Journal of Industrial and Management Optimization

https://www.aimspress.com/journal/jimo

JIMO, 22(8): 3761–3788.
DOI:10.3934/jimo.2026135
Received: 30 April 2026
Revised: 08 July 2026
Accepted: 15 July 2026
Published: 20 July 2026

Research article

Robust portfolio selection with regret aversion under HARA utility

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Abstract: This paper studies a robust portfolio optimization problem for a regret-averse investor under model ambiguity within a hyperbolic absolute risk aversion (HARA) framework. Wealth is allocated across multiple risky assets whose price dynamics are driven by a multi-dimensional Brownian motion. A dynamic programming approach is used to derive explicit expressions for the robust optimal investment strategy, the density generator associated with the worst-case probability measure, and the corresponding value function. Sensitivity analysis is conducted to illustrate the economic implications of the theoretical results. The results indicate that, in the presence of regret aversion, the robust optimal portfolio exhibits reference dependence, which is further amplified as the degree of regret aversion increases. In addition, greater ambiguity aversion toward a given source of risk leads to a reduction in exposure to assets that are more sensitive to that source.

Keywords: robust portfolio selection, regret aversion, ambiguity, HARA utility, dynamic programming
Mathematics Subject Classification: 91G10, 93E20, 90C17

1. Introduction

The seminal work of Markowitz [1] established the static mean–variance paradigm, which laid the theoretical foundation for modern portfolio theory. Ever since its inception, portfolio selection has remained a central topic in financial economics, drawing sustained interest from both industry practitioners and academic researchers. Merton [2] extended the portfolio selection problem to a continuous-time framework and used dynamic programming techniques to derive the optimal investment strategies under the expected utility maximization criterion. Building on this contribution, Merton [3] introduced hyperbolic absolute risk aversion (HARA) utility into a continuous-time portfolio selection framework, encompassing both the constant relative risk aversion (CRRA) and constant absolute risk aversion (CARA) specifications, and derived the corresponding optimal strategy in closed form.

Merton's framework has been extensively extended in subsequent research. From a control-theoretic perspective, Lim and Zhou [4] presented a control methodology based on HARA utility function as an alternative to the exponential-of-an-integral approach to finding robust controllers. In the field portfolio selection, Ma and Zhu [5] revisited the Merton problem by exploiting the HARA utility framework and derived closed-form solutions for the optimal investment and consumption problem under CARA utility. Liu et al. [6] examined optimal portfolio selection for defined contribution (DC) pension plans in the presence of asset mispricing and derived explicit optimal strategies under HARA utility. Chang and Chen [7] investigated a DC pension fund investment problem with return-of-premium clauses in an environment characterized by stochastic interest rates and stochastic volatility, obtaining a closed-form optimal strategy under HARA utility. Zhang [8] studied an optimal asset–liability management problem and derived closed-form expressions for both the optimal investment strategy and the associated value function under HARA utility. More recently, Liang et al. [9] investigated the asset allocation problem for HARA-type investors and derived a unified closed-form formula for the optimal portfolio. Jeon and Kwak [10] explored the optimal consumption and investment decisions for an economic agent facing a welfare constraint and characterized the optimal strategies under HARA utility. Shen et al. [11] contributed further to the literature by examining optimal dynamic portfolio choice in flexible incomplete-market environments with wealth-dependent HARA utility.

The previous studies considered a portfolio selection problem under the assumption of fully rational preferences. In practice, however, investors' behavior is often influenced by emotions and deviates from full rationality. When making portfolio decisions, investors typically consider a reference point, which may correspond to the terminal wealth or utility of a competitor, a benchmark strategy, or other potential investment strategies. They prefer their terminal wealth to exceed this reference level or, at the very least, to avoid falling short of it. Otherwise, they may experience regret or disappointment. Motivated by these considerations, an expanding strand of the literature integrates behavioral preferences, particularly regret aversion, into portfolio choice frameworks. For instance, Doreleijers and Hambel [12] studied the optimal investment problem for a regret-averse investor who experiences regret when wealth falls below a benchmark and rejoicing when it exceeds that benchmark; they obtained a closed-form solution for the optimal strategy under CRRA utility. Similarly, under CRRA preferences, Van Bilsen et al. [13] explicitly derived and analyzed the optimal consumption and portfolio policies for a loss-averse investor with a reference point. Overall, the existing literature on behavioral portfolio choice has predominantly been developed under CRRA or CARA utility specifications, with comparatively less attention given to more general preference classes such as HARA.

In the classical portfolio selection literature discussed above, investors are typically assumed to be ambiguity-neutral, with the model parameters treated as known or accurately estimated. In practice, the true underlying model is difficult to specify, as its calibration and statistical estimation are often challenging, leading to model uncertainty, commonly referred to as ambiguity. Accordingly, recent research has focused on robust portfolio optimization under model ambiguity. For example, Lin et al. [14] analyzed a robust portfolio choice problem with correlated risky assets, and derived closed-form optimal strategies under CRRA preferences. Wang and Zhang [15] studied a nonzero-sum stochastic differential game among multiple competing asset–liability managers with CARA utilities, and characterized explicit solutions for the robust Nash equilibrium strategies, the density generator associated with the worst-case probability measure, and the corresponding value functions. Furthermore, Wang and Zhang [16] investigated a robust asset–liability management problem with ambiguity aversion under a multivariate stochastic covariance model characterized by hybrid stochastic volatility and stochastic interest rates, and obtained explicit robust optimal strategies within the HARA utility framework. More recently, Chu et al. [17] studied robust portfolio choice with stochastic factors and market frictions under mean–variance preference, further highlighting the importance of robustness in dynamic investment decisions.

The abovementioned literature can therefore be viewed as three related but distinct streams. The first stream develops HARA utility and its applications in stochastic control and portfolio selection. The second stream incorporates regret or behavioral preferences into investment decisions. The third stream studies robust portfolio selection under model ambiguity. However, these streams have not been fully integrated. Existing HARA-based portfolio models generally do not incorporate regret aversion, whereas regret-averse portfolio models are primarily developed under CRRA-type preferences and often neglect model ambiguity. Moreover, robust HARA portfolio models typically focus on uncertainty aversion without considering regret arising from forgone investment opportunities. Therefore, the joint effect of regret aversion and model ambiguity within a HARA portfolio framework remains insufficiently understood. This gap motivates the present study.

Building on the abovementioned analysis, a robust portfolio optimization problem with regret aversion is studied under a HARA utility framework. An ambiguity-averse and regret-averse investor maximizes the expected utility of terminal wealth in a continuous-time market with multidimensional Brownian motion. Regret theory is incorporated following Doreleijers and Hambel [12], while ambiguity is modeled as a set of alternative probability measures in the spirit of Wang and Zhang [16]. Closed-form solutions for the robust optimal strategy, the density generator associated with the worst-case probability measure, and the value function are obtained via dynamic programming. Sensitivity analysis is conducted to assess the impact of regret and ambiguity aversion on the robust optimal portfolio allocation.

This study contributes to the portfolio selection literature in three aspects. First, it extends regret-averse portfolio choice from the commonly used CRRA-type utility specifications to the more general HARA utility framework. Second, it further extends the regret-averse HARA framework by incorporating model ambiguity and derives explicit solutions for the robust optimal strategy, the worst-case density generator, and the value function. Third, it characterizes the joint economic effects of regret aversion and ambiguity aversion. Regret aversion affects the investor's response to forgone investment opportunities, whereas source-specific ambiguity aversion reduces exposure to assets that are more sensitive to the corresponding sources of risk.

The remainder of this paper is organized as follows. In Section 2, the financial market is described, a HARA utility specification incorporating regret aversion is introduced, the model ambiguity is specified, and the corresponding portfolio selection problem with regret aversion and ambiguity is formulated for terminal utility maximization. In Section 3, a dynamic programming approach is used to derive the robust optimal strategy, the associated worst-case density generator, and the value function. In Section 4, the sensitivity of the robust optimal strategy to the model parameters is analyzed. Conclusions are provided in Section 5.

2. Model formulation

2.1. The financial market

Consider a financial market with $n+1$ risky assets, whose prices at time t are denoted by $S_0(t), S_1(t), \ldots, S_n(t)$. Let $\mathbf{A}^\top$ represent the transpose of a matrix or vector $\mathbf{A}$. Throughout this paper, we define a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, on which an m-dimensional standard Brownian motion $\mathbf{Z}(t) = (Z_1(t), Z_2(t), \ldots, Z_m(t))^\top$ is defined. The filtration $\mathcal{F} = \{\mathscr{F}_t; t \geq 0\}$ is such that $\mathscr{F}_t = \sigma\{\mathbf{Z}(u); 0 \leq u \leq t\}$, which is generated by the Brownian motion, and $\mathbb{P}$ denotes the probability measure. The filtration $\mathcal{F}_t$ can be interpreted as the information set available up to time t. The prices of these risky assets are assumed to follow the geometric Brownian motion dynamics, i.e.,

$$\begin{cases} \mathrm{d}S_i(t) = S_i(t)\Big[b_i(t)\mathrm{d}t + \sum_{j=1}^m \sigma_{ij}(t)\mathrm{d}Z_j(t)\Big], \\ S_i(0) = s_i^0, \quad i = 0, 1, 2, \ldots, n, \end{cases} \tag{2.1}$$

where $b_i(t)$ and $\boldsymbol{\sigma}_i(t) = (\sigma_{i1}(t), \sigma_{i2}(t), \ldots, \sigma_{im}(t))$ represent the appreciation rate and the volatility vector for the i-th risky asset, respectively. The condition $m \geq n$ indicates that the number of risk sources is at least as large as the number of risky assets. Additionally, $\sigma_{ij}(t) > 0$ denotes the volatility of the i-th asset's price due to the j-th risk source. Let $\boldsymbol{\sigma}(t) = (\sigma_{ij}(t))_{n\times m}$, and assume that for all $t \in [0, T]$, a constant $\lambda > 0$ exists such that $\boldsymbol{\Sigma}(t) \triangleq \boldsymbol{\sigma}(t)\boldsymbol{\sigma}(t)^\top \geq \lambda \mathbf{I}_{n\times n}$, where $\mathbf{I}_{n\times n}$ is the identity matrix. Matrices and vectors are represented in bold throughout the text.

Remark 1. *In the proposed framework, the* 0*-th asset is generally considered a risky asset. However, it degenerates into a risk-free asset if $b_0(t) = r(t)$ and $\sigma_{0j}(t) = 0$ for $j = 1, 2, \ldots, m$, where $r(t)$ denotes the risk-free interest rate.*

Suppose an investor with initial wealth w_0 can freely trade risky assets in the financial market described earlier. Let $\boldsymbol{\pi} \triangleq \{\boldsymbol{\pi}(t) \mid \boldsymbol{\pi}(t) = (\pi_1(t), \pi_2(t), \ldots, \pi_n(t))^\top, t \in [0, T]\}$ denote the investment strategy, where $\pi_i(t)$ represents the proportion of wealth allocated to the i-th risky asset at time t. The remaining portion, $1 - \sum_{i=1}^n \pi_i(t)$, is allocated to the 0-th asset. Let $W(t)$ denote the investor's wealth at time t. Then, under the self-financing condition, the dynamics of $\{W(t)\}_{t\in[0,T]}$ under the strategy $\boldsymbol{\pi}$ can be expressed as follows:

$$\begin{cases} \dfrac{\mathrm{d}W(t)}{W(t)} & = \Big[1 - \sum_{i=1}^n \pi_i(t)\Big]\dfrac{\mathrm{d}S_0(t)}{S_0(t)} + \sum_{i=1}^n \pi_i(t)\dfrac{\mathrm{d}S_i(t)}{S_i(t)} \\ & = \Big[b_0(t) + \boldsymbol{\pi}(t)^\top \boldsymbol{\xi}(t)\Big]\mathrm{d}t + \Big[\boldsymbol{\sigma}_0(t) + \boldsymbol{\pi}(t)^\top \boldsymbol{\theta}(t)\Big]\mathrm{d}\mathbf{Z}(t), \\ W(0) & = w_0, \end{cases} \tag{2.2}$$

where

$$\begin{cases} \boldsymbol{\xi}(t) & \triangleq \mathbf{b}(t) - \mathbf{1}b_0(t), \\ \boldsymbol{\theta}(t) & \triangleq \boldsymbol{\sigma}(t) - \mathbf{1}\boldsymbol{\sigma}_0(t), \\ \mathbf{1} & \triangleq (1, 1, ..., 1) \in \mathbb{R}^n, \\ \mathbf{b}(t) & \triangleq \Big(b_1(t), b_2(t), ..., b_n(t)\Big)^\top. \end{cases} \tag{2.3}$$

2.2. HARA utility

In this paper, the family of HARA utility functions is considered, defined by

$$U(w) = \frac{1-p}{pq}\left(\frac{q}{1-p}w + \beta\right)^p, \tag{2.4}$$

with $q > 0, \beta > 0$, $p < 1$, $p \neq 0$, and $-\frac{\beta(1-p)}{q} < w < \infty$. The absolute risk-aversion coefficient in Eq. (2.4) is

$$ARA(w) = -\frac{U''(w)}{U'(w)} = \frac{(1-p)q}{qw + (1-p)\beta}, \tag{2.5}$$

while the relative risk-aversion coefficient in that is

$$RRA(w) = wARA(w) = \frac{(1-p)qw}{qw + (1-p)\beta}. \tag{2.6}$$

It is well known that for $p > 1$, satiation occurs when $w \geq \frac{(p-1)\beta}{q}$, which represents a non-rational behavior for a financial agent. Therefore, we focus solely on the case $p < 1$.

The HARA family is rather rich, since, by suitable adjustments of the parameters, one can obtain utility functions with the absolute or relative risk aversion increasing, decreasing, or constant.

Remark 2. *If $\beta = 0$ and $q = 1 - p$, Eq.* (2.4) *is transformed into*

$$U(w) = \frac{w^p}{p}, \tag{2.7}$$

yielding the CRRA utility function.

Remark 3. *When $\beta = 1$ and $p \to -\infty$, after an appropriate normalization of Eq.* (2.4)*, the HARA utility converges to*

$$U(w) = -\frac{e^{-qw}}{q}, \tag{2.8}$$

which corresponds to the CARA utility function.

2.3. Regret aversion

Consider an investor who is both risk-averse and regret-averse and seeks to maximize the expected utility of their terminal wealth. Specifically, the investor experiences satisfaction (or regret) based on whether the realized wealth from the chosen investment strategy outperforms (or underperforms) that of an alternative strategy. This forgone alternative can be viewed as a benchmark portfolio or reference point against which the investor evaluates performance, thus capturing the psychological aspect of regret or rejoicing in investment decisions. Referring to the utility specification proposed by Doreleijers and Hambel [12], the utility function that incorporates regret and rejoicing can be described as

$$\tilde{U}(w, \hat{w}) = \frac{1-p}{pq}\left(\frac{q}{1-p}w + \beta\right)^p \left(\frac{\hat{w}}{w}\right)^\kappa, \tag{2.9}$$

with κ being the regret aversion parameter and $\hat{w}$ being the wealth of the benchmark, $\kappa > 1$. A larger (smaller) value of κ indicates a higher (lower) sensitivity to regret arising from the forgone alternative.

