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Research article

Critical blowup in coupled Parity-Time-symmetric nonlinear Schrödinger equations

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Abstract: In this article, we obtain sufficient conditions to obtain finite time blowup in a system of two coupled nonlinear Schrödinger (NLS) equations in the critical case. This system mainly considered here in dimension 2, couples one equation including gain and the other one including losses, constituting a generalization of the model of pulse propagation in birefringent optical fibers. In the spirit of the seminal work of Glassey, the proofs used the virial technique arguments.

Keywords: Coupled nonlinear Schrödinger equations; Parity-Time symmetry; finite time blowup; generalized Manakov model

1. Introduction

Several applications need to solve and study nonlinear Schrödinger (NLS) equations. Hereafter, a basic model of propagation of weakly dispersive waves is considered by a system of coupled NLS equations which reads as follows:

$$\left\{ \begin{array}{rcl} \imath u_t & = & -\Delta u + \kappa v + \imath\gamma u - (g_{11}|u|^2 + g_{12}|v|^2)u, \\ \imath v_t & = & -\Delta v + \kappa u - \imath\gamma v - (g_{12}|u|^2 + g_{22}|v|^2)v, \end{array} \right. \quad (1)$$

with the coefficients of the nonlinear parts being real. The $\imath$ is the complex such that $\imath^2 = -1$ and the coefficients κ and γ are positive constants that characterize gain and loss in wave components. This model is known for its pertinence for several applications of nonlinear optics (as birefringent optics fiber) and has been studied by several authors [1, 2, 3, 4, 5, 6, 7, 8]. The parameter γ does not influence the model by a damping phenomenon as it appears in each component of the system by opposite sign (see for example some studies of damped NLS [9, 10, 11]).

In addition, one recalls that if the parameter γ was equal to zero, the density and the energy would be time invariants, and the model is known as the Hamiltonian version of the generalized Manakov system [5, 6]. In this paper, we focus onto giving sufficient conditions in order to predict that the solution of the Cauchy problem in the critical case blows up in finite time.

Let us recall that the system (1) is considered as a Parity-Time (PT) symmetric system as soon as the coefficients $g_{11} = g_{22}$, which means that if $g_{11} = g_{22}$, the following formal property holds: if $(u(x,t), v(x,t))$ solve the system (1), then the pair $(u_{PT}(x,t), v_{PT}(x,t) := (\bar{v}(x,-t), \bar{u}(x,t))$ also solves the same system (hereafter an overbar stands for the complex conjugation).

The concept of PT-symmetry first emerged from quantum mechanics with the study of pure real spectra of non-Hermitian operators [12]. But for around one decade this PT-symmetry property gained a particular relevance due to its importance in several other branches in Physics: optics, Bose-Einstein Condensates, plasmonic waveguides, electronic circuits, superconductivity,... [13, 7, 14] (and references therein). Indeed, the interplay between this property and the nonlinearity of the system seems to be at the heart of several phenomenon as the behaviour of a single atomic specie in two different ground states of Bose-Einstein Condensates [15], or the modulational instability of the carrier wave between two waveguides [14].

The existence of a unique global solution $(u(t), v(t)) \in C(\mathbf{R}, (H^1(\mathbf{R}))^2)$ of the Cauchy problem for the generalized Manakov system (1) in dimension 1 with $(u(0), v(0)) = (u_0, v_0) \in (H^1(\mathbf{R}))^2$ is known [6]. Also in 1D, we proved recently that if the symmetry is *unbroken* ($\gamma < \kappa$) the H^1-norm of the solution cannot blow up in finite time [8]. But in higher dimension, the answer is fully different as the problem is more complex. In the supercritical case, which corresponds to dimension greater or equal to 3, finite time blowup for the system (1) is obtained under suffisient conditions on the parameters [16].

