Journal of Industrial and
Management Optimization

https://www.aimspress.com/journal/jimo

JIMO, 22(6): 2683–2696.
DOI:10.3934/jimo.2026098
Received: 16 January 2026
Revised: 28 April 2026
Accepted: 07 May 2026
Published: 13 May 2026

Research article

Some bounds of solutions to the polynomial complementarity problem

Jun He[1,*], **Shi-Liang Wu**[2], **Dongsheng Luo**[1] **and Qingyu Zeng**[1]

[1] School of Mathematics, Zunyi Normal College, Zunyi, Guizhou, 563006, P.R. China
[2] Yunnan Key Laboratory of Modern Analytical Mathematics and Applications, Yunnan Normal University, Kunming, Yunnan, 650500, P.R. China

* **Correspondence:** Email: hejunfan1@163.com.

Abstract: The polynomial complementarity problem (PCP) is to find a vector $\mathbf{x} \in \mathbb{R}^n_+$ such that $\sum_{h=1}^{m-1} \mathcal{A}_h\mathbf{x}^{m-h} + \mathbf{q} \geq \mathbf{0}, \mathbf{x}^\top(\sum_{h=1}^{m-1} \mathcal{A}_h\mathbf{x}^{m-h} + \mathbf{q}) = 0$. In this paper, we further investigate the lower bound for the solution set of the PCP with $\mathbf{q} \in \mathbb{R}^n_{++}$. When $\mathcal{A}_1$ is an R_0-tensor, we present an improved upper bound for the solution set of the PCP, which is tighter than the bound given by Xu et al. in [Bounds of the solution set to the polynomial complementarity problem. J. Optim. Theory Appl. 203 (2024) 146-164]. Finally, we prove that the proposed lower and upper bounds for the solution set of the PCP with partially symmetric tensor tuples are attainable. Numerical examples are given to show the efficiency of the proposed results.

Keywords: upper bounds; lower bounds; tensor; polynomial complementarity problem
Mathematics Subject Classification: 90C33, 90C26

1. Introduction

The tensor complementarity problem, denoted by TCP($\mathcal{A}, \mathbf{q}$), is to find a vector $\mathbf{x} \in \mathbb{R}^n_+$ such that

$$\mathbf{x} \geq \mathbf{0}, \mathbf{q} + \mathcal{A}\mathbf{x}^{m-1} \geq \mathbf{0}, \mathbf{x}^\top(\mathcal{A}\mathbf{x}^{m-1} + \mathbf{q}) = 0,$$

where $\mathcal{A} \in \mathbb{R}^{[m,n]}$ and $\mathbf{q} \in \mathbb{R}^n$ [1]. So far, many researchers have paid attention to this topic [2, 3] because of its applications in DNA micro-arrays, communication, and n-person non-cooperative games [4].Numerical algorithms for solving tensor complementarity problems have been also proposed recently [5].

The polynomial complementarity problem, denoted by PCP($\Theta, \mathbf{q}$), is to find a vector $\mathbf{x} \in \mathbb{R}^n_+$ such that

$$\mathbf{x} \geq \mathbf{0}, F(\mathbf{x}) := \sum_{h=1}^{m-1} \mathcal{A}_h\mathbf{x}^{m-h} + \mathbf{q} \geq \mathbf{0}, \mathbf{x}^\top(\sum_{h=1}^{m-1} \mathcal{A}_h\mathbf{x}^{m-h} + \mathbf{q}) = 0, \tag{1.1}$$

where $\Theta := (\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{m-1}) \in \mathbb{F}^{[m,n]} := \mathbb{R}^{[m,n]} \times \ldots \times \mathbb{R}^{[2,n]}$ with $\mathcal{A}_1 \in \mathbb{R}^{[m,n]}, \ldots, \mathcal{A}_{m-1} \in \mathbb{R}^{[2,n]}$, $\mathbf{q} \in \mathbb{R}^n$, and

$$\left(\mathcal{A}_h \mathbf{x}^{m-h}\right)_i = \sum_{i_2,\ldots,i_{m-h+1}=1}^{n} a_{ii_2\ldots i_{m-h+1}} x_{i_2} \ldots x_{i_{m-h+1}}, i \in [n].$$

The polynomial complementarity problem (PCP), first introduced by Gowda [6], is an extension of the tensor complementarity problem [1, 7]. Several applications, such as hypergraph clustering problems, multi-person noncooperative games, and traffic equilibrium problems, were considered in [4], and can be modeled as polynomial complementarity problems. As a natural extension of the polynomial complementarity problems, generalized polynomial complementarity problems over a polyhedral cone were introduced in [8].

Gowda proved that the solution set of the PCP is nonempty and compact when zero is the only solution of TCP($\mathcal{A}_1, \mathbf{0}$) and the topological degree of $\min\{\mathbf{x}, \mathcal{A}_1 \mathbf{x}^{m-1}\}$ at the origin is nonzero [6]. By using the squared slack variables technique, Hieu et al. discussed the univariate representations of solutions to generic polynomial complementarity problems [9]. The solutions of the polynomial complementarity problem were investigated based the concepts of S-tensor tuples and Q-tensor tuples [10, 11].

An important property in theory is the bounds of the solution set of the PCP, which can be used to find the numerical solution of PCP [12]. Ling et al. have shown that the solution set of PCP with a leading ER-tensor is nonempty and compact, and proposed lower bounds of the solution set of the PCP when Θ is strictly semicopositive [13]. Li et al. presented lower bounds of the solution set of the PCP for Θ being an $\alpha - \mathbf{q}$ tensor tuple [14]. When Θ is a commonly strictly semipositive tensor tuple, upper bounds for the solution set of the PCP were obtained in [15]. Xu et al. presented lower and upper bounds with the condition $\mathbf{q} \in \mathbb{R}^n \backslash \mathbb{R}^n_+$ [16].

All of the lower bounds for the solution norm of the PCP($\Theta, \mathbf{q}$) discussed above are based on the condition $\mathbf{q} \in \mathbb{R}^n \backslash \mathbb{R}^n_+$. When $\mathbf{q} \in \mathbb{R}^n_{++}$, obviously, zero is a lower bound on the norm of the solution set of PCP($\Theta, \mathbf{q}$). However, the lower bound zero has limited practical utility. Hence, a natural question is, *how to obtain tighter lower bound for the solution set of the PCP*($\Theta, \mathbf{q}$) *with* $\mathbf{q} \in \mathbb{R}^n_{++}$?

Based on the above motivation, we further study the lower bound for the solution set of the PCP($\Theta, \mathbf{q}$) with $\mathbf{q} \in \mathbb{R}^n_{++}$. In Section 2, some basic concepts and notations are introduced. In Section 3, we obtain the lower bound for the solution set of PCP($\Theta, \mathbf{q}$) with $\mathbf{q} \in \mathbb{R}^n_{++}$. In Section 4, an improved upper bound for the solution set with $\pi(\mathcal{A}_1)$ is also presented. In Section 5, we prove that the proposed lower and upper bounds of the solution set with partially symmetric tensor tuples are attainable. Finally, in Section 6, some conclusions are given.

2. Preliminaries

Throughout this paper, let $\mathbb{R}$ and $\mathbb{R}^n$ be real field and n-dimensional real Euclidean space, and let $\mathbb{R}_+$ and $\mathbb{R}^n_+$ ($\mathbb{R}^n_{++}$) be the sets of all nonnegative real numbers and nonnegative (positive) vectors in $\mathbb{R}^n$. For the vector $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x}^{[m]}$ denotes a vector defined as $(\mathbf{x}^{[m]})_i = (x_i)^m$ for $i \in [n] := \{1, \ldots, n\}$, $\mathbf{x}_+ := (\max\{0, x_1\}, \ldots, \max\{0, x_n\})^\top$, and $\|\mathbf{x}\|_\infty := \max\{|x_i|, i \in [n]\}$.