In what follows, we restrict our attention to the case $p < 0$ for ease of exposition. The utility function in Eq. (2.9) (i) does not rely on the hypothetical ex-post best forgone outcome $w_{\max}$ but rather on an achievable benchmark outcome $\hat{w}$; (ii) induces regret in the agent if she/he misses profits relative to the benchmark outcome; and (iii) induces rejoicing if she/he outperforms the benchmark.

Remark 4. *Note that when $\kappa = 0$, indicating regret neutrality, the model in Eq.* (2.9) *reduces to the HARA utility function.*

Comparing the HARA utility function in Eq. (2.4) with the proposed utility function in Eq. (2.9), the following cases can be distinguished:

1. If $w < \hat{w}$, then $\tilde{U}(w, \hat{w}) < U(w)$, as $\left(\frac{\hat{w}}{w}\right)^{\kappa} \geq 1$ for $\kappa > 1$, $p < 0$. The investor experiences regret (disutility) for missing out on the foregone option, resulting in the utility incorporating regret and rejoicing being lower than the utility in a rational state.
2. If $w = \hat{w}$, then $\tilde{U}(w, \hat{w}) = U(w)$, as $\left(\frac{\hat{w}}{w}\right)^{\kappa} = 1$ for $\kappa > 1$, $p < 0$. The investor experiences neither regret nor rejoicing, as the realized choice is identical to the forgone choice, resulting in the utility incorporating regret and rejoicing being equal to the utility in a rational state.
3. If $w > \hat{w}$, then $\tilde{U}(w, \hat{w}) > U(w)$, as $\left(\frac{\hat{w}}{w}\right)^{\kappa} \leq 1$ for $\kappa > 1$, $p < 0$. The investor experiences rejoicing from choosing the better option compared with the foregone option, resulting in the utility incorporating regret and rejoicing being higher than the utility in a rational state.

2.4. Model ambiguity

In traditional dynamic portfolio selection problems, investors are typically assumed to be neutral to ambiguity, and the model parameters are presumed to be known under a reference probability measure (denoted $\mathbb{P}$). In practice, however, investors often face difficulties in determining the precise underlying model attributable to challenges in refinement through calibration or statistical estimation, which gives rise to model ambiguity. As a consequence, ambiguity aversion is naturally induced, since strategies derived under a misspecified model may deviate from those that are truly optimal. To address this issue, a family of alternative probability measures is introduced, under which an optimal robust strategy is sought.

The risky asset process is defined under a reference probability measure $\mathbb{P}$. Let $\mathbb{Q}$ be any probability measure equivalent to $\mathbb{P}$ (i.e., $\mathbb{Q} \sim \mathbb{P}$), and denote the set of all such equivalent measures as $\mathcal{Q} = \{\mathbb{Q}\,|\mathbb{Q} \sim \mathbb{P}\}$. By Girsanov's theorem, for each $\mathbb{Q} \in \mathcal{Q}$, there is a progressively measurable process $\boldsymbol{\phi}(t) = (\phi_1(t), \phi_2(t), \cdots, \phi_m(t))'$ such that $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{F}_t} = \Lambda(t)$, and the associated $\Lambda(t)$ is given by

$$\Lambda(t) = \exp\left\{-\sum_{i=1}^{m}\int_0^t \phi_i(s)\mathrm{d}Z_i(s) - \frac{1}{2}\sum_{i=1}^{m}\int_0^t \phi_i{}^2(s)\mathrm{d}s\right\}, \tag{2.10}$$

where $\{\phi_i(t)\}_{t\in[0,T]}, i = 1, 2, \cdots, m$ satisfies (i) $\phi_i(t)$ is $\mathcal{F}_t$-measurable for all t; and (ii) $\mathbb{E}^{\mathbb{P}}\left[\frac{1}{2}\int_0^T \|\phi_i(t)\|^2\mathrm{d}t\right] < \infty$. Let Φ denote the corresponding admissible class.

The density process defined in Eq. (2.10) is thus a martingale. By Girsanov's theorem, the Brownian motion under the alternative measure $\mathbb{Q}$ is represented via an appropriate drift transformation:

$$\mathrm{d}Z_i{}^{\mathbb{Q}}(t) = \mathrm{d}Z_i(t) + \phi_i(t)\mathrm{d}t,\ i = 1, 2, \cdots, m.$$

Consequently, the wealth process $\{W(t)\}_{t\in[0,T]}$ for an ambiguity-averse investor reformulated under measure $\mathbb{Q}$ is given by

$$\frac{\mathrm{d}W(t)}{W(t)} = \Big[b_0(t) + \boldsymbol{\pi}(t)^\top \boldsymbol{\xi}(t) - \Big(\sigma_0(t) + \boldsymbol{\pi}(t)^\top \boldsymbol{\theta}(t)\Big)\boldsymbol{\phi}(t)\Big]\mathrm{d}t + \Big[\sigma_0(t) + \boldsymbol{\pi}(t)^\top \boldsymbol{\theta}(t)\Big]\mathrm{d}\mathbf{Z}^{\mathbb{Q}}. \tag{2.11}$$

2.5. The robust optimization problem

In this section, the robust optimization problem is formulated to characterize the robust optimal investment strategy under regret aversion and model ambiguity, where the expected utility of terminal wealth is evaluated relative to a forgone benchmark.

Let $L^2_{\mathcal{F}}(0,T;\mathbb{R}^n)$ denote the set of $\mathbb{R}^n$-valued, progressively measurable stochastic processes $f(t)$ that are adapted to the filtration $\{\mathscr{F}_t\}_{0\le t\le T}$, with $\mathbb{E}\left[\int_0^T |f(t)|^2\,dt\right] < +\infty$. A strategy $\boldsymbol{\pi}(\cdot) = \{\boldsymbol{\pi}(t), t \in [0,T]\}$ is called an admissible strategy if $\boldsymbol{\pi}(\cdot) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^n)$, and the tuple $(W(\cdot),\boldsymbol{\pi}(\cdot))$ satisfies Eq. (2.2). In this case, we refer to $(W(\cdot),\boldsymbol{\pi}(\cdot))$ as an admissible tuple. The set of all admissible tuples over $[0,T]$ is denoted $\boldsymbol{\Pi}$. Taking into account the forgone benchmark decision $\hat{\boldsymbol{\pi}}$ and outcome $\hat{w}$, the investor's objective is to find a strategy $\boldsymbol{\pi}$ from the admissible strategies that maximizes the expected utility, incorporating regret aversion. The formulation is as follows:

$$\sup_{\pi\in\boldsymbol{\Pi}}\inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}^{\mathbb{Q}}\left[\tilde{U}\left(W(T),\hat{W}(T)\right) + P\left(\mathbb{P}\,\|\mathbb{Q}\right)\Big|W(t) = w, \hat{W}(t) = \hat{w}\right], \tag{2.12}$$

where $\boldsymbol{\Pi}$ denotes the set of admissible investment strategies and $\mathcal{Q}$ denotes the set of alternative probability measures. The term $P\left(\mathbb{P}\,\|\mathbb{Q}\right) \ge 0$ is introduced as a penalty to quantify the deviation of an alternative measure from the reference measure. For analytical tractability, as with Maenhout [18] and Uppal and Wang [19], the penalty function is assumed to take the form

$$P\left(\mathbb{P}\,\|\mathbb{Q}\right) \triangleq \sum_{i=1}^{m}\int_t^T \psi_i\left(u,\phi_i(u),W(u),\hat{W}(u)\right)\mathrm{d}u, \tag{2.13}$$

where $\psi_i\left(u,\phi_i(u),W(u),\hat{W}(u)\right) = \frac{\phi_i{}^2(u)}{2\varphi_i\left(u,W(u),\hat{W}(u)\right)}, i = 1,2,\cdots,m$. Let $\boldsymbol{\psi} \triangleq (\psi_1,\psi_2,\cdots,\psi_m)^\top$. It is also assumed that $\varphi_i(\cdot,\cdot,\cdot)$ is given by

$$\varphi_i\left(t,w,\hat{w}\right) = \frac{\varepsilon_i}{pH(t,w,\hat{w})}, \tag{2.14}$$

where $\varepsilon_i > 0, i = 1,2,\cdots,m$ represents the investor's aversion to ambiguity with respect to the i-th source of risk, capturing the degree of aversion to uncertainty of diffusion. Leaving $\boldsymbol{\varepsilon} \triangleq \mathrm{diag}(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_m)$, it immediately follows that $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^\top$.

Remark 5. *Consider the utility function that incorporates regret and rejoicing in Eq.* (2.9)*. When $p < 0$, the term $(1-p)/pq$ is negative, while the remaining factors are positive on the admissible wealth domain. Hence, the utility is negative. Since the value function $H(t,w,\hat{w})$ represents the optimized continuation utility and inherits the same sign structure as the terminal utility in the HARA setting, $H(t,w,\hat{w})$ is also negative when $p < 0$. Therefore, the product $pH(t,w,\hat{w})$ remains positive. Thus, in the admissible parameter region considered in the paper, $pH(t,w,\hat{w}) > 0$. This shows that the penalty term is non-negative and well defined under the parameter settings considered in the paper.*

Remark 6. *Note that when $\boldsymbol{\varepsilon} = \mathbf{0}_{m\times m}$, indicating a ambiguity-neutral stance, in which the model reduces to the standard stochastic control problem without ambiguity penalization.*

We assume that the agent can observe the outcome of the forgone benchmark, i.e., forgone wealth $\hat{W} = \left\{\hat{W}(t)\right\}_{t\in[0,T]}$, associated with a benchmark investment strategy $\hat{\boldsymbol{\pi}}$. Given the outcome of the forgone benchmark, the agent's value function for utility from terminal wealth is

$$H(t, w, \hat{w}) = \sup_{\boldsymbol{\pi}\in\boldsymbol{\Pi}} \inf_{\mathbb{Q}\in Q} \mathbb{E}^{\mathbb{Q}}\left[\tilde{U}\left(W(T), \hat{W}(T)\right) + P\left(\mathbb{P}\,\|\mathbb{Q}\right) \middle| W(t) = w, \hat{W}(t) = \hat{w}\right], \tag{2.15}$$

with the terminal condition

$$H(T, w, \hat{w}) = \tilde{U}\left(W(T), \hat{W}(T)\right). \tag{2.16}$$

3. Solution to the robust optimization problem

In this section, a stochastic dynamic programming approach is used to derive explicit solutions for the robust optimal investment strategy, the density generator associated with the worst-case probability measure, and the corresponding value function for Problem (2.12). The baseline model (2.12) accounts for both regret aversion and model ambiguity. As special cases, the model reduces to: (i) a regret-averse portfolio optimization problem without model ambiguity; (ii) a regret-neutral portfolio optimization problem with model ambiguity; and (iii) the classical portfolio selection problem under HARA utility, without regret aversion or model ambiguity.

Within the stochastic dynamic programming framework, we first impose the necessary assumption. We then derive the associated Hamilton–Jacobi–Bellman (HJB) equation and analytically characterize the robust optimal solution to the control problem (2.12).

Assumption 1. *The market coefficients $b_0(t)$, $\boldsymbol{\xi}(t)$, $\boldsymbol{\sigma}_0(t)$, and $\boldsymbol{\theta}(t)$, together with the benchmark strategy $\widehat{\boldsymbol{\pi}}(t)$, are continuous and uniformly bounded on $[0, T]$. Moreover, define*

$$\overline{\boldsymbol{M}}(t, w) \triangleq \frac{(w\alpha)^2}{p}\boldsymbol{\theta}(t)\boldsymbol{\varepsilon}\boldsymbol{\theta}^\top(t) - w^2\delta\boldsymbol{\theta}(t)\boldsymbol{\theta}^\top(t),$$

where α and δ are defined in Eq. (3.3). We assume that $\overline{\boldsymbol{M}}(t, w)$ is nonsingular for every $(t, w) \in [0, T] \times (0, \infty)$ and that its inverse is uniformly bounded, i.e., a constant $C_M > 0$ exists such that

$$\sup_{(t,w)\in[0,T]\times(0,\infty)} \left\|\overline{\boldsymbol{M}}(t, w)^{-1}\right\| \leq C_M.$$

Theorem 1. *For any $t \in [0, T]$, the optimal value function $H(t, w, \hat{w})$ of the optimization problem (2.12) is given by*

$$H(t, w, \hat{w}) = \frac{1 - p}{pq}\left(\frac{q}{1 - p}w + \beta\right)^p \left(\frac{\hat{w}}{w}\right)^\kappa \exp\left[\int_t^T g(u)\,\mathrm{d}u\right], \tag{3.1}$$

where the function $g(\cdot)$ is defined by

$$
\begin{aligned}
g(t) \triangleq{}& w\alpha b_0(t) + \kappa\big[b_0(t) + \hat{\boldsymbol{\pi}}^\top(t)\boldsymbol{\xi}(t)\big] + \frac{1}{2}w^2\delta\,\boldsymbol{\sigma}_0(t)\boldsymbol{\sigma}_0^\top(t) \\
&+ \frac{1}{2}\kappa(\kappa-1)\Big[\boldsymbol{\sigma}_0(t)\boldsymbol{\sigma}_0^\top(t) + \boldsymbol{\sigma}_0(t)\boldsymbol{\theta}^\top(t)\hat{\boldsymbol{\pi}}(t) + \hat{\boldsymbol{\pi}}^\top(t)\boldsymbol{\theta}(t)\boldsymbol{\sigma}_0^\top(t) + \hat{\boldsymbol{\pi}}^\top(t)\boldsymbol{\theta}(t)\boldsymbol{\theta}^\top(t)\hat{\boldsymbol{\pi}}(t)\Big] \\
&+ w\kappa\alpha\boldsymbol{\sigma}_0(t)\big[\boldsymbol{\sigma}_0(t) + \hat{\boldsymbol{\pi}}^\top(t)\boldsymbol{\theta}(t)\big]^\top - \frac{w^2\alpha^2}{2p}\boldsymbol{\sigma}_0(t)\boldsymbol{\varepsilon}\boldsymbol{\sigma}_0^\top(t) \\
&+ \frac{1}{2}\Big[\alpha\boldsymbol{\xi}^\top(t) + w\delta\boldsymbol{\sigma}_0(t)\boldsymbol{\theta}^\top(t) + \kappa\alpha\big[\boldsymbol{\sigma}_0(t) + \hat{\boldsymbol{\pi}}^\top(t)\boldsymbol{\theta}(t)\big]\boldsymbol{\theta}^\top(t) - \frac{w\alpha^2}{p}\boldsymbol{\sigma}_0(t)\boldsymbol{\varepsilon}\boldsymbol{\theta}^\top(t)\Big] \\
&\times\Big[\frac{\alpha^2}{p}\boldsymbol{\theta}(t)\boldsymbol{\varepsilon}\boldsymbol{\theta}^\top(t) - \delta\boldsymbol{\theta}(t)\boldsymbol{\theta}^\top(t)\Big]^{-1} \\
&\times\Big[\alpha\boldsymbol{\xi}(t) + w\delta\boldsymbol{\theta}(t)\boldsymbol{\sigma}_0^\top(t) + \kappa\alpha\boldsymbol{\theta}(t)\big[\boldsymbol{\sigma}_0(t) + \hat{\boldsymbol{\pi}}^\top(t)\boldsymbol{\theta}(t)\big]^\top - \frac{w\alpha^2}{p}\boldsymbol{\theta}(t)\boldsymbol{\varepsilon}\boldsymbol{\sigma}_0^\top(t)\Big],
\end{aligned}
\tag{3.2}
$$

with

$$
\begin{cases}
\alpha \triangleq \dfrac{pq}{qw + (1-p)\beta} - \dfrac{\kappa}{w}, \\
\delta \triangleq \dfrac{pq^2(p-1)}{[qw + (1-p)\beta]^2} + \dfrac{\kappa(\kappa+1)}{w^2} - \dfrac{2pq\kappa}{w[qw + (1-p)\beta]}.
\end{cases}
\tag{3.3}
$$