But in the critical case, as far as we know, only partial results are available until now [14]. The existence of finite-time blowup for solutions of a single NLS equation in dimension $n \geq 2$ is well known [17]. Analysis of global existence and blow-up of solutions of the Hamiltonian version of the generalized Manakov system have been studied [18, 19]. And solutions of this Hamiltonian version without linear coupling are known to blow up in finite time in a specific way that is called L^2-concentration (mass concentration) [20]. In our focusing case, as soon as the PT-symmetry is unbroken ($\gamma < \kappa$) and the nonlinear coefficients g_{11}, g_{22} and g_{12} equals to 1, a global solution in $(H^1(\mathbf{R}^2))^2$ exists for initial solutions that provide a density remaining limited all over time [6].

Our main contribution is to adapt the sketch of proof that has been developed for the supercritical case by other authors [16]. But to develop it, we had to modify slightly the second time-derivative expression by substituting the first integral Stokes variable using the energy equality. Following that, the proof by contradiction can be led in the spirit of the technique that has been introduced by Glassey [21]. Assuming that the nonlinear coefficients are such that the quadradic form

$$\phi(x, y) = g_{11}x^4 + g_{22}y^4 + 2g_{12}x^2y^2$$

is negative definite, one obtains the non existence of the solution of the Cauchy problem in the interval $[0, T_0]$.

The outline of the paper is the following. The main results consist of gathering in the proposition, the computations of the time-derivative of the density, the time-derivative of the energy, and the first and second time-derivative of the mean square momentum. In a second proposition, we focus onto initial conditions of the Cauchy problem that provide the suffisient blowup conditions of the main theorem that will follow.

2. Main results

Let us consider here a local solution in energy space

$$(u(t), v(t)) \in C\left([-t_0, t_0], H^1(\mathbf{R}^2) \times H^1(\mathbf{R}^2)\right)$$

of the Cauchy problem (1)-$(u(0), v(0)) = (u_0, v_0)$, that is obtained by a modified contraction method with Stritcharz estimates [22].

Let us recall that in the Hamiltonian case (for $\gamma = 0$), it is known that for $(u, v) \in C(\mathbf{R}; H^1(\mathbf{R}^n) \times H^1(\mathbf{R}^n))$, the two following quantities are conserved, namely the density, which is defined by,

$$Q(t) = \int (|u|^2 + |v|^2) dx,$$

and the total energy of the system (1)

$$E(t) = \int \left(|\nabla u|^2 + |\nabla v|^2 + \kappa(\bar{u}v + u\bar{v}) - \frac{g_{11}}{2}|u|^4 - \frac{g_{22}}{2}|v|^4 - g_{12}|u|^2|v|^2\right) dx.$$

In the sequel, we also need to introduce the mean square momentum, and its time derivative

$$X(t) = \int |x|^2(|u|^2 + |v|^2) dx, \quad \text{and } Y(t) = \frac{dX(t)}{dt}.$$

In the next proposition some results are recalled for a convenient self-consistent corpus of this paper.

Proposition 2.1. *For $(u(t), v(t)) \in C(\mathbf{R}, (H^1(\mathbf{R}^n))^2)$ the solution of the Cauchy problem for the generalized Manakov system (1) with $(u(0), v(0)) = (u_0, v_0) \in (H^1(\mathbf{R}^n))^2$,*

$$\frac{dQ}{dt} = 2\gamma \int (|u|^2 - |v|^2) dx, \tag{2}$$

$$\frac{dE}{dt} = 2\gamma \int (|\nabla u|^2 - |\nabla v|^2 - g_{11}|u|^4 + g_{22}|v|^4) dx, \tag{3}$$

$$Y(t) = 4\mathcal{I}m \int ((\overline{u}x.\nabla u) + (\overline{v}x.\nabla v)) dx + 2\gamma \int |x|^2(|u|^2 - |v|^2) dx. \tag{4}$$

$$\begin{aligned} \frac{dY}{dt} = 4nE(t) + 4(2-n) \int (|\nabla u|^2 + |\nabla v|^2) dx + 4\gamma^2 X(t) + \\ 8\gamma \mathcal{I}m \int ((\overline{u}x.\nabla u) - (\overline{v}x.\nabla v)) dx + 8\kappa\gamma \mathcal{I}m \int |x|^2(v\overline{u}) dx - 8\kappa n \mathcal{R}e \int u\overline{v} dx. \end{aligned} \tag{5}$$

