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Research article

The Bedrosian Identity for L^p Function and the Hardy Space on Tube

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Abstract: In this paper, we are devoted to establishing several necessary and sufficient conditions for $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$ with $\frac{1}{p}+\frac{1}{q} \leq 1$ to satisfy the Bedrosian identity $H(fg) = fHg$, where H denotes the n-dimensional Hilbert transform. In addition, we also show that the distribution $f \in \mathcal{D}'_{L^p}(\mathbb{R}^n)$ can be represented by functions in the Hardy space on tube.

Keywords: Bedrosian identity; Fourier transform; Hilbert transform; Distribution

1. Introduction

The complex signal method especially analytic signal method is a classical way of defining the phase and amplitude of signals, which plays an important role in meteorological as well as atmospheric applications, ocean engineering, structural science, and imaging processing, one can refer to [16, 17, 18, 22, 28, 24] for details. This results in the widely used empirical mode decomposition and the Hilbert-Huang transform, see for instance [7, 16, 17].

The Hilbert transform is a well-known and useful concept in harmonic analysis and signal processing (see for instance [13, 2, 9, 15]). The 1-dimensional Hilbert transform H for functions $f \in L^p(\mathbb{R})$ $(1 \leq p < \infty)$ can be stated as follows

$$(Hf)(x) := \text{p.v.}\frac{1}{\pi}\int_{\mathbb{R}} \frac{f(y)}{x-y}dy = \lim_{\epsilon\to 0}\frac{1}{\pi}\int_{|y-x|\geq\epsilon}\frac{f(y)}{x-y}dy, \ \ x \in \mathbb{R}.$$

Regarding to the Hilbert transform defined above: If $f, g \in L^2(\mathbb{R})$ satisfy either supp$\hat{f} \subseteq \mathbb{R}_+(\mathbb{R}_+ = [0,\infty))$, supp$\hat{g} \subseteq \mathbb{R}_+$ or supp$\hat{f} \subseteq [-a, a]$, supp$\hat{g} \subseteq (-\infty, -a]\bigcup[a, \infty)$ for some positive number a, then the following identity holds true

$$[H(fg)](x) = f(x)(Hg)(x), \ \ x \in \mathbb{R},$$

which is named as the Bedrosian identity to honor Bedrosian for his contribution [1]. Later, some efforts were devoted to obtaining more general sufficient conditions (see e.g., [3, 21]). The Bedrosian identity simplifies the calculation of the Hilbert transform of a product of functions. In recent years, the Bedrosian identity has attracted considerable attention and progress has been made. There is a large number of documents for the studies of the Bedrosian identity, see for example [6, 5, 8, 25, 26, 30, 31, 32, 34, 35, 36, 37]. It is worthwhile to state that an observation in [33] implies that the Hilbert transform is essentially the only operator satisfying the Bedrosian theorem.

It is well known that the complex signal method via the Hilbert transform has already become a significant tool in signal analysis and processing, especially in the time-frequency analysis (see, e.g., [4, 1, 9, 14, 21]). Imaging and other applications to multidimensional signals call for extension of the method to higher dimensions. Therefore, it is natural to establish the Bedrosian identity in n-dimension case. To the best knowledge of the authors, there are only very few results on multidimensional Bedrosian identities. Actually, in some special case, that is $p = 2$, one can refer to [33, 38, 19] consulting the multidimensional Bedrosian identity.

Now let us give the definition of the total Hilbert transform as well as the partial Hilbert transform.

Definition 1.1. *The partial Hilbert transform for $f \in L^p(\mathbb{R}^n)(1 \leq p < \infty)$, $H_j f$ is given by*

$$(H_j f)(x) := \text{p.v.}\frac{1}{\pi}\int_{\mathbb{R}} \frac{f(y)}{(x_j - y_j)} dy_j.$$

The total Hilbert transform H of a function is given by

$$(Hf)(x) := \text{p.v.}\frac{1}{(\pi)^n}\int_{\mathbb{R}^n} \frac{f(y)}{\prod_{j=1}^n (x_j - y_j)} dy \triangleq (H_1 H_2 \cdots H_n) f(x),\ x \in \mathbb{R}^n.$$

The Fourier transform $\hat{f}$ of $f \in L^1(\mathbb{R}^n)$ is defined as

$$\hat{f}(x) = \int_{\mathbb{R}^n} f(t) e^{-ix\cdot t} dt, \quad x \in \mathbb{R}^n.$$

Next, let us give some basic notation. Let $\mathcal{D}(\mathbb{R}^n)$ be the space of infinitely differentiable functions on $\mathbb{R}^n$ with compact support and $\mathcal{D}'(\mathbb{R}^n)$ the space of distributions, namely, the dual of $\mathcal{D}(\mathbb{R}^n)$. A distribution T is said to vanish on an open subset $\Omega \subseteq \mathbb{R}^n$ as long as for each $\phi \in \mathcal{D}(\mathbb{R}^n)$ with supp$\phi \subseteq \Omega$, $< T, \phi >= T(\phi)$ equals zero. The support of $T \in \mathcal{D}'(\mathbb{R}^n)$, denoted by supp$T$, is defined to be the complement of the largest open subset of $\mathbb{R}^n$ on which T vanishes. This definition is consistent with the ordinary one when T is a continuous function. Set

$$D_+ = \{x : x = (x_1, \cdots, x_n) \in \mathbb{R}^n, sgn(-x) = \prod_{j=1}^n sgn(-x_j) = 1\},$$

$$D_- = \{x : x = (x_1, \cdots, x_n) \in \mathbb{R}^n, sgn(-x) = \prod_{j=1}^n sgn(-x_j) = -1\}$$

and

$$D_0 = \{x : x = (x_1, \cdots, x_n) \in \mathbb{R}^n, sgn(-x) = \prod_{j=1}^n sgn(-x_j) = 0\}.$$

We denote by $\mathcal{D}_{D_+}(\mathbb{R}^n)$, $\mathcal{D}_{D_-}(\mathbb{R}^n)$ and $\mathcal{D}_{D_0}(\mathbb{R}^n)$ the set of functions in $\mathcal{D}(\mathbb{R}^n)$ that are supported on D_+, D_- and D_0, respectively.