Then, the robust optimal investment strategy $\boldsymbol{\pi}^*(t)$ and the density generator process $\boldsymbol{\phi}^*(t)$ corresponding to the worst-case scenario are, respectively, given by

$$
\begin{aligned}
\boldsymbol{\pi}^*(t) ={}& \Big[\frac{w\alpha^2}{p}\boldsymbol{\theta}(t)\boldsymbol{\varepsilon}\boldsymbol{\theta}^\top(t) - w\delta\boldsymbol{\theta}(t)\boldsymbol{\theta}^\top(t)\Big]^{-1^\top} \\
&\times\Big[\alpha\boldsymbol{\xi}(t) + w\delta\boldsymbol{\theta}(t)\boldsymbol{\sigma}_0^\top(t) + \kappa\alpha\boldsymbol{\theta}(t)\big[\boldsymbol{\sigma}_0(t) + \hat{\boldsymbol{\pi}}^\top(t)\boldsymbol{\theta}(t)\big]^\top - \frac{w\alpha^2}{p}\boldsymbol{\theta}(t)\boldsymbol{\varepsilon}\boldsymbol{\sigma}_0^\top(t)\Big],
\end{aligned}
\tag{3.4}
$$

and

$$
\begin{aligned}
\boldsymbol{\phi}^*(t) ={}& \frac{w\alpha}{p}\boldsymbol{\varepsilon}\bigg\{\boldsymbol{\sigma}_0^\top(t) + \boldsymbol{\theta}^\top(t)\Big[\frac{w\alpha^2}{p}\boldsymbol{\theta}(t)\boldsymbol{\varepsilon}\boldsymbol{\theta}^\top(t) - w\delta\boldsymbol{\theta}(t)\boldsymbol{\theta}^\top(t)\Big]^{-1^\top} \\
&\times\Big[\alpha\boldsymbol{\xi}(t) + w\delta\boldsymbol{\theta}(t)\boldsymbol{\sigma}_0^\top(t) + \kappa\alpha\boldsymbol{\theta}(t)\big[\boldsymbol{\sigma}_0(t) + \hat{\boldsymbol{\pi}}^\top(t)\boldsymbol{\theta}(t)\big]^\top - \frac{w\alpha^2}{p}\boldsymbol{\theta}(t)\boldsymbol{\varepsilon}\boldsymbol{\sigma}_0^\top(t)\Big]\bigg\}.
\end{aligned}
\tag{3.5}
$$

Proof In accordance with the principle of dynamic programming, the HJB equation governing the value function $H(t, w, \hat{w})$ can be expressed as

$$0 = \sup_{\boldsymbol{\pi}\in\boldsymbol{\Pi}} \inf_{\mathbb{Q}\in Q} \Big\{ H_t + wH_w\Big[b_0(t) + \boldsymbol{\pi}^\top(t)\boldsymbol{\xi}(t) - [\boldsymbol{\sigma}_0(t) + \boldsymbol{\pi}(t)^\top\boldsymbol{\theta}(t)]\,\boldsymbol{\phi}(t)\Big] \\ + \hat{w}H_{\hat{w}}\left[b_0(t) + \hat{\boldsymbol{\pi}}^\top(t)\boldsymbol{\xi}(t)\right] \\ + \frac{1}{2}w^2H_{ww}\left[\boldsymbol{\sigma}_0(t) + \boldsymbol{\pi}^\top(t)\boldsymbol{\theta}(t)\right]\left[\boldsymbol{\sigma}_0(t) + \boldsymbol{\pi}^\top(t)\boldsymbol{\theta}(t)\right]^\top \\ + \frac{1}{2}\hat{w}^2H_{\hat{w}\hat{w}}\left[\boldsymbol{\sigma}_0(t) + \hat{\boldsymbol{\pi}}^\top(t)\boldsymbol{\theta}(t)\right]\left[\boldsymbol{\sigma}_0(t) + \hat{\boldsymbol{\pi}}^\top(t)\boldsymbol{\theta}(t)\right]^\top \\ + w\hat{w}H_{w\hat{w}}\left[\boldsymbol{\sigma}_0(t) + \boldsymbol{\pi}^\top(t)\boldsymbol{\theta}(t)\right]\left[\boldsymbol{\sigma}_0(t) + \hat{\boldsymbol{\pi}}^\top(t)\boldsymbol{\theta}(t)\right]^\top \\ + \frac{1}{2}H\boldsymbol{\phi}^\top(t)\boldsymbol{\varepsilon}\boldsymbol{\phi}(t)\Big\}, \tag{3.6}$$

with the boundary condition $H(T, w, \hat{w}) = \tilde{U}\left(W(T), \hat{W}(T)\right)$, where the partial derivatives are defined by

$$\begin{cases} H_t = \dfrac{\partial H(t,w,\hat{w})}{\partial t}, \quad H_w = \dfrac{\partial H(t,w,\hat{w})}{\partial w}, \quad H_{ww} = \dfrac{\partial^2 H(t,w,\hat{w})}{\partial w^2}, \\ H_{\hat{w}} = \dfrac{\partial H(t,w,\hat{w})}{\partial \hat{w}}, \quad H_{\hat{w}\hat{w}} = \dfrac{\partial^2 H(t,w,\hat{w})}{\partial \hat{w}^2}, \quad H_{w\hat{w}} = \dfrac{\partial^2 H(t,w,\hat{w})}{\partial w \partial \hat{w}}, \end{cases} \tag{3.7}$$

The first-order condition for $\boldsymbol{\phi}(t)$ in Eq. (3.6) leads to

$$0 = -wH_w[\boldsymbol{\sigma}_0(t) + \boldsymbol{\pi}^\top(t)\boldsymbol{\theta}(t)] + pH\boldsymbol{\phi}^*(t)^{-1}\boldsymbol{\varepsilon}.$$

By rearranging the expressions above, the density generator process $\boldsymbol{\phi}^*(t)$ corresponding to the worst-case scenario is obtained in closed form as follows:

$$\boldsymbol{\phi}^*(t) = \frac{wH_w}{pH}\,\boldsymbol{\varepsilon}[\boldsymbol{\sigma}_0(t) + \boldsymbol{\pi}^\top(t)\boldsymbol{\theta}(t)]^\top. \tag{3.8}$$

Substituting Eq. (3.8) into the HJB equation Eq. (3.6) and rearranging yields

$$0 = H_t + wH_w\left[b_0(t) + \boldsymbol{\pi}^\top(t)\boldsymbol{\xi}(t)\right] + \hat{w}H_{\hat{w}}\left[b_0(t) + \hat{\boldsymbol{\pi}}^\top(t)\boldsymbol{\xi}(t)\right] \\ + \frac{1}{2}w^2H_{ww}\left[\boldsymbol{\sigma}_0(t) + \boldsymbol{\pi}^\top(t)\boldsymbol{\theta}(t)\right]\left[\boldsymbol{\sigma}_0(t) + \boldsymbol{\pi}^\top(t)\boldsymbol{\theta}(t)\right]^\top \\ + \frac{1}{2}\hat{w}^2H_{\hat{w}\hat{w}}\left[\boldsymbol{\sigma}_0(t) + \hat{\boldsymbol{\pi}}^\top(t)\boldsymbol{\theta}(t)\right]\left[\boldsymbol{\sigma}_0(t) + \hat{\boldsymbol{\pi}}^\top(t)\boldsymbol{\theta}(t)\right]^\top \\ + w\hat{w}H_{w\hat{w}}\left[\boldsymbol{\sigma}_0(t) + \boldsymbol{\pi}^\top(t)\boldsymbol{\theta}(t)\right]\left[\boldsymbol{\sigma}_0(t) + \hat{\boldsymbol{\pi}}^\top(t)\boldsymbol{\theta}(t)\right]^\top \\ + \frac{w^2H_w^2}{2pH}\left[\boldsymbol{\sigma}_0(t) + \boldsymbol{\pi}^\top(t)\boldsymbol{\theta}(t)\right]\boldsymbol{\varepsilon}[\boldsymbol{\sigma}_0(t) + \boldsymbol{\pi}^\top(t)\boldsymbol{\theta}(t)]^\top. \tag{3.9}$$

By imposing the first-order condition for $\boldsymbol{\pi}(t)$ in Eq. (3.9), it follows that

$$0 = wH_w\boldsymbol{\xi}^\top(t) + w^2H_{ww}[\boldsymbol{\sigma}_0(t)\boldsymbol{\theta}^\top(t) + \boldsymbol{\pi}^{\top *}(t)\boldsymbol{\theta}(t)\boldsymbol{\theta}^\top(t)] + w\hat{w}\hat{H}_{w\hat{w}}[\boldsymbol{\sigma}_0(t) + \hat{\boldsymbol{\pi}}^\top(t)\boldsymbol{\theta}(t)]\boldsymbol{\theta}^\top(t) \\ - \frac{w^2H_w^2}{pH}[\boldsymbol{\sigma}_0(t)\boldsymbol{\varepsilon}\boldsymbol{\theta}^\top(t) + \boldsymbol{\pi}^{\top *}(t)\boldsymbol{\theta}(t)\boldsymbol{\varepsilon}\boldsymbol{\theta}^\top(t)].$$

Rearranging the expressions above yields the robust optimal investment strategy in closed form as follows:

$$\begin{aligned}\boldsymbol{\pi}^*(t) = &\left[\frac{wH_w^2}{pH}\boldsymbol{\theta}(t)\boldsymbol{\varepsilon}\boldsymbol{\theta}^\top(t) - wH_{ww}\boldsymbol{\theta}(t)\boldsymbol{\theta}^\top(t)\right]^{-1\top} \\ &\times\left[H_w\boldsymbol{\xi}(t) + wH_{ww}\boldsymbol{\theta}(t)\boldsymbol{\sigma}_0^\top(t) + \hat{w}H_{w\hat{w}}\boldsymbol{\theta}(t)[\boldsymbol{\sigma}_0(t) + \hat{\boldsymbol{\pi}}^\top(t)\boldsymbol{\theta}(t)]^\top - \frac{wH_w^2}{pH}\boldsymbol{\theta}(t)\boldsymbol{\varepsilon}\boldsymbol{\sigma}_0^\top(t)\right].\end{aligned} \tag{3.10}$$

Obviously, Eq. (3.6) is a nonlinear partial differential equation that is analytically intractable. A candidate solution is therefore conjectured in the following form:

$$H(t, w, \hat{w}) = \frac{1-p}{pq}\left(\frac{q}{1-p}w + \beta\right)^p \left(\frac{\hat{w}}{w}\right)^\kappa f(t). \tag{3.11}$$

under the boundary condition $f(T) = 1$. It follows from Eq. (3.11) that

$$\begin{cases} H_t = \dfrac{f_t(t)}{f(t)}H, \quad H_w = \left[\dfrac{pq}{qw + (1-p)\beta} - \dfrac{\kappa}{w}\right]H, \\ H_{ww} = \left\{\dfrac{pq^2(p-1)}{[qw + (1-p)\beta]^2} + \dfrac{\kappa(\kappa+1)}{w^2} - \dfrac{2pq\kappa}{w\,[qw + (1-p)\beta]}\right\}H, \\ H_{\hat{w}} = \dfrac{\kappa}{\hat{w}}H, \quad H_{\hat{w}\hat{w}} = \dfrac{\kappa(\kappa-1)}{\hat{w}^2}H, \quad H_{w\hat{w}} = \dfrac{\kappa}{\hat{w}}\left[\dfrac{pq}{qw + (1-p)\beta} - \dfrac{\kappa}{w}\right]H. \end{cases} \tag{3.12}$$

Let $\alpha \triangleq \frac{pq}{qw+(1-p)\beta} - \frac{\kappa}{w}$, $\delta \triangleq \frac{pq^2(p-1)}{[qw+(1-p)\beta]^2} + \frac{\kappa(\kappa+1)}{w^2} - \frac{2pq\kappa}{w[qw+(1-p)\beta]}$. Substituting the expression for the derivative of the value function in Eq. (3.12) into the robust optimal investment strategy in Eq. (3.10) yields

$$\begin{aligned}\boldsymbol{\pi}^*(t) = &\left[\frac{w\alpha^2}{p}\boldsymbol{\theta}(t)\boldsymbol{\varepsilon}\boldsymbol{\theta}^\top(t) - w\delta\boldsymbol{\theta}(t)\boldsymbol{\theta}^\top(t)\right]^{-1\top} \\ &\times\left[\alpha\boldsymbol{\xi}(t) + w\delta\boldsymbol{\theta}(t)\boldsymbol{\sigma}_0^\top(t) + \kappa\alpha\boldsymbol{\theta}(t)[\boldsymbol{\sigma}_0(t) + \hat{\boldsymbol{\pi}}^\top(t)\boldsymbol{\theta}(t)]^\top - \frac{w\alpha^2}{p}\boldsymbol{\theta}(t)\boldsymbol{\varepsilon}\boldsymbol{\sigma}_0^\top(t)\right].\end{aligned}$$

