Journal of Industrial and Management Optimization

https://www.aimspress.com/journal/jimo

JIMO, 22(6): 2866–2877.
DOI:10.3934/jimo.2026105
Received: 15 February 2026
Revised: 11 May 2026
Accepted: 14 May 2026
Published: 25 May 2026

Research article

From offline to online: Sequentially distributing points on a sphere

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Abstract: We study the online version of the point distribution problem on a sphere, where the points arrive sequentially and must be placed irrevocably. The underlying offline problems, such as the Fekete problem, Tammes problem, Thomson problem, and Fejes Toth problem, are computationally intractable. To address this, we establish a unified model and propose a novel online algorithmic framework. The algorithm can provide a theoretical approximation ratio, bounding the performance gap between our online solution and the offline optimum. As an application, we derive concrete approximation guarantees for the aforementioned classical problems.

Keywords: online distribution; sphere; approximation algorithm
Mathematics Subject Classification: 90C26, 90C59

1. Introduction

The problem of distributing the points over the surface of a sphere is an ancient and famous optimization problem in physics, chemistry, and mechanics [1]. In particular, uniformly distributed points on a sphere are practically important in numerical integration. A key application is the approximation of Earth-surface integrals from satellite data, which is performed by calculating averages at well-chosen points. The problem can be generally described as how the n points would distribute themselves on the sphere such that the potential energy is maximized (or minimized). These problems are highly challenging due to their nonconvex objective functions and geometric complexity in high dimensions. Furthermore, they are characterized more by open conjectures than by established theorems. The fifth open problem in [1] presents four specific problems corresponding to different energy functions. Reference [2] reviews some methods for distributing multiple points on a sphere.

1.1. Background and classical problems

Given a unit sphere in $\mathbb{R}^m$, we denote by $x_1, x_2, \ldots, x_n$ the n distinct points in $\mathbb{R}^m$ and $\|\cdot\|$ the usual Euclidean norm. In this paper, we can unify the energy issues into the following nonconvex optimization problem:

$$(\mathrm{S_n}) \quad \max \quad f_n(\|x_1 - x_2\|, \|x_1 - x_3\|, \ldots, \|x_{n-1} - x_n\|)$$
$$\text{s. t.} \quad \|x_1\| = \|x_2\| = \ldots = \|x_n\| = 1.$$

In the following text, we use d_{ij} to denote the Euclidean distance of two distinct points x_i and x_j for $1 \le i < j \le n$, that is, $d_{ij} = \|x_i - x_j\|$. The objective function $f_n(d_{12}, d_{13}, \ldots, d_{n-1,n})$ is usually called *energy function* and it is usually nonconvex. Assume the function f_n satisfies the following two conditions:

(1) $f_n(d_{12}, d_{13}, \ldots, d_{n-1,n})$ is monotonically increasing in terms of d_{ij} for every pair (i, j), where $1 \le i < j \le n$.

(2) $f_n(cd_{12}, cd_{13}, \ldots, cd_{n-1,n}) \ge g(c) \cdot f_n(d_{12}, d_{13}, \ldots, d_{n-1,n})$ for each constant c, where $g(c)$ is a function with respect to c.

It is not difficult to see that when f_n takes different energy expressions, the above model corresponds to the following specific problems.

(i) When $f_n(d_{12}, d_{13}, \ldots, d_{n-1,n}) = \prod_{1\le i<j\le n} d_{ij}$, the problem $(\mathrm{S_n})$ is simplified to the Fekete problem:

$$(\mathrm{F_n}) \quad \max \quad F_n(x_1, \ldots, x_n) := \prod_{1\le i<j\le n} \|x_i - x_j\|$$
$$\text{s. t.} \quad \|x_i\| = 1, \ i = 1, \ldots, n.$$

Then, Condition (1) above is satisfied obviously and $g(c) = c^{\frac{n(n-1)}{2}}$ in Condition (2). Sometimes, $(\mathrm{F_n})$ is stated in the following equivalent form:

$$(\mathrm{V_n}) \quad \min \quad -\log F_n(x_1, \ldots, x_n) := \sum_{1\le i<j\le n} \log \frac{1}{\|x_i - x_j\|}$$
$$\text{s. t.} \quad \|x_i\| = 1, \ i = 1, \ldots, n.$$

The solutions of Fekete problem $(\mathrm{F_n})$ are called the elliptic (or logarithmic) Fekete points.