To apply the Fourier transform, we also introduce the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ and its dual $\mathcal{S}'(\mathbb{R}^n)$, the space of temperate distributions. The Schwartz class $\mathcal{S}(\mathbb{R}^n)$ consists of infinitely differentiable function φ on $\mathbb{R}^n$ for all $\alpha, \beta \in \mathbb{Z}_+^n$ satisfies

$$\sup_{x\in\mathbb{R}^n} |x^\alpha D^\beta \varphi(x)| < \infty,$$

where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n), \beta = (\beta_1, \beta_2, \cdots, \beta_n)$, α_j and β_j are nonnegative integers. The Fourier transform $\hat{\varphi}$ is a linear homeomorphism from $\mathcal{S}(\mathbb{R}^n)$ onto itself. Meanwhile, the following identity holds

$$(H\varphi)^\wedge(x) = (-i)\mathrm{sgn}(x)\hat{\varphi}(x), \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^n).$$

The Fourier transform $\mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ defined as

$$\langle \hat{\psi}, \varphi \rangle = \langle \psi, \hat{\varphi} \rangle, \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^n)$$

is a linear isomorphism from $\mathcal{S}'(\mathbb{R}^n)$ onto itself. For $\psi \in \mathcal{S}'(\mathbb{R}^n)$, $\forall \varphi \in \mathcal{S}(\mathbb{R}^n)$, it is easy to check that

$$\langle \tilde{\check{\psi}}, \varphi \rangle = \langle \check{\psi}, \tilde{\varphi} \rangle = \langle \psi, \check{\tilde{\varphi}} \rangle = \langle \psi, \hat{\varphi} \rangle = \langle \hat{\psi}, \varphi \rangle.$$

Therefore in the sense of distribution, we obtain

$$\tilde{\check{\psi}} = \hat{\psi},$$

where $\tilde{\varphi}(x) = \varphi(-x)$. $\check{\psi}$ is the inverse Fourier transform defined as

$$\langle \check{\psi}, \varphi \rangle = \langle \psi, \check{\varphi} \rangle.$$

For the detail properties of $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$, see for example [27, 11, 14].

A function f defined on $\mathbb{R}^n$ belongs to $\mathcal{D}_{L^p}(\mathbb{R}^n)$, $1 < p < \infty$ if and only if

(1) $f \in C^\infty(\mathbb{R}^n)$,

(2) $D^k f \in L^p(\mathbb{R}^n)$, $k = 0, 1, \cdots$.

In the sequel for $1 < p < \infty$, we denote by $\mathcal{D}'_{L^p}(\mathbb{R}^n)$ the dual of $\mathcal{D}_{L^{p'}}(\mathbb{R}^n)$, where $\frac{1}{p} + \frac{1}{p'} = 1$.

Our second main component of this paper is to consider the boundary values of holomorphic functions in distribution (see for instance [20]). More precisely, we show that the distribution $f \in \mathcal{D}'_{L^p}(\mathbb{R}^n)$ can be represented by functions in the Hardy space on tube.

Let B denote an open connected subset of $\mathbb{R}^n$. The tube is defined by $T_B \doteq \{x + iy : x \in \mathbb{R}^n, y \in B\}$. The Hardy space [29] on tube T_B is defined as

$$H^p(T_B) := \{f \in H(T_B) : \|f\|_{H^p} < \infty\},$$

where $\|f\|_{H^p} = \sup_{y\in B} \left(\int_{\mathbb{R}^n} |f(x + iy)|^p dx \right)^{\frac{1}{p}}$, and $H(T_B)$ consists of all the holomorphic functions on T_B.

Definition 1.2. *Let $f \in \mathcal{D}'_{L^p}(\mathbb{R}^n)$, $1 < p < \infty$. The distributional differentiation and Hilbert transform of f are defined as*

$$\left\langle D^k f, \varphi \right\rangle = \left\langle f, (-1)^{|k|} D^k \varphi \right\rangle,$$

and

$$\left\langle Hf, \varphi \right\rangle = \left\langle f, (-1)^n H\varphi \right\rangle, \quad \forall \varphi \in \mathcal{D}_{L^{p'}}(\mathbb{R}^n)$$

respectively.

Here we want to mention that Pandey [23] proved that $D^k f \in \mathcal{D}'_{L^p}(\mathbb{R}^n),\ Hf \in \mathcal{D}'_{L^p}(\mathbb{R}^n)$ and Hilbert transform H defined on $L^p(\mathbb{R}^n)$ $(p > 1)$ is isomorphism from $\mathcal{D}_{L^p}(\mathbb{R}^n)$ onto itself.

The present paper is structured as follows. In section 2, we characterize the Bedrosian Identity of total Hilbert tranform, which consists of several lemmas. In section 3, we prove distribution $f \in \mathcal{D}'_{L^p}$ can be represented by functions in the Hardy space on tube.

