big(-\frac{\lambda_1 + \frac{\alpha k_1^2}{l_1} + \frac{\beta l_1^2}{k_1}}{(k_1^3 + l_1^3)(\frac{2m}{2-m} + 1)\frac{2m}{2-m}}\Big)^{1/2}(k_1 x + l_1 y + \lambda_1 t) + C_{23}\Big]^{2/(m-2)}, \\
u(x, y, t) &= \frac{k_1}{l_1}\Big[\pm\Big(-\frac{\lambda_1 + \frac{\alpha k_1^2}{l_1} + \frac{\beta l_1^2}{k_1}}{(k_1^3 + l_1^3)(\frac{2m}{2-m} + 1)\frac{2m}{2-m}}\Big)^{1/2}(k_1 x + l_1 y + \lambda_1 t) + C_{23}\Big]^{2/(m-2)},
\end{aligned}$$

where C_{23} is an arbitrary constant and $k_1^3 + l_1^3 \neq 0$.
Remark 1: When $\deg(g(x)) = \deg(a_0(x)) = 2$ and $2 - \frac{2m-4}{n-m} = i, (i \in Z, i < 4)$, there is no solution for them by using the method as that of $2 - \frac{2m-4}{n-m} = 4$.
Remark 2: When $\deg(g(x)) = \deg(a_0(x)) = j, (j \in Z, j > 2)$, there is no exact solution of (1.1) by using the method as that of $\deg(g(x)) = \deg(a_0(x)) = 2$.

3.2. $m = n$

Similarly, we propose a transformation denoted by $w = \phi^{\frac{2}{2-m}}$. Then, (2.3) can be converted to

$$\lambda_1\phi^2 + \Big(\frac{\alpha k_1^2}{l_1} + \frac{\beta l_1^2}{k_1}\Big)\phi^4 + (k_1^3 + l_1^3)(\frac{2m}{2-m} - 1)\frac{2m}{2-m}(\phi')^2 + (k_1^3 + l_1^3)\frac{2m}{2-m}\phi\phi'' - g\phi^{2-\frac{2m}{2-m}} = 0. \quad (3.11)$$

Let $x = \phi, y = \frac{d\phi}{d\xi}$, thus (3.11) is equivalent to the two dimensional autonomous system

$$\begin{cases} x' = y, \\ y' = -\Bigg(\frac{\lambda_1 x^2 + (\frac{\alpha k_1^2}{l_1} + \frac{\beta l_1^2}{k_1})x^4 - gx^{2-\frac{2m}{2-m}} + (k_1^3+l_1^3)(\frac{2m}{2-m}-1)\frac{2m}{2-m}y^2}{(k_1^3+l_1^3)\frac{2m}{2-m}x}\Bigg). \end{cases} \quad (3.12)$$

Making the transformation $d\eta = \frac{d\xi}{(k_1^3+l_1^3)\frac{2m}{n-m}x}$, then, (3.12) becomes

$$\begin{cases} \frac{dx}{d\eta} = (k_1^3 + l_1^3)\frac{2m}{2-m}xy, \\ \frac{dy}{d\eta} = -\Big(\lambda_1 x^2 + (\frac{\alpha k_1^2}{l_1} + \frac{\beta l_1^2}{k_1})x^4 - gx^{2-\frac{2m}{2-m}} + (k_1^3 + l_1^3)(\frac{2m}{2-m} - 1)\frac{2m}{2-m}y^2\Big). \end{cases} \quad (3.13)$$

Similarly, let $M = 1$, we have

$$(k_1^3 + l_1^3)\frac{2m}{2-m}xa_1'(x) = h(x)a_1(x) + (k_1^3 + l_1^3)(\frac{2m}{2-m} - 1)\frac{2m}{2-m}a_1(x), \quad (3.14)$$

$$(k_1^3 + l_1^3)\frac{2m}{2-m}xa_0'(x) = g(x)a_1(x) + h(x)a_0(x), \quad (3.15)$$

$$g(x)a_0(x) = -\Bigg(\lambda_1 x^2 + (\frac{\alpha k_1^2}{l_1} + \frac{\beta l_1^2}{k_1})x^4 - gx^{2-\frac{2m}{2-m}}\Bigg)a_1(x). \quad (3.16)$$