(ii) When $f_n(d_{12}, d_{13}, \ldots, d_{n-1,n}) = \min\{d_{12}, d_{13}, \ldots, d_{n-1,n}\}$, the problem $(\mathrm{S_n})$ reduces to Tammes problem, which can be reformulated as follows:

$$(\mathrm{T_n}) \quad \max \quad T_n(x_1, \ldots, x_n) := \min_{1\le i<j\le n} \|x_i - x_j\|$$
$$\text{s. t.} \quad \|x_i\| = 1, \ i = 1, \ldots, n.$$

It is trivial to see that Condition (1) is satisfied and $g(c) = c$ in Condition (2) for this case.

Tammes problem $(\mathrm{T_n})$ is proposed by a Dutch botanist Tammes when studying the pores in spherical pollen grains [3]. For $(\mathrm{T_n})$, exact solutions are known only for $n \le 12$ and $n = 24$ even for $m = 3$. Later, Wang and Xia [4] prove that the optimal value of semidefinite programming relaxation (SDP) for the Tammes problem is equivalent to Rankin's first bound [5] and the SDP relaxation is tight if and only if $n \le m + 1$. The research on the upper bound of $(\mathrm{T_n})$ has attracted the attention of many scholars, see [6].

(iii) When $f_n(d_{12}, d_{13}, \ldots, d_{n-1,n}) = -\sum_{1\le i<j\le n} \frac{1}{d_{ij}}$, the problem ($\mathrm{S_n}$) is reduced to the following Thomson problem:

$$(\mathrm{H_n}) \quad \max \quad H_n(x_1, \ldots, x_n) := -\sum_{1\le i<j\le n} \frac{1}{\|x_i - x_j\|}$$
$$\text{s. t.} \quad \|x_i\| = 1, \; i = 1, \ldots, n.$$

Here, Condition (1) is also satisfied and $g(c) = \frac{1}{c}$ in Condition (2). A general case of ($\mathrm{H_n}$) depends on $p > 0$, and

$$H_n(x_1, \ldots, x_n) := -\sum_{1\le i<j\le n} \frac{1}{\|x_i - x_j\|^p}.$$

For this case, when the number n is fixed, ($\mathrm{H_n}$) is asymptotically equivalent to ($\mathrm{T_n}$) as $p \to \infty$ [1].

(iv) When $f_n(d_{12}, d_{13}, \ldots, d_{n-1,n}) = \sum_{1\le i<j\le n}(d_{ij})^q$, $q > 0$, the problem ($\mathrm{S_n}$) becomes Fejes Toth problem:

$$(\mathrm{F_{n,q}}) \quad \max \quad F_{n,q}(x_1, \ldots, x_n) := \sum_{1\le i<j\le n} \|x_i - x_j\|^q$$
$$\text{s. t.} \quad \|x_i\| = 1, \; i = 1, \ldots, n.$$

Condition (1) is valid and $g(c) = c^q$ in this case. Actually, the problem ($\mathrm{F_{n,q}}$) is interesting only for $q < 2$, and ($\mathrm{F_{n,q}}$) is an open problem of long standing in convex geometry even for $q = 1$. If $q \ge 2$ and n is even, ($\mathrm{F_{n,q}}$) is easy to solve: an optimal configuration can be obtained by putting half of the n points together at the north pole and the other half concentrated at the south pole [6].

1.2. Related work

Before presenting our online algorithm, we briefly review the relevant existing work and highlight the differences between our work and the literature.

Recently, Chen et al.[7] introduced the online dispersion problem, where the points arrive and depart over time in a polytope, and the goal is to maximize the minimum pairwise distance. They provided the competitive algorithms for one-dimensional segments, squares, and higher-dimensional polytopes. Their work focuses on the max-min objective and relies on deterministic constructions using prefixed positions and grid-based recursion. In contrast, our work uses an online algorithm as a constructive tool to generate the approximate solutions for offline problems. We then compare our online solution to the offline optimum by providing an approximation ratio.

The offline energy minimization problems, including the Thomson problem and its generalization have been extensively studied. The moment methods and semidefinite programming can be used to derive bounds for Riesz minimal energy problems [8]. Matzke et al. [9] studied polarization and greedy energy sequences on the sphere.

Recent work on online hitting of unit balls by De and Singh [10] and online sphere packing by Lintzmayer, Miyazawa, and Xavier [11] shares the online sequential decision-making aspect with our work but differs in objectives and geometric constraints. Our work complements this growing body of literature by addressing a different class of online geometric optimization problems.