The tensor $\mathcal{A} = (a_{i_1 i_2 \ldots i_m})$ represents a multidimensional array whose entries are represented as $a_{i_1 i_2 \ldots i_m}$ with $i_j = 1, \ldots, n_j$ and $j = 1, \ldots, m$, where m represents the order of $\mathcal{A}$, and $(n_1, \ldots, n_m)$ represents the dimension of $\mathcal{A}$. Unless otherwise stated, all tensors involved are limited to the field of real numbers.

$\mathcal{A}$ is called an m order n dimensional tensor if $n_1 = \ldots = n_m = n$, denoted by $\mathcal{A} \in \mathbb{R}^{[m,n]}$. Let O be zero tensors. If $\mathcal{A} = (a_{i_1 i_2 \ldots i_m}) \in \mathbb{R}^{[m,n]}$ is symmetric on modes $i_2, \ldots, i_m$, then $\mathcal{A}$ is called partially symmetric [17]. $\Theta = (\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{m-1}) \in \mathbb{F}^{[m,n]}$ is called a partially symmetric tensor tuple if $\mathcal{A}_h$ is partially symmetric for all $h \in [m-1]$, denoted by $\Theta \in \mathbb{PF}^{[m,n]}$.

Some useful definitions and results are given below.

Definition 2.1. *[15] $\Theta = (\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{m-1}) \in \mathbb{F}^{[m,n]}$ is called a strictly semipositive tensor tuple if for any $\mathbf{x} \in \mathbb{R}^n_+ \backslash \{\mathbf{0}\}$, there exists $t \in [n]$ such that $x_t > 0$ and $(\sum\limits_{h=1}^{m-1} \mathcal{A}_h \mathbf{x}^{m-h})_t > 0$; likewise, it is called a strictly copositive tensor tuple if and only if $\sum\limits_{h=1}^{m-1} \mathcal{A}_h \mathbf{x}^{m-h+1} > 0$ for all $\mathbf{x} \in \mathbb{R}^n_+ \backslash \{\mathbf{0}\}$.*

Definition 2.2. *[7] $\mathcal{A} \in \mathbb{R}^{[m,n]}$ is called an R_0-tensor if TCP$(\mathcal{A}, \mathbf{0})$ has a unique solution.*

Definition 2.3. *[18] Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$. For each $i \in [n]$, we define*

$$r_i^-(A) := \sum_{i_2,\ldots,i_n \neq i,i,\ldots,i} (-a_{ii_2\ldots i_n})_+.$$

Then, $\mathcal{A}$ is said to be a GRSDD-tensor (i.e., generalized row strictly diagonally dominant tensor) if $|a_{ii\ldots i}| - r_i^-(A) > 0$ for all $i \in [n]$.

In particular, for $\mathcal{A}_1$ being an R_0-tensor, the following result, Theorem 2.1, was obtained in [16].

Theorem 2.1. [16, Theorem 4.1] *Let $\Theta = (\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{m-1}) \in \mathbb{F}^{[m,n]}$, $\mathbf{x}^*$ be a solution of PCP$(\Theta, \mathbf{q})$ and $\mathbf{q} \in \mathbb{R}^n$. Assume that $\mathcal{A}_1$ is an R_0-tensor, $(\mathcal{A}_2, \ldots, \mathcal{A}_{m-1}) \neq (O, \ldots, O)$. Then,*

$$\|\mathbf{x}^*\|_\infty \leq \kappa_1(\Theta) = \max_{d \in \{\iota(\Theta), m\}} \left(\frac{\max\limits_{i \in [n]} \{\sum_{h=2}^{m-1} \sum_{i_2,\ldots,i_{m-h+1}=1}^{n} |a_{ii_2\ldots i_{m-h+1}}| + |q_i|\}}{\pi(\mathcal{A}_1)} \right)^{\frac{1}{d-1}},$$

where

$$\iota(\Theta) := \min\{h | \mathcal{A}_h \neq O, h \in [m-1], h \neq 1\},$$

$$\pi(\mathcal{A}_1) = \min_{\mathbf{x} \geq 0, \|\mathbf{x}\|_\infty = 1} \max \left\{ \max_{i \in [n]} x_i (\mathcal{A}_1 \mathbf{x}^{m-1})_i, \|(-\mathcal{A}_1 \mathbf{x}^{m-1})_+\|_\infty \right\}.$$

For $\mathbf{q} \in \mathbb{R}^n_+$, a lower bound of the solution set of PCP$(\Theta, \mathbf{q})$ was presented in [14] by Li et al. That is:

Theorem 2.2. [14, Theorem 1] *Let $\Theta = (\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{m-1}) \in \mathbb{F}^{[m,n]}$ and $\mathbf{q} \in \mathbb{R}^n_+$. Then, zero is a tight lower bound on the norm of the solutions of PCP$(\Theta, \mathbf{q})$.*

3. Lower bound of the solution set with $\mathbf{q} \in \mathbb{R}^n_{++}$

In this section, we present the lower bound of the solution set of PCP$(\Theta, \mathbf{q})$ with $\mathbf{q} \in \mathbb{R}^n_{++}$. For this goal, we require the following lemma.

Lemma 3.1. *Let $\mathcal{A} = (a_{i_1 i_2 \ldots i_m}) \in \mathbb{R}^{[m,n]}$ and $\mathbf{q} \in \mathbb{R}^n$. If PCP$(\Theta, \mathbf{q})$ has a nonzero solution $\mathbf{x}^*$ and $\|\mathbf{x}^*\|_\infty = x^*_t$, then $(\sum_{h=1}^{m-1} \mathcal{A}_h \mathbf{x}^{m-h})_t + q_t = 0$. Moreover, if $q_t > 0$, there exist $h \in [m-1]$ and $i_2 \ldots i_{m-h+1} \in [n]$ such that $a_{ti_2\ldots i_{m-h+1}} < 0$; if $q_t < 0$, there exist $h \in [m-1]$ and $i_2 \ldots i_{m-h+1} \in [n]$ such that $a_{ti_2\ldots i_{m-h+1}} > 0$.*

Proof. Suppose that PCP$(\Theta, \mathbf{q})$ has a nonzero solution $\mathbf{x}^*$ and $\|\mathbf{x}^*\|_\infty = x_t^*$. Then, $x_t^* > 0$ and

$$x_t^*((\sum_{h=1}^{m-1} \mathcal{A}_h(\mathbf{x}^*)^{m-h})_t + q_t) = 0,$$

which implies that

$$(\sum_{h=1}^{m-1} \mathcal{A}_h(\mathbf{x}^*)^{m-h})_t + q_t = 0. \tag{3.1}$$

If $q_t > 0$, by $\mathbf{x} \in \mathbb{R}_+^n$, we have

$$q_t = -(\sum_{h=1}^{m-1} \mathcal{A}_h(\mathbf{x}^*)^{m-h})_t > 0,$$

which means that there exist $h \in [m-1]$ and $i_2 \dots i_{m-h+1} \in [n]$ such that $a_{ti_2\dots i_{m-h+1}} < 0$.

If $q_t < 0$, by $\mathbf{x} \in \mathbb{R}_+^n$, we have

$$q_t = -(\sum_{h=1}^{m-1} \mathcal{A}_h(\mathbf{x}^*)^{m-h})_t < 0,$$

which means that there exist $h \in [m-1]$ and $i_2 \dots i_{m-h+1} \in [n]$ such that $a_{ti_2\dots i_{m-h+1}} > 0$. □

If $\mathbf{0}$ is a solution of PCP$(\Theta, \mathbf{q})$, then zero is a tight lower and upper bound on the norm of the solution set of PCP$(\Theta, \mathbf{q})$. Therefore, we only consider PCP$(\Theta, \mathbf{q})$ with nonzero solutions. For each $i \in [n]$,

$$\underline{M_i}(\Theta) := \min\{h | \exists a_{ii_2\dots i_{m-h+1}} < 0, h \in [m-1]\},$$

$$\overline{M_i}(\Theta) := \max\{h | \exists a_{ii_2\dots i_{m-h+1}} < 0, h \in [m-1]\},$$

and

$$\|\mathbf{q}\|_{\min} := \min\{q_i : i \in [n], \mathbf{q} \in \mathbb{R}_{++}^n\},$$

we can present the lower bound of the solution set of TCP$(\mathcal{A}, \mathbf{q})$ with $\mathbf{q} \in \mathbb{R}_{++}^n$; see Theorem 3.1.