Proof. The equalities (2) and (3) are obvious (see [6, 8]).
To prove equality (4), one needs to take the scalar product of the first equality of the system (1) with $2u$ and the second one with $2v$.

Taking the imaginary part of the first computation, one obtains:

$$\frac{\partial}{\partial t}|u|^2 = -2\mathcal{I}m(\overline{u}\Delta u) + 2\gamma|u|^2.$$

But, one can aslo write:

$$\frac{\partial}{\partial t}|u|^2 = -2\nabla.(\mathcal{I}m(\overline{u}\nabla u)) + 2\gamma|u|^2.$$

Taking the product with $|x|^2$ and after space integration over $\mathbf{R}^n$, one gets:

$$\frac{\partial}{\partial t}\int |x|^2|u|^2 dx = -2\int \nabla.(\mathcal{I}m(\overline{u}\nabla u))|x|^2 dx + 2\gamma\int |x|^2|u|^2 dx.$$

Following an integration by parts of the first integral of the right hand side, the previous equality becomes:

$$\frac{\partial}{\partial t}\int |x|^2|u|^2 dx = 4\int \sum_{k=1}^{n}(\mathcal{I}m(\overline{u}\nabla u))_k x_k dx + 2\gamma\int |x|^2|u|^2 dx.$$

In other words,

$$\frac{\partial}{\partial t}\int |x|^2|u|^2 dx = 4\mathcal{I}m\int (\overline{u}x.\nabla u)dx + 2\gamma\int |x|^2|u|^2 dx. \tag{6}$$

Processing in a similar way with the second equality of the system (1), one gets the following equality.

$$\frac{\partial}{\partial t}\int |x|^2|v|^2 dx = 4\mathcal{I}m\int (\overline{v}x.\nabla v)dx - 2\gamma\int |x|^2|v|^2 dx. \tag{7}$$

Adding the equalities (6) and (7), one recovers the equality (4).

To prove equality (5), one needs to start by taking the scalar product of the first equality of the system (1) with $(2x.\nabla u)$ and the second one with $(2x.\nabla v)$. Following that, after integrating over $\mathbf{R}^n$ each term of the expression that has been obtained, one takes the real part of their sum to obtain:

$$\begin{aligned}2\mathcal{R}e\int \imath\left[(x.\nabla\overline{u})\frac{\partial u}{\partial t} + (x.\nabla\overline{v})\frac{\partial v}{\partial t}\right]dx &= -2\mathcal{R}e\int \left[(x.\nabla\overline{u})\Delta u + (x.\nabla\overline{v})\Delta v\right]dx \\ &+ 2\kappa\mathcal{R}e\int \left[(x.\nabla\overline{u})v + (x.\nabla\overline{v})u\right]dx + 2\gamma\mathcal{R}e\int \imath\left[(x.\nabla\overline{u})u + (x.\nabla\overline{v})v\right]dx \\ &- 2\mathcal{R}e\int \left[(x.\nabla\overline{u})(g_{11}|u|^2 + g_{12}|v|^2)u + (x.\nabla\overline{v})(g_{22}|v|^2 + g_{12}|v|^2)v\right]dx.\end{aligned} \tag{8}$$

First, let us denote by R_1, R_2, R_3 and R_4, each term of the right hand side of the equality (8) (in the same order of appearance), such that the right hand side equals the sum of each R_i.
By several integration by parts, similarly as been done ([22] p. 125, for example), we get

$$R_1 = -2\mathcal{R}e\int \left[(x.\nabla\overline{u})\Delta u + (x.\nabla\overline{v})\Delta v\right]dx = (2-n)\int (|\nabla u|^2 + |\nabla v|^2)dx.$$

