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Research article

Monotone Dynamical Systems with Polyhedral Order Cones and Dense Periodic Points

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Abstract: Let $X \subset \mathbb{R}^n$ be a set whose interior is connected and dense in X, ordered by a closed convex cone $K \subset \mathbb{R}^n$ having nonempty interior. Let $T\colon\ X \approx X$ be an order-preserving homeomorphism. The following result is proved: Assume the set of periodic points of T is dense in X, and K is a polyhedron. Then T is periodic.

Keywords: Dynamical systems; ordered spaces; convex cones; periodic orbits

1. Introduction

The following postulates and notation are used throughout:

- $K \subset \mathbb{R}^n$ (Euclidean n-space) is a *solid order cone*: a closed convex cone that has nonempty interior $\mathsf{Int}\,(K)$ and contains no affine line.
- $\mathbb{R}^n$ has the (partial) order $\geq$ determined by K:

$$y \geq x \iff y - x \in K,$$

  referred to as the K-order.
- $X \subset \mathbb{R}^n$ is a nonempty set whose $\mathsf{Int}\,(X)$ is connected and dense in X.
- $T\colon\ X \approx X$ is homeomorphism that is *monotone* for the K-order:

$$x \geq y \implies Tx \geq Ty.$$

A point $x \in X$ has *period* k provided k is a positive integer and $T^k x = x$. The set of such points is $\mathcal{P}_k = \mathcal{P}_k(T)$, and the set of periodic points is $\mathcal{P} = \mathcal{P}(T) = \bigcup_k \mathcal{P}_k$. T is *periodic* if $X = \mathcal{P}_k$, and *pointwise periodic* if $X = \mathcal{P}$.

Our main concern is the following speculation:

Conjecture. *If $\mathcal{P}$ is dense in X, then T is periodic.*

The assumptions on X show that T is periodic iff $T|\,\mathsf{Int}\,(X)$ is periodic. Therefore we assume henceforth:

- X is connected and open $\mathbb{R}^n$.

We prove the conjecture under the additional assumption that K is a *polyhedron*, the intersection of finitely many closed affine halfspaces of $\mathbb{R}^n$:

Theorem 1 (Main). *Assume K is a polyhedron, $T\colon\ X \approx X$ is monotone for the K-order, and $\mathcal{P}$ is dense in X. Then T is periodic.*

For analytic maps there is an interesting contrapositive:

Theorem 2. *Assume K is a polyhedron and $T\colon\ X \approx X$ is monotone for the K-order. If T is analytic but not periodic, $\mathcal{P}$ is nowhere dense.*

Proof. As X is open and connected but not contained in any of the closed sets $\mathcal{P}_k$, analyticity implies each $\mathcal{P}_k$ is nowhere dense. Since $\mathcal{P} = \bigcup_{k=1}^{\infty} \mathcal{P}_k$, a well known theorem of Baire [1] implies $\mathcal{P}$ is nowhere dense. ∎

The following result of D. Montgomery [4]* is crucial for the proof of the Main Theorem:

Theorem 3 (Montgomery). *Every pointwise periodic homeomorphism of a connected manifold is periodic.*

Notation

i, j, k, l denote positive integers. Points of $\mathbb{R}^n$ are denoted by $a, b, p, q, u, v, w, x, y, z$.
$x \preceq y$ is a synonym for $y \succeq x$. If $x \preceq y$ and $x \neq y$ we write $x \prec$ or $y \succ x$.
The relations $x \ll y$ and $y \gg x$ mean $y - x \in \mathsf{Int}\,(K)$.
A set S is *totally ordered* if $x, y \in S \implies x \preceq y$ or $x \succeq y$.
If $x \preceq y$, the *order interval* $[x, y]$ is $\{z\colon\ x \preceq z \preceq y\} = K_x \cap -K_y$.
The translation of K by $x \in \mathbb{R}^n$ is $K_x := \{w + x, w \in K.\}$
The image of a set or point ξ under a map H is denoted by $H\xi$ or $H(\xi)$. A set S is *positively invariant* under H if $HS \subset S$, *invariant* if $H\xi = \xi$, and *periodically invariant* if $H^k\xi = \xi$.

2. Proof of the Main Theorem

The following four topological consequences of the standing assumptions are valid even if K is not polyhedral.

Proposition 4. *Assune $p, q \in \mathcal{P}_k$ are such that*

$$p \ll q, \qquad p, q \in \mathcal{P}_k. \qquad [p, q] \subset X.$$

Then $T^k([p, q] = [p, q]$.

*See also S. Kaul [3].