Theorem 3.1. *Let* $\Theta \in \mathbb{F}^{[m,n]}$*, and let* $\mathbf{q} \in \mathbb{R}_{++}^n$, $\mathbf{x}^*$ *be a nonzero solution of PCP*$(\Theta, \mathbf{q})$. *Then,*

$$\begin{aligned} \|\mathbf{x}^*\|_\infty \geq \omega_1(\Theta) = \min\Bigg(&\min_{i\in[n]} \left(\frac{\|\mathbf{q}\|_{\min}}{\sum_{h=1}^{m-1} \sum_{i_2,\dots,i_{m-h+1}=1}^{n} (-a_{ii_2\dots i_{m-h+1}})_+} \right)^{\frac{1}{m-\underline{M_i}(\Theta)}}, \\ &\min_{i\in[n]} \left(\frac{\|\mathbf{q}\|_{\min}}{\sum_{h=1}^{m-1} \sum_{i_2,\dots,i_{m-h+1}=1}^{n} (-a_{ii_2\dots i_{m-h+1}})_+} \right)^{\frac{1}{m-\overline{M_i}(\Theta)}} \Bigg). \end{aligned} \tag{3.2}$$

Proof. If $\mathbf{q} \in \mathbb{R}_{++}^n$, $\mathbf{x}^* \in \mathbb{R}^n$ is a solution of PCP$(\Theta, \mathbf{q})$ and $\mathbf{x}^* \neq \mathbf{0}$. By Lemma 3.1, there exists $t \in [n]$ such that $(\sum_{h=1}^{m-1} \mathcal{A}_h(\mathbf{x}^*)^{m-h})_t + q_t = 0$, and

$$\begin{aligned} \|\mathbf{q}\|_{\min} \leq q_t &= -\sum_{h=1}^{m-1} \sum_{i_2,\dots,i_{m-h+1}=1}^{n} a_{ti_2\dots i_{m-h+1}} x_{i_2}^* \dots x_{i_{m-h+1}}^* \\ &= -\sum_{h=1}^{m-1} \sum_{a_{ti_2\dots i_{m-h+1}} \geq 0} a_{ti_2\dots i_{m-h+1}} x_{i_2}^* \dots x_{i_{m-h+1}}^* - \sum_{h=1}^{m-1} \sum_{a_{ti_2\dots i_{m-h+1}} < 0} a_{ti_2\dots i_{m-h+1}} x_{i_2}^* \dots x_{i_{m-h+1}}^* \end{aligned}$$

$$\leq -\sum_{k=\underline{M}_i(\Theta)}^{\overline{M}_i(\Theta)} \sum_{a_{ti_2\dots i_{m-h+1}}<0} a_{ti_2\dots i_{m-h+1}} x^*_{i_2}\dots x^*_{i_{m-h+1}}. \tag{3.3}$$

Assume that $\|\mathbf{x}^*\|_\infty \geq 1$; then, we can obtain

$$\begin{aligned}\frac{\|\mathbf{q}\|_{\min}}{\|\mathbf{x}^*\|_\infty^{m-\underline{M}_i(\Theta)}} &\leq \sum_{h=\underline{M}_i(\Theta)}^{\overline{M}_i(\Theta)} \sum_{i_2,\dots,i_{m-h+1}=1}^{n} (-a_{ti_2\dots i_{m-h+1}})_+ \frac{x^*_{i_2}\dots x^*_{i_{m-h+1}}}{\|\mathbf{x}^*\|_\infty^{m-\underline{M}_i(\Theta)}} \\ &\leq \sum_{h=1}^{m-1} \sum_{i_2,\dots,i_{m-h+1}=1}^{n} (-a_{ti_2\dots i_{m-h+1}})_+.\end{aligned}$$

So,

$$\|\mathbf{x}^*\|_\infty \geq \min_{t\in[n]} \left(\frac{\|\mathbf{q}\|_{\min}}{\sum_{h=1}^{m-1}\sum_{i_2,\dots,i_{m-h+1}=1}^{n}(-a_{ti_2\dots i_{m-h+1}})_+} \right)^{\frac{1}{m-\underline{M}_i(\Theta)}}.$$

Similarly, if $\|\mathbf{x}^*\|_\infty < 1$, then by (3.3), we have

$$\begin{aligned}\frac{\|\mathbf{q}\|_{\min}}{\|\mathbf{x}^*\|_\infty^{m-\overline{M}_i(\Theta)}} &\leq \sum_{h=\underline{M}_i(\Theta)}^{\overline{M}_i(\Theta)} \sum_{i_2,\dots,i_{m-h+1}=1}^{n} (-a_{ti_2\dots i_{m-h+1}})_+ \frac{x^*_{i_2}\dots x^*_{i_{m-h+1}}}{\|\mathbf{x}^*\|_\infty^{m-\overline{M}_i(\Theta)}} \\ &\leq \sum_{h=1}^{m-1} \sum_{i_2,\dots,i_{m-h+1}=1}^{n} (-a_{ti_2\dots i_{m-h+1}})_+.\end{aligned}$$

So,

$$\|\mathbf{x}^*\|_\infty \geq \min_{t\in[n]} \left(\frac{\|\mathbf{q}\|_{\min}}{\sum_{h=1}^{m-1}\sum_{i_2,\dots,i_{m-h+1}=1}^{n}(-a_{ti_2\dots i_{m-h+1}})_+} \right)^{\frac{1}{m-\overline{M}_i(\Theta)}}.$$

□

Remark 3.1. *If* $\mathbf{q} \in \mathbb{R}^n_{++}$, *then* $\|\mathbf{q}\|_{\min} > 0$ *and* $\omega_1(\Theta) > 0$, *which means that the lower bound obtained by Theorem 3.1 can be tighter than the result in Theorem 2.2 when* $\mathbf{q} \in \mathbb{R}^n_{++}$.

Remark 3.2. *All the bounds for the solution norm of the PCP*$(\Theta, \mathbf{q})$ *discussed in [19] are based on the condition* $\mathbf{q} \in \mathbb{R}^n \setminus \mathbb{R}^n_+$. *If* $\mathbf{q} \in \mathbb{R}^n_{++}$, *all the bounds obtained in [19] are useless in this case.*

In order to show the efficiency of Theorem 3.1, we give the following numerical example.

Example 3.1. *Consider PCP*$(\Theta, \mathbf{q})$, *where* $\Theta = (\mathcal{A}_1, \mathcal{A}_2)$, $\mathcal{A}_1 = (a_{i_1i_2i_3}) \in \mathbb{R}^{[3,2]}$ *with entries*

$$a_{111} = a_{222} = 1$$

and other entries of $\mathcal{A}_1$ *are zero,* $\mathcal{A}_2 \in \mathbb{R}^{[2,2]}$ *with*

$$a_{11} = a_{22} = -2$$

and other entries of $\mathcal{A}_2$ *are zero, and* $\mathbf{q} = (1, 1)^\top$. *It is easy to see that*

$$\{(1,1)^\top, (0,1)^\top, (1,0)^\top\}$$

are nonzero solutions of PCP$(\Theta, \mathbf{q})$*. By the result in Theorem 2.2, one can get*

$$\|\mathbf{x}^*\|_\infty \geq 0,$$

which is useless in practical applications. By Theorem 3.1, one can get

$$\|\mathbf{x}^*\|_\infty \geq 0.5 > 0,$$

which is sharper than the result in Theorem 2.2.