Also, the second term of the right-hand-side of equality (8) can be rewritten as

$$R_2 = 2\kappa\sum_{k=1}^{n}\int \mathcal{R}e x_k\frac{\partial}{\partial x_k}(u\overline{v})dx = -2\kappa n\mathcal{R}e\int u\overline{v}dx.$$

By transforming the third term of the right-hand-side of (8), one obtains:

$$R_3 = -2\gamma\mathcal{I}m\int \left[(x.\nabla\overline{u})u - (x.\nabla\overline{v})v\right]dx.$$

Expressing
$$\frac{\partial \overline{u}}{\partial x_k} = \frac{\partial |u|^2}{\partial x_k} - \overline{u}\frac{\partial u}{\partial x_k},$$
and similarly for the second component, the first part being real, one obtains by taking the imaginary part,

$$\begin{aligned} R_3 &= -2\gamma \mathcal{I}m \sum_{k=1}^{n} \int x_k \left(-\overline{u}\frac{\partial u}{\partial x_k} + \overline{v}\frac{\partial v}{\partial x_k} \right) dx \\ &= 2\gamma \mathcal{I}m \int \left[(\overline{u}x.\nabla u) - (\overline{v}x.\nabla v) \right] dx. \end{aligned}$$

The last term of the right-hand-side of (8) can be formulated as follows.

$$\begin{aligned} R_4 = -\sum_{k=1}^{n} \int x_k \left[g_{11}|u|^2 (2\mathcal{R}e \frac{\partial \overline{u}}{\partial x_k} u) + g_{22}|v|^2 (2\mathcal{R}e \frac{\partial \overline{v}}{\partial x_k} v) \right] dx \\ + g_{12} \sum_{k=1}^{n} \int x_k \left[|v|^2 (2\mathcal{R}e \frac{\partial \overline{u}}{\partial x_k} u) + |u|^2 (2\mathcal{R}e \frac{\partial \overline{v}}{\partial x_k} v) \right] dx. \end{aligned}$$

Hence,

$$R_4 = -\sum_{k=1}^{n} \int x_k \left[g_{11}|u|^2 \frac{\partial}{\partial x_k}|u|^2 + g_{22}|v|^2 \frac{\partial}{\partial x_k}|v|^2 \right] dx + g_{12} \sum_{k=1}^{n} \int x_k \left[|v|^2 \frac{\partial}{\partial x_k}|u|^2 + |u|^2 \frac{\partial}{\partial x_k}|v|^2 \right] dx.$$

Finally, one gets:

$$\begin{aligned} R_4 &= -\sum_{k=1}^{n} \int x_k \left[\frac{1}{2}\frac{\partial}{\partial x_k}(g_{11}|u|^4 + g_{22}|v|^4) + g_{12}\frac{\partial}{\partial x_k}(|u|^2|v|^2) \right] dx \\ &= \frac{n}{2} \int (g_{11}|u|^4 + g_{22}|v|^4) dx + n g_{12} \int |u|^2|v|^2 dx. \end{aligned}$$

The left-hand-side of the equality (8) can be also rewritten as follows:

$$\begin{aligned} \mathcal{R}e \int \imath \sum_{k=1}^{n} x_k \left[\left(\frac{\partial \overline{u}}{\partial x_k} u \right)_t - \frac{\partial}{\partial x_k}(u\overline{u}_t) + \left(\frac{\partial \overline{v}}{\partial x_k} v \right)_t - \frac{\partial}{\partial x_k}(u\overline{u}_t) \right] dx \\ = -\frac{\partial}{\partial t} \mathcal{I}m \int \left[(x.\nabla \overline{u})u + (x.\nabla \overline{v})v \right] dx - \mathcal{R}e \int \imath \sum_{k=1}^{n} x_k \frac{\partial}{\partial x_k}(u\overline{u}_t + v\overline{v}_t) dx, \end{aligned}$$

