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Research article

Analysis of a queueing-inventory system with Client Choice Service under a modified (s, S) reorder policy

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Abstract: This study investigates a novel queueing-inventory system governed by a newly proposed Client Choice Policy, which extends the modified (s, S) ordering policy. The system considers a maximum capacity of S units of raw material. Upon the arrival of a client, the raw material is processed into finished goods after some random time. Client arrivals follow a Markovian arrival process (MAP) and a finite waiting platform is available. A reorder point is fixed as s. At the time of replenishment, the inventory gets filled up to the level of S units. In addition to that, the clients in the waiting platform may choose to purchase the raw materials. Clients who accept this offer leave the system immediately with the raw material, while the others remain in the queue to receive the finished goods. The numerical cost analysis estimates the expected total cost using steady-state probabilities derived from the finite generator matrix. The total cost is computed as the weighted sum of holding, replenishment, processing, and waiting costs. By evaluating these costs over different parameter combinations, the optimal policy that minimizes the expected total cost is identified numerically.

Keywords: clients choice policy; dual choice service; modified (s, S) policy; Markovian arrival process
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1. Introduction

In this article, the researchers introduce the Client Choice Policy (CCP), which extends the modified (s, S) ordering policy. In today's world, the proposed CCP can be observed across different industries. For example, in electrical supply stores, if a ready made electrical panel or lighting unit is temporarily out of stock, some clients, particularly electricians, may choose to purchase individual components (raw materials) at the time of replenishment and assemble the product themselves. Similar behavior is seen in e-commerce grocery systems where clients typically demand finished products, yet at the time of replenishment, they may be offered the option to directly purchase raw items such as unpacked produce instead of waiting for processing and packaging. Similarly, car repair workshops maintain inventories of raw spare parts that can be processed into finished components, but during replenishment, some clients may prefer to purchase raw parts immediately for self-repair rather than wait for full processing by the technician. Another instance can be found in restaurants, where clients generally expect prepared dishes, but during restocking or peak hours, they may be offered the option to directly buy raw ingredients such as marinated meat or dough for home preparation instead of waiting for service. These examples underline how the CCP mirrors real-world client flexibility.

Client Choice Policy (CCP)

The Client Choice Policy (CCP) is a state-dependent decision mechanism in a queueing–inventory system that allows clients to choose between available service options at specific decision points. In this policy, when inventory is replenished, clients who are waiting in the system are given a special opportunity: They may either continue waiting for the finished goods or choose to purchase the raw material directly. Clients who accept this option leave the system immediately with the raw material, while those who decline remain in the queue to receive the finished goods. This choice is offered only at replenishment times. This structured approach, which actively involves clients in the service decision process during replenishment epochs, is referred to as the CCP. The assembly cost of each inventory is assumed to be negligible in magnitude relative to the other cost components considered in this model.

To efficiently handle inventory replenishment in stochastic queueing-inventory systems, numerous researchers have used various reorder policies. [1] investigated a stochastic queueing-inventory framework operating under two restocking strategies, (s, S) and (s, Q). In this model, arriving clients who encounter a stockout enter an orbit of unlimited capacity and retry for service after random intervals. Once the inventory is replenished, these orbiting clients receive service based on availability. The study compared the efficiency of the two ordering policies by analyzing system performance measures and identifying the more effective replenishment approach. [2] studied a multiserver Markovian queue where servers are treated as inventory replenished under the (s, S) policy. Using a two-dimensional quasi-birth-and-death (QBD) process, the study analyzed steady-state behavior and average inventory cycle time. Additionally, it dealt with cost optimization, reducing waiting, holding, and ordering costs. [3] presented a mathematical model that incorporates storage limits, backlogs, and lost sales into the (R, Q) policy. It aimed to minimize total costs related to ordering, shortages, and overstocking under probabilistic demand and lead times. The study compared continuous and periodic review systems, showing how capacity constraints and review frequency influence optimal inventory decisions. [4–6] These studies developed (s, Q) and (r, Q) policies for inventory systems with perishable inventory, lead times, and limited resources, focusing on cost reduction and efficient service.