2. The bedrosian identity for $L^p(\mathbb{R}^n)$ function

This part is motivated by the need of defining multidimensional complex signals. We define the complex signal of $f \in L^p$ through total Hilbert transform H as $f + iHf$. In this section we investigate the multidimensional Bedrosian identity $H(fg) = fHg$ for $f \in L^p(\mathbb{R}^n),\ g \in L^q(\mathbb{R}^n)$ with $1 < p,\ q < \infty$. In particular, several necessary and sufficient conditions to guarantee the Bedrosian identity to be valid are obtained.

Lemma 2.1. *Let $f \in L^p(\mathbb{R}^n)$ for $1 < p < \infty$, then*

$$H(f * g)(x) = (Hf * g)(x) \ \ a.e. \ \ g \in L^1(\mathbb{R}^n).$$

Proof. According to the properties of Hilbert transform and convolution, it is not difficult to show that $f * g,\ Hf * g$ and $H(f * g)$ all belong to $L^p(\mathbb{R}^n)$. Thus both sides of the above equality are well defined. By Fubini's theorem, for all $h \in L^{p'}(\mathbb{R}^n)$, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} H(f * g)(\xi)h(\xi)d\xi &= (-1)^n \int_{\mathbb{R}^n} (Hh)(\xi)(f * g)(\xi)d\xi \\
&= (-1)^n \int_{\mathbb{R}^n} g(u) \int_{\mathbb{R}^n} (Hh)(\xi)f(\xi - u)d\xi du \\
&= \int_{\mathbb{R}^n} g(u) \int_{\mathbb{R}^n} h(\xi)(Hf)(\xi - u)d\xi du \\
&= \int_{\mathbb{R}^n} h(\xi)(Hf * g)(\xi)d\xi
\end{aligned}$$

The proof of the lemma is completed. □

Lemma 2.2. *Let $1 \le p \le 2, f \in L^p(\mathbb{R}^n)$. If there exists $g \in L^p(\mathbb{R}^n)$ such that*

$$(-i)^n sgn(x)\hat{f}(x) = \hat{g}(x), \tag{1}$$

then $Hf = g$ a.e..

Proof. As $f \in L^p(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$ satisfy (1), we can choose a sequence of functions $\{\phi_j, j \in \mathbb{N}\}$ that are infinitely differentiable with compact support satisfying for each $f, g \in L^p(\mathbb{R}^n)$

$$\lim_{j\to\infty} ||\phi_j * f - f||_{L^p(\mathbb{R}^n)} = 0, \quad \lim_{j\to\infty} ||\phi_j * g - g||_{L^p(\mathbb{R}^n)} = 0,$$

where $\phi_j * f$ is convolution of ϕ_j and f given by

$$\phi_j * f(x) = \int_{\mathbb{R}^n} \phi_j(x - t)f(t)dt = \int_{\mathbb{R}^n} \phi_j(t)f(x - t)dt.$$

It is clear that

$$(\phi_j * f)^\wedge(x) = \hat{\phi}_j(x)\hat{f}(x).$$

The above identity together with (1) implies that

$$(-i)^n sgn(x)(\phi_j * f)^\wedge(x) = \hat{\phi}_j(x)\hat{g}(x) \;\; a.e.\; x \in \mathbb{R}^n.$$

Thus

$$Hf * \phi_j = H(\phi_j * f) = \phi_j * g \tag{2}$$

According to the property of Hilbert transform

$$\|H(\phi_j * f) - Hf\|_{L^p(\mathbb{R}^n)} \le A_p \|\phi_j * f - f\|_{L^p(\mathbb{R}^n)} \to 0 \quad (j \to \infty).$$

Therefore,

$$\|Hf - g\|_{L^p(\mathbb{R}^n)} \le \|Hf - H(\phi_j * f)\|_{L^p(\mathbb{R}^n)} + \|g * \phi_j - g\|_{L^p(\mathbb{R}^n)} \to 0 \quad (j \to \infty).$$

This fact yields that $Hf = g$ *a.e.* We thus complete the proof. □

Lemma 2.3. *Assume that* $p, q, r \in (1, \infty]$ *satisfy* $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} \le 1$. *Let* $f \in L^p(\mathbb{R}^n)$ *and* $g \in L^q(\mathbb{R}^n)$, *then* $(\mathrm{supp}\hat{f}) \bigcup (\mathrm{supp}\hat{g}) \subseteq D_+ \bigcup D_0$ *and* $(\mathrm{supp}\hat{f}) \bigcup (\mathrm{supp}\hat{g}) \subseteq D_- \bigcup D_0$ *imply* $\mathrm{supp}(fg)^\wedge \subseteq D_+ \bigcup D_0$ *and* $\mathrm{supp}(fg)^\wedge \subseteq D_- \bigcup D_0$, *respectively.*

Proof. The Hölder inequality

$$\|fg\|_{L^r(\mathbb{R}^n)} \le \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)} \tag{3}$$

implies that $fg \in L^r(\mathbb{R}^n)$. Thus, for each $\phi \in \mathcal{S}(\mathbb{R}^n)$ we get that

$$\left\langle (fg)^\wedge, \phi \right\rangle = \left\langle fg, \hat{\phi} \right\rangle = \int_{\mathbb{R}^n} f(t)g(t)\hat{\phi}(t)dt.$$

Choose a function $\psi \in \mathcal{D}(\mathbb{R}^n)$ such that $\hat{\psi}(0) = 1$ and set

$$f_j(x) := \int_{\mathbb{R}^n} \psi_j(x - t) f(t) dt, \quad j \in \mathbb{N},$$

where $\psi_j(t) = j^n \psi(jt), t \in \mathbb{R}^n$. For each $j \in \mathbb{N}$, the function f_j enjoys the property that $f_j \in \mathbb{C}^\infty(\mathbb{R}^n), D^k f_j \in L^\infty(\mathbb{R}^n)$ for each nonnegative integer k. Furthermore, if $p < \infty$, then f_j converges to f in $L^p(\mathbb{R}^n)$ as j goes to infinity. This fact as well as (3) yields for $p < \infty$ that

$$\|f_j g - fg\|_{L^r(\mathbb{R}^n)} \le \|f_j - f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)} \to 0 \; (j \to \infty).$$