1.3. Our contribution and organization

To the best of our knowledge, this paper provides the first unified online algorithmic framework for the Fekete, Tammes, Thomson, and Fejes Toth problems. While online dispersion has been studied in computational geometry [7] and energy minimization has been extensively analyzed in offline settings [8, 9], the online versions of these specific classical problems with their distinct objective functions have not been previously addressed.

In this work, if a feasible solution can be generated in polynomial time satisfying

$$v(\mathrm{S_n}) \geq f_n(\|\widetilde{x}_1 - \widetilde{x}_2\|, \|\widetilde{x}_1 - \|\widetilde{x}_3\|, \ldots, \|\widetilde{x}_{n-1} - \widetilde{x}_n\|) \geq \varphi \cdot v(\mathrm{S_n}), \tag{1.1}$$

where $v(\cdot)$ denotes the optimal value of $(\cdot)$, then $(\widetilde{x}_1, \ldots, \widetilde{x}_n)$ is called a φ-approximation solution for problem $(\mathrm{S_n})$ and φ is referred to as the approximation ratio. In particular, for the solution of $(\mathrm{S_n})$ produced by Algorithm 1, we have $\varphi = g(c_0)$, where

$$c_0 = \left(\frac{1 - \frac{\alpha_0}{\sqrt{m}}}{2} \right)^{\frac{1}{2}} \tag{1.2}$$

with $\alpha_0 = \sqrt{2 \ln \frac{n-1}{\rho}}$ and $\rho \in (0, 1)$.

The main contributions are summarized as follows. We propose a general optimization problem $(\mathrm{S_n})$ that subsumes the four classical problems as special cases. Then, we design a randomized online algorithm (Algorithm 1) that places points sequentially on the sphere. The algorithm is simple and efficient. For each of the four classical problems, we can derive the explicit approximation ratios relative to the offline optimum. At last, we demonstrate how our results apply to the training point selection problem $(\mathrm{SP_{n,\epsilon}})$, where the incremental nature of our algorithm is particularly advantageous.

The remainder of this paper is organized as follows. In Section 2, we formally introduce the problem of online distributing the points on a sphere and present a conjecture related to the Tammes problem $(\mathrm{T_n})$. Section 3 proposes an online algorithm for solving the general model $(\mathrm{S_n})$ and derives the approximation ratio between online solution and the offline optimum. Section 4 demonstrates an application to $(\mathrm{SP_{n,\epsilon}})$. Last, conclusions are made in Section 5.

Throughout this paper, we denote by $\mathbb{R}^m$ the m-dimensional real vector space, $\mathbb{N}_+$ denotes the positive integer set, and $\Pr(\cdot)$ represents the probability.

2. Online distributing the points on the sphere

In practice, the relation between the model accuracy and the distance among the training points is often considered, that is, the new added point is remote from the existing evaluated points [12, 13]. This leads to the following problem:

$$(\mathrm{SP_{n,\epsilon}}) \quad \min \quad n \qquad \text{s. t.} \quad v(S_n) \leq \epsilon,$$

where ϵ is the given error bound. To solve $(\mathrm{SP_{n,\epsilon}})$, the problem $(\mathrm{S_n})$ must be computed repeatedly for each distinct n. This process incurs substantial computational complexity as $(\mathrm{S_n})$ itself is nontrivial to solve. For this reason, an online version of ($\mathrm{S_n}$) could be considered, in which problem the points are placed on the sphere sequentially (i.e., only one point is added at each step) until the error bound ϵ is satisfied.

In recent years, online optimization has been a hot topic of interest in many fields, including computer science and operations research. Some practical problems, such as online routing problem [14, 15] and online adwords problem [16, 17], can be formulated as online linear programming problem, whose constraint matrix is revealed column by column corresponding to the objective coefficient [18]. Specifically, at each time t, the previous $t-1$ decision variables are given and the coefficients are revealed. Then, the t-th decision variable has to be chosen satisfying the constraints.