In queueing-inventory models, client choice among service options based on observable information like queue length, priority class, or waiting time has been integrated to understand its effect on inventory depletion, delays, and costs. For example, in a queueing-inventory model with skeptical and trusting clients, [7] examine how clients who prefer full service versus those willing to leave during service phases impact system performance and inventory control. Similarly, stability of queueing-inventory systems with clients of different priorities [8] explores how allowing different client priority classes alters admission control and lost sales when inventory stockouts occur. The paper "Queueing-inventory system with return of purchased items and client feedback" [9] introduces feedback behavior: Clients may return items or make repeat purchases after a thinking time, thereby affecting both queueing and inventory dynamics. Another related work, "Analysis of M/M/1/N Stochastic Queueing—Inventory System with Discretionary Priority Service and Retrial Facility" [10] includes clients who retry after being blocked or find the server busy and priority service rules guide which clients are served first. The survey "Queueing-inventory Systems" [11] summarizes many models, including those with multi class clients, priority service, retrials, and heterogeneous client behaviour, emphasizing how client choice mechanisms are becoming central to modern models. Further related works can be found in [12, 13], where researchers examined impulsive arrivals and order cancellations, analyzing their impact on system stability and performance. Additionally, they investigated online reservations where clients choose between distinct service modes, highlighting how differentiated client types influence queue dynamics and inventory utilization.

The integration of MAP in queueing-inventory models helps analyze performance measures like the probability of stockouts, waiting times, and inventory levels more accurately under stochastic fluctuations. Researchers have used MAP based frameworks to study systems with varying operational complexities. [14] examined a perishable inventory system with MAP arrivals and phase-type service times, emphasizing how perishability influences service and replenishment dynamics. [15] considered random replenishment opportunities under MAP-driven demand, showing that correlated arrivals significantly affect inventory availability. [16] analyzed an n-policy inventory system with impatient clients and MAP arrivals, highlighting how demand correlation impacts waiting time and service quality. Similarly, [17] investigated a system operating in a random environment, revealing that environmental variations alter service efficiency and inventory levels. Collectively, these studies demonstrate that MAP-based modeling provides a more flexible and realistic approach to analyzing queueing-inventory systems. Collectively, these studies suggest that MAP-based modeling gives a more flexible and realistic method to analyzing queueing-inventory systems.

In recent years, [18–20] examined more realistic and sophisticated queueing models that incorporate priority structures, capacity limits, and Markovian arrival patterns. These systems are typically analyzed using multidimensional Markov chain techniques to evaluate their long-run performance. The findings emphasize that well-designed threshold policies, controlled simultaneous service, and flexible service mechanisms play a significant role in enhancing overall system effectiveness and stability.

Several researchers have studied queueing–inventory systems under different operational settings. [21] analyzed an (s, S) inventory model with priority clients, retrial behavior, and impatience, focusing on steady-state system performance. [22] examined priority multi server retrial inventory systems with MAP arrivals and explored service optimization strategies. [23] considered a reservation-based queueing–inventory model with perishable items, retrial mechanisms, and cancellation features, along with performance evaluation and optimization analysis. [24] investigated a disaster queue with impatient clients in a random environment, primarily emphasizing analytical modeling approaches. In addition, [25] developed a two-stage queueing–inventory system for pre booked orders under an (s, Q) policy.

Despite these significant contributions, to the best of my knowledge, the existing literature does not consider client decision-making behavior at replenishment epochs. In practice, client behavior may change when inventory is replenished, yet this interaction between replenishment dynamics and client choice has not been adequately explored. This gap motivates the present study.

To address this limitation, we develop a single-server queueing–inventory system that incorporates a Client Choice Policy (CCP). In this framework, clients generally request finished goods. However, at replenishment times, those waiting in the system are offered an alternative: They may choose to purchase raw materials immediately instead of continuing to wait for the finished product. Clients who accept this offer exit the system after purchasing the raw materials, thereby introducing a new behavioral dimension into the model. For clients, the main purpose of accepting the raw material is to avoid extended waiting periods in the system and purchase inventory items on time. The product can be assembled after, during their own time.

By embedding this client choice mechanism within a modified (s,S) policy inventory structure, the proposed model explicitly connects client decision-making with inventory replenishment dynamics. This integration bridges an important gap in the literature and enables a more realistic and behaviorally responsive queueing–inventory analysis.