As a result, if $p < \infty$, then there holds

$$\lim_{j \to \infty} \int_{\mathbb{R}^n} f_j(t)g(t)\hat{\phi}(t)dt = \int_{\mathbb{R}^n} f(t)g(t)\hat{\phi}(t)dt.$$

The above equality remains true for $p = \infty$ because in this case we have that $f_j \in L^\infty(\mathbb{R}^n)$ converges almost everywhere to f and thus that $g\hat{\phi} \in L^1(\mathbb{R}^n)$.

Now we suppose that $(\text{supp}\hat{f})\bigcup(\text{supp}\hat{g}) \subseteq D_+\bigcup D_0$. To show that $\text{supp}(fg)^\wedge \subseteq D_+\bigcup D_0$, it is sufficient to show that $\int_{\mathbb{R}^n} f_j(t)g(t)\hat{\phi}(t)dt = 0$ for each $j\in\mathbb{N}$ and $\phi\in\mathcal{D}_{D_-}(\mathbb{R}^n)$. For $\phi\in\mathcal{D}_{D_-}(\mathbb{R}^n)$, the properties of the functions f_j ensure that $f_j\phi\in\mathcal{S}(\mathbb{R}^n)$, $j\in\mathbb{N}$. Therefore,

$$\int_{\mathbb{R}^n} g(t)f_j(t)\hat{\phi}(t)dt = \Big\langle \hat{g}, (f_j\hat{\phi})^\vee\Big\rangle,\ \ j\in\mathbb{N}.$$

A direct computation shows that

$$\begin{aligned}(f_j\hat{\phi})^\vee(x) &= \int_{\mathbb{R}^n}\hat{\psi}_j(-t)\hat{f}(-t)\phi(x-t)dt\\ &= \int_{\mathbb{R}^n}\hat{\psi}\Big(\frac{-t}{j}\Big)\hat{f}(-t)\phi(x-t)dt\\ &= \int_{\mathbb{R}^n}\check{\psi}\Big(\frac{-t}{j}\Big)\hat{f}(t)\phi(x+t)dt\\ &= \Big\langle \hat{f}, \check{\psi}(\frac{\cdot}{j})\phi(x+\cdot)\Big\rangle.\end{aligned}$$

The above conclusion together with $\phi\in\mathcal{D}_{D_-}(\mathbb{R}^n)$, $(\text{supp}\hat{f})\subseteq D_+\bigcup D_0$ implies that $\text{supp}(f_j\hat{\phi})^\vee\subseteq D_-$. It follows by the assumption $(\text{supp}\hat{g})\subseteq D_+\bigcup D_0$ that

$$\Big\langle \hat{g}, (f_j\hat{\phi})^\vee\Big\rangle = 0,\quad j\in\mathbb{N}.$$

Therefore, we get $\text{supp}(fg)\hat{} \subseteq D_+\bigcup D_0$. The other case can be dealt with likewise. This immediately completes the proof. □

Lemma 2.4. *If* $f\in L^p(\mathbb{R}^n)$ *for* $1<p<\infty$, *then* $\text{supp}(f-(-i)^nHf)^\wedge\subseteq D_-\bigcup D_0$.

Proof. According to the definition, we need to show for each $\phi\in\mathcal{D}_{D_+}(\mathbb{R}^n)$ that

$$\langle (f-(-i)^nHf)^\wedge, \phi\rangle = 0.$$

For this purpose, we point out that if $1\le p<\infty$ then for each $\psi\in\mathcal{S}(\mathbb{R}^n)$ there holds

$$\Big\langle Hf, \psi\Big\rangle = \Big\langle f, (-1)^nH\psi\Big\rangle. \tag{4}$$

Let $\phi\in\mathcal{D}_{D_+}(\mathbb{R}^n)$, then

$$\begin{aligned}\Big\langle (f-(-i)^nHf)^\wedge, \phi\Big\rangle &= \Big\langle (f-(-i)^nHf), \hat{\phi}\Big\rangle\\ &= \Big\langle f, \hat{\phi}-(i)^nH\hat{\phi}\Big\rangle\\ &= \Big\langle \hat{f}, \phi-(-1)^n sgn(\cdot)\phi\Big\rangle\\ &= 0,\end{aligned}$$

thereby completing the proof. □

Lemma 2.5. *Let* $f\in L^p(\mathbb{R}^n)$ *for* $1<p<\infty$, *then on* $D_+\bigcup D_-$, $\text{supp}\hat{f}\subseteq D_+\bigcup D_0$ *and* $\text{supp}\hat{f}\subseteq D_-\bigcup D_0$ *are equivalent to* $Hf=(i)^nf$ *and* $Hf=-(i)^nf$, *respectively.*

Proof. Suppose $f \in L^p(\mathbb{R}^n)$ satisfies that $Hf = (i)^n f$. Then we have for each $\phi \in \mathcal{D}_{D_-}(\mathbb{R}^n)$ that

$$\langle \hat{f}, \phi \rangle = \langle f, \hat{\phi} \rangle = \frac{1}{2}\Big(\langle f, \hat{\phi} \rangle + (-i)^n \langle (i)^n f, \hat{\phi} \rangle\Big) = \frac{1}{2}\Big(\langle f, \hat{\phi} \rangle + (-i)^n \langle Hf, \hat{\phi} \rangle\Big). \tag{5}$$

According to (4), we get that

$$\langle f, \hat{\phi} \rangle + (-i)^n \langle Hf, \hat{\phi} \rangle = \langle f, \hat{\phi} + (i)^n H\hat{\phi} \rangle = \Big\langle \hat{f}, \phi(1 + (-1)^n sgn(\cdot)) \Big\rangle = 0. \tag{6}$$

Combining (5) with (6) proves that $\langle \hat{f}, \phi \rangle = 0$ for each $\phi \in \mathcal{D}_{D_-}(\mathbb{R}^n)$, that is supp$\hat{f} \subseteq D_+ \bigcup D_0$.