4. Improved upper bound for the solution set with $\pi(\mathcal{A}_1)$

The solution set of the PCP is nonempty and compact when $\mathcal{A}_1$ is an R_0-tensor [6]. In this section, we further study the solution set of the PCP for $\mathcal{A}_1$ being an R_0-tensor, and gain an improved upper bound for the solution set of the PCP with $\pi(\mathcal{A}_1)$.

Theorem 4.1. *Let* $\Theta = (\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{m-1}) \in \mathbb{F}^{[m,n]}$*,* $\mathbf{x}^*$ *be a solution of PCP*$(\Theta, \mathbf{q})$ *and* $\mathbf{q} \in \mathbb{R}^n$*. Assume that* $\mathcal{A}_1$ *is an* R_0*-tensor and* $(\mathcal{A}_2, \ldots, \mathcal{A}_{m-1}) \neq (\mathcal{O}, \ldots, \mathcal{O})$*. Then,*

$$\|\mathbf{x}^*\|_\infty \leq \kappa_2(\Theta) = \max_{d\in\{\iota(\Theta),m\}} \left(\max \left\{ \frac{\max_{i\in[n]}\{\sum_{h=2}^{m-1} \sum_{i_2,\ldots,i_{m-h+1}=1}^{n} (a_{ii_2\ldots i_{m-h+1}})_+ + (\mathbf{q}_+)_i\}}{\pi(\mathcal{A}_1)}, \right.\right.$$
$$\left.\left. \frac{\max_{i\in[n]}\{\sum_{h=2}^{m-1} \sum_{i_2,\ldots,i_{m-h+1}=1}^{n} (-a_{ii_2\ldots i_{m-h+1}})_+ + ((-\mathbf{q})_+)_i\}}{\pi(\mathcal{A}_1)} \right\} \right)^{\frac{1}{d-1}}. \tag{4.1}$$

Proof. If $\mathcal{A}_1 \in \mathbb{R}^{[m,n]}$ is an R_0-tensor, from Theorem 8 in [20], we have $\pi(\mathcal{A}_1) > 0$. If $\mathbf{x}^*$ is a solution of PCP$(\Theta, \mathbf{q})$, then we consider the following two cases:

(a) If $\|\mathbf{x}^*\|_\infty \geq 1$, then by (1.1), for each $i \in [n]$, we have

$$\begin{aligned}
\frac{-(\mathcal{A}_1(\mathbf{x}^*)^{m-1})_i}{\|\mathbf{x}^*\|_\infty^{m-1}} &\leq \frac{\sum_{h=2}^{m-1} \sum_{i_2,\ldots,i_{m-h+1}=1}^{n} a_{ii_2\ldots i_{m-h+1}} x_{i_2}^* \ldots x_{i_{m-h+1}}^* + q_i}{\|\mathbf{x}^*\|_\infty^{m-1}} \\
&\leq \frac{\sum_{h=2}^{m-1} \sum_{i_2,\ldots,i_{m-h+1}=1}^{n} (a_{ii_2\ldots i_{m-h+1}})_+ x_{i_2}^* \ldots x_{i_{m-h+1}}^* + q_i}{\|\mathbf{x}^*\|_\infty^{m-1}} \\
&\leq \frac{\sum_{h=2}^{m-1} \sum_{i_2,\ldots,i_{m-h+1}=1}^{n} (a_{ii_2\ldots i_{m-h+1}})_+ x_{i_2}^* \ldots x_{i_{m-h+1}}^* + (\mathbf{q}_+)_i}{\|\mathbf{x}^*\|_\infty^{m-1}} \\
&\leq \frac{(\sum_{k=2}^{m-1} \sum_{i_2,\ldots,i_{m-h+1}=1}^{n} (a_{ii_2\ldots i_{m-h+1}})_+ + (\mathbf{q}_+)_i)\|\mathbf{x}^*\|_\infty^{m-l(\Theta)}}{\|\mathbf{x}^*\|_\infty^{m-1}}.
\end{aligned}$$

So,

$$\frac{\|(-\mathcal{A}_1(\mathbf{x}^*)^{m-1})_+\|_\infty}{\|\mathbf{x}^*\|_\infty^{m-1}} \leq \frac{\max_{i\in[n]}\{\sum_{h=2}^{m-1} \sum_{i_2,\ldots,i_{m-h+1}=1}^{n} (a_{ii_2\ldots i_{m-h+1}})_+ + (\mathbf{q}_+)_i\}}{\|\mathbf{x}^*\|_\infty^{\iota(\Theta)-1}}. \tag{4.2}$$

On the other hand, by (1.1), for each $i \in [n]$, we also have

$$\frac{x_i^*(\mathcal{A}_1(\mathbf{x}^*)^{m-1})_i}{\|\mathbf{x}^*\|_\infty^{m}} = \frac{-\sum_{h=2}^{m-1} \sum_{i_2,\ldots,i_{m-h+1}=1}^{n} a_{ii_2\ldots i_{m-h+1}} x_i^* x_{i_2}^* \ldots x_{i_{m-h+1}}^* - x_i^* q_i}{\|\mathbf{x}^*\|_\infty^{m}}$$

$$\begin{aligned}&\le \frac{\sum_{h=2}^{m-1}\sum_{i_2,\ldots,i_{m-h+1}=1}^{n}(-a_{ii_2\ldots i_{m-h+1}})_+ x_i^* x_{i_2}^* \ldots x_{i_{m-h+1}}^* + x_i^*((-\mathbf{q})_+)_i}{\|\mathbf{x}^*\|_\infty^m}\\ &\le \frac{(\sum_{h=2}^{m-1}\sum_{i_2,\ldots,i_{m-h+1}=1}^{n}(-a_{ii_2\ldots i_{m-h+1}})_+ + ((-\mathbf{q})_+)_i)\|\mathbf{x}^*\|_\infty^{m-l(\Theta)+1}}{\|\mathbf{x}^*\|_\infty^m}.\end{aligned}$$

So,

$$\max_{i\in[n]}\frac{x_i^*(\mathcal{A}_1(\mathbf{x}^*)^{m-1})_i}{\|\mathbf{x}^*\|_\infty^m} \le \frac{\max_{i\in[n]}\{\sum_{h=2}^{m-1}\sum_{i_2,\ldots,i_{m-h+1}=1}^{n}(-a_{ii_2\ldots i_{m-h+1}})_+ + ((-\mathbf{q})_+)_i\}}{\|\mathbf{x}^*\|_\infty^{\iota(\Theta)-1}}. \tag{4.3}$$

Combining (4.2) and (4.3), we can obtain

$$\begin{aligned}0<\pi(\mathcal{A}_1)\le \max\Bigg\{&\frac{\max_{i\in[n]}\{\sum_{h=2}^{m-1}\sum_{i_2,\ldots,i_{m-h+1}=1}^{n}(a_{ii_2\ldots i_{m-h+1}})_+ + (\mathbf{q}_+)_i\}}{\|\mathbf{x}^*\|_\infty^{\iota(\Theta)-1}},\\ &\frac{\max_{i\in[n]}\{\sum_{h=2}^{m-1}\sum_{i_2,\ldots,i_{m-h+1}=1}^{n}(-a_{ii_2\ldots i_{m-h+1}})_+ + ((-\mathbf{q})_+)_i\}}{\|\mathbf{x}^*\|_\infty^{\iota(\Theta)-1}}\Bigg\}.\end{aligned}$$