1.1. Structure of the paper

The paper's structure is as follows: The model is described and the key findings are discussed and the model's transitions are analyzed in Section 2; the steady-state analysis is the subject of Section 3; the essential metrics of the model are described in Section 4; the numerical illustration is interpreted in Section 5; and the conclusion and recommendations are finally presented in Section 6.

2. Overview of the system

The proposed model considers a single-server queueing-inventory system governed by the newly introduced Client Choice Policy (CCP), which extends the modified (s, S) ordering policy. The system maintains a maximum inventory of S raw material units, with replenishment initiated whenever the inventory level decreases to the reorder point s. A finite waiting platform of capacity N is provided, including one at the service point. All arriving clients enter into the platform via the Markovian arrival process (MAP). The MAP is represented by a generator matrix F = $F_0 + F_1$ of order w x w, where F_0 controls no-arrival transitions, and F_1 controls arrival transitions. The arriving client's stationary rate λ is defined as $\lambda = \eta F_1 e$, where the stationary row vector η of size 1 x w is generated using ηe = 0 and ηe = 1.

Every arriving client places a demand exclusively for finished goods. To fulfil this demand, the server converts raw materials into finished products, with each service consuming one unit of raw material. The service times are assumed to be independent exponentially distributed with parameter μ. If the server is idle at the time of a client arrival, the client is served immediately; otherwise, the client joins the waiting platform. When the waiting platform is already full, the arriving client is denied entry and leaves the system. The service discipline is based on the first-come-first-served (FCFS) rule.

We introduce a novel feature into the system through the CCP. At the time of replenishment, if no clients are present in the waiting platform, the inventory is simply refilled to the maximum level S, and the replenishment lead time is assumed to follow an exponential distribution with parameter β. However, if clients are waiting at the moment of replenishment, a special offer is provided. The waiting platform has a maximum capacity of N. Suppose there are j clients waiting in the waiting platform ($j \leq N$). We assume that each waiting client independently accepts the raw-material offer with probability p. Consequently, the number of clients who accept the offer follows a binomial distribution with parameters (j, p).

Thus, the probability that exactly i out of the j waiting clients accept the raw-material offer is given by

$$p_i = \binom{j}{i} p^i (1 - p)^{j-i}, \quad i = 0, 1, 2, \ldots, j,$$

which satisfies

$$\sum_{i=0}^{j} p_i = 1.$$

If i clients accept the offer, each of them immediately receives one unit of raw material and is removed from the waiting platform with rate $p_i\beta$, while the remaining $j - i$ clients continue to wait for finished goods.

2.1. Main results

The system assumptions indicate that the three-dimensional stochastic process $\{(V_1(t), V_2(t), V_3(t); t \geq 0\}$ is a continuous-time Markov chain (CTMC) with the state space
K=$\{(v_1,v_2,v_3)/0 \leq v_1 \leq S; 0 \leq v_2 \leq N; 1 \leq v_3 \leq w\}$.

In this process, we utilize the following arguments:

$v_1(t)$: the count of items in the inventory at time t.
$v_2(t)$: the count of clients in the waiting platform at time t.
$v_3(t)$: phase of clients arrival process at a given time t.

Transition

- **Transitions due to arrivals:**
  $* \; (v_1,v_2,v_3) \xrightarrow{F_1} (v_1, v_2 + 1, v_3)$
  $(0 \leq v_1 \leq S; 0 \leq v_2 \leq N - 1; 1 \leq v_3 \leq w)$

- **Transitions due to service:**
  $* \; (v_1,v_2,v_3) \xrightarrow{\mu I_w} (v_1 - 1, v_2 - 1, v_3)$
  $(1 \leq v_1 \leq S; 1 \leq v_2 \leq N; 1 \leq v_3 \leq w)$

- **Transitions due to replenishment:**
  $* \; (v_1,0,v_3) \xrightarrow{\beta I_w} (S, 0, v_3)$
  $(0 \leq v_1 \leq s; 1 \leq v_3 \leq w)$

  $* \; (v_1,v_2,v_3) \xrightarrow{p_i\beta I_w} (S, v_2 - i, v_3)$
  $(0 \leq v_1 \leq s; 1 \leq v_2 \leq N; 1 \leq v_3 \leq w;$ $i-$ number of clients choosing to accept raw materials)