Conversely, suppose that we have $f \in L^p(\mathbb{R}^n)$ with supp$\hat{f} \subseteq D_+ \bigcup D_0$. By lemma 2.4, there holds supp$(f + (-i)^n Hf)^\wedge \subseteq D_+ \bigcup D_0$, then supp$(Hf)^\wedge \subseteq D_+ \bigcup D_0$. Therefore, to show that $Hf = (i)^n f$ it suffices to show for each $\phi \in \mathcal{D}_{D_+}(\mathbb{R}^n)$ that

$$\Big\langle (f - (-i)^n Hf)^\wedge, \phi \Big\rangle = 0.$$

Lemma 2.4 work for this purpose. This ends the proof. □

Now we introduce that each $f \in L^p(\mathbb{R}^n)$ for $1 < p < \infty$ can be decomposed as $f = f_+ + f_-$, where

$$f_+ = \frac{f + (-i)^n Hf}{2}, \quad f_- = \frac{f - (-i)^n Hf}{2}.$$

Theorem 2.6. *Let $f \in L^p(\mathbb{R}^n), g \in L^q(\mathbb{R}^n)$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Then f, g satisfy the Bedrosian identity $H(fg) = fHg$ on $D_+ \bigcup D_-$ if and only if*

$$\text{supp}(f_+ g_-)^\wedge \subseteq D_- \bigcup D_0 \text{ and } \text{supp}(f_- g_+)^\wedge \subseteq D_+ \bigcup D_0.$$

Proof. By using the above decomposition $f = f_+ + f_-$ and $g = g_+ + g_-$, we can rewrite the Bedrosian identity as

$$H(f_+ g_+ + f_- g_- + f_+ g_- + f_- g_+) = f_+ Hg_+ + f_- Hg_- + f_+ Hg_- + f_- Hg_+. \tag{6}$$

For each $\phi \in L^q(\mathbb{R}^n)$, it holds

$$\begin{aligned} \Big\langle Hf_+, \phi \Big\rangle &= \Big\langle f_+, (-1)^n H\phi \Big\rangle \\ &= \Big\langle \frac{f + (-i)^n Hf}{2}, (-1)^n H\phi \Big\rangle \\ &= \frac{1}{2}\Big\langle f, (-1)^n H\phi \Big\rangle + \frac{(-i)^n}{2} \langle f, H^2\phi \rangle \\ &= \frac{1}{2}\Big\langle Hf, \phi \Big\rangle + \frac{(-i)^n}{2}\Big\langle f, H^2\phi \Big\rangle \\ &= \Big\langle (i)^n \frac{f + (-i)^n Hf}{2}, \phi \Big\rangle \\ &= \Big\langle (i)^n f_+, \phi \Big\rangle. \end{aligned}$$

That is $Hf_+ = (i)^n f_+$. Adopting the same argument, one may conclude that $Hf_- = -(i)^n f_-$. The above fact leads us to the following equivalent of (6)

$$H(f_+ g_+ + f_- g_- + f_+ g_- + f_- g_+) = (i)^n f_+ g_+ - (i)^n f_- g_- + (i)^n f_- g_+ - (i)^n f_+ g_-. \tag{7}$$

According to Lemma 2.5, it is easy to check that

$$\text{supp}\hat{f}_+ \subseteq D_+ \bigcup D_0, \ \ \text{supp}\hat{f}_- \subseteq D_- \bigcup D_0,$$

and similarly

$$\text{supp}\hat{g}_+ \subseteq D_+ \bigcup D_0, \ \ \text{supp}\hat{g}_- \subseteq D_- \bigcup D_0.$$

An application of Lemma 2.3 then yields that

$$\text{supp}(f_+g_+)^\wedge \subseteq D_+ \bigcup D_0, \ \ \text{supp}(f_-g_-)^\wedge \subseteq D_- \bigcup D_0. \tag{8}$$

By the Hölder inequality, functions $f_+g_+, f_+g_-, f_-g_+, f_-g_-$ are all in $L^r(\mathbb{R}^n)$. The results relation (8) hence imply by Lemma 2.5 that

$$H(f_+g_+) = (i)^n f_+g_+, \ \ H(f_-g_-) = -(i)^n f_-g_-.$$

Therefore, equation (7) holds if and only if

$$H(f_+g_- + f_-g_+) = (i)^n f_-g_+ - (i)^n f_+g_-. \tag{9}$$

If $\text{supp}(f_+g_-)^\wedge \subseteq D_- \bigcup D_0$ and $\text{supp}(f_-g_+)^\wedge \subseteq D_+ \bigcup D_0$ holds true, then (9) is valid. On the other hand, we suppose that (9) is true. By applying the Hilbert transform to both sides of equation (9), one gets that

$$(-1)^n(f_+g_- + f_-g_+) = (i)^n H(f_-g_+) - (i)H(f_+g_-). \tag{10}$$

Combining (10) and (9), it follows that

$$H(f_-g_+) = (i)^n f_-g_+, \ \ H(f_+g_-) = -(i)^n f_+g_-.$$

The above is equivalent to $\text{supp}(f_+g_-)^\wedge \subseteq D_- \bigcup D_0$ and $\text{supp}(f_-g_+)^\wedge \subseteq D_+ \bigcup D_0$. The proof is complete. □