The original problem ($\mathrm{S_n}$) can be viewed as an offline problem. For ($\mathrm{S_n}$), we can introduce the following online subproblem at the n-th step:

$$(\widetilde{\mathrm{S}}_\mathrm{n}) \quad \max \quad f_n(\|\widetilde{x}_1-\widetilde{x}_2\|,\ldots,\|\widetilde{x}_1-x_n\|,\ldots,\|\widetilde{x}_{n-1}-x_n\|) \qquad \text{s. t.} \quad \|x_n\|=1,$$

where $\widetilde{x}_1,\ldots,\widetilde{x}_{n-1}$ are obtained at the $(n-1)$-th step. It is trivial that $\frac{v(\widetilde{S}_n)}{v(S_n)} \le 1$ when the objective function is non-negative and $\frac{v(\widetilde{S}_n)}{v(S_n)} \ge 1$ when it is negative. In Section 3, we will analyze the lower bound of this ratio for the non-negative case and the upper bound for the negative case accordingly.

In this section, we first consider the Tammes problem ($\mathrm{T_n}$) with $m=2$ and $n=3$ as an instance. The original problem ($\mathrm{T_n}$) is maximizing the minimal distance between each two points among the n points on a unit sphere in $\mathbb{R}^m$. It is straightforward that the optimal offline solution is achieved by uniformly distributing three points on the sphere, denoted as x_1^*, x_2^*, x_3^*. Using a sequential (online) method, we obtain an alternative configuration $\widetilde{x}_1,\widetilde{x}_2,\widetilde{x}_3$. This method proceeds as follows: First, $\widetilde{x}_1$ is placed arbitrarily. Then, $\widetilde{x}_2$ is chosen to maximize the distance $d(\widetilde{x}_1,\widetilde{x}_2)$, forming a diameter of the unit circle. Finally, $\widetilde{x}_3$ is placed at the midpoint of the arc connecting $\widetilde{x}_1$ and $\widetilde{x}_2$. It can be observed that

$$\varphi = \frac{\|\widetilde{x}_1-\widetilde{x}_3\|}{\|x_1^*-x_3^*\|} = \frac{\sqrt{2}}{\sqrt{3}}.$$

However, determining the optimal value of φ in the general case, that is, finding the largest possible φ such that inequality (1.1) holds for all n, remains an open question. The following theorem provides an answer for the case of $m=2$.

Theorem 2.1. *For the Tammes problem* ($\mathrm{T_n}$) *with $m=2$, consider the following online strategy:*

- *The first point is placed arbitrarily.*
- *Each subsequent point is placed at the midpoint of the largest arc between the consecutive existing points.*

Then, the optimal value of φ for this strategy is $\frac{1}{2}$.

Proof. First we note that $\varphi = 1$ when $n=2^k$, where $k \in \mathbb{N}_+$. That is to say, the optimal objective function values for both offline and online problems are equal in this case. The case of $n=2$ is trivial, and the general result follows by induction. Thus, we omit the proof here.

Next, we only consider $2^k+1 \le n < 2^{k+1}$, where k is some given positive integer. For the offline problem,

$$v(\mathrm{T_n}) = 2\sin\frac{\pi}{n}, \tag{2.1}$$

and the optimal value is obtained when the n points distribute on the sphere uniformly. The online strategy first places 2^k equally spaced points by recursive bisection and then places the remaining points at the midpoint of the arc of two adjacent points. Then, we can find $\widetilde{x}_1, \ldots, \widetilde{x}_n$ such that

$$\mathrm{T_n}(\widetilde{x}_1, \ldots, \widetilde{x}_n) = 2\sin\frac{\pi}{2^{k+1}}. \tag{2.2}$$

It follows from (2.1) and (2.2) that the approximation ratio for this n is

$$\varphi(n) = \frac{\sin\frac{\pi}{2^{k+1}}}{\sin\frac{\pi}{n}} \geq \frac{\sin\frac{\pi}{2^{k+1}}}{\sin\frac{\pi}{2^k+1}},$$

where the last inequality holds since $2^k + 1 \leq n < 2^{k+1}$. Therefore, we can get the optimal value of φ

$$\varphi_{opt} = \inf_{k=1,2,\ldots} \frac{\sin\frac{\pi}{2^{k+1}}}{\sin\frac{\pi}{2^k+1}}$$

and

$$\frac{\sin\frac{\pi}{2^{k+1}}}{\sin\frac{\pi}{2^k+1}} > \frac{\sin\frac{\pi}{2^{k+1}}}{\sin\frac{\pi}{2^k}} = \frac{1}{2\cos\frac{\pi}{2^{k+1}}} \geq \frac{1}{2}.$$