According to $m \neq n$, we have $h(x) = -(k_1^3 + l_1^3)(\frac{2m}{2-m} - 1)\frac{2m}{2-m}$, $a_1(x) = 1$ and $\deg(g(x)) = \deg(a_0(x)) = j, (j \in Z^+, j \geq 2)$. Considering all cases, only when $\deg(g(x)) = \deg(a_0(x)) = 3$, i.e., $2 - \frac{2m}{2-m} = 6$ $(m = n = 4)$, there exist solutions of (1). We suppose that

$$\begin{aligned} g(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3, \\ a_0(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3, \quad (a_3 \neq 0, b_3 \neq 0), \end{aligned} \quad (3.17)$$

where $a_i, b_i, (i = 0, 1, 2, 3)$ are all constants to be determined. Substituting (3.17) into (3.15), we obtain

$$g(x) = 4(k_1^3 + l_1^3)\left(5b_0 + 4b_1 x + 3b_2 x^2 + 2b_3 x^3\right).$$

Substituting $a_0(x), a_1(x)$ and $g(x)$ into (3.16), and setting all the coefficients of powers x to be zero, we have

$$b_0 = b_2 = 0, \;\; 16(k_1^3 + l_1^3)b_1^2 = -\lambda_1, \;\; 8(k_1^3 + l_1^3)b_3^2 = g, \;\; 24(k_1^3 + l_1^3)b_1 b_3 = -(\frac{\alpha k_1^2}{l_1} + \frac{\beta l_1^2}{k_1}).$$

Solving it, we find

$$b_0 = b_2 = 0, \quad b_1 = \pm\sqrt{\frac{-\lambda_1}{16(k_1^3 + l_1^3)}}, \quad b_3 = \pm\sqrt{\frac{g}{8(k_1^3 + l_1^3)}}, \quad -9\lambda_1 g = 2(\frac{\alpha k_1^2}{l_1} + \frac{\beta l_1^2}{k_1})^2. \quad (3.18)$$

Using the conditions (3.18) in $p(x,y) = a_0(x) + a_1(x)y = 0$, we obtain

$$y = \pm\sqrt{\frac{-\lambda_1}{16(k_1^3+l_1^3)}}x \pm \sqrt{\frac{g}{8(k_1^3+l_1^3)}}x^3. \tag{3.19}$$

Then, (3.19) can be reduced to

$$\frac{d\phi}{d\xi} = \pm\sqrt{\frac{-\lambda_1}{16(k_1^3+l_1^3)}}\xi \pm \sqrt{\frac{g}{8(k_1^3+l_1^3)}}\xi^3. \tag{3.20}$$

Solving (3.20), we obtain

$$\phi(\xi) = \pm\left(\pm\sqrt{\frac{-2g}{\lambda_1}} + C_{24}e^{\pm\frac{1}{2}\sqrt{\frac{-\lambda_1}{k_1^3+l_1^3}}\xi}\right)^{-1/2}.$$

Thus, we can have the exact solution

$$\begin{aligned}
w(x,y,t) &= \pm\left(\pm\sqrt{\frac{-2g}{\lambda_1}} + C_{24}e^{\pm\frac{1}{2}\sqrt{\frac{-\lambda_1}{k_1^3+l_1^3}}(k_1x+l_1y+\lambda_1 t)}\right)^{1/2},\\
v(x,y,t) &= \pm\frac{l_1}{k_1}\left(\pm\sqrt{\frac{-2g}{\lambda_1}} + C_{24}e^{\pm\frac{1}{2}\sqrt{\frac{-\lambda_1}{k_1^3+l_1^3}}(k_1x+l_1y+\lambda_1 t)}\right)^{1/2},\\
u(x,y,t) &= \pm\frac{k_1}{l_1}\left(\pm\sqrt{\frac{-2g}{\lambda_1}} + C_{24}e^{\pm\frac{1}{2}\sqrt{\frac{-\lambda_1}{k_1^3+l_1^3}}(k_1x+l_1y+\lambda_1 t)}\right)^{1/2},
\end{aligned}$$

where C_{24} is an arbitrary constant and $\alpha k_1^3 + \beta l_1^3 \neq 0$, $\lambda_1(k_1^3 + l_1^3) < 0$.

4. Conclusion

This paper considered the generalized (2+1)-dimensional BKP equation, by the aid of the G'/G-expansion method and the first integral method. Rational function solutions, periodic function solutions and hyperbolic function solutions are obtained under some parametric conditions and the values of m and n in several cases. In [10], authors gave some exact solutions of system (1.1) under some parametric conditions by using the bifurcation theory of dynamical systems. Here, we make a simple comparison:

1. When $m = 2, n = 3, g = 0$, in [10], authors gave the exact solution (3.20) in P2443 under the parametric conditions $\alpha + \beta < 0, c < 0$ and in this paper, we get $w(x,y,t) = \frac{-20(k_1^3+l_1^3)}{\lambda_1}\left(\frac{C_1}{C_1(k_1x+l_1y+\lambda_1 t)+C_2}\right)^2$ under the parametric condition $\alpha k_1^3 + \beta l_1^3 = 0$.