Theorem 2.7. *If* $f \in L^p(\mathbb{R}^n), g \in L^q(\mathbb{R}^n)$ *satisfy either* $(\text{supp}\hat{f}) \bigcup (\text{supp}\hat{g}) \subseteq D_+ \bigcup D_0$ *or* $(\text{supp}\hat{f}) \bigcup (\text{supp}\hat{g}) \subseteq D_- \bigcup D_0$ *then the Bedrosian identity holds on* $D_+ \bigcup D_-$ *.*

Proof. If $(\text{supp}\hat{f}) \bigcup (\text{supp}\hat{g}) \subseteq D_+ \bigcup D_0$, then by Lemma 2.5, $Hf = (i)^n f$ and $Hg = -(i)^n g$. Therefore, according to the definition of f_+ and f_-, we have that

$$f_+ = \frac{f + (-i)^n Hf}{2} = \frac{f + (-i)^n (i)^n f}{2} = f, \quad f_- = 0.$$

Similarly $g_+ = g, \ g_- = 0$. The desired result then follows by a trivial application of Theorem 2.6. The other case can be proved in the same way. The proof is thus completed. □

Theorem 2.8. *Suppose* $f \in \mathcal{S}(\mathbb{R}^n), g \in L^2(\mathbb{R}^n)$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *satisfy the Bedrosian identity if and only if*

$$2\int_{D_-} \hat{f}(\xi - \eta)\hat{g}(\eta)d\eta + \int_{D_0} \hat{f}(\xi - \eta)\hat{g}(\eta)d\eta = 0, \ \ \xi \in D_+ \bigcup D_0 \tag{11}$$

and

$$2\int_{D_+} \hat{f}(\xi - \eta)\hat{g}(\eta)d\eta + \int_{D_0} \hat{f}(\xi - \eta)\hat{g}(\eta)d\eta = 0, \ \ \xi \in D_- \bigcup D_0. \tag{12}$$

Proof. Since $fg, fHg \in L^1(\mathbb{R}^n)$, $H(fg) = fHg$ *a.e.* if and only if

$$(fHg)^\wedge(\xi) = (-i)^n sgn(\xi)(fg)^\wedge(\xi),\ \xi \in \mathbb{R}^n. \tag{13}$$

The equation (13) admits the form

$$(-i)^n sgn(\xi) \int_{\mathbb{R}^n} \hat{f}(\xi-\eta)\hat{g}(\eta)d\eta = \int_{\mathbb{R}^n} \hat{f}(\xi-\eta)(-i)^n sgn(\eta)\hat{g}(\eta)d\eta,$$

which gives

$$\int_{\mathbb{R}^n} \hat{f}(\xi-\eta)\hat{g}(\eta)(sgn(\xi)-sgn(\eta))d\eta = 0,\ \xi \in \mathbb{R}^n.$$

Clearly, the above integral can be divided into the following three parts

$$2\int_{D_-} \hat{f}(\xi-\eta)\hat{g}(\eta)d\eta + \int_{D_0} \hat{f}(\xi-\eta)\hat{g}(\eta)d\eta = 0,\ \ \xi \in D_+,$$

$$2\int_{D_+} \hat{f}(\xi-\eta)\hat{g}(\eta)d\eta + \int_{D_0} \hat{f}(\xi-\eta)\hat{g}(\eta)d\eta = 0,\ \ \xi \in D_-,$$

$$\int_{D_-} \hat{f}(\xi-\eta)\hat{g}(\eta)d\eta = \int_{D_+} \hat{f}(\xi-\eta)\hat{g}(\eta)d\eta,\quad \xi \in D_0.$$

By the continuity of $\int_{D_+} \hat{f}(\cdot-\eta)\hat{g}(\eta)d\eta$ and $\int_{D_-} \hat{f}(\cdot-\eta)\hat{g}(\eta)d\eta$, it is not hard to show that f and g satisfy the three integrals above if and only they satisfy (11) and (12). We thus conclude the proof. □

3. Distribution $f \in L^p(\mathbb{R}^n)$ and the Hardy space on tube

In this section, we give some lemmas which will be used in the proof process of the main result.

Lemma 3.1 (see [10])**.** *Let $u \in L^p(\mathbb{R})$ for $1 < p < \infty$. The function $G(u)(z)$ is defined as*

$$G(u)(z) = \frac{1}{\pi i}\int_{\mathbb{R}} \frac{u(t)}{t-z}dt.$$

Then $G(u) \in H^p(\mathbb{C}_+)$. Moreover,

$$\int_{\mathbb{R}} |G(u)(x+iy)|^p dx \le A_p \int_{\mathbb{R}} |u(t)|^p dt,$$

where $A_p = \max\{\frac{p2^p}{p-1}, 2^p p^{p-1}\}$.