Moreover, taking the limit $k \to \infty$ and using $\sin x \sim x$ as $x \to 0$:

$$\lim_{k\to\infty} \frac{\sin\frac{\pi}{2^{k+1}}}{\sin\frac{\pi}{2^k+1}} = \lim_{k\to\infty} \frac{\frac{\pi}{2^{k+1}}}{\frac{\pi}{2^k+1}} = \lim_{k\to\infty} \frac{2^k+1}{2^{k+1}} = \frac{1}{2}.$$

Thus, the infimum is exactly $\frac{1}{2}$. Consequently, the optimal approximation ratio for this specific online strategy is $\frac{1}{2}$. □

Remark 2.2. Theorem 2.1 analyzes a particular online strategy for the Tammes problem when $m = 2$. The value $\frac{1}{2}$ is the best approximation ratio that this algorithm can guarantee. This does not imply that no other online algorithm can achieve a better ratio. Whether a better online algorithm exists remains an open question.

From the above analysis, we conjecture that the following result holds for general m, although a rigorous proof is still lacking.

Conjecture 2.3. *The optimal value of φ for Tammes problem* ($\mathrm{T_n}$) *is* $\frac{1}{2}$.

3. The approximation ratio between online solution and the offline optimum

In this section, we assume that the total number of points n is known to the algorithm in advance. The points arrive sequentially, and at each step the algorithm must place the current point irrevocably. The approximation ratio is defined with respect to the offline optimum for the same n. Next, we propose an online approximation algorithm to solve ($\mathrm{S_n}$).

Algorithm 1: an approximation algorithm for $(\mathrm{S_n})$
1. Generate $\widetilde{x}_1$ uniformly on the surface of an m-dimensional unit sphere.
2. Input $\rho \in (0,1)$ and let $\alpha = \sqrt{2\ln\frac{1}{\rho}}$.
3. Repeatedly generate $\widetilde{x}_2$ uniformly on the surface of an m-dimensional unit sphere until $\widetilde{x}_2^T\widetilde{x}_1 < \frac{\alpha}{\sqrt{m}}$.
4. While $j = 3,\dots,n$ repeats steps (a) and (b).
   (a) Update $\alpha = \sqrt{2\ln\frac{j-1}{\rho}}$.
   (b) Repeatedly uniformly generate $\widetilde{x}_j \in \mathbb{R}^m$ on the surface of an m-dimensional sphere until $\widetilde{x}_j^T\widetilde{x}_i < \frac{\alpha}{\sqrt{m}}$ for $i \in \{1,\dots,j-1\}$.
5. Output $(\widetilde{x}_1,\dots,\widetilde{x}_n)$.

The existence of $\widetilde{x}_j,\ j = 2,\dots,n$ in Algorithm 1 is based on the following well-known result [19]:

Lemma 3.1. *Let $b = \{b_1,\dots,b_m\} \in \mathbb{R}^m$ and $\|b\| > 0$. Let $\eta = (\eta_1,\dots,\eta_m)^T$ be uniformly distributed over the sphere of radius $\sqrt{m}$ in $\mathbb{R}^m$ (i.e., $\|\eta\| = \sqrt{m}$). Then, for any $\alpha > 0$ and $m \geq 2$, we have*

$$\Pr\left(b^T\eta \geq \alpha\|b\|\right) < e^{-0.5\alpha^2},\ \forall n \geq 2,\ \forall \alpha > 0. \tag{3.1}$$

Remark 3.2. Reference [13] gives a comprehensive proof of the following result:

$$\Pr\left(b^T\eta \geq \alpha\|b\|\right) = \begin{cases} \frac{\int_{\alpha/\sqrt{m}}^1\left(\sqrt{1-t^2}\right)^{m-3}\mathrm{d}t}{2\int_0^1\left(\sqrt{1-t^2}\right)^{m-3}\mathrm{d}t}, & \text{if } \alpha \leq \sqrt{m}, \\ 0, & \text{if } \alpha > \sqrt{m}. \end{cases}$$

and

$$\Pr\left(b^T\eta \geq \alpha\|b\|\right) < e^{-0.45\alpha^2},\ \forall n \geq 2,\ \forall \alpha > 0.$$

Since our algorithm samples points on the unit sphere, we restate Lemma 3.1 in the following equivalent scaled form.