(b) If $0<\|\mathbf{x}^*\|_\infty<1$, then by (1.1), for each $i\in[n]$, we have

$$\begin{aligned}\frac{-(\mathcal{A}_1(\mathbf{x}^*)^{m-1})_i}{\|\mathbf{x}^*\|_\infty^{m-1}} &\le \frac{\sum_{h=2}^{m-1}\sum_{i_2,\ldots,i_{m-h+1}=1}^{n} a_{ii_2\ldots i_{m-h+1}} x_{i_2}^* \ldots x_{i_{m-h+1}}^* + q_i}{\|\mathbf{x}^*\|_\infty^{m-1}}\\ &\le \frac{\sum_{h=2}^{m-1}\sum_{i_2,\ldots,i_{m-h+1}=1}^{n}(a_{ii_2\ldots i_{m-h+1}})_+ x_{i_2}^* \ldots x_{i_{m-h+1}}^* + q_i}{\|\mathbf{x}^*\|_\infty^{m-1}}\\ &\le \frac{\sum_{h=2}^{m-1}\sum_{i_2,\ldots,i_{m-h+1}=1}^{n}(a_{ii_2\ldots i_{m-h+1}})_+ x_{i_2}^* \ldots x_{i_{m-h+1}}^* + (\mathbf{q}_+)_i}{\|\mathbf{x}^*\|_\infty^{m-1}}\\ &\le \frac{\sum_{h=2}^{m-1}\sum_{i_2,\ldots,i_{m-h+1}=1}^{n}(a_{ii_2\ldots i_{m-h+1}})_+ + (\mathbf{q}_+)_i}{\|\mathbf{x}^*\|_\infty^{m-1}}.\end{aligned}$$

So,

$$\frac{\|(-\mathcal{A}_1(\mathbf{x}^*)^{m-1})_+\|_\infty}{\|\mathbf{x}^*\|_\infty^{m-1}} \le \frac{\max_{i\in[n]}\{\sum_{h=1}^{m-1}\sum_{i_2,\ldots,i_{m-h+1}=1}^{n}(a_{ii_2\ldots i_{m-h+1}})_+ + (\mathbf{q}_+)_i\}}{\|\mathbf{x}^*\|_\infty^{m-1}}. \tag{4.4}$$

On the other hand, by (1.1), for each $i\in[n]$, we also have

$$\begin{aligned}\frac{x_i^*(\mathcal{A}_1(\mathbf{x}^*)^{m-1})_i}{\|\mathbf{x}^*\|_\infty^m} &= \frac{-\sum_{h=2}^{m-1}\sum_{i_2,\ldots,i_{m-h+1}=1}^{n} a_{ii_2\ldots i_{m-h+1}} x_i^* x_{i_2}^* \ldots x_{i_{m-h+1}}^* - x_i^* q_i}{\|\mathbf{x}^*\|_\infty^m}\\ &\le \frac{\sum_{h=2}^{m-1}\sum_{i_2,\ldots,i_{m-h+1}=1}^{n}(-a_{ii_2\ldots i_{m-h+1}})_+ x_i^* x_{i_2}^* \ldots x_{i_{m-h+1}}^* + x_i^*((-\mathbf{q})_+)_i}{\|\mathbf{x}^*\|_\infty^m}\\ &\le \frac{(\sum_{h=2}^{m-1}\sum_{i_2,\ldots,i_{m-h+1}=1}^{n}(-a_{ii_2\ldots i_{m-h+1}})_+ + ((-\mathbf{q})_+)_i)\|\mathbf{x}^*\|_\infty}{\|\mathbf{x}^*\|_\infty^m}.\end{aligned}$$

So,

$$\max_{i\in[n]}\frac{x_i^*(\mathcal{A}_1(\mathbf{x}^*)^{m-1})_i}{\|\mathbf{x}^*\|_\infty^m} \le \frac{\max_{i\in[n]}\{\sum_{h=2}^{m-1}\sum_{i_2,\ldots,i_{m-h+1}=1}^{n}(-a_{ii_2\ldots i_{m-h+1}})_+ + ((-\mathbf{q})_+)_i\}}{\|\mathbf{x}^*\|_\infty^{m-1}}. \tag{4.5}$$

Combining (4.4) and (4.5), we obtain

$$0 < \pi(\mathcal{A}_1) \le \max\left\{\frac{\max_{i\in[n]}\{\sum_{h=2}^{m-1}\sum_{i_2,\ldots,i_{m-h+1}=1}^{n}(a_{ii_2\ldots i_{m-h+1}})_+ + (\mathbf{q}_+)_i\}}{\|\mathbf{x}^*\|_\infty^{m-1}}, \frac{\max_{i\in[n]}\{\sum_{h=2}^{m-1}\sum_{i_2,\ldots,i_{m-h+1}=1}^{n}(-a_{ii_2\ldots i_{m-h+1}})_+ + ((-\mathbf{q})_+)_i\}}{\|\mathbf{x}^*\|_\infty^{m-1}}\right\}.$$

□

Remark 4.1. *By direct computation, we have*

$$\begin{aligned}\max\Bigg\{&\max_{i\in[n]}\left\{\sum_{h=2}^{m-1}\sum_{i_2,\ldots,i_{m-h+1}=1}^{n}(a_{ii_2\ldots i_{m-h+1}})_+ + (\mathbf{q}_+)_i\right\},\\ &\max_{i\in[n]}\left\{\sum_{k=2}^{m-1}\sum_{i_2,\ldots,i_{m-h+1}=1}^{n}(-a_{ii_2\ldots i_{m-h+1}})_+ + ((-\mathbf{q})_+)_i\right\}\Bigg\}\\ &\le \max_{i\in[n]}\left\{\sum_{h=2}^{m-1}\sum_{i_2,\ldots,i_{m-h+1}=1}^{n}|a_{ii_2\ldots i_{m-h+1}}| + |q_i|\right\}.\end{aligned}$$

Then, $\kappa_2(\Theta) \le \kappa_1(\Theta)$, which means that the upper bound obtained by Theorem 4.1 is always better than the result in Theorem 2.1.

Remark 4.2. *(i) All the bounds for the solution norm of the PCP$(\Theta, \boldsymbol{q})$ discussed in [19] are based on the condition $\mathbf{q} \in \mathbb{R}^n \setminus \mathbb{R}^n_+$. If $\mathbf{q} \in \mathbb{R}^n$, the bounds obtained in [19] may be useless.*

(ii) If $\mathbf{q} \in \mathbb{R}^n \setminus \mathbb{R}^n_+$, assume that $\mathcal{A}_1$ is a GRSDD-tensor with all diagonal entries being positive. $(\mathcal{A}_2, \ldots, \mathcal{A}_{m-1}) = (O, \ldots, O)$, $\|(-\mathbf{q})_+\|_\infty = \|\mathbf{q}\|_\infty$, $\pi(\mathcal{A}_1) > \sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\ldots i_m}$ for some $i \in \Omega(\mathbf{q})$, $\Omega(\mathbf{q}) = \{i \in [n] : -q_i = \|(-\mathbf{q})_+\|_\infty\}$ when $\mathbf{q} \in \mathbb{R}^n \setminus \mathbb{R}^n_+$. By the analysis in Remark 6 in [20], the upper bound obtained by Theorem 4.1 is better than the one by Theorem 3.2 in [19].

The following example is given to illustrate Theorem 4.1.

Example 4.1. *Consider PCP$(\Theta, \mathbf{q})$, where $\Theta = (\mathcal{A}_1, \mathcal{A}_2)$ and $\mathcal{A}_1 = (a_{i_1i_2i_3}) \in \mathbb{R}^{[3,2]}$ with entries*

$$a_{111} = a_{222} = -1$$

and other entries of $\mathcal{A}_1$ are zero, $\mathcal{A}_2 \in \mathbb{R}^{[2,2]}$ with

$$a_{11} = a_{22} = -2, a_{12} = a_{21} = 2,$$

and $\mathbf{q} = \mathbf{0}$. By direct computation, we have

$$\begin{aligned}\pi(\mathcal{A}_1) &= \min_{\mathbf{x}\ge 0, \|\mathbf{x}\|_\infty = 1}\max\left\{\max_{i\in[n]} x_i(\mathcal{A}_1\mathbf{x}^{m-1})_i, \|(-\mathcal{A}_1\mathbf{x}^{m-1})_+\|_\infty\right\}\\ &= \min_{\mathbf{x}\ge 0, \|\mathbf{x}\|_\infty = 1}\|(-\mathcal{A}_1\mathbf{x}^{m-1})_+\|_\infty\\ &= \min_{\mathbf{x}\ge 0, \|\mathbf{x}\|_\infty = 1}\|(x_1^2, x_2^2)^\top\|_\infty.\end{aligned}$$

For any $\mathbf{x} \in \{\mathbb{R}^2 : x_1 = 1, x_2 \in [0, 1]\}$,

$$\|(x_1^2, x_2^2)^\top\|_\infty = x_1^2 = 1.$$

For any $\mathbf{x} \in \{\mathbb{R}^2 : x_2 = 1, x_1 \in [0, 1]\}$,

$$\|(x_1^2, x_2^2)^\top\|_\infty = x_2^2 = 1.$$

Thus, $\pi(\mathcal{A}_1) = 1$.