Lemma 3.2. *Let $\Gamma = \{y = (y_1, y_2) \in \mathbb{R}^2 : y_1 > 0, y_2 > 0\}$. Suppose $f \in L^p(\mathbb{R}^2)$ $(1 < p < \infty)$. The Cauchy integral of f is given by*

$$F(z) = \frac{1}{(2\pi i)^2}\int_{\mathbb{R}^2} K(z-t)f(t)dt.$$

Then $F(z) \in H^p(T_\Gamma)$. Moreover, there exists a constant A such that

$$\|F\|_{H^p(T_\Gamma)} \le A\|f\|_p.$$

Proof. We consider the first octant $\Gamma = \{y = (y_1, y_2) \in \mathbb{R}^n : y_1 > 0, y_2 > 0\}$, then we can show $\Gamma^* = \overline{\Gamma}$ ($\overline{\Gamma}$ is the closure of Γ). The Cauchy kernel associated with the tube $T_\Gamma = \{z = x + iy : x \in \mathbb{R}^2, y \in \Gamma\}$ is

$$K(z) = \frac{1}{(2\pi i)^2} \int_{\Gamma^*} e^{2\pi i z\cdot t} dt = \int_0^\infty \int_0^\infty e^{2\pi i(z_1 t_1 + z_2 t_2)} dt_1 dt_2 = \prod_{j=1}^{2} \frac{-1}{2\pi i z_j}.$$

Direct calculation yields that

$$\begin{aligned} F(z) &= \frac{1}{(2\pi i)^2} \int_{\mathbb{R}^2} K(z-t) f(t) dt \\ &= \frac{1}{(2\pi i)^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{f(t_1, t_2)}{(t_1 - z_1)(t_2 - z_2)} dt_1 dt_2. \end{aligned}$$

Now we denote

$$(Cf)(z_1, t_2) = \int_{\mathbb{R}} \frac{f(t_1, t_2)}{t_1 - z_1} dt_1.$$

By using Lemma 3.1 as well as the fact $f \in L^p(\mathbb{R}^2)$, we know that

$$(Cf)_{t_2}(z_1) = (Cf)(z_1, t_2) \in H^p(\mathbb{C}_+),$$

and

$$\int_{\mathbb{R}} |(Cf)_{t_2}(x_1 + iy_1)|^p dx_1 \le A_p \int_{\mathbb{R}} |f(t_1, t_2)|^p dt_1 < \infty,$$

where A_p is a constant and $\mathbb{C}_+ = \{z = x + iy : x \in \mathbb{R}, y > 0\}$ is the upper half-plane in $\mathbb{C}$. It thus gives

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |(Cf)(z_1, t_2)|^p dx_1 dt_2 \le A_p \int_{\mathbb{R}} \int_{\mathbb{R}} |f(t_1, t_2)|^p dt_1 dt_2 < \infty.$$

As a result, we have $(Cf)_{z_1}(t_2) = (Cf)(z_1, t_2) \in L^p(\mathbb{R})$. Keeping in mind the following fact

$$F(z) = \frac{1}{(2\pi i)^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{f(t_1, t_2)}{(t_1 - z_1)(t_2 - z_2)} dt_1 dt_2 = \int_{\mathbb{R}} \frac{(Cf)_{z_1}(t_2)}{t_2 - z_2} dt_2,$$

as well as lemma 3.1, we can show that for such fixed z_1

$$\int_{\mathbb{R}} \frac{(Cf)_{z_1}(t_2)}{t_2 - z_2} dt_2 \in H^p(\mathbb{C}_+).$$

Thus, it gives

$$\int_{\mathbb{R}} \Big| \int_{\mathbb{R}} \frac{(Cf)_{z_1}(t_2)}{t_2 - z_2} dt_2 \Big|^p dx_2 \le A_p \int_{\mathbb{R}} |(Cf)_{z_1}(t_2)|^p dt_2.$$

Then, we can conclude

$$\begin{aligned} \int_{\mathbb{R}} \int_{\mathbb{R}} \Big| \int_{\mathbb{R}} \frac{(Cf)_{z_1}(t_2)}{t_2 - z_2} dt_2 \Big|^p dx_2 dx_1 &\le A_p \int_{\mathbb{R}} \int_{\mathbb{R}} |(Cf)_{z_1}(t_2)|^p dt_2 dx_1 \\ &\le A_p^2 \int_{\mathbb{R}} \int_{\mathbb{R}} |f(t_1, t_2)|^p dt_1 dt_2 \\ &< \infty. \end{aligned}$$

Therefore, $F(z) \in H^p(T_\Gamma)$. Moreover, there exists a constant A such that

$$\|F\|_{H^p} \leq A\|f\|_p.$$

Thus, we complete the proof.

□

Remark 3.3. *Adopting the induction, we can get the same conclusion when n is a finite and positive integer.*

Corollary 3.4. *Let* Γ *be an open cone in* $\mathbb{R}^n$ *and* $F \in H^p(T_\Gamma)$ *for* $1 \leq p \leq 2$, *then* $F(z)$ *has the form*

$$F(z) = \int_{\mathbb{R}^n} K(z-\xi)F(\xi)d\xi,$$

where $F(\xi) = \lim_{\eta\to 0 \eta\in\Gamma} F(\xi + i\eta)$ *in* $L^p(\mathbb{R}^n)$.

Proof. By using [12, Theorem 3.3.5], the above Corollary follows.

□

With these lemmas at our dispersal, we move to prove the main results.