Consequently, the closed-form analytical expression for the robust optimal value function in Eq. (3.4) is derived. Substituting Eq. (3.4) and Eq. (3.11)–(3.12) into Eq. (3.8) leads to

$$\begin{aligned}\boldsymbol{\phi}^*(t) = \frac{w\alpha}{p}\boldsymbol{\varepsilon}\Big\{&\boldsymbol{\sigma}_0^\top(t) + \boldsymbol{\theta}^\top(t)\Big[\frac{w\alpha^2}{p}\boldsymbol{\theta}(t)\boldsymbol{\varepsilon}\boldsymbol{\theta}^\top(t) - w\delta\boldsymbol{\theta}(t)\boldsymbol{\theta}^\top(t)\Big]^{-1\top} \\ &\times\left[\alpha\boldsymbol{\xi}(t) + w\delta\boldsymbol{\theta}(t)\boldsymbol{\sigma}_0^\top(t) + \kappa\alpha\boldsymbol{\theta}(t)[\boldsymbol{\sigma}_0(t) + \hat{\boldsymbol{\pi}}^\top(t)\boldsymbol{\theta}(t)]^\top - \frac{w\alpha^2}{p}\boldsymbol{\theta}(t)\boldsymbol{\varepsilon}\boldsymbol{\sigma}_0^\top(t)\right]\Big\}.\end{aligned}$$

Thus, the explicit expression for the density generator $\boldsymbol{\phi}^*(t)$ associated with the worst-case probability measure in Eq. (3.5) is derived. Substituting Eq. (3.11)–(3.12) and Eq. (3.4)–(3.5) into the HJB equation

(3.9) yields

$$
\begin{aligned}
0 = {} & \frac{f_t(t)}{f(t)} + w\alpha b_0(t) + \kappa\big[b_0(t) + \hat{\boldsymbol{\pi}}^\top(t)\boldsymbol{\xi}(t)\big] + \frac{1}{2}w^2\delta\boldsymbol{\sigma}_0(t)\boldsymbol{\sigma}_0^\top(t) \\
& + \frac{1}{2}\kappa(\kappa-1)\Big[\boldsymbol{\sigma}_0(t)\boldsymbol{\sigma}_0^\top(t) + \boldsymbol{\sigma}_0(t)\boldsymbol{\theta}^\top(t)\hat{\boldsymbol{\pi}}(t) + \hat{\boldsymbol{\pi}}^\top(t)\boldsymbol{\theta}(t)\boldsymbol{\sigma}_0^\top(t) + \hat{\boldsymbol{\pi}}^\top(t)\boldsymbol{\theta}(t)\boldsymbol{\theta}'(t)\hat{\boldsymbol{\pi}}(t)\Big] \\
& + w\kappa\alpha\boldsymbol{\sigma}_0(t)[\boldsymbol{\sigma}_0(t) + \hat{\boldsymbol{\pi}}^\top(t)\boldsymbol{\theta}(t)]^\top - \frac{w^2\alpha^2}{2p}\boldsymbol{\sigma}_0(t)\boldsymbol{\varepsilon}\boldsymbol{\sigma}_0^\top(t) \\
& + \frac{1}{2}\Big[\alpha\boldsymbol{\xi}^\top(t) + w\delta\boldsymbol{\sigma}_0(t)\boldsymbol{\theta}^\top(t) + \kappa\alpha[\boldsymbol{\sigma}_0(t) + \hat{\boldsymbol{\pi}}^\top(t)\boldsymbol{\theta}(t)]\boldsymbol{\theta}^\top(t) - \frac{w\alpha^2}{p}\boldsymbol{\sigma}_0(t)\boldsymbol{\varepsilon}\boldsymbol{\theta}^\top(t)\Big] \\
& \times\Big[\frac{\alpha^2}{p}\boldsymbol{\theta}(t)\boldsymbol{\varepsilon}\boldsymbol{\theta}^\top(t) - \delta\boldsymbol{\theta}(t)\boldsymbol{\theta}^\top(t)\Big]^{-1} \\
& \times\Big[\alpha\boldsymbol{\xi}(t) + w\delta\boldsymbol{\theta}(t)\boldsymbol{\sigma}_0'(t) + \kappa\alpha\boldsymbol{\theta}(t)[\boldsymbol{\sigma}_0(t) + \hat{\boldsymbol{\pi}}^\top(t)\boldsymbol{\theta}(t)]^\top - \frac{w\alpha^2}{p}\boldsymbol{\theta}(t)\boldsymbol{\varepsilon}\boldsymbol{\sigma}_0^\top(t)\Big].
\end{aligned} \tag{3.13}
$$

After simplification, the purely time-dependent function $f(t)$ is shown to satisfy the following ordinary differential equation (ODE):

$$
\frac{f_t(t)}{f(t)} + g(t) = 0, \tag{3.14}
$$

with the terminal condition $f(T) = 1$ and $g(t)$ given by the following expression

$$
\begin{aligned}
g(t) = {} & w\alpha b_0(t) + \kappa\big[b_0(t) + \hat{\boldsymbol{\pi}}^\top(t)\boldsymbol{\xi}(t)\big] + \frac{1}{2}w^2\delta\boldsymbol{\sigma}_0(t)\boldsymbol{\sigma}_0^\top(t) \\
& + \frac{1}{2}\kappa(\kappa-1)\Big[\boldsymbol{\sigma}_0(t)\boldsymbol{\sigma}_0^\top(t) + \boldsymbol{\sigma}_0(t)\boldsymbol{\theta}^\top(t)\hat{\boldsymbol{\pi}}(t) + \hat{\boldsymbol{\pi}}^\top(t)\boldsymbol{\theta}(t)\boldsymbol{\sigma}_0^\top(t) + \hat{\boldsymbol{\pi}}^\top(t)\boldsymbol{\theta}(t)\boldsymbol{\theta}'(t)\hat{\boldsymbol{\pi}}(t)\Big] \\
& + w\kappa\alpha\boldsymbol{\sigma}_0(t)[\boldsymbol{\sigma}_0(t) + \hat{\boldsymbol{\pi}}^\top(t)\boldsymbol{\theta}(t)]^\top - \frac{w^2\alpha^2}{2p}\boldsymbol{\sigma}_0(t)\boldsymbol{\varepsilon}\boldsymbol{\sigma}_0^\top(t) \\
& + \frac{1}{2}\Big[\alpha\boldsymbol{\xi}^\top(t) + w\delta\boldsymbol{\sigma}_0(t)\boldsymbol{\theta}^\top(t) + \kappa\alpha[\boldsymbol{\sigma}_0(t) + \hat{\boldsymbol{\pi}}^\top(t)\boldsymbol{\theta}(t)]\boldsymbol{\theta}^\top(t) - \frac{w\alpha^2}{p}\boldsymbol{\sigma}_0(t)\boldsymbol{\varepsilon}\boldsymbol{\theta}^\top(t)\Big] \\
& \times\Big[\frac{\alpha^2}{p}\boldsymbol{\theta}(t)\boldsymbol{\varepsilon}\boldsymbol{\theta}^\top(t) - \delta\boldsymbol{\theta}(t)\boldsymbol{\theta}^\top(t)\Big]^{-1} \\
& \times\Big[\alpha\boldsymbol{\xi}(t) + w\delta\boldsymbol{\theta}(t)\boldsymbol{\sigma}_0'(t) + \kappa\alpha\boldsymbol{\theta}(t)[\boldsymbol{\sigma}_0(t) + \hat{\boldsymbol{\pi}}^\top(t)\boldsymbol{\theta}(t)]^\top - \frac{w\alpha^2}{p}\boldsymbol{\theta}(t)\boldsymbol{\varepsilon}\boldsymbol{\sigma}_0^\top(t)\Big].
\end{aligned}
$$

The ODE has a closed-form solution which is given by

$$
f(t) = \exp\left[\int_t^T g(s)\,\mathrm{d}\,s\right], \tag{3.15}
$$

with $g(t)$ as denoted earlier. Substituting Eq. (3.15) into Eq. (3.11) yields the closed-form analytical expression of the robust optimal value function, thereby completing the proof.

We next establish a verification theorem to show that the solution of the HJB equation (3.6) indeed represents the value function of the robust optimization problem (2.12) under HARA preferences.

Theorem 2 (Verification theorem)**.** *Let the differential operator $\mathcal{L}^{\pi,\phi}$ be given by*

$$\begin{aligned}\mathcal{L}^{\pi,\phi}\widetilde{H}(t,w,\hat{w}) \triangleq & \widetilde{H}_t + w\widetilde{H}_w\left[b_0(t) + \boldsymbol{\pi}^\top(t)\boldsymbol{\xi}(t) - \left[\boldsymbol{\sigma}_0(t) + \boldsymbol{\pi}(t)^\top\boldsymbol{\theta}(t)\right]\boldsymbol{\phi}(t)\right] \\ & + \hat{w}\widetilde{H}_{\hat{w}}\left[b_0(t) + \hat{\boldsymbol{\pi}}^\top(t)\boldsymbol{\xi}(t)\right] \\ & + \frac{1}{2}w^2\widetilde{H}_{ww}\left[\boldsymbol{\sigma}_0(t) + \boldsymbol{\pi}^\top(t)\boldsymbol{\theta}(t)\right]\left[\boldsymbol{\sigma}_0(t) + \boldsymbol{\pi}^\top(t)\boldsymbol{\theta}(t)\right]^\top \\ & + \frac{1}{2}\hat{w}^2\widetilde{H}_{\hat{w}\hat{w}}\left[\boldsymbol{\sigma}_0(t) + \hat{\boldsymbol{\pi}}^\top(t)\boldsymbol{\theta}(t)\right]\left[\boldsymbol{\sigma}_0(t) + \hat{\boldsymbol{\pi}}^\top(t)\boldsymbol{\theta}(t)\right]^\top \\ & + w\hat{w}\widetilde{H}_{w\hat{w}}\left[\boldsymbol{\sigma}_0(t) + \boldsymbol{\pi}^\top(t)\boldsymbol{\theta}(t)\right]\left[\boldsymbol{\sigma}_0(t) + \hat{\boldsymbol{\pi}}^\top(t)\boldsymbol{\theta}(t)\right]^\top.\end{aligned}$$

Suppose that a real-valued function $\widetilde{H}(t,w,\hat{w}) \in C^{1,2,2}([0,T]\times\mathbb{R}\times\mathbb{R})$ and an admissible Markovian control pair $(\boldsymbol{\pi}^*,\boldsymbol{\phi}^*) \in \Pi\times\Phi$ exist such that the following conditions hold:

(i) For any $\boldsymbol{\phi} \in \Phi$,

$$\mathcal{L}^{\pi^*,\phi}\widetilde{H}(t,w,\hat{w}) + \mathbf{1}_m^\top\boldsymbol{\psi}\left(t,\boldsymbol{\phi},w,\hat{w}\right) \geq 0.$$

(ii) For any $\boldsymbol{\pi} \in \Pi$,

$$\mathcal{L}^{\pi,\phi^*}\widetilde{H}(t,w,\hat{w}) + \mathbf{1}_m^\top\boldsymbol{\psi}\left(t,\boldsymbol{\phi}^*,w,\hat{w}\right) \leq 0.$$

(iii) At the candidate controls $(\boldsymbol{\pi}^*,\boldsymbol{\phi}^*)$,

$$\mathcal{L}^{\pi^*,\phi^*}\widetilde{H}(t,w,\hat{w}) + \mathbf{1}_m^\top\boldsymbol{\psi}\left(t,\boldsymbol{\phi}^*,w,\hat{w}\right) = 0.$$

(iv) For all $(\boldsymbol{\pi},\boldsymbol{\phi}) \in \Pi\times\Phi$, the terminal condition satisfies

$$\lim_{t\to T}\widetilde{H}(t,W(t),\hat{W}(t)) = \widetilde{U}\left(W(T),\hat{W}(T)\right).$$

(v) The families $\left\{\widetilde{H}\left(\tau,W(\tau),\hat{W}(\tau)\right)\right\}_{\tau\in\mathcal{T}}$ and $\left\{\psi_i\left(\tau,\phi_i(\tau),W(\tau),\hat{W}(\tau)\right)\right\}_{\tau\in\mathcal{T}}$, $i = 1,2,\ldots,m$, are uniformly integrable, where $\mathcal{T}$ denotes the set of all stopping times taking values in $[0,T]$.

Then $\boldsymbol{\pi}^*$ is an optimal strategy, $\boldsymbol{\phi}^*$is the density generator associated with the worst-case probability measure, and $\widetilde{H}(t,w,\hat{w}) = H(t,w,\hat{w})$ is the corresponding value function.

The proof follows the verification approach in [20, 21] and is given in Appendix B.

To further clarify the theoretical implications of the model, the following corollaries provide analytical solutions for several special cases. Specifically, analytical expressions for the optimal strategy and value function are first provided for an ambiguity-neutral investor with regret aversion. Next, the robust optimal investment strategy and the associated worst-case density generator are characterized for a regret-neutral investor with ambiguity aversion. Finally, analytical expressions for the optimal strategy and value function are presented for an investor who is neutral to both ambiguity and regret. For brevity, a detailed analysis of these cases is omitted, as it can be inferred directly from the general solution in Theorem 1.