and one can still use the system (1) to obtain:

$$\begin{aligned} \mathcal{R}e \int \imath \sum_{k=1}^{n} x_k \left[\left(\frac{\partial \overline{u}}{\partial x_k} u \right)_t - \frac{\partial}{\partial x_k}(u\overline{u}_t) + \left(\frac{\partial \overline{v}}{\partial x_k} v \right)_t - \frac{\partial}{\partial x_k}(u\overline{u}_t) \right] dx \\ = \frac{\partial}{\partial t} \mathcal{I}m \int \left[(\overline{u}x.\nabla u) + (\overline{v}x.\nabla v) \right] dx - n\mathcal{R}e \int (|\nabla u|^2 + |\nabla v|^2) dx \end{aligned}$$

$$-n\kappa\int(v\overline{u}+u\overline{v})dx+n\int(g_{11}|u|^4+g_{22}|v|^4+2g_{12}|u|^2|v|^2)dx.$$

Consequently, using all the previous computations, equation (8) can be formulated in the following way, by recovering the energy expression,

$$\begin{aligned}\frac{\partial}{\partial t}\mathcal{I}m\int\left[(\overline{u}x.\nabla u)+(\overline{v}x.\nabla v)\right]dx &= nE(t)+(2-n)\int(|\nabla u|^2+|\nabla v|^2)dx\\ &\quad -2\kappa n\mathcal{R}e\int u\overline{v}dx+\gamma\mathcal{I}m\int\left[(\overline{u}x.\nabla u)-(\overline{v}x.\nabla v)\right]dx. \quad (9)\end{aligned}$$

In other hand, let us take the difference between the scalar product of the first equation of the system (1) with $|x|^2u$ and the second equation with $|x|^2v$. Then, the imaginary part of the computations gives:

$$\frac{1}{2}\frac{\partial}{\partial t}\int|x|^2(|u|^2-|v|^2)dx=2\kappa\mathcal{I}m\int|x|^2v\overline{u}dx+\gamma X(t). \quad (10)$$

By summing side by side 4 times equation (9) with 4γ times equation (10), we obtain the final formulation of the second time derivative of X. □

In order to state the main result of this paper, let us now introduce several notations. Following the introduction, we will consider the quadratic form $\phi(X,Y)$,

$$\phi(X,Y):=g_{11}X^4+g_{22}Y^4+2g_{12}X^2Y^2, \quad (11)$$

that we assume negative definite. This property is valid as soon as the coefficients g_{11},g_{22},g_{12} are all strictly negative, but also if $g_{11},g_{22}<0$ and $g_{12}\leq\sqrt{g_{11}g_{22}}$.
Let us define $\rho(t)$, $F(t)$, $M(t)$ and $G(t)$ four real maps by,

$$\begin{aligned}\rho(t) &= \int_0^t\int_0^\sigma(|\nabla u|^2+|\nabla v|^2+|u|^4+|v|^4)d\tau d\sigma,\\ F(t) &= X(0)+Y(0)t+6E(0)t^2+\frac{3\kappa}{\gamma^2}Q(0)(e^{2\gamma t}-2\gamma t-1),\\ M(t) &= \sup_{\tau\in[0,t]}F(\tau)+1,\\ G(t) &= M(t)\left(\frac{c_1t}{2}+\exp(\frac{\gamma c_3t}{c_2})-1\right),\end{aligned}$$

where the real constants c_1,c_2 and c_3 being as follows

$$\begin{aligned}c_1 &= 4\gamma^2(\frac{\kappa}{\gamma}+3),\\ c_2 &= \begin{cases}\min(2,-2g_{11},-2g_{22}) & \text{if } g_{12}<0\\ \min(2,(-2g_{11}+g_{12}),(-2g_{22}+g_{12})) & \text{if } g_{12}>0,\end{cases}\\ c_3 &= 24\max(1,|g_{11}|,|g_{22}|).\end{aligned}$$