Lemma 3.3. *Let $b = \{b_1,\dots,b_m\} \in \mathbb{R}^m$ and $\|b\| = 1$. Let $\eta = (\eta_1,\dots,\eta_m)^T$ be uniformly distributed over the unit sphere in $\mathbb{R}^m$ (i.e., $\|\eta\| = 1$). Then, for any $\alpha > 0$ and $m \geq 2$, we have*

$$\Pr\left(b^T\eta \geq \frac{\alpha}{\sqrt{m}}\right) < e^{-0.5\alpha^2},\ \forall n \geq 2,\ \forall \alpha > 0. \tag{3.2}$$

Following from Lemma 3.3, we can get the following tail estimation.

Corollary 3.4. *For any given $x_1,\dots,x_{j-1}$ on the unit sphere in $\mathbb{R}^m$, let x_j be uniformly distributed over the unit sphere in $\mathbb{R}^m$. Set $\alpha_0 = \sqrt{2\ln\frac{n-1}{\rho}}$. Then, for any fixed $\rho \in (0,1)$ and $j \in \{2,\dots,n\}$,*

$$\Pr\left(x_j^Tx_i < \frac{\alpha_0}{\sqrt{m}},\ i \in \{1,\dots,j-1\}\right) \geq 1-\rho > 0.$$

Proof. According to Lemma 3.3, for any $j \in \{2, \ldots, n\}$, we have

$$\Pr\left(x_j^T x_i \geq \frac{\alpha_0}{\sqrt{m}}\right) < \frac{\rho}{n-1},\ i \in \{1, \ldots, j-1\}.$$

Then, it holds that

$$\begin{aligned}
& \Pr\left(x_j^T x_i < \frac{\alpha_0}{\sqrt{m}},\ i \in \{1, \ldots, j-1\}\right) \\
= \;& 1 - \Pr\left(x_j^T x_i \geq \frac{\alpha_0}{\sqrt{m}} \text{ for some } i \in \{1, \ldots, j-1\}\right) \\
\geq \;& 1 - \sum_{i \in \{1, \ldots, j-1\}} \Pr\left(x_j^T x_i \geq \frac{\alpha_0}{\sqrt{m}}\right) \\
= \;& 1 - \sum_{i=1}^{j-1} \frac{\rho}{n-1} \\
= \;& 1 - \frac{j-1}{n-1}\rho \\
\geq \;& 1 - \rho > 0.
\end{aligned}$$

The proof is complete. □

Before deriving the approximation guarantee, we first analyze the expected runtime of Algorithm 1. At each step j ($j = 2, \ldots, n$), suppose the previously accepted points $x_1, \ldots, x_{j-1}$ are fixed. By Corollary 3.4, a uniformly random candidate point x_j on the unit sphere satisfies the required constraints with probability at least $1 - \rho > 0$. Hence, the expected number of trials satisfies

$$\mathbb{E}[T_j] \leq \frac{1}{1-\rho}.$$

The expected total number of trials over all steps is

$$\mathbb{E}[T_{\text{total}}] = \sum_{j=2}^{n} \mathbb{E}[T_j] \leq \frac{n-1}{1-\rho}.$$

Each trial requires checking the inner products with at most $j - 1$ existing points, summing the expected cost over all steps yields a total expected time complexity of $O(n^2)$. Consequently, Algorithm 1 yields an approximate solution for the offline problem ($\mathrm{S_n}$) in expected polynomial time.

Theorem 3.5. *Let* $\alpha_0 = \sqrt{2 \ln \frac{n-1}{\rho}}$*. For the solution* $(\widetilde{x}_1, \ldots, \widetilde{x}_n)$ *returned by Algorithm 1, we have*

$$v(\mathrm{S_n}) \geq f_n(\|\widetilde{x}_1 - \widetilde{x}_2\|, \|\widetilde{x}_1 - \widetilde{x}_3\|, \ldots, \|\widetilde{x}_{n-1} - \widetilde{x}_n\|) \geq g(c_0) \cdot v(\mathrm{S_n}),$$

where c_0 *is defined as (1.2).*

Proof. For any feasible solution $(y_1, \ldots, y_n)$ for $(\mathrm{S_n})$, we have

$$\|y_i - y_j\| = (2 - 2y_i^T y_j)^{\frac{1}{2}} \leq 2, \ 1 \leq i < j \leq n, \tag{3.3}$$

where the last inequality holds according to the Cauchy-Schwarz inequality. For the solution $(\widetilde{x}_1, \ldots, \widetilde{x}_n)$ returned by Algorithm 1,