Corollary 1. *Consider a regret-averse but ambiguity-neutral investor. For any $t \in [0,T]$, the optimal value function is given by*

$$H(t,w,\hat{w}) = \frac{1-p}{pq}\left(\frac{q}{1-p}w + \beta\right)^p \exp\left[\int_t^T g(s)\,\mathrm{d}s\right], \tag{3.16}$$

where

$$
\begin{aligned}
g(t) = {} & w\alpha b_0(t) + \kappa[b_0(t) + \hat{\boldsymbol{\pi}}^\top(t)\boldsymbol{\xi}(t)] + \frac{1}{2}w^2\delta\,\boldsymbol{\sigma}_0(t)\boldsymbol{\sigma}_0^\top(t) \\
& + \frac{1}{2}\kappa(\kappa-1)\Big[\boldsymbol{\sigma}_0(t)\boldsymbol{\sigma}_0^\top(t) + \boldsymbol{\sigma}_0(t)\boldsymbol{\theta}^\top(t)\hat{\boldsymbol{\pi}}(t) + \hat{\boldsymbol{\pi}}^\top(t)\boldsymbol{\theta}(t)\boldsymbol{\sigma}_0^\top(t) \\
& + \hat{\boldsymbol{\pi}}^\top(t)\boldsymbol{\theta}(t)\boldsymbol{\theta}^\top(t)\hat{\boldsymbol{\pi}}(t)\Big] + w\kappa\alpha\,\boldsymbol{\sigma}_0(t)[\boldsymbol{\sigma}_0(t) + \hat{\boldsymbol{\pi}}^\top(t)\boldsymbol{\theta}(t)]^\top \\
& - \frac{1}{2}\Big[\alpha\boldsymbol{\xi}^\top(t) + w\delta\boldsymbol{\sigma}_0(t)\boldsymbol{\theta}^\top(t) + \kappa\alpha[\boldsymbol{\sigma}_0(t) + \hat{\boldsymbol{\pi}}^\top(t)\boldsymbol{\theta}(t)]\boldsymbol{\theta}^\top(t)\Big] \times [\delta\boldsymbol{\theta}(t)\boldsymbol{\theta}^\top(t)]^{-1} \\
& \times \Big[\alpha\boldsymbol{\xi}(t) + w\delta\boldsymbol{\theta}(t)\boldsymbol{\sigma}_0^\top(t) + \kappa\alpha\boldsymbol{\theta}(t)[\boldsymbol{\sigma}_0(t) + \hat{\boldsymbol{\pi}}^\top(t)\boldsymbol{\theta}(t)]^\top\Big].
\end{aligned}
\tag{3.17}
$$

Correspondingly, the optimal strategy can be expressed as

$$
\boldsymbol{\pi}^*(t) = -[\boldsymbol{\theta}(t)\boldsymbol{\theta}^\top(t)]^{-1}\left[\frac{\alpha\boldsymbol{\xi}(t) + \alpha\kappa\boldsymbol{\theta}(t)[\boldsymbol{\sigma}_0^\top(t) + \boldsymbol{\theta}^\top(t)\hat{\boldsymbol{\pi}}(t)]}{w\delta} + \boldsymbol{\theta}(t)\boldsymbol{\sigma}_0^\top(t)\right]. \tag{3.18}
$$

Ambiguity aversion is absent when $\boldsymbol{\varepsilon} = \mathbf{0}_{m\times m}$, implying that the investor is ambiguity-neutral. The optimization problem reduces to maximizing the expected utility of terminal wealth under risk aversion and regret aversion. Corollary 1 is obtained from Theorem 1 by specializing the parameters to $\boldsymbol{\varepsilon} = \mathbf{0}_{m\times m}$.

Corollary 2. *Consider an ambiguity-averse but regret-neutral investor. Then, for any $t \in [0, T]$, the optimal value function is given by*

$$
H(t, w, \hat{w}) = \frac{1-p}{pq}\left(\frac{q}{1-p}w + \beta\right)^p \exp\left[\int_t^T g(s)\,\mathrm{d}s\right], \tag{3.19}
$$

where

$$
\begin{aligned}
g(t) \triangleq {} & w\alpha b_0(t) + \frac{1}{2}w^2\delta\,\boldsymbol{\sigma}_0(t)\boldsymbol{\sigma}_0^\top(t) - \frac{w^2\alpha^2}{2p}\boldsymbol{\sigma}_0(t)\boldsymbol{\varepsilon}\boldsymbol{\sigma}_0^\top(t) \\
& + \frac{1}{2}\Big[\alpha\boldsymbol{\xi}^\top(t) + w\delta\boldsymbol{\sigma}_0(t)\boldsymbol{\theta}^\top(t) - \frac{w\alpha^2}{p}\boldsymbol{\sigma}_0(t)\boldsymbol{\varepsilon}\boldsymbol{\theta}^\top(t)\Big] \\
& \times \Big[\frac{\alpha^2}{p}\boldsymbol{\theta}(t)\boldsymbol{\varepsilon}\boldsymbol{\theta}^\top(t) - \delta\boldsymbol{\theta}(t)\boldsymbol{\theta}^\top(t)\Big]^{-1} \\
& \times \Big[\alpha\boldsymbol{\xi}(t) + w\delta\boldsymbol{\theta}(t)\boldsymbol{\sigma}_0^\top(t) - \frac{w\alpha^2}{p}\boldsymbol{\theta}(t)\boldsymbol{\varepsilon}\boldsymbol{\sigma}_0^\top(t)\Big],
\end{aligned}
\tag{3.20}
$$

with

$$
\begin{cases}
\alpha \triangleq \dfrac{pq}{qw + (1-p)\beta}, \\
\delta \triangleq \dfrac{pq^2(p-1)}{[qw + (1-p)\beta]^2}.
\end{cases}
\tag{3.21}
$$

Moreover, the corresponding robust optimal investment strategy $\boldsymbol{\pi}^*(t)$ and the density generator process $\boldsymbol{\phi}^*(t)$ under the worst-case scenario are, respectively, given by

$$
\boldsymbol{\pi}^*(t) = \Big[\frac{w\alpha^2}{p}\boldsymbol{\theta}(t)\boldsymbol{\varepsilon}\boldsymbol{\theta}^\top(t) - w\delta\boldsymbol{\theta}(t)\boldsymbol{\theta}^\top(t)\Big]^{-1^\top} \times \Big[\alpha\boldsymbol{\xi}(t) + w\delta\boldsymbol{\theta}(t)\boldsymbol{\sigma}_0^\top(t) - \frac{w\alpha^2}{p}\boldsymbol{\theta}(t)\boldsymbol{\varepsilon}\boldsymbol{\sigma}_0^\top(t)\Big], \tag{3.22}
$$

and

$$\phi^*(t) = \frac{w\alpha}{p}\,\varepsilon\Big\{\sigma_0^\top(t) + \theta^\top(t)\Big[\frac{w\alpha^2}{p}\theta(t)\varepsilon\theta^\top(t) - w\delta\theta(t)\theta^\top(t)\Big]^{-1^\top} \times \left[\alpha\xi(t) + w\delta\theta(t)\sigma_0^\top(t) - \frac{w\alpha^2}{p}\theta(t)\varepsilon\sigma_0^\top(t)\right]\Big\}. \tag{3.23}$$

where α and δ are as defined above.

When $\kappa = 0$, regret aversion is absent; the investor is thus regret-neutral and evaluates outcomes solely on the basis of realized wealth, without reference to forgone alternatives. The problem therefore reduces to determining the robust optimal strategy under model ambiguity. Corollary 2 follows from Theorem 1 by setting $\kappa = 0$.

Corollary 3. *Consider an investor who is both ambiguity-neutral and regret-neutral. For any $t \in [0, T]$, the optimal value function is given by*

$$H(t, w, \hat{w}) = \frac{1-p}{pq}\left(\frac{q}{1-p}w + \beta\right)^p \exp\left[\int_t^T g(s)\,\mathrm{d}s\right], \tag{3.24}$$

where

$$g(t) = w\alpha b_0(t) + \frac{1}{2}w^2\delta\sigma_0(t)\sigma_0^\top(t) - \frac{1}{2}[\alpha\xi^\top(t) + w\delta\sigma_0(t)\theta^\top(t)] \times [\delta\theta(t)\theta^\top(t)]^{-1} \times [\alpha\xi(t) + w\delta\theta(t)\sigma_0^\top(t)], \tag{3.25}$$

with α and δ as defined in Eq. (3.21). *The corresponding optimal investment strategy is given by*

$$\pi^*(t) = -\left[\frac{wpq^2(p-1)}{(qw+(1-p)\beta)^2}\theta(t)\theta^\top(t)\right]^{-1} \times \left[\frac{pq}{qw+(1-p)\beta}\xi(t) + \frac{wpq^2(p-1)}{[qw+(1-p)\beta]^2}\theta(t)\sigma_0^\top(t)\right]. \tag{3.26}$$

Regret aversion vanishes when $\kappa = 0$, and ambiguity aversion vanishes when $\varepsilon = \mathbf{0}_{m\times m}$. The investor is both regret-neutral and ambiguity-neutral such that $\kappa = 0$ and $\varepsilon = \mathbf{0}_{m\times m}$. In this case, the robust portfolio selection problem with regret aversion under HARA utility reduces to the classical portfolio selection problem under HARA utility. Corollary 3 is obtained from Theorem 1 by setting $\kappa = 0$ and $\varepsilon = \mathbf{0}_{m\times m}$.

Remark 7. *As a special case of Corollary 3, suppose that an ambiguity-neutral and regret-neutral investor with CRRA peference. For any $t \in [0, T]$, the optimal portfolio strategy is*

$$\pi^*(t) = [\theta(t)\theta(t)^\top]^{-1}\left[\frac{\xi(t)}{1-p} - \theta(t)\sigma_0(t)^\top\right]. \tag{3.27}$$

The corresponding value function is

$$H(t, w, \widehat{w}) = \frac{w^p}{p}\exp\left\{\int_t^T g(s)\,ds\right\}, \tag{3.28}$$

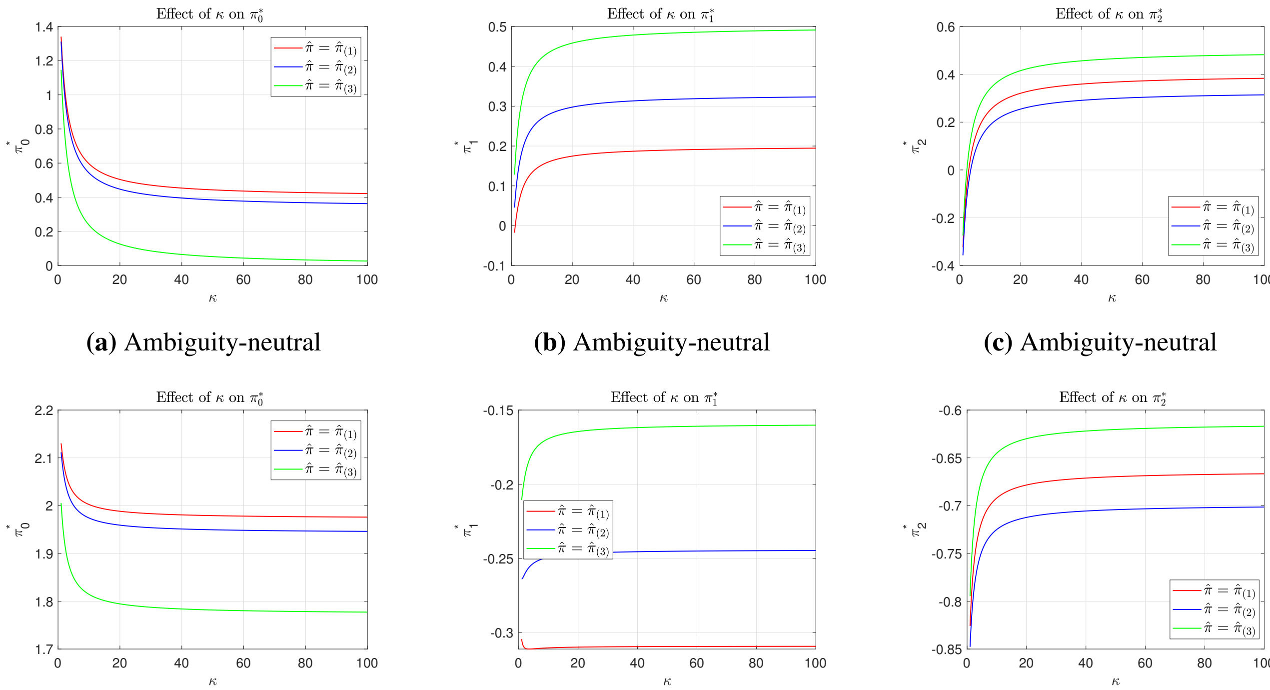
where

$$g(t) = b_0(t) - \frac{1}{2}(1-p)\sigma_0(t)\sigma_0(t)^\top + \frac{1}{2}\Big[\xi(t)^\top - (1-p)\sigma_0(t)\theta(t)^\top\Big]\Big[(1-p)\theta(t)\theta(t)^\top\Big]^{-1}\Big[\xi(t) - (1-p)\theta(t)\sigma_0(t)^\top\Big]. \tag{3.29}$$

4. Sensitivity analysis

This section presents a sensitivity analysis for the robust optimal investment strategy derived in Theorem 1 to illustrate representative comparative-static patterns under parameter variations. The effects of market parameters and the investor's preference parameters on the robust optimal portfolio allocation and the corresponding expected return and variance are examined. In particular, the interaction effects among regret aversion, ambiguity aversion, and risk aversion are investigated to further illustrate the economic implications of the proposed framework.

For expositional clarity, a market with three risky assets, indexed by (0,1,2), and two sources of risk is considered. The proportions of wealth allocated to risky assets 1 and 2 are denoted $\pi_1^*(t)$ and $\pi_2^*(t)$, respectively. Their expected returns are $b_1(t)$ and $b_2(t)$, and their volatility vectors are $\boldsymbol{\sigma}_1(t) = (\sigma_{11}(t), \sigma_{12}(t))$ and $\boldsymbol{\sigma}_2(t) = (\sigma_{21}(t), \sigma_{22}(t))$. The remaining proportion, $(1 - \pi_1^*(t) - \pi_2^*(t))$, is allocated to risky asset 0, whose expected return is $b_0(t)$ and whose volatility vector is $\boldsymbol{\sigma}_0(t) = (\sigma_{01}(t), \sigma_{02}(t))$. To ensure robustness, the analysis is repeated under three benchmark strategies. The benchmark portfolios allocate wealth among risky assets 0, 1, and 2 with proportions $(0.4, 0.2, 0.4)$, $(0.33, 0.33, 0.33)$, and $(0, 0.5, 0.5)$, respectively. Unless otherwise stated, the baseline parameters are set as $w = 1$, $\hat{w} = 1$, $b_0 = 0.05$, $b_1 = 0.07$, $b_2 = 0.09$, $\sigma_{01} = 0.35$, $\sigma_{02} = 0.25$, $\sigma_{11} = 0.25$, $\sigma_{12} = 0.45$, $\sigma_{21} = 0.60$, $\sigma_{22} = 0.30$, $p = -0.5$, $q = 0.3$, $\beta = 0.8$, $\kappa = 2$, $\varepsilon_1 = 0.5$, and $\varepsilon_2 = 0.5$.



(a) Ambiguity-neutral **(b)** Ambiguity-neutral **(c)** Ambiguity-neutral

(d) Ambiguity-averse **(e)** Ambiguity-averse **(f)** Ambiguity-averse

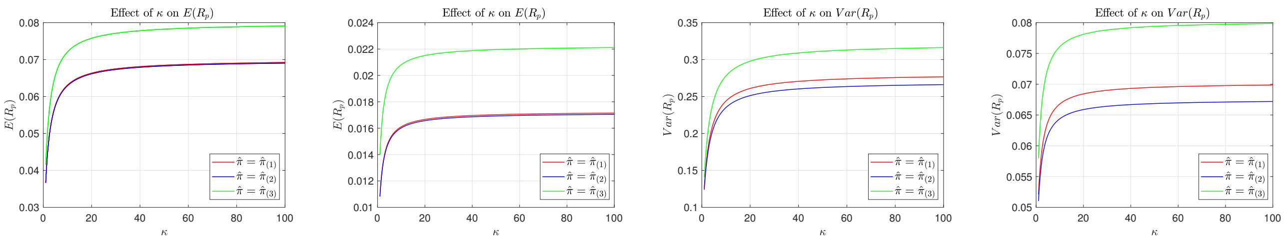
Figure 1. The effect of the regret aversion coefficient κ on the optimal investment strategy π_i^*.