By Theorem 2.1, one can get

$$\|\mathbf{x}^*\|_\infty \le \kappa_1(\Theta) = 4.$$

By Theorem 4.1, we have

$$\|\mathbf{x}^*\|_\infty \le \kappa_2(\Theta) = 2,$$

which is tighter than the result in Theorem 2.1.

5. Bounds of the solution set with partially symmetric tensor tuples

For $\Theta = (\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_{m-1}) \in \mathbb{F}^{[m,n]}$, there exists a corresponding partially symmetric tensor tuple $\widetilde{\Theta} = (\widetilde{\mathcal{A}}_1, \widetilde{\mathcal{A}}_2, \dots, \widetilde{\mathcal{A}}_{m-1}) \in \mathbb{PF}^{[m,n]}$ such that

$$F(\mathbf{x}) = \sum_{h=1}^{m-1} \mathcal{A}_h \mathbf{x}^{m-h} + \mathbf{q} = \sum_{h=1}^{m-1} \widetilde{\mathcal{A}}_h \mathbf{x}^{m-h} + \mathbf{q}.$$

Denote

$$S_F := \Big\{\Theta = (\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_{m-1}) \in \mathbb{F}^{[m,n]} | \mathbf{x} \ge \mathbf{0}, F(\mathbf{x}) = \sum_{h=1}^{m-1} \mathcal{A}_h \mathbf{x}^{m-h} + \mathbf{q} \ge \mathbf{0}\Big\}.$$

Next, we will give the evidence that bounds for PCP$(\Theta, \mathbf{q})$ with partially symmetric tensor tuples can reach the extreme values.

Theorem 5.1. *If* $\Theta = (\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_{m-1}) \in \mathbb{F}^{[m,n]}$, $\widetilde{\Theta} = (\widetilde{\mathcal{A}}_1, \widetilde{\mathcal{A}}_2, \dots, \widetilde{\mathcal{A}}_{m-1}) \in \mathbb{PF}^{[m,n]}$ *is the corresponding partially symmetric tensor tuple with* $\widetilde{\mathcal{A}}_h = (\widetilde{a}_{i_1 i_2 \dots i_{m-h+1}})$, *and* $\mathbf{q} \in \mathbb{R}^n_{++}$*; then* $\omega_1(\widetilde{\Theta}) = \max_{\Theta \in S_F} \omega_1(\Theta)$, *where the definition of* $\omega_1(\cdot)$ *is introduced in (3.2).*

Proof. First, we show that $\omega_1(\widetilde{\Theta}) \ge \omega_1(\Theta)$ for $\forall \Theta \in S_F$, where

$$\begin{aligned} \omega_1(\widetilde{\Theta}) = \min\Bigg(& \min_{i\in[n]} \left(\frac{\|\mathbf{q}\|_{\min}}{\sum_{h=1}^{m-1} \sum_{i_2,\dots,i_{m-h+1}=1}^{n} (-\widetilde{a}_{ii_2\dots i_{m-h+1}})_+} \right)^{\frac{1}{m-\underline{M_i}(\widetilde{\Theta})}}, \\ & \min_{i\in[n]} \left(\frac{\|\mathbf{q}\|_{\min}}{\sum_{h=1}^{m-1} \sum_{i_2,\dots,i_{m-h+1}=1}^{n} (-\widetilde{a}_{ii_2\dots i_{m-h+1}})_+} \right)^{\frac{1}{m-\overline{M_i}(\widetilde{\Theta})}} \Bigg), \end{aligned}$$

and

$$\omega_1(\Theta) = \min\Bigg(\min_{i\in[n]} \left(\frac{\|\mathbf{q}\|_{\min}}{\sum_{h=1}^{m-1} \sum_{i_2,\dots,i_{m-h+1}=1}^{n} (-a_{ii_2\dots i_{m-h+1}})_+} \right)^{\frac{1}{m-\underline{M_i}(\Theta)}},$$

$$\min_{i\in[n]}\left(\frac{\|\mathbf{q}\|_{\min}}{\sum_{h=1}^{m-1}\sum_{i_2,\ldots,i_{m-h+1}=1}^{n}(-a_{ii_2\ldots i_{m-h+1}})_+}\right)^{\frac{1}{m-\overline{M}_i(\Theta)}}\Bigg).$$

From the definitions of $\underline{M_i}(\cdot)$, $\overline{M_i}(\cdot)$, and partial symmetry of tensors, we have

$$\underline{M_i}(\widetilde{\Theta}) \geq \underline{M_i}(\Theta),\ \ \overline{M_i}(\widetilde{\Theta}) \leq \overline{M_i}(\Theta),$$

and

$$\sum_{h=1}^{m-1}\sum_{i_2,\ldots,i_{m-h+1}=1}^{n}(-\widetilde{a}_{ii_2\ldots i_{m-h+1}})_+ \leq \sum_{h=1}^{m-1}\sum_{i_2,\ldots,i_{m-h+1}=1}^{n}(-a_{ii_2\ldots i_{m-h+1}})_+.$$

If

$$\frac{\|\mathbf{q}\|_{\min}}{\sum_{h=1}^{m-1}\sum_{i_2,\ldots,i_{m-h+1}=1}^{n}(-\widetilde{a}_{ii_2\ldots i_{m-h+1}})_+} \geq 1,$$

then we have

$$\begin{aligned}\omega_1(\widetilde{\Theta}) &= \min_{i\in[n]}\left(\frac{\|\mathbf{q}\|_{\min}}{\sum_{kh=1}^{m-1}\sum_{i_2,\ldots,i_{m-h+1}=1}^{n}(-\widetilde{a}_{ii_2\ldots i_{m-h+1}})_+}\right)^{\frac{1}{m-\underline{M_i}(\widetilde{\Theta})}}\\ &\geq \min_{i\in[n]}\left(\frac{\|\mathbf{q}\|_{\min}}{\sum_{h=1}^{m-1}\sum_{i_2,\ldots,i_{m-h+1}=1}^{n}(-a_{ii_2\ldots i_{m-h+1}})_+}\right)^{\frac{1}{m-\underline{M_i}(\Theta)}} = \omega_1(\Theta).\end{aligned}$$

If

$$\frac{\|\mathbf{q}\|_{\min}}{\sum_{h=1}^{m-1}\sum_{i_2,\ldots,i_{m-h+1}=1}^{n}(-\widetilde{a}_{ii_2\ldots i_{m-h+1}})_+} < 1,$$

then we have

$$\begin{aligned}\omega_1(\widetilde{\Theta}) &= \min_{i\in[n]}\left(\frac{\|\mathbf{q}\|_{\min}}{\sum_{kh=1}^{m-1}\sum_{i_2,\ldots,i_{m-h+1}=1}^{n}(-\widetilde{a}_{ii_2\ldots i_{m-h+1}})_+}\right)^{\frac{1}{m-\overline{M_i}(\widetilde{\Theta})}}\\ &\geq \min_{i\in[n]}\left(\frac{\|\mathbf{q}\|_{\min}}{\sum_{h=1}^{m-1}\sum_{i_2,\ldots,i_{m-h+1}=1}^{n}(-a_{ii_2\ldots i_{m-h+1}})_+}\right)^{\frac{1}{m-\overline{M_i}(\Theta)}} = \omega_1(\Theta).\end{aligned}$$