Proposition 2.2. *Let C_0 a real constant defined later according to the initial conditions $(u_0(x), v_0(x))$ of the Cauchy problem for the system (1). Let $\beta = \frac{\gamma c_3}{c_2}$ be another constant, and let us define the real map*

$$\widetilde{M}(t) = 1 + X(0) + C_0(e^{2\gamma t} - 1),$$

and

$$\begin{aligned} T_0^+ &= \frac{1}{\beta}\ln\left(1 + \frac{\beta^2}{(1+X(0))(\beta^2+c_1)}\right) \\ T_0^- &= \frac{1}{\beta}\ln\left(1 + \frac{\beta^2}{\widetilde{M}(T_0^+)(\beta^2+c_1)}\right). \end{aligned} \tag{12}$$

If

$$E(0) < -\frac{\widetilde{M}(T_0^+)}{6(T_0^-)^2}, \tag{13}$$

we fix $C_0 = \dfrac{|Y(0)|}{2\gamma} + \dfrac{3\kappa}{\gamma^2}Q(0)$. *And if*

$$Y(0) < \frac{6\kappa}{\gamma}Q(0) - \frac{\widetilde{M}(T_0^+)}{T_0^-}, \tag{14}$$

we fix $C_0 = \dfrac{6|E(0)|}{\gamma^2} + \dfrac{3\kappa}{\gamma^2}Q(0)$.
Then, there exists $T_0 \in [T_0^-, T_0^+]$ such that the following conditions are satisfied:

$$F(T_0) + 1 < 0, \tag{15}$$

$$G(T_0) < 1. \tag{16}$$

The proof of this proposition is similar to that have been done by Dias *et. al.* [16].

Proof. Let us introduce

$$\widetilde{G}(t) := M(t)\left(1 + \frac{c_1}{\beta^2}\right)(e^{\beta t} - 1) \quad \text{for } t > 0.$$

It is obvious that for $t > 0$, $\widetilde{G}(t) > G(t)$ and it can be also easily viewed that $\lim_{0^+}\widetilde{G} = 0$, $\lim_{+\infty}\widetilde{G} = +\infty$. By intermediate value theorem, let us then define T_0 as the smallest solution of the equation $\widetilde{G}(T_0) = 1$. This automatically implies condition (16), and also,

$$\forall 0 < t < T_0, \qquad T_0 < \frac{1}{\beta}\ln\left(\frac{\beta^2}{M(t)(\beta^2+c_1)}\right).$$

Using the fact that $M(t) \geq F(0) + 1 = 1 + X(0)$ one obtains that $T_0 \leq T_0^+$.

Moreover, let us first consider that $E(0) \leq 0$, then $F(t) + 1 - 6E(0)t^2 \leq \widetilde{M}(t)$ with $t \geq 0$. Therefore, for all $t \in [0, T_0^+]$ one obtains that

$$M(t) \leq \sup_{\tau\in[0,t]} F(\tau) + 1 - 6E(0)\tau^2 \leq \sup_{\tau\in[0,t]} \widetilde{M}(\tau) = \widetilde{M}(T_0^+),$$

which implies that $T_0 \geq T_0^-$.

Condition (13) implies that $E(0) < -\dfrac{\widetilde{M}(T_0)}{6T_0^2}$ from which (15) is obtained.

Secondly, let us consider now that condition (14) holds.
Thus, $F(t) - Y(0)t + \frac{6\kappa}{\gamma}Q(0)t \geq F(t)$ for any $Y(0) \leq \frac{6\kappa}{\gamma}Q(0)$. Therefore, for all $t \in [0, T_0^+]$ one obtains also that

$$M(t) \leq \sup_{\tau\in[0,t]} F(\tau) + 1 - Y(0)\tau + \frac{6\kappa}{\gamma}Q(0)\tau \leq \sup_{\tau\in[0,t]} \widetilde{M}(\tau) = \widetilde{M}(T_0^+),$$

which implies that $T_0 \geq T_0^-$.