The sensitivity results are presented in Figures 1–13, and the corresponding summary statistics are reported in Appendix A. The parameter intervals considered in the sensitivity analysis are chosen to provide economically meaningful comparative-static illustrations. They are not intended to cover the full feasible ranges of the parameters but rather to highlight their qualitative effects on the model outcomes.

Figure 1 illustrates the impact of the regret aversion coefficient κ on the optimal investment strategy $(\pi_0^*, \pi_1^*, \pi_2^*)$ under different benchmark portfolios. In this figure, (a)–(c) correspond to ambiguity-neutral investors, whereas (d)–(f) consider ambiguity-averse investors. Several observations can be obtained. (1) For a fixed regret aversion level, the benchmark portfolio exerts a positive influence on the optimal allocation. Specifically, if the benchmark assigns a larger proportion of wealth to a risky asset, the investor is more likely to allocate additional wealth to that asset in order to mitigate potential regret arising from underinvestment. (2) In the ambiguity-neutral case, the optimal strategy converges to the benchmark strategy as κ increases, i.e., $\lim_{\kappa\to\infty} \boldsymbol{\pi}^*(t) = \hat{\boldsymbol{\pi}}(t)$ where $\boldsymbol{\varepsilon} = \mathbf{0}_{m\times m}$. Therefore, for a fixed benchmark portfolio, a higher degree of regret aversion induces a stronger tendency to align the optimal strategy with the benchmark allocation, as shown in (a)–(c). (3) As κ increases, the robust optimal investment strategy $\boldsymbol{\pi}^*$ converges to a constant policy, since

$$\lim_{\kappa\to\infty} \boldsymbol{\pi}^*(t) = \left[\boldsymbol{\theta}(t)\boldsymbol{\varepsilon}\boldsymbol{\theta}^\top(t) + p\boldsymbol{\theta}(t)\boldsymbol{\theta}^\top(t)\right]^{-1} \times \left[p\boldsymbol{\theta}(t)\boldsymbol{\theta}^\top(t)\hat{\pi}(t) - \boldsymbol{\theta}(t)\boldsymbol{\varepsilon}\boldsymbol{\sigma}_0^\top(t)\right].$$

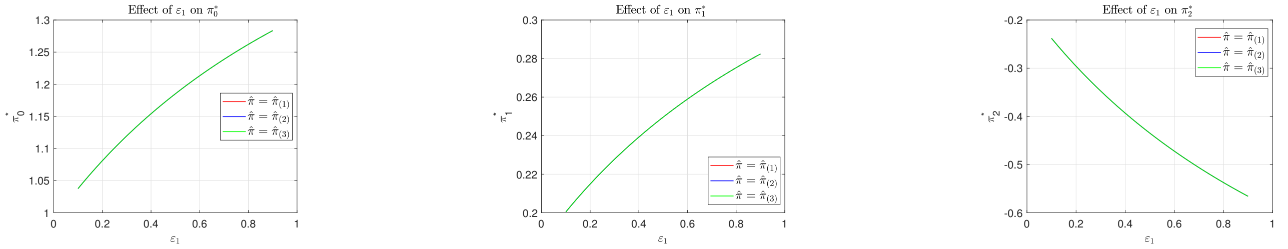
This convergence behavior is illustrated in (d)–(f), indicating that stronger regret aversion reduces the incentive to deviate from the benchmark strategy. (4) A comparison between ambiguity-neutral and ambiguity-averse investors (e.g., (a) and (d)) shows that regret aversion and ambiguity aversion jointly affect portfolio decisions. Specifically, ambiguity-averse investors tend to reduce their exposure to assets with higher uncertainty, leading to more conservative investment strategies.



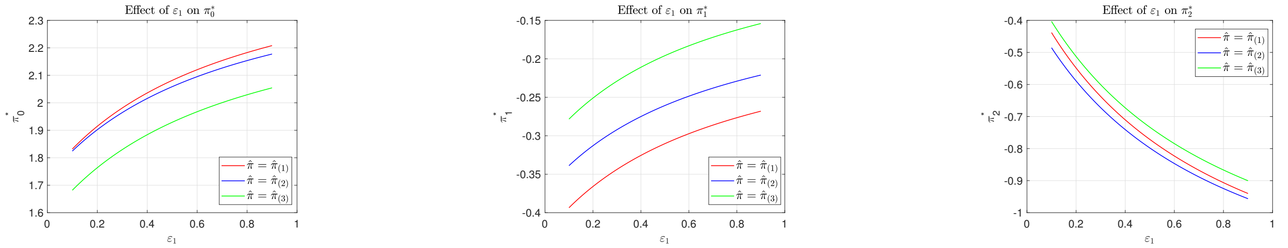
(a) Ambiguity-neutral **(b)** Ambiguity-averse **(c)** Ambiguity-neutral **(d)** Ambiguity-averse

Figure 2. The effect of the regret aversion coefficient κ on the expected return $\mathrm{E}(R_p)$ and the variance $\mathrm{Var}(R_p)$ of the optimal portfolio.

Figure 2 illustrates the effects of the regret aversion coefficient κ on the expected return $E(R_p)$ and variance $\mathrm{Var}(R_p)$ of the optimal portfolio under different benchmark strategies. Several observations can be obtained. (1) The expected return and variance of the optimal portfolio vary across different benchmark strategies. This is because the benchmark allocation directly affects the reference-dependent optimal strategy, thereby altering the resulting portfolio return–risk characteristics. (2) A comparison between ambiguity-neutral and ambiguity-averse cases (e.g., (a) and (c)) indicates that regret aversion and ambiguity aversion jointly influence the portfolio's performance. In particular, ambiguity aversion reduces the investor's exposure to assets with higher uncertainty, leading to a lower expected return accompanied by lower portfolio variance.



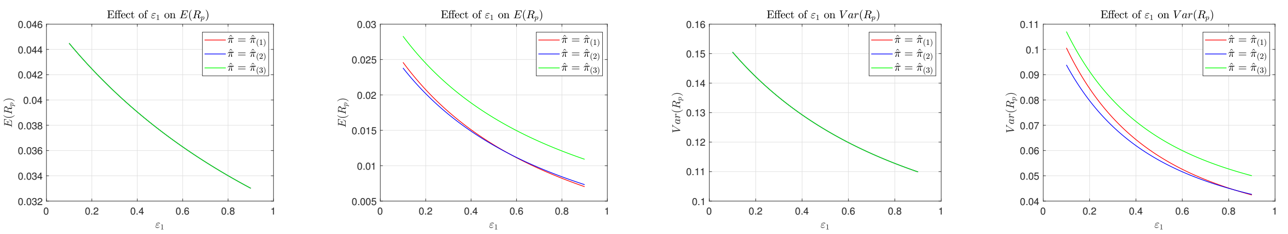
(a) Regret-neutral **(b)** Regret-neutral **(c)** Regret-neutral



(d) Regret-averse **(e)** Regret-averse **(f)** Regret-averse

Figure 3. The effect of the ambiguity aversion coefficient ε_1 on the optimal investment strategy π_i^*.

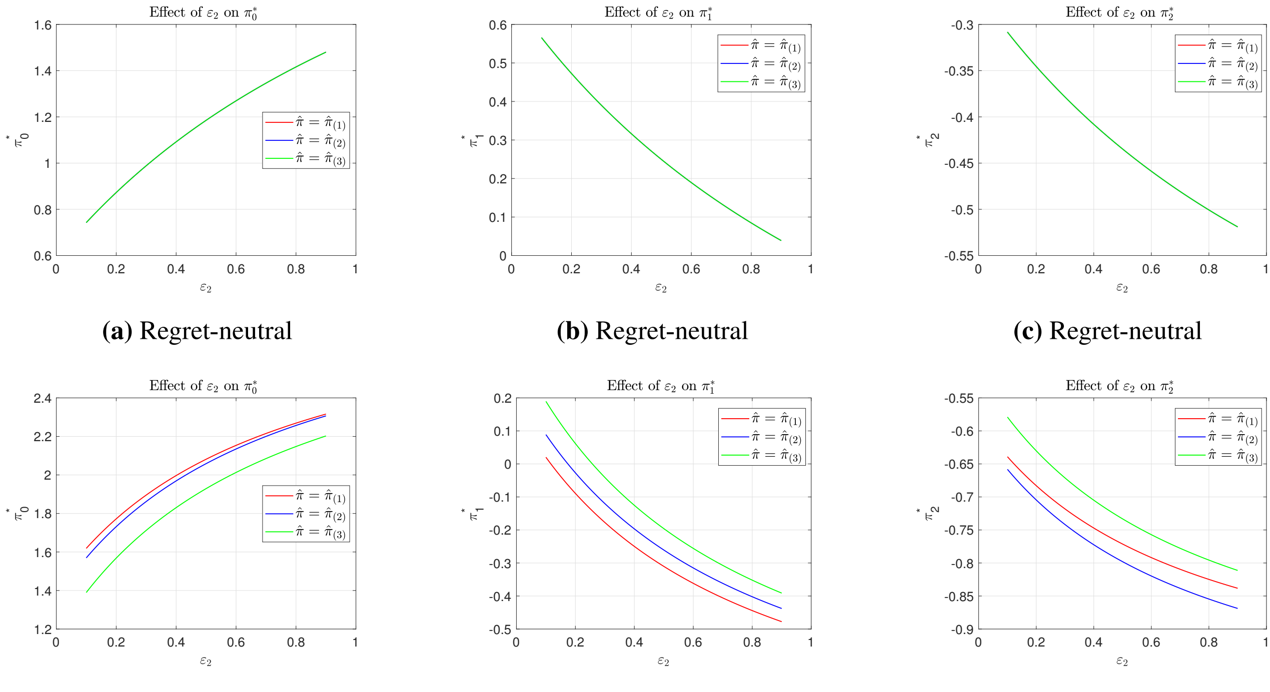
Figure 3 illustrates the effect of the ambiguity aversion parameter ε_1 on the robust optimal investment strategy under different benchmark strategies. Several observations are obtained. (1) The allocations to risky assets 0 and 1 increase with ε_1, whereas the allocation to risky asset 2 decreases. This result can be explained by the heterogeneous exposure of assets to the first risk source. Since risky assets 0 and 1 exhibit lower exposure to the first risk source than risky asset 2, an increase in ambiguity aversion toward this source induces investors to reduce the allocation to risky asset 2 and reallocate wealth toward risky assets 0 and 1. This result indicates that ambiguity aversion affects portfolio allocation according to the exposure of each asset to the corresponding uncertain risk source. (2) When regret aversion is incorporated, the benchmark strategy affects the optimal allocation. For example, in (d), the allocation to risky asset 0 is highest when the benchmark is $\hat{\pi}_{(1)}$, followed by $\hat{\pi}_{(3)}$ and $\hat{\pi}_{(3)}$, which is consistent with the ranking of the corresponding benchmark allocations. Similar patterns are observed in (e) and (f). (3) Parts (a)–(c) show that, in the absence of regret aversion, the optimal investment strategies are identical across different benchmark strategies. Therefore, benchmark dependence arises from regret aversion.



(a) Regret-neutral **(b)** Regret-averse **(c)** Regret-neutral **(d)** Regret-averse

Figure 4. The effect of the ambiguity aversion coefficient ε_1 on the expected return $\mathrm{E}(R_p)$ and the variance $\mathrm{Var}(R_p)$ of the optimal portfolio.

Figure 4 illustrates the effect of the ambiguity aversion parameter ε_1 on the expected return $E(R_p)$ and the variance $\mathrm{Var}(R_p)$ of the robust optimal portfolio under different benchmark strategies. Several observations can be made as follows. (1) Both the expected return and variance of the robust optimal portfolio decrease with ε_1. As discussed previously, an increase in ambiguity aversion toward the first risk source induces a reallocation away from risky asset 2 toward risky assets 0 and 1. Since risky asset 2 provides a higher expected return with greater exposure to the first risk source, while risky assets 0 and 1 exhibit relatively lower return and risk characteristics, stronger ambiguity aversion leads to a reduction in both portfolio return and risk. (2) In the presence of regret aversion, (b) and (d) show that the benchmark strategy $\hat{\pi}_{(3)}$ generates the highest expected return and variance. This is because $\hat{\pi}_{(3)}$ assigns the largest allocation to risky asset 2. Under regret aversion, the robust optimal strategy becomes reference-dependent and inherits part of the benchmark allocation pattern, resulting in higher exposure to risky asset 2. Although different benchmark strategies lead to distinct return–risk levels, the overall decreasing trend with respect to ε_1 remains unchanged, indicating the interaction between regret aversion and ambiguity aversion. (3) Parts (a) and (c) demonstrate that, in the absence of regret aversion, the expected return and variance are identical across different benchmark strategies. This result follows from the fact that benchmark allocations do not affect the optimal strategy when regret aversion is absent.



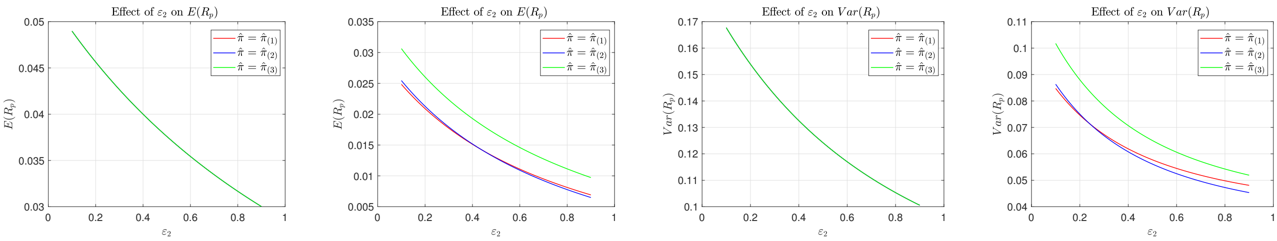
(a) Regret-neutral **(b)** Regret-neutral **(c)** Regret-neutral

(d) Regret-averse **(e)** Regret-averse **(f)** Regret-averse

Figure 5. The effect of the ambiguity aversion coefficient ε_2 on the optimal investment strategy π_i^*.