Therefore, $\omega_1(\widetilde{\Theta}) \geq \omega_1(\Theta)$ for $\forall\Theta \in S_F$. □

Theorem 5.2. *If* $\Theta = (\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{m-1}) \in \mathbb{F}^{[m,n]}$, $\widetilde{\Theta} = (\widetilde{\mathcal{A}}_1, \widetilde{\mathcal{A}}_2, \ldots, \widetilde{\mathcal{A}}_{m-1}) \in \mathbb{PF}^{[m,n]}$ *is the corresponding partially symmetric tensor tuple with* $\widetilde{\mathcal{A}}_h = (\widetilde{a}_{i_1i_2\ldots i_{m-h+1}})$*, and* $\mathbf{q} \in \mathbb{R}^n$*; then* $\kappa_2(\widetilde{\Theta}) = \min_{\Theta\in S_F}\kappa_2(\Theta)$*, where the definition of* $\kappa_2(\cdot)$ *is introduced in (4.1).*

Proof. First, we show that $\kappa_2(\widetilde{\Theta}) \leq \kappa_2(\Theta)$ for $\forall\Theta \in S_F$, where

$$\begin{aligned}\kappa_2(\widetilde{\Theta}) = \max_{d\in\{\iota(\widetilde{\Theta}),m\}}\Bigg(\max\Bigg\{&\frac{\max_{i\in[n]}\{\sum_{h=2}^{m-1}\sum_{i_2,\ldots,i_{m-h+1}=1}^{n}(\widetilde{a}_{ii_2\ldots i_{m-h+1}})_+ + (\mathbf{q}_+)_i\}}{\pi(\widetilde{\mathcal{A}}_1)},\\ &\frac{\max_{i\in[n]}\{\sum_{h=2}^{m-1}\sum_{i_2,\ldots,i_{m-h+1}=1}^{n}(-\widetilde{a}_{ii_2\ldots i_{m-h+1}})_+ + ((-\mathbf{q})_+)_i\}}{\pi(\widetilde{\mathcal{A}}_1)}\Bigg\}\Bigg)^{\frac{1}{d-1}},\end{aligned}$$

and

$$\kappa_2(\Theta) = \max_{d\in\{\iota(\Theta),m\}}\left(\max\left\{\frac{\max_{i\in[n]}\{\sum_{h=2}^{m-1}\sum_{i_2,\ldots,i_{m-h+1}=1}^{n}(a_{ii_2\ldots i_{m-h+1}})_+ + (\mathbf{q}_+)_i\}}{\pi(\mathcal{A}_1)},\right.\right.$$
$$\left.\left.\frac{\max_{i\in[n]}\{\sum_{h=2}^{m-1}\sum_{i_2,\ldots,i_{m-h+1}=1}^{n}(-a_{ii_2\ldots i_{m-h+1}})_+ + ((-\mathbf{q})_+)_i\}}{\pi(\mathcal{A}_1)}\right\}\right)^{\frac{1}{d-1}}.$$

From the definitions of $\iota(\cdot)$ and partial symmetry of tensors, we have $\iota(\widetilde{\Theta}) \geq \iota(\Theta)$,

$$\begin{aligned}&\sum_{h=2}^{m-1}\sum_{i_2,\ldots,i_{m-h+1}=1}^{n}(\widetilde{a}_{ii_2\ldots i_{m-h+1}})_+ + (\mathbf{q}_+)_i\\ &\leq \sum_{h=2}^{m-1}\sum_{i_2,\ldots,i_{m-h+1}=1}^{n}(a_{ii_2\ldots i_{m-h+1}})_+ + (\mathbf{q}_+)_i,\end{aligned}$$

and

$$\begin{aligned}&\sum_{h=2}^{m-1}\sum_{i_2,\ldots,i_{m-h+1}=1}^{n}(-\widetilde{a}_{ii_2\ldots i_{m-h+1}})_+ + ((-\mathbf{q})_+)_i\\ &\leq \sum_{h=2}^{m-1}\sum_{i_2,\ldots,i_{m-h+1}=1}^{n}(-a_{ii_2\ldots i_{m-h+1}})_+ + ((-\mathbf{q})_+)_i.\end{aligned}$$

If

$$\max\left\{\frac{\max_{i\in[n]}\{\sum_{h=2}^{m-1}\sum_{i_2,\ldots,i_{m-h+1}=1}^{n}(\widetilde{a}_{ii_2\ldots i_{m-h+1}})_+ + (\mathbf{q}_+)_i\}}{\pi(\mathcal{A}_1)},\right.$$
$$\left.\frac{\max_{i\in[n]}\{\sum_{h=2}^{m-1}\sum_{i_2,\ldots,i_{m-h+1}=1}^{n}(-\widetilde{a}_{ii_2\ldots i_{m-h+1}})_+ + ((-\mathbf{q})_+)_i\}}{\pi(\mathcal{A}_1)}\right\} \geq 1,$$

then we have

$$\begin{aligned}\kappa_2(\widetilde{\Theta}) &= \max\left\{\frac{\max_{i\in[n]}\{\sum_{h=2}^{m-1}\sum_{i_2,\ldots,i_{m-h+1}=1}^{n}(\widetilde{a}_{ii_2\ldots i_{m-h+1}})_+ + (\mathbf{q}_+)_i\}}{\pi(\mathcal{A}_1)},\right.\\ &\qquad\left.\frac{\max_{i\in[n]}\{\sum_{h=2}^{m-1}\sum_{i_2,\ldots,i_{m-h+1}=1}^{n}(-\widetilde{a}_{ii_2\ldots i_{m-h+1}})_+ + ((-\mathbf{q})_+)_i\}}{\pi(\mathcal{A}_1)}\right\})^{\frac{1}{\iota(\widetilde{\Theta})-1}}\\ &\leq \max\left\{\frac{\max_{i\in[n]}\{\sum_{h=2}^{m-1}\sum_{i_2,\ldots,i_{m-h+1}=1}^{n}(\widetilde{a}_{ii_2\ldots i_{m-h+1}})_+ + (\mathbf{q}_+)_i\}}{\pi(\mathcal{A}_1)},\right.\\ &\qquad\left.\frac{\max_{i\in[n]}\{\sum_{h=2}^{m-1}\sum_{i_2,\ldots,i_{m-h+1}=1}^{n}(-a_{ii_2\ldots i_{m-h+1}})_+ + ((-\mathbf{q})_+)_i\}}{\pi(\mathcal{A}_1)}\right\})^{\frac{1}{\iota(\Theta)-1}} = \kappa_2(\Theta).\end{aligned}$$