Theorem 3.5. *Let* $f \in \mathcal{D}'_{L^p}(\mathbb{R}^n)$ *for* $1 < p < \infty$. *We set*

$$F(z) = \frac{1}{(2\pi i)^n}\Big\langle f(t),\ \frac{1}{\prod_{i=1}^n (t_i - z_i)}\Big\rangle,$$

then

$$|F(x+iy)|^p \leq \frac{A_\delta}{|y_1 y_2 \cdots y_n|}, \quad |y_i| > \delta_i \ (i = 1, 2, \cdots, n),$$

where A_δ *is a constant,* $\delta = (\delta_1, \cdots, \delta_n)$, $\delta_i > 0$ $(i = 1, 2 \cdots, n)$. *Moreover, there exist nonnegative integer* j, *and functions* $F_\alpha(z) \in H^p(T_\Gamma)$ *such that*

$$F(z) = \sum_{|\alpha|\leq j} D^\alpha F_\alpha(z),$$

where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ *is a multi-index notation and* $\Gamma = \{y = (y_1, y_2, \cdots, y_n \in \mathbb{R}^n) : y_1 > 0, y_2 > 0, \cdots, y_n > 0\}$ *is the first octant.*

Proof. For $f \in \mathcal{D}'_{L^p}(\mathbb{R}^n)$, $1 < p < \infty$, by Structure Formula [27] there exists a function $g_\alpha \in L^p(\mathbb{R}^n)$, such that the distribution f admits the form

$$f = \Sigma_{|\alpha|\leq j} D^\alpha g_\alpha.$$

Therefore, we have

$$\begin{aligned} |F(z)| &= \Big|\frac{1}{(2\pi i)^n}\Big\langle \sum_{|\alpha|\leq j} D_t^\alpha g_\alpha, \frac{1}{\prod_{i=1}^n (t_i - z_i)}\Big\rangle\Big| \\ &\leq \frac{1}{(2\pi)^n} \sum_{|\alpha|\leq j} \int_{\mathbb{R}^n} |g_\alpha| \Big|\prod_{i=1}^n \frac{\alpha_i!}{(t_i - z_i)^{\alpha_i+1}}\Big| dt \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{(2\pi)^n}\sum_{|\alpha|\leq j}\Big[\|g_\alpha\|_p\prod_{i=1}^{n}(\alpha_i!)\Big(\int_{\mathbb{R}}\frac{1}{(t_i^2+1)^{\frac{p'(\alpha_i+1)}{2}}|y_i|^{(\alpha_i+1)p'-1}}dt_i\Big)^{\frac{1}{p'}}\Big]\\
&= \frac{1}{|y_1y_2\cdots y_n|^{\frac{1}{p}}}\frac{1}{(2\pi)^n}\sum_{|\alpha|\leq j}\Big(\|g_\alpha\|_p\prod_{i=1}^{n}(\alpha_i!)\Big(\int_{\mathbb{R}}\frac{1}{(t_i^2+1)^{\frac{p'(\alpha_i+1)}{2}}}\frac{1}{|y_i|^{\alpha_i}}dt_i\Big)^{\frac{1}{p'}}\Big)\\
&\leq \frac{B_\delta}{|y_1y_2\cdots y_n|^{\frac{1}{p}}},
\end{aligned}$$

where

$$B_\delta=\frac{1}{|y_1y_2\cdots y_n|^{\frac{1}{p}}}\frac{1}{(2\pi)^n}\sum_{|\alpha|\leq j}\left(\|g_\alpha\|_p\prod_{i=1}^{n}(\alpha_i!)\Big(\int_{\mathbb{R}}\frac{1}{(t_i^2+1)^{\frac{p'(\alpha_i+1)}{2}}}\frac{1}{|\delta_i|^{\alpha_i}}dt_i\Big)^{\frac{1}{p'}}\right),$$

$|y_i|\geq\delta_i$, $\alpha_i\geq 0$, $(i=1,2,\cdots,n)$, $|\alpha|=\alpha_1+\alpha_2+\cdots+\alpha_n$, $D_x^\alpha=\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}$ and $\frac{1}{p}+\frac{1}{p'}=1$. Denoting $A_\delta=B_\delta^p$, we get

$$|F(x+iy)|^p\leq\frac{A_\delta}{|y_1y_2\cdots y_n|},\quad |y_i|\geq\delta_i,\quad i=1,2,\cdots,n.$$

Now we start to compute $F(z)$:

$$\begin{aligned}
F(z) &= \frac{1}{(2\pi i)^n}\Big\langle\sum_{|\alpha|\leq j}D_t^\alpha g_\alpha,\ \frac{1}{\prod_{i=1}^n(t_i-z_i)}\Big\rangle\\
&= \frac{1}{(2\pi i)^n}\sum_{|\alpha|\leq j}\int_{\mathbb{R}^n}g_\alpha(t)D_z^\alpha\Big(\frac{1}{\prod_{i=1}^n(t_i-z_i)}\Big)dt\\
&= \sum_{|\alpha|\leq j}D_z^\alpha F_\alpha(z),
\end{aligned}$$

where

$$F_\alpha(z)=\frac{1}{(2\pi i)^n}\int_{\mathbb{R}^n}\frac{g_\alpha(t)}{\prod_{i=1}^n(t_i-z_i)}dt.$$

In view of lemma 3.1 and Remark 3.3, we have

$$F_\alpha(z)\in H^p(T_\Gamma).$$

This completes the proof of Theorem 3.5. □

Corollary 3.6. *Let $1<p\leq 2$. Assume that j is a nonnegative integer and $F_\alpha(z)\in H^p(T_\Gamma)$. We denote*

$$F(z)=\sum_{|\alpha|\leq j}D^\alpha F_\alpha(z),$$

then there exists $f(x)\in\mathcal{D}'_{L^p}(\mathbb{R}^n)$ such that

$$F(z)=\frac{1}{(2\pi i)^n}\Big\langle f(t),\ \frac{1}{\prod_{i=1}^n(t_i-z_i)}\Big\rangle,$$

where α is a multi-index notation and Γ is an open convex cone.

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Conflict of Interest

We declare that we have no conflict of interest.

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