Condition (14) also implies that $Y(0) \leq \frac{6\kappa}{\gamma}Q(0)$ from which (15) is obtained. □

The Theorem 2.1 will explain how conditions (13) or (14) (which imply (15) and (16) thanks to Proposition 2.2) can be considered as suffisient blowup conditions as soon as the quadratic form ϕ defined by (11) is negative definite.

Theorem 2.1. *Let $n = 2$ and let us assume that the coefficients g_{11} and g_{22} are strictly negative and either $g_{12} < 0$ or $g_{12} \leq \sqrt{g_{11}g_{22}}$.*
Assuming moreover that the initial data $(u_0(x), v_0(x))$ of the Cauchy problem for the system (1) are chosen such that there exists $T_0 > 0$ for which the two conditions (15) and (16) hold. Then, the solution of the Cauchy problem for the system (1) does not exist in the interval $[0, T_0]$.

Proof. The proof is done by contradiction. With the notations which have been previously introduced, the first step of the proof is to establish the viriel estimate:

$$X(t) + c_2\rho(t) \leq c_1 \int_0^t \int_0^\sigma X(\tau)d\tau d\sigma + \gamma c_3 \int_0^t \rho(\tau)d\tau + F(t), \tag{17}$$

and secondly, by continuation technique we obtain the non positivity which is unacceptable for a viriel.

Using equality (5) of Proposition 2.1 for $n = 2$, after decomposition of the last term, we get:

$$\frac{d^2X}{dt^2} = 8E(t) + 4\gamma^2 X(t) + 8\gamma \mathcal{I}m \int ((\overline{u}x.\nabla u) - (\overline{v}x.\nabla v))dx$$
$$+ 8\kappa\gamma \int |x|^2(v\overline{u})dx - 24\kappa\mathcal{R}e \int u\overline{v}dx + 8\kappa\mathcal{R}e \int u\overline{v}dx.$$

In order to substitute the last term of the previous rhs, one can use the total energy definition by this way,

$$4\kappa \int (u\overline{v} + \overline{u}v)dx = 8\kappa\mathcal{R}e \int u\overline{v}dx = 4E(t) - 4\int(|\nabla u|^2 + |\nabla v|^2)dx$$
$$+ 2\int (g_{11}|u|^4 + g_{22}|v|^4 + 2g_{12}|u|^2|v|^2)dx,$$

and one obtains

$$\frac{d^2X}{dt^2} + 4\int(|\nabla u|^2 + |\nabla v|^2)dx - 2\int(g_{11}|u|^4 + g_{22}|v|^4 + 2g_{12}|u|^2|v|^2)dx$$
$$= 12E(t) + 4\gamma^2X(t) + 8\gamma\mathcal{I}m\int((\bar{u}x.\nabla u) - (\bar{v}x.\nabla v))dx$$
$$+ 8\kappa\gamma\mathcal{I}m\int|x|^2(v\bar{u})dx - 24\kappa\mathcal{R}e\int u\bar{v}dx.$$

According to equality (3) of Proposition 2.1, the following upper bound is straightforward:

$$\frac{dE}{dt} \leq 2\gamma\max(1, |g_{11}|, |g_{22}|)(\|\nabla u\|^2 + \|\nabla v\|^2 + \|u\|_{L^4}^4 + \|v\|_{L^4}^4),$$

and after integration,

$$E(t) \leq E(0) + \frac{\gamma c_3}{12}\int_0^t(\|\nabla u\|^2 + \|\nabla v\|^2 + \|u\|_{L^4}^4 + \|v\|_{L^4}^4)d\tau.$$