$$\|\widetilde{x}_i - \widetilde{x}_j\| = (2 - 2\widetilde{x}_i^T \widetilde{x}_j)^{\frac{1}{2}} \geq \left[2\left(1 - \frac{\alpha_0}{\sqrt{m}}\right)\right]^{\frac{1}{2}}, \ 1 \leq i < j \leq n, \tag{3.4}$$

where the last inequality holds following from Corollary 3.3. Combining the inequality (3.3) and (3.4) yields

$$\|\widetilde{x}_i - \widetilde{x}_j\| \geq \left(\frac{1 - \frac{\alpha_0}{\sqrt{m}}}{2}\right)^{\frac{1}{2}} \cdot \|y_i - y_j\|, \ 1 \leq i < j \leq n.$$

It follows from the definition of f_n that

$$\begin{aligned} & f_n(\|\widetilde{x}_1 - \widetilde{x}_2\|, \|\widetilde{x}_1 - \widetilde{x}_3\|, \ldots, \|\widetilde{x}_{n-1} - \widetilde{x}_n\|) \\ \geq \ & f_n(c_0\|y_1 - y_2\|, c_0\|y_1 - y_3\|, \ldots, c_0\|y_{n-1} - y_n\|) \\ \geq \ & g(c_0) f_n(\|y_1 - y_2\|, \|y_1 - y_3\|, \ldots, \|y_{n-1} - y_n\|). \end{aligned} \tag{3.5}$$

If $(y_1, \ldots, y_n)$ is the optimal solution of $(\mathrm{S_n})$, then

$$f_n(\|y_1 - y_2\|, \|y_1 - y_3\|, \ldots, \|y_{n-1} - y_n\|) = v(\mathrm{S_n})$$

and Equality (3.5) still holds. Thus, we complete the proof. □

Remark 3.6. Theorem 3.5 indicates that Algorithm 1 can find a $g(\frac{\sqrt{2}}{2})$ asymptotic approximation solution for $(\mathrm{S_n})$ as $\frac{m}{\ln(n-1)}$ increases to infinity. From the definition of c_0, the ratio is well defined only when $m > 2\ln\frac{n-1}{\rho}$. If this condition fails, the bound does not provide a useful guarantee. This remains an open direction for future research.

Following from Theorem 3.5, we can derive concrete approximation guarantees for the aforementioned classical problems.

Corollary 3.7. *For the problem* $(\mathrm{F_n})$ *with* $g(c) = c^{\frac{n(n-1)}{2}}$, *Algorithm 1 yields a feasible solution* $(\widetilde{x}_1, \ldots, \widetilde{x}_n)$ *satisfying*

$$v(\mathrm{F_n}) \geq F_n(\widetilde{x}_1, \ldots, \widetilde{x}_n) \geq c_0^{\frac{n(n-1)}{2}} \cdot v(\mathrm{F_n}).$$

Corollary 3.8. *For the problem* $(\mathrm{T_n})$ *with* $g(c) = c$, *a feasible solution* $(\widetilde{x}_1, \ldots, \widetilde{x}_n)$ *can be obtained by Algorithm 1 and satisfies*

$$v(\mathrm{T_n}) \geq T_n(\widetilde{x}_1, \ldots, \widetilde{x}_n) \geq c_0 \cdot v(\mathrm{T_n}).$$

Remark 3.9. Corollary 3.8 can guarantee that $(\mathrm{T_n})$ admits a $\frac{1}{2}$ approximation ratio when $n \leq \rho e^{\frac{m}{8}} + 1 < e^{\frac{m}{8}} + 1$; this ratio however, is only a conjecture, see Conjecture 2.3.

Corollary 3.10. *For the problem* $(\mathrm{H_n})$ *with* $g(c) = \frac{1}{c}$*, a feasible solution* $(\widetilde{x}_1, \ldots, \widetilde{x}_n)$ *can be obtained by Algorithm 1 and satisfies*

$$v(\mathrm{H_n}) \geq H_n(\widetilde{x}_1, \ldots, \widetilde{x}_n) \geq \frac{1}{c_0} \cdot v(\mathrm{H_n}).$$

Corollary 3.11. *For the problem* $(\mathrm{F_{n,q}})$ *with* $g(c) = c^q$*, the feasible solution* $(\widetilde{x}_1, \ldots, \widetilde{x}_n)$ *obtained via Algorithm 1 satisfies*

$$v(\mathrm{F_{n,q}}) \geq F_{n,q}(\widetilde{x}_1, \ldots, \widetilde{x}_n) \geq c_0{}^q \cdot v(\mathrm{F_{n,q}}).$$