Figure 5 illustrates the effect of the ambiguity aversion parameter ε_2 on the robust optimal investment strategy under different benchmark strategies. Several observations can be made as follows. (1) The allocation to risky asset 0 increases with ε_2, whereas the allocations to risky assets 1 and 2 decrease. This result is explained by the heterogeneous exposure of assets to the second risk source. Since risky asset

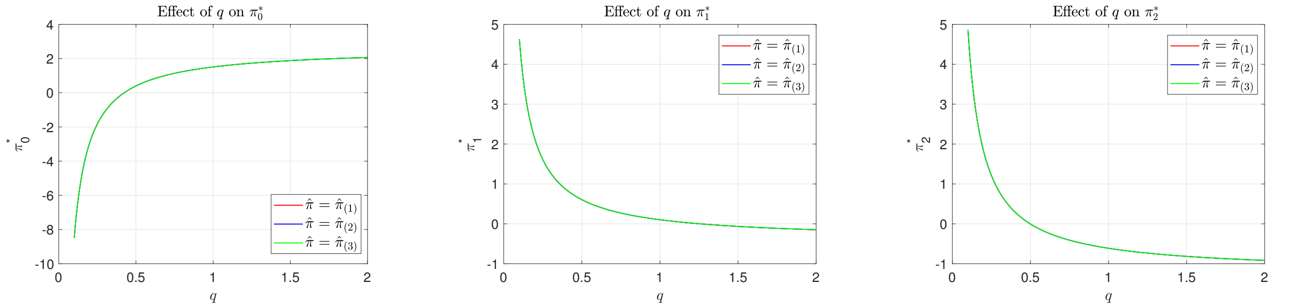
0 has lower exposure to the second risk source than risky assets 1 and 2, stronger ambiguity aversion toward this source leads to a reduction in the allocations to risky assets 1 and 2 and an increase in the allocation to risky asset 0. This further demonstrates that ambiguity aversion generates asset-specific portfolio adjustments through heterogeneous exposure to different risk sources. (2) Parts (a)–(c) show that, in the absence of regret aversion, the robust optimal investment strategies are identical across different benchmark strategies. Similar results are observed in the subsequent sensitivity analyses. (3) When regret aversion is incorporated, the benchmark strategy affects the robust optimal allocation. For example, in (d), the allocation to risky asset 0 is highest when the benchmark is $\hat{\pi}_{(1)}$, followed by $\hat{\pi}_{(2)}$ and $\hat{\pi}_{(3)}$, which is consistent with the corresponding benchmark allocations. Similar results are observed in (e)–(f). This indicates that regret aversion makes the robust optimal investment strategy sensitive to the selected benchmark strategy.



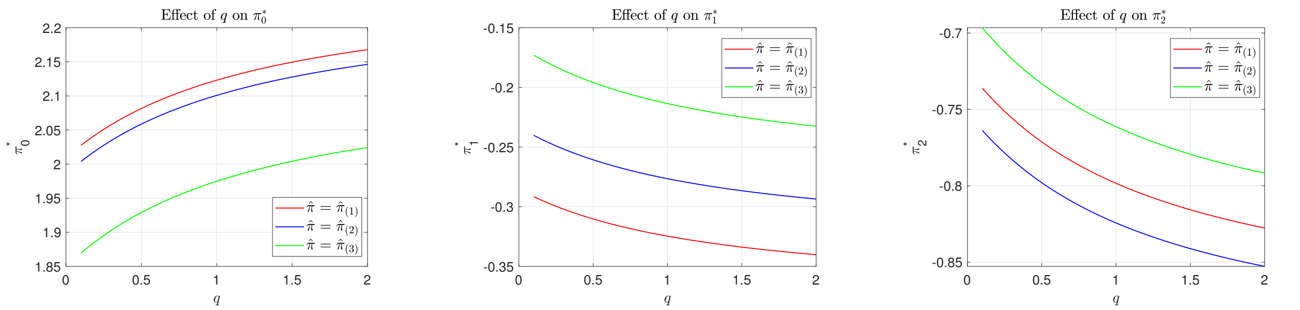
(a) Regret-neutral **(b)** Regret-averse **(c)** Regret-neutral **(d)** Regret-averse

Figure 6. The effect of the ambiguity aversion coefficient ε_2 on the expected return $\mathrm{E}(R_p)$ and the variance $\mathrm{Var}(R_p)$ of the optimal portfolio.

Figure 6 illustrates the effect of the ambiguity aversion parameter ε_2 on the expected return $E(R_p)$ and the variance $\mathrm{Var}(R_p)$ of the robust optimal portfolio under different benchmark strategies. Several observations can be made as follows. (1) Both $E(R_P)$ and $\mathrm{Var}(R_P)$ decrease monotonically with ε_2. This is driven by a portfolio reallocation effect: Higher ambiguity aversion toward the second risk source induces investors to reduce their exposure to risky assets 1 and 2, while increasing allocation to risky asset 0. Since risky assets 1 and 2 are associated with a higher return and higher risk relative to asset 0, this reallocation leads to a simultaneous decline in both portfolio return and risk. (2) In the presence of regret aversion ((b) and (d)), the benchmark strategy based on $\hat{\pi}_{(3)}$ yields the highest expected return and variance. This is because $\hat{\pi}_{(3)}$ places the largest weight on risky asset 2. Under regret aversion, the robust optimal strategy becomes reference-dependent and partially inherits the benchmark allocation structure, resulting in a higher exposure to risky asset 2. Although different benchmark strategies generate distinct levels of return and risk, the monotonic decreasing pattern with respect to ε_2 remains unchanged, highlighting the interaction between regret aversion and ambiguity aversion. (3) In the absence of regret aversion ((a) and (c)), the expected return and variance are identical across benchmark strategies. This follows from the fact that, without considerations of regret, benchmark allocations do not influence the robust optimal portfolio choice. Similar results are observed in the subsequent sensitivity analyses.



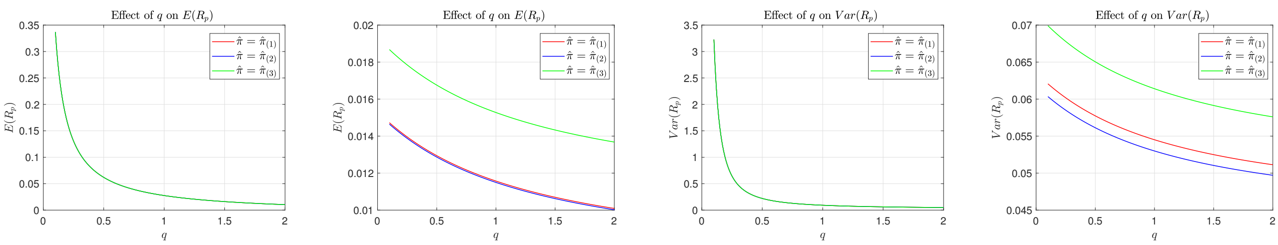
(a) Ambiguity-neutral, regret-neutral **(b)** Ambiguity-neutral, regret-neutral **(c)** Ambiguity-neutral, regret-neutral



(d) Ambiguity-averse, regret-averse **(e)** Ambiguity-averse, regret-averse **(f)** Ambiguity-averse, regret-averse

Figure 7. The effect of the risk-aversion coefficient q on the optimal investment strategy π_i^*.

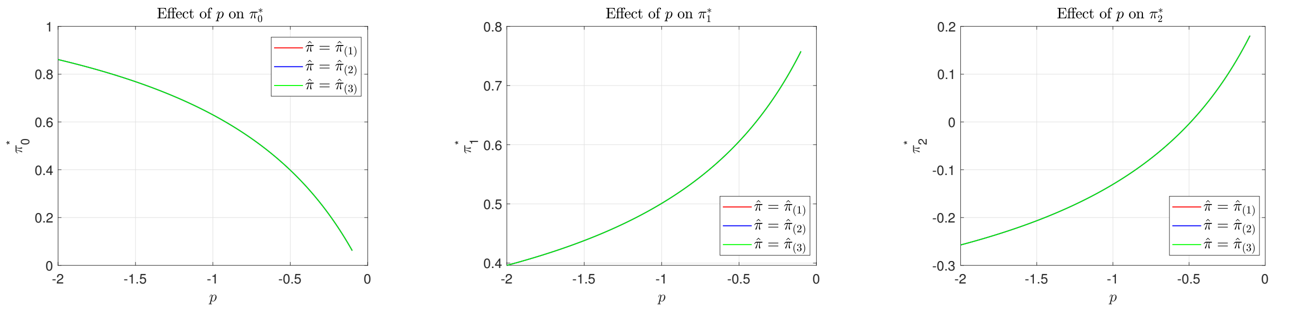
Figure 7 reports the effect of the parameter q on optimal portfolio allocation under different benchmark strategies. Several observations can be made as follows. (1) The allocation to risky asset 0 increases with q, whereas allocations to risky assets 1 and 2 decrease. This pattern is driven by increasing absolute risk aversion with respect to q, i.e., $\partial ARA/\partial q > 0$, which induces a systematic reallocation from high-risk assets toward the low-risk asset. (2) Under (d)–(f), the same monotonic patterns persist, and assets with larger benchmark weights tend to receive higher robust optimal allocations. This effect is consistent across sensitivity analyses (not reported for brevity).



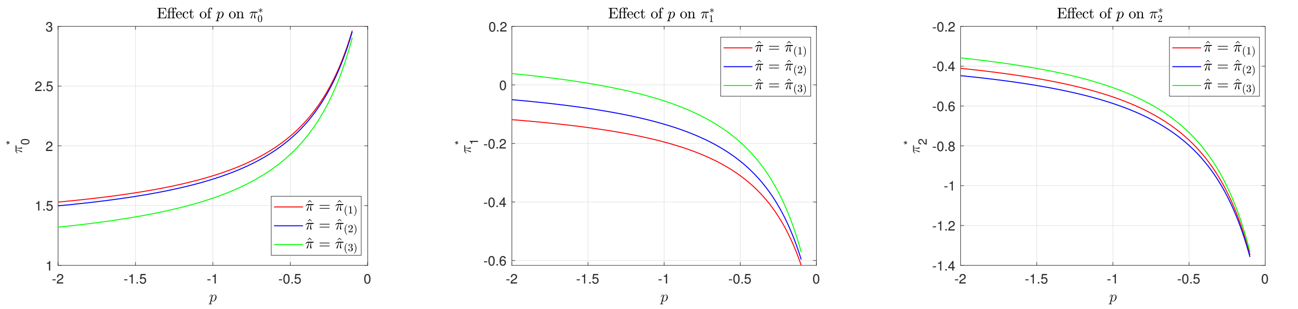
(a) Ambiguity-neutral, regret-neutral **(b)** Ambiguity-averse, regret-averse **(c)** Ambiguity-neutral, regret-neutral **(d)** Ambiguity-averse, regret-averse

Figure 8. The effect of the risk-aversion coefficient q on the expected return $\mathrm{E}(R_p)$ and the variance $\mathrm{Var}(R_p)$ of the optimal portfolio.

Figure 8 reports the effect of the parameter q on the expected return $E(R_p)$ and the variance $\mathrm{Var}(R_p)$ of the optimal portfolio under different benchmark strategies. Several observations can be made as follows. (1) Both $E(R_p)$ and $\mathrm{Var}(R_p)$ decrease with q. This is driven by an increase in risk aversion induced by q, which leads to a systematic shift from high-risk to low-risk assets, thereby reducing both return and risk. (2) In (b) and (d), the benchmark strategy based on $\hat{\pi}_{(3)}$ yields the highest expected return and variance, due to its relatively larger exposure to risky asset 2. Although benchmark-specific differences exist in levels, all cases exhibit a consistent decreasing pattern with respect to q, reflecting the joint effect of risk, ambiguity, and regret aversion. This pattern is robust across sensitivity analysis (not reported).



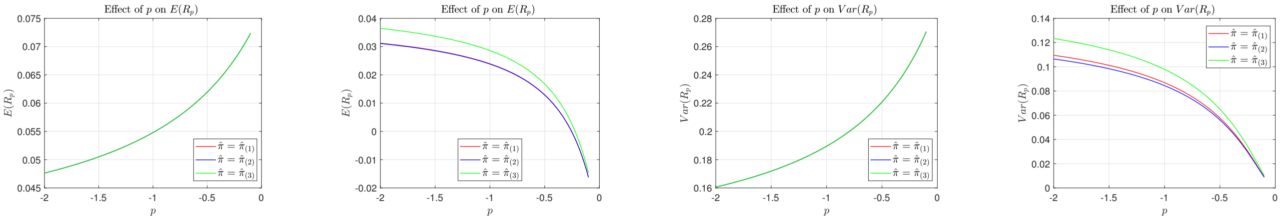
(a) Ambiguity-neutral, regret-neutral **(b)** Ambiguity-neutral, regret-neutral **(c)** Ambiguity-neutral, regret-neutral



(d) Ambiguity-averse, regret-averse **(e)** Ambiguity-averse, regret-averse **(f)** Ambiguity-averse, regret-averse

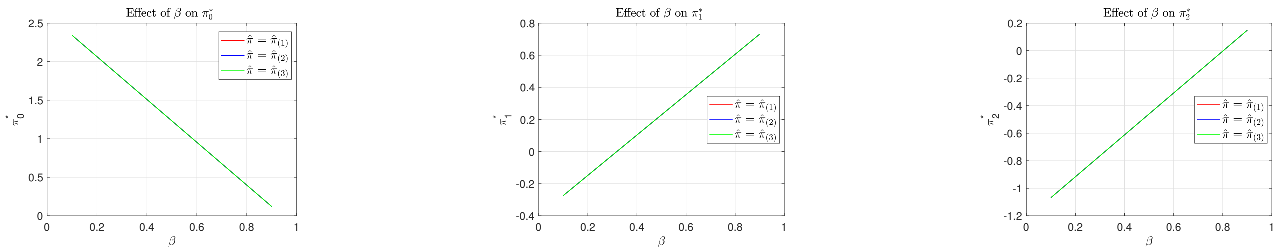
Figure 9. The effect of the risk-aversion coefficient p on the optimal investment strategy π_i^*.

Figure 9 illustrates the impact of the parameter p on optimal portfolio allocation under different benchmark settings. Several observations can be made as follows. (1) In panels (a)–(c), the allocation to risky asset 0 decreases with p, whereas allocations to risky assets 1 and 2 increase. This is consistent with the standard HARA framework, where a higher p implies lower risk aversion and thus a greater allocation to risky assets. (2) By contrast, panels (d)–(f) exhibit the opposite pattern, in which allocations to risky asset 0 increase with p, whereas allocations to risky assets 1 and 2 decrease. This pattern can be explained by the interaction between risk preferences and behavioral distortions when both regret aversion and ambiguity aversion are incorporated. Regret aversion introduces a penalty associated with deviations from a reference wealth level, thereby increasing investors' sensitivity to unfavorable outcomes and wealth fluctuations. Ambiguity aversion further induces a more cautious evaluation of uncertain risk sources through a robust adjustment over ambiguity sets, leading to more conservative portfolio choices. As a result, risk-taking behavior in the robust portfolio optimization model is jointly determined by standard risk preference effects and behavioral components. In this setting, changes in p do not translate directly into monotonic changes in risk exposure, as the impact of p may be partially offset by regret and ambiguity considerations. In particular, under certain parameter configurations, an increase in p may be associated with a reduction in risky asset allocations.