2. When $m = 2, n = 2(k+1), (k \in Z^+), g = 0$, in [10], authors gave the solitary wave solutions (3.9) in P2441 under the parametric conditions $\alpha + \beta < 0, c < 0$ and in this paper, we get $w(x,y,t) = \left(\sqrt{\frac{-2(k+2)(k_1^3+l_1^3)}{\lambda_1 k^2}}\frac{C_1}{C_1(k_1x+l_1y+\lambda_1 t)+C_2}\right)^{1/k}$ under the parametric condition $\alpha k_1^3 + \beta l_1^3 = 0$.

3. When $m = 3, n = 4, g = 0$, in [10], authors gave the compacton solution (3.23) in P2443 under the parametric conditions $\alpha + \beta < 0, c < 0$ and in this paper, we get $w(x,y,t) = \frac{-42(k_1^3+l_1^3)}{\lambda_1}\left(\frac{C_1}{C_1(k_1x+l_1y+\lambda_1 t)+C_2}\right)^2$ under the parametric condition $\alpha k_1^3 + \beta l_1^3 = 0$.

4. When $m = 3, n = 5, g = 0$, in [10], authors gave the exact solution (3.27) in P2443 under the parametric conditions $c < 0$ and in this paper, we get $w(x, y, t) = \sqrt{\frac{-12(k_1^3+l_1^3)}{\lambda_1}\frac{C_1}{C_1(k_1x+l_1y+\lambda_1t)+C_2}}$ under the parametric condition $\alpha k_1^3 + \beta l_1^3 = 0$.

5. When $m = 4, n = 6, g = 0$, in [10], authors gave the periodic cusp wave solutions (3.6) in P2441 under the parametric conditions $\alpha + \beta < 0, c < 0$ and in this paper, we get $w(x, y, t) = \sqrt{\frac{-20(k_1^3+l_1^3)}{\lambda_1}\frac{C_1}{C_1(k_1x+l_1y+\lambda_1t)+C_2}}$ under the parametric condition $\alpha k_1^3 + \beta l_1^3 = 0$.

6. When $m = 4, n = 2k + 1, (k \in Z^+), g = 0$, in [10], authors gave the exact solutions (3.17) and (3.18) in P2442 under the parametric conditions $\alpha + \beta < 0, c < 0$ and in this paper, we get $w(x, y, t) = \left(\frac{-8(2k+5)(k_1^3+l_1^3)}{\lambda_1(2k-3)^2}\right)^{\frac{1}{2k-3}}\left(\frac{C_1}{C_1(k_1x+l_1y+\lambda_1t)+C_2}\right)^{\frac{2}{2k-3}}$ under the parametric condition $\alpha k_1^3 + \beta l_1^3 = 0$.

7. When $m = 2, n = 4, g \neq 0$, in [10], authors gave the exact solutions (3.30) in P2443 under the parametric conditions $g < \frac{(\alpha+\beta)^2}{4c}, c > 0, g > 0, \alpha + \beta > 0$, (3.33) in P2444 under the parametric conditions $g < \frac{(\alpha+\beta)^2}{4c}, c < 0, g > 0, \alpha + \beta > 0$ or $g > \frac{(\alpha+\beta)^2}{4c}, c < 0, g > 0, \alpha + \beta < 0$ and (3.41), (3.43) in P2445 under the parametric conditions $g > \frac{(\alpha+\beta)^2}{4c}, c < 0, g < 0, \alpha + \beta < 0$ and in this paper, we get (2.32) under the parametric conditions $\lambda_1(k_1^3 + l_1^3) < 0$, $\frac{\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1}}{2(k_1^3+l_1^3)} > 0$ and (2.33) under the parametric conditions $\lambda_1(k_1^3 + l_1^3) < 0$, $\frac{\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1}}{2(k_1^3+l_1^3)} < 0$.

8. When $m = 2, n = 2, g \neq 0$, in [10], authors gave the exact solutions (3.36) and (3.38) in P2444 under the parametric conditions $\alpha + \beta - c > 0, g > 0$ and in this paper, we get the exact solutions (2.38) under the parametric conditions $\frac{\lambda_1+\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1}}{k_1^3+l_1^3} < 0$, (2.39) under the parametric conditions $\frac{\lambda_1+\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1}}{k_1^3+l_1^3} > 0$ and (2.40) under the parametric conditions $\frac{\lambda_1+\frac{\alpha k_1^2}{l_1}+\frac{\beta l_1^2}{k_1}}{k_1^3+l_1^3} = 0$.

In addition, when let m, n be other values, we have got other exact solutions of (1.1) under some parametric conditions that haven't been given in [10]. Certainly, system (1.1) should be studied further, which will be left to a further discussion.

Conflict of Interest

All authors declare no conflicts of interest in this paper.

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