If

$$\max\left\{\frac{\max_{i\in[n]}\{\sum_{h=2}^{m-1}\sum_{i_2,\ldots,i_{m-h+1}=1}^{n}(\widetilde{a}_{ii_2\ldots i_{m-h+1}})_+ + (\mathbf{q}_+)_i\}}{\pi(\mathcal{A}_1)},\right.$$

$$\left.\frac{\max_{i\in[n]}\{\sum_{h=2}^{m-1}\sum_{i_2,\ldots,i_{m-h+1}=1}^{n}(-\widetilde{a}_{ii_2\ldots i_{m-h+1}})_+ + ((-\mathbf{q})_+)_i\}}{\pi(\mathcal{A}_1)}\right\} < 1,$$

then we have

$$\begin{aligned}
\kappa_2(\widetilde{\Theta}) &= \max\left\{\frac{\max_{i\in[n]}\{\sum_{h=2}^{m-1}\sum_{i_2,\ldots,i_{m-h+1}=1}^{n}(\widetilde{a}_{ii_2\ldots i_{m-h+1}})_+ + (\mathbf{q}_+)_i\}}{\pi(\mathcal{A}_1)},\right.\\
&\qquad \left.\left.\frac{\max_{i\in[n]}\{\sum_{h=2}^{m-1}\sum_{i_2,\ldots,i_{m-h+1}=1}^{n}(-\widetilde{a}_{ii_2\ldots i_{m-h+1}})_+ + ((-\mathbf{q})_+)_i\}}{\pi(\mathcal{A}_1)}\right\}\right)^{\frac{1}{m-1}}\\
&\leq \max\left\{\frac{\max_{i\in[n]}\{\sum_{h=2}^{m-1}\sum_{i_2,\ldots,i_{m-h+1}=1}^{n}(\widetilde{a}_{ii_2\ldots i_{m-h+1}})_+ + (\mathbf{q}_+)_i\}}{\pi(\mathcal{A}_1)},\right.\\
&\qquad \left.\left.\frac{\max_{i\in[n]}\{\sum_{h=2}^{m-1}\sum_{i_2,\ldots,i_{m-h+1}=1}^{n}(-a_{ii_2\ldots i_{m-h+1}})_+ + ((-\mathbf{q})_+)_i\}}{\pi(\mathcal{A}_1)}\right\}\right)^{\frac{1}{m-1}} = \kappa_2(\Theta).
\end{aligned}$$

Therefore, $\kappa_2(\widetilde{\Theta}) \leq \kappa_2(\Theta)$ for $\forall \Theta \in S_F$. □

Remark 5.1. *If* $\widetilde{\Theta} = (\widetilde{\mathcal{A}}_1, \widetilde{\mathcal{A}}_2, \ldots, \widetilde{\mathcal{A}}_{m-1}) \in \mathbb{PF}^{[m,n]}$ *is the corresponding partially symmetric tensor tuple of* $\Theta = (\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{m-1}) \in \mathbb{F}^{[m,n]}$, *then,* $\omega_1(\widetilde{\Theta})$ *and* $\kappa_2(\widetilde{\Theta})$ *are the best lower or upper bounds for PCP*$(\Theta, \mathbf{q})$. *Therefore, when the tensor tuple in Theorems 3.1 and 4.1 is partially symmetric, the corresponding optimal bounds can be obtained.*

6. Conclusion

In this paper, we further study the lower and upper bounds for the solution norm of PCP$(\Theta, \mathbf{q})$. When $\mathcal{A}_1$ is an R_0-tensor, an improved upper bound for the solution set with $\pi(\mathcal{A}_1)$ is gained, which is always tighter than the upper bound obtained in [16]. Finally, we prove that the proposed lower and upper bounds for the solution set of the PCP with partially symmetric tensor tuples are attainable (i.e., they reach the extreme values).

Author contributions

These authors contributed equally to this work.

Acknowledgments

This work is supported by Zunyi Science and Technology Cooperation Plan (Zun Shi Ke He HZ Zi(2024)142), Foundation of Zunyi Normal University for Doctor BS [2021]06, ZunshiBS[2023]4, Guizhou Provincial Basic Research Program (MS[2026]851), National Natural Science Foundation of China (11961082), Cross-integration Innovation Team of Modern Applied Mathematics and Life Sciences in Yunnan Province, China (202405AS350003), and Yunnan Key Laboratory of Modern Analytical Mathematics and Applications (202302AN360007).

Declaration of Use of Generative-AI Tools

The authors declare that they did not utilize any artificial intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that they have no competing interests.

References

1. Y. S. Song, L. Q. Qi, Properties of some classes of structured tensors, *J. Optim. Theory Appl.,* **165** (2015), 854–873. https://doi.org/10.1007/s10957-014-0616-5
2. M. L. Che, L. Q. Qi, Y. M. Wei, Positive-definite tensors to nonlinear complementarity problemss, *J. Optim. Theory Appl.,* **168** (2016), 475–487. https://doi.org/10.1007/s10957-015-0773-1
3. S. Q. Du, W. Y. Ding, Y. M. Wei, Acceptable solutions and backward errors for tensor complementarity problems, *J. Optim. Theory Appl.,* **188** (2021), 260–276. https://doi.org/10.1007/s10957-020-01774-y
4. Z. H. Huang, L. Q. Qi, Tensor complementarity problems-part III: applications, *J. Optim. Theory Appl.,* **183** (2019), 771–791. https://doi.org/10.1007/s10957-019-01573-0
5. X. Z. Wang, M. L. Che, Y. M. Wei, Randomized Kaczmarz methods for tensor complementarity problems, *Comput. Optim. Appl.,* **82** (2022), 595–615. https://doi.org/10.1007/s10589-022-00382-y
6. M. S. Gowda, Polynomial complementarity problems, *arXiv:1609.05267,* 2017. https://doi.org/10.48550/arXiv.1609.05267
7. Y. S. Song, L. Q. Qi, Properties of tensor complementarity problem and some classes of structured tensors, *arXiv:1412.0113,* 2017. https://doi.org/10.48550/arXiv.1412.0113
8. T. T. Shang, J. Yang, G. J. Tang, Generalized polynomial complementarity problems over a polyhedral cone, *J. Optim. Theory Appl.,* **192** (2022), 443–483. https://doi.org/10.1007/s10957-021-01969-x
9. V. T. Hieu, A. N. Iusem, P. H. Schmolling, T. Akiko, Univariate representations of solutions to generic polynomial complementarity problems, *J. Optim. Theory Appl.,* **207** (2025), 26. https://doi.org/10.1007/s10957-025-02788-0
10. X. L. Li, G. J. Tang, Strict feasibility for the polynomial complementarity problem, *J. Glob. Optim.,* **89** (2024), 57–71. https://doi.org/10.1007/s10898-023-01339-z
11. T. T. Shang, W. S. Jia, Q tensor tuples to polynomial complementarity problems, *Operations Research Lett.,* **61** (2025), 107297. https://doi.org/10.1016/j.orl.2025.107297
12. R. Andreani, L. D. Secchin, P. J. Silva, Convergence properties of a second order augmented Lagrangian method for mathematical programs with complementarity constraints, *SIAM J. Optim.,* **28** (2018), 2574–2600. https://doi.org/10.1137/17M1125698

13. L. Y. Ling, H. J. He, C. Ling, On error bounds of polynomial complementarity problems with structured tensors, *Optimization,* **67** (2018), 341–358. https://doi.org/10.1080/02331934.2017.1391254
14. X. L. Li, T. T. Shang, G. J. Tang, Lower bounds of the solution set of the polynomial complementarity problem, *Optim. Lett.,* **18** (2024), 497–511. https://doi.org/10.1007/s11590-023-02004-w
15. T. T. Shang, G. J. Tang, Structured tensor tuples to polynomial complementarity problems, *J. Glob. Optim.,* **86** (2023), 867–883. https://doi.org/10.1007/s10898-023-01302-y
16. Y. Xu, G. Y. Ni, M. S. Zhang, Bounds of the solution set to the polynomial complementarity problem, *J. Optim. Theory Appl.,* **203** (2024), 146–164. https://doi.org/10.1007/s10957-024-02484-5
17. T. G. Kolda, B. W. Bader, Tensor decompositions and applications, *SIAM Rev.,* **51** (2009), 455–500. https://doi.org/10.1137/07070111X
18. Y. Xu, W. Gu, Z. H. Huang, Estimations on upper and lower bounds of solutions to a class of tensor complementarity problems, *Front. Math. China,* **14** (2019), 661–671. https://doi.org/10.1007/s11464-019-0770-z
19. X. L. Li, G. J. Tang, The bounds of solutions to polynomial complementarity problems, *J. Optim. Theory Appl.,* **203** (2024), 2560–2591. https://doi.org/10.1007/s10957-024-02511-5
20. Y. Xu, Z. H. Huang, Bounds of the solution set of the tensor complementarity problem, *Optim. Lett.,* **15** (2021), 2701–2718. https://doi.org/10.1007/s11590-020-01697-7

AIMS Press