Step by step let us formulate an upper bound for the second derivative of the viriel.
Using Cauchy-Schwarz inequality and Young inequality with $\varepsilon = 2\gamma$, we obtain

$$8\gamma\mathcal{I}m\int((\bar{u}x.\nabla u) - (\bar{v}x.\nabla v))dx \leq 8\gamma\int(|\bar{u}x.\nabla u| + |\bar{v}x.\nabla v|)dx,$$

then,

$$8\gamma\mathcal{I}m\int((\bar{u}x.\nabla u) - (\bar{v}x.\nabla v))dx \leq 2(\|\nabla u\|^2 + \|\nabla v\|^2) + 8\gamma^2X. \tag{18}$$

Also, we get

$$8\kappa\gamma\mathcal{I}m\int|x|^2(v\bar{u})dx \leq 4\kappa\gamma X, \tag{19}$$

and using the density upper bound,

$$24\kappa\mathcal{R}e\int u\bar{v}dx = 12\kappa\int(u\bar{v} + \bar{u}v)dx \leq 12\kappa Q(0)e^{2\gamma t}. \tag{20}$$

With the notations that have been introduced before, and inequalities (18), (19) and (20) the second derivative of X can be estimated as follows,

$$\frac{d^2X}{dt^2} + c_2(|\nabla u|^2 + |\nabla v|^2 + |u|^4 + |v|^4) \leq c_1X(t)$$
$$+ \gamma c_3\int_0^t(|\nabla u|^2 + |\nabla v|^2 + |u|^4 + |v|^4)d\tau + 12E(0) + 12\kappa Q(0)e^{2\gamma t}.$$

After successively integrate two times in time, the inequality (17) is obtained.

Now, one can continue as Dias *et al.* did in [16] by assuming that the solution of the Cauchy problem for the system (1) exists for all $t \in [0, T_0]$ with $T_0 > 0$ such that conditions (15) and (16) hold. One can then define

$$T_1 = \sup\{t \in [0, T_0] : X(s) \leq M(T_0) \text{ for any } s \in [0, t]\}.$$

Due to the viriel estimate (17) and condition (15), one obtains that

$$X(t) + c_2\rho(t) \leq c_1 M(T_0)\frac{T_0^2}{2} + \gamma c_3 \int_0^t \rho(\tau)d\tau + M(T_0) - 1$$
$$< M(T_0) + \gamma c_3 \int_0^t \rho(\tau)d\tau. \tag{21}$$

Then, the positivity of the viriel and Gronwall's inequality provides the inequality as follows,

$$\rho(t) \leq \frac{M(T_0)}{c_2}\exp(\frac{\gamma c_3}{c_2}t).$$

Using this upper bound back into inequality (17) and due to the definition of $G(t)$, we find that

$$X(t) + c_2\rho(t) \leq M(T_0)\left(c_1\frac{T_0^2}{2} + \exp(\frac{\gamma c_3}{c_2}T_0) - 1\right) + F(t)$$
$$\leq G(T_0) + F(t).$$

As the mapping $\rho(t)$ is positive,

$$X(t) \leq G(T_0) + F(t), \tag{22}$$

then by taking the least upper bound and using the assumption (16), one obtains

$$\sup_{t\in[0,T_0]} X(t) \leq M(T_0) - 1 + G(T_0) < M(T_0).$$

Thus, $T_1 = T_0$, hence using (22) and assumption (16),

$$X(T_0) \leq F(T_0) + 1 < 0.$$

This non positivity being impossible for a viriel, the proof is done. □

3. Concluding remarks

This paper aims to study blowup phenomenon in finite time for solution of a coupled system of NLS equations (in critical case) which statisfies a PT-symmetry property. In particular, two sufficient conditions of finite time blowup have been obtained by adapting a method used recently by Dias *et al.* for the problem in the supercritical case. Several complements should be done in the future about this phenomenon, like estimates of the finite blowup times and a numerical procedure suitable to provide some illustrations.

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Conflict of Interest

All authors declare no conflicts of interest in this paper.

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