Proof. It suffices to take $k = 1$. Evidently $T\mathcal{P} = \mathcal{P}$, and $T[p,q] \subset [p,q]$ because T is monotone, whence $\mathsf{Int}([p,q]) \cap \mathcal{P}$ is positively invariant under T. The conclusion follows because $\mathsf{Int}([p,q]) \cap \mathcal{P}$ is dense in $[p,q]$ and T is continuous. ∎

Proposition 5. *Assume $a, b \in \mathcal{P}_k, a \ll b$, and $[a,b] \subset X$. There is a compact arc $J \subset \mathcal{P}_k \cap [a,b]$ that joins a to b, and is totally ordered by $\ll$.*[†]

Proof. An application of Zorn's Lemma yields a maximal set $J \subset [a,b] \cap P$ such that: J is totally ordered by $\ll$, $a = \max J$, $b = \min J$. Maximality implies J is compact and connected and $a, b \in J$, so J is an arc (Wilder [7], Theorem I.11.23). ∎

Proposition 6. *Let $M \subset X$ be a homeomorphically embedded topological manifold of dimension $n-1$, with empty boundary.*

(i) *$\mathcal{P}$ is dense in M.*

(ii) *If M is periodically invariant, it has a neighborhood base $\mathcal{B}$ of periodically invariant open sets.*

Proof. (i) M locally separates X, by Lefschetz duality [5] (or dimension theory [6]. Therefore we can choose a family $\mathcal{V}$ of nonempty open sets in X that the family of sets $\mathcal{V}_M := \{V \cap M\colon\ V \in \mathcal{V})$ satisfies:

- $\mathcal{V}_M$ is a neighborhood basis of M,
- each set $V \cap M$ separates V.

By Proposition 5, for each $V \in \mathcal{V}$ there is a compact arc $J_V \cap \mathcal{P} \cap V$ whose endpoints a_V, b_v lie in different components of $V \setminus M$. Since J_V is connected, it contains a point in $V \cap M \cap \mathcal{P}$. This proves (i).

(ii) With notation as above, let $B_V := [a_V, b_V] \setminus \partial[a_V, b_V]$. The desired neighborhood basis is $\mathcal{B} := \{B_V\colon\ V \in \mathcal{V}\}$. ∎

From Propositions 4 and 6 we infer:

Proposition 7. *Suppose $p, q \in \mathcal{P}$, $p \ll q$ and $[p,q] \subset X$. Then $\mathcal{P}$ is dense in $\partial[p,q]$.* ∎

Let $\mathcal{T}(m)$ stand for the statement of Theorem 1 for the case $n = m$. Then $\mathcal{T}(0)$ is trivial, and we use the following inductive hypothesis:

Hypothesis (Induction)**.** $n \geq 1$ and $\mathcal{T}(n-1)$ holds.

Let $Q \subset \mathbb{R}^n$ be a compact n-dimensional polyhedron. Its boundary ∂Q is the union of finitely many convex compact $(n-1)$-cells, the *faces* of Q. Each face F is the intersection of $\partial[p,q]$ with a unique affine hyperplane E^{n-1}. The corresponding *open face* $F^\circ := F \setminus \partial F$ is an open $(n-1)$-cell in E^{n-1}. Distinct open faces are disjoint, and their union is dense and open in ∂Q.

Proposition 8. *Assume $p, q \in \mathcal{P}_k,\ p \ll q,\ [p,q] \subset X$. Then $T|\partial[p,q]$ is periodic.*

[†]This result is adapted from Hirsch & Smith [2], Theorems 5,11 & 5,15.

Proof. $[p, q]$ is a compact, convex n-dimensional polyhedron, invariant under T^k (Proposition 4). By Proposition 6 applied to $M := \partial[p, q]$, there is a neighborhood base $\mathcal{B}$ for $\partial[p, q]$ composed of periodically invariant open sets. Therefore if $F^\circ \subset \partial[p, q]$ is an open face of $[p, q]$, the family of sets

$$\mathcal{B}_{F^\circ} := \{W \in \mathcal{B}: \ W \subset F^\circ\}$$

is a neighborhood base for F°, and each $W \in \mathcal{B}_{F^\circ}$ is a periodically invariant open set in which $\mathcal{P}$ is dense.

For every face F of $[p, q]$ the Induction Hypothesis shows that $F^\circ \subset \mathcal{P}$. Therefore Montgomery's Theorem implies $T|F^\circ$ is periodic, so $T|F$ is periodic by continuity. Since $\partial[p, q]$ is the union of the finitely many faces, it follows that $T|\partial[p, q]$ is periodic. ∎

To complete the inductive proof of the Main Theorem, it suffices by Montgomery's theorem to prove that an arbitrary $x \in X$ is periodic. As X is open in $\mathbb{R}^n$ and $\mathcal{P}$ is dense in X, there is an order interval $[a, b] \subset X$ such that

$$a \ll x \ll b, \qquad a, b \in \mathcal{P}_k.$$

By Proposition 5, a and b are the endpoints of a compact arc $J \subset \mathcal{P}_k \cap [a, b]$, totally ordered by $\ll$. Define $p, q \in J$:

$$p := \sup\{y \in J: \ y \leq x\}, \qquad q := \inf\{y \in J: \ y \geq x\}.$$

If $p = q = x$ then $x \in \mathcal{P}_k$. Otherwise $p \ll q$, implying $x \in \partial[p, q]$, whence $x \in \mathcal{P}$ by Proposition 8. ∎

Conflict of Interest

The author declares no conflicts of interest in this paper.

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