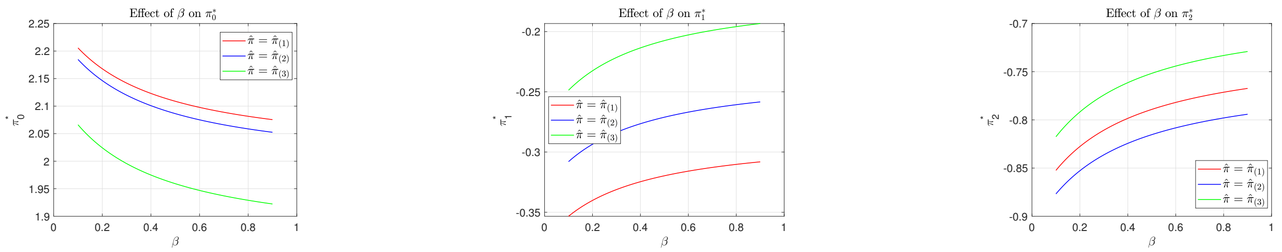


(a) Ambiguity-neutral, regret-neutral **(b)** Ambiguity-averse, regret-averse **(c)** Ambiguity-neutral, regret-neutral **(d)** Ambiguity-averse, regret-averse

Figure 10. The effect of the risk-aversion coefficient p on the expected return $\mathrm{E}(R_p)$ and the variance $\mathrm{Var}(R_p)$ of the optimal portfolio.



(a) Ambiguity-neutral, regret-neutral **(b)** Ambiguity-neutral, regret-neutral **(c)** Ambiguity-neutral, regret-neutral

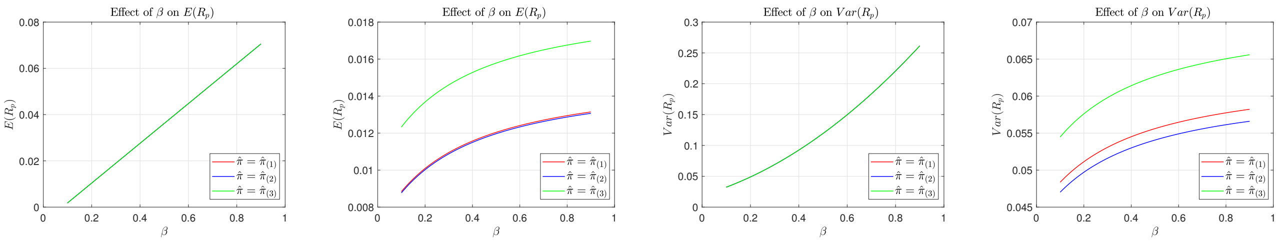


(d) Ambiguity-averse, regret-averse **(e)** Ambiguity-averse, regret-averse **(f)** Ambiguity-averse, regret-averse

Figure 11. The effect of the risk-aversion coefficient β on the optimal investment strategy π_i^*

Figure 10 reports the effect of the parameter p on the expected return $E(R_p)$ and the variance $\mathrm{Var}(R_p)$ under different benchmark strategies. The sensitivity results show that under the joint presence of regret and ambiguity aversion, the effect of p on the portfolio's risk–return characteristics differs from that in the standard HARA framework. In the classical HARA model, a higher p implies lower risk aversion and thus higher exposure to risky assets. In contrast, in the portfolio optimization model with regret and ambiguity aversion, investors' decisions are jointly influenced by the curvature of the HARA utility, psychological losses induced by deviations from a reference wealth level, and uncertainty over return distributions. Regret aversion increases sensitivity to wealth fluctuations and potential underperformance, while ambiguity aversion further amplifies the aversion to parameter uncertainty and model misspecification. As a result, portfolio choices are no longer solely governed by the standard risk–return trade-off implied by risk preferences. Instead, investors adopt more conservative allocation strategies under the combined behavioral distortions and uncertainty constraints. This leads to a systematic reallocation toward lower-risk assets, implying that both the expected return and variance of the optimal portfolio decrease with p, even when p increases the implied willingness to bear risk in the standard HARA setting.

Figure 11 reports the effect of the parameter β on optimal portfolio allocation under different benchmark strategies. The results show that the allocation to risky asset 0 decreases with β, whereas allocations to risky assets 1 and 2 increase. This monotonic pattern is driven by the fact that the absolute risk aversion $ARA(\cdot)$ is decreasing in β, implying that higher values of β correspond to lower risk aversion and therefore stronger risk-taking behavior.



(a) Ambiguity-neutral, regret-neutral

(b) Ambiguity-averse, regret-averse

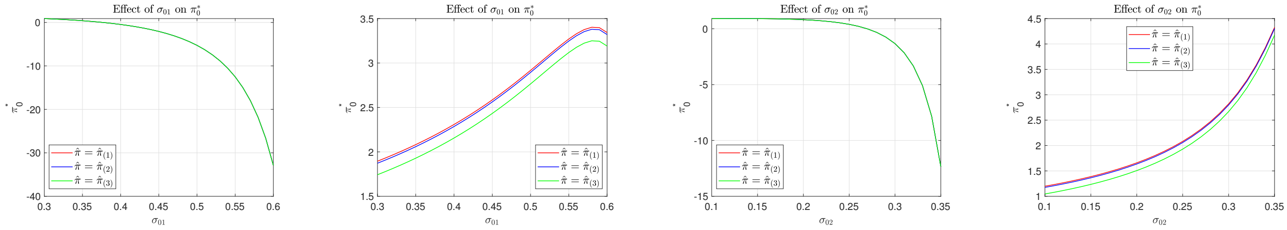
(c) Ambiguity-neutral, regret-neutral

(d) Ambiguity-averse, regret-averse

Figure 12. The effect of the risk-aversion coefficient β on the expected return $\mathrm{E}(R_p)$ and the variance $\mathrm{Var}(R_p)$ of the optimal portfolio.

Figure 12 reports the effect of the parameter β on the expected return $E(R_p)$ and the variance $\mathrm{Var}(R_p)$ under different benchmark strategies. The sensitivity results show that both the expected return and variance of the optimal portfolio increase with β. This is driven by a reduction in the absolute risk aversion as β increases, which induces a systematic reallocation toward higher-risk, higher-return assets and away from low-risk assets.

Figure 13 illustrates the effects of the volatility parameters σ_{ij} on the optimal portfolio allocations π_i^* under different benchmark strategies. Specifically, (a)–(d) correspond to the allocation of risky asset 0, (e)–(h) correspond to risky asset 1, and (i)–(l) correspond to risky asset 2. Under regret-neutral and ambiguity-neutral preferences, the optimal allocations generally decrease with the corresponding volatility parameters. Specifically, π_0^*, π_1^*, and π_2^* exhibit overall declining trends as σ_{01} and σ_{02}, σ_{11} and σ_{12}, and σ_{21} and σ_{22} increase, respectively, although minor fluctuations are observed due to the interaction among multiple risk sources. When regret aversion and ambiguity aversion are incorporated, the volatility effects become more heterogeneous. Specifically, π_0^* initially increases and then decreases with σ_{01}, while increasing with σ_{02}; π_1^* decreases with σ_{11} but initially decreases and then increases with σ_{12}; and π_2^* increases with both σ_{21} and σ_{22}. These results indicate that volatility parameters do not simply represent isolated risk levels in a multifactor environment. Instead, each σ_{ij} captures the exposure to a specific risk source and affects both the marginal variance contributions and cross-asset dependence structures. Therefore, volatility changes generate a trade-off between increased direct risk exposure and potential diversification benefits, resulting in heterogeneous portfolio adjustments. With regret and ambiguity aversion, investors evaluate volatility changes not only via conventional risk exposure but also via sensitivity to unfavorable outcomes and uncertainty regarding risk sources. These behavioral adjustments modify the marginal impact of different volatility components, strengthening the heterogeneity of portfolio responses and, for some risk sources, altering the direction of optimal allocation adjustments.

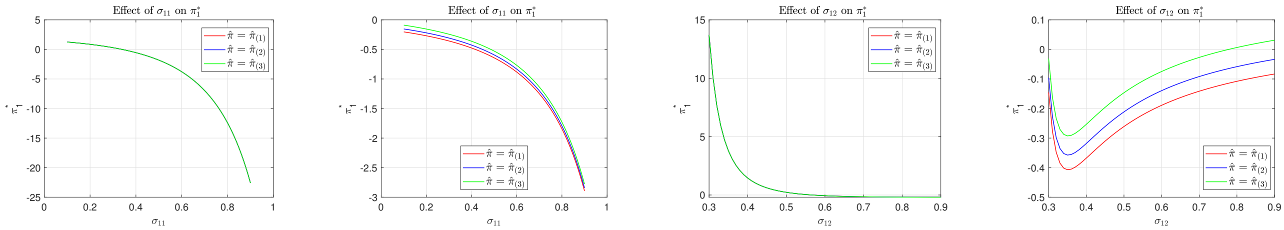


(a) Ambiguity-neutral, regret-neutral

(b) Ambiguity-averse, regret-averse

(c) Ambiguity-neutral, regret-neutral

(d) Ambiguity-averse, regret-averse

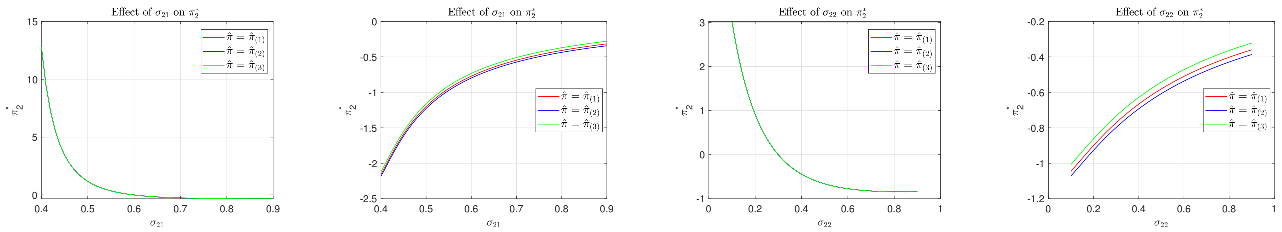


(e) Ambiguity-neutral, regret-neutral

(f) Ambiguity-averse, regret-averse

(g) Ambiguity-neutral, regret-neutral

(h) Ambiguity-averse, regret-averse



(i) Ambiguity-neutral, regret-neutral

(j) Ambiguity-averse, regret-averse

(k) Ambiguity-neutral, regret-neutral

(l) Ambiguity-averse, regret-averse

Figure 13. The effect of the volatility parameters σ_{ij} on the optimal investment strategy π_i^*.

5. Conclusion

We investigate a robust optimal investment problem for a regret-averse investor under model ambiguity within a HARA utility framework. The investor operates in a financial market consisting of multiple risky assets whose price dynamics follow multidimensional geometric Brownian motions. Regret arises when the realized performance falls short of a benchmark, whereas rejoicing occurs when the benchmark is outperformed. Model ambiguity reflects uncertainty about the underlying data-generating process; an ambiguity-averse investor therefore evaluates decisions under a worst-case scenario to guard against potential model misspecification. By applying dynamic programming, closed-form expressions are derived for the robust optimal investment strategy, the density generator associated with the worst-case scenario, and the corresponding value function. Sensitivity analyses are also conducted to illustrate the economic implications of the theoretical results.

The theoretical and sensitivity results reveal several important economic implications. (1) Regret aversion introduces a reference-dependent investment mechanism. Unlike conventional risk preferences that evaluate portfolios solely on the basis of realized wealth, regret-averse investors adjust their allocations toward benchmark portfolios to mitigate potential regret from inferior performance relative to forgone alternatives. Consequently, stronger regret aversion increases the importance of benchmark allocations in determining the optimal investment decisions. (2) Ambiguity aversion affects portfolio choice through a risk-source-specific channel. Rather than uniformly reducing risky investments, ambiguity-averse investors adjust their exposure according to the sensitivity of assets to different sources of uncertainty. This finding highlights that model uncertainty influences portfolio allocation through heterogeneous risk exposures. (3) The interaction among risk aversion, regret aversion, and ambiguity aversion generates a trade-off among risk taking, benchmark tracking, and uncertainty avoidance. Risk aversion governs the overall willingness to hold risky assets, regret aversion discourages deviations from the reference portfolios, and ambiguity aversion tilts allocations away from assets with greater exposure to ambiguous risk sources. Therefore, the optimal portfolio decisions are shaped not by a single preference channel but by the joint and potentially competing effects of risk preferences, regret-rejoicing behavior, and uncertainty considerations. (4) Increases in volatility do not necessarily reduce optimal risky allocations. In a multifactor setting, each level of volatility is a risk loading rather than an isolated risk level, so its variation affects both marginal variance and cross-asset covariance. The resulting diversification or hedging benefits may offset direct risk increases, yielding nonmonotone allocation responses. Regret and ambiguity aversion further strengthen this mechanism by valuing assets according to their roles in mitigating benchmark deviations and ambiguous risk exposures, which can amplify or even reverse the sensitivity of optimal allocations to volatility.

Despite its contributions, the proposed framework has several limitations that point to promising directions for future research. First, although the model provides analytical insights into the joint effects of regret aversion and ambiguity aversion, it abstracts from important market frictions such as transaction costs and short-selling constraints. In practice, these frictions may hinder the implementation of the theoretically robust optimal strategies, as transaction costs can attenuate the incentive to make benchmark-oriented adjustments driven by regret aversion, while trading constraints may further alter the sensitivity of portfolio allocations to ambiguity. Incorporating such realistic frictions would therefore enhance the practical applicability of the framework and yield deeper insights into behavioral portfolio choice in more realistic market environments. Second, the baseline market parameters are assumed to be constant to ensure analytical tractability and facilitate clear a sensitivity analysis. However, financial markets are inherently dynamic and evolve over time. Future research may therefore extend the present framework by allowing for stochastic or time-dependent parameters, such as stochastic interest rates and stochastic volatility processes, which would better capture evolving market conditions and further improve the practical relevance of the resulting robust portfolio strategies.

Author contributions

Shiyin Mo: Conceptualization, formal analysis, investigation, methodology, visualization, and writing–original draft; Huainian Zhu: Supervision, funding acquisition, validation, and writing–review and editing.

Acknowledgments

This project was supported by the Natural Science Foundation of Guangdong Province of China (Grant No. 2023A1515012335).

Conflict of interest

The authors declare no conflict of interest.

Use of generative-AI tools declaration

In the preparation of this work, the authors used Generative AI tools such as ChatGPT to assist in improving the clarity of the language. All AI-assisted content was carefully reviewed and revised by the authors, who take full responsibility for the final version of the manuscript.

Data availability

No data was used for the research described in the article.

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