4. Application to the training point selection problem $(\mathrm{SP_{n,\epsilon}})$

With Algorithm 1 and its approximation guarantee established in Section 3, we now return to the problem $(\mathrm{SP_{n,\epsilon}})$ introduced in Section 2. Recall that $(\mathrm{SP_{n,\epsilon}})$ seeks the smallest n such that the approximation error does not exceed a given error bound ϵ. If all the points are constrained to lie on the surface of the unit sphere in $\mathbb{R}^m$, then from the approximation guarantee (1.1), for the configuration $(\widetilde{x}_1, \ldots, \widetilde{x}_n)$ produced by Algorithm 1, we have:

$$v(S_n) \geq f_n(\|\widetilde{x}_1 - \widetilde{x}_2\|, \ldots, \|\widetilde{x}_{n-1} - \widetilde{x}_n\|) \geq \varphi \cdot v(S_n), \tag{4.1}$$

where φ is the approximation ratio derived in Theorem 3.4 (e.g., $\varphi = c_0$ for the Tammes problem, $\varphi = c_0^{n(n-1)/2}$ for the Fekete problem, etc.).

Now, suppose that for the configuration $(\widetilde{x}_1, \ldots, \widetilde{x}_n)$ produced by Algorithm 1 and

$$f_n(\|\widetilde{x}_1 - \widetilde{x}_2\|, \ldots, \|\widetilde{x}_{n-1} - \widetilde{x}_n\|) \leq \varphi \cdot \epsilon. \tag{4.2}$$

Then, combining (4.1) and (4.2), we obtain

$$\varphi \cdot v(S_n) \leq f_n(\|\widetilde{x}_1 - \widetilde{x}_2\|, \ldots, \|\widetilde{x}_{n-1} - \widetilde{x}_n\|) \leq \varphi \cdot \epsilon, \tag{4.3}$$

which implies $v(S_n) \leq \epsilon$. Hence, the offline optimum satisfies the error bound. Since Algorithm 1 is incremental (each new point is added without modifying previous points), we can efficiently search for the minimal feasible n as follows:

1. Set $n = 1$ and initialize an empty configuration.
2. While the stopping condition is not met:
   (a) Use Algorithm 1 to add a new point $\widetilde{x}_n$ to the existing configuration.
   (b) Compute the current objective value $f_n(\|\widetilde{x}_1 - \widetilde{x}_2\|, \ldots, \|\widetilde{x}_{n-1} - \widetilde{x}_n\|)$.
   (c) If $f_n(\cdots) \leq \varphi \cdot \epsilon$, then stop and output n as a feasible solution to $(\mathrm{SP_{n,\epsilon}})$.
   (d) Otherwise, increment n and continue.

This incremental procedure avoids repeatedly solving the intractable offline problem $(\mathrm{SP_{n,\epsilon}})$ for each candidate n. Instead, it leverages the online nature of Algorithm 1 to build the point set progressively, checking the feasibility condition at each step. The first n satisfying (4.2) yields a feasible solution, and the theoretical guarantee ensures that the corresponding offline optimum $\mathrm{v(S_n)}$ indeed meets the error bound ϵ. The incremental nature of Algorithm 1 makes it particularly well-suited for problems where the required number of points is unknown a priori and must be determined adaptively.

5. Conclusions

In this paper, we studied the online version of the point distribution problem on a sphere. By introducing a unified model that generalizes several classical offline problems, we proposed an online algorithmic framework with provable approximation guarantees. Specifically, we established theoretical bounds on the performance gap between our online solution and the offline optimum, and derived concrete approximation ratios for each of these classical problems. At last, we apply the proposed algorithm to the training point selection problem. To the best of our knowledge, this work provides the first unified online algorithmic framework for these problems.

Author contributions

S. Wang wrote the main manuscript text. X.L. Cen made revisions to this paper. All authors read and approved the final manuscript.

Use of Generative-AI tools declaration

AI tools used: Deepseek-V3.2
Purpose of use: Language refinement.

Acknowledgments

This research was supported by the Fundamental Research Funds for the Central Universities (2025MS076) and the Fundamental Research Program of Shanxi Province (No.202203021222245).

Conflict of interest

The authors declare that they have no conflict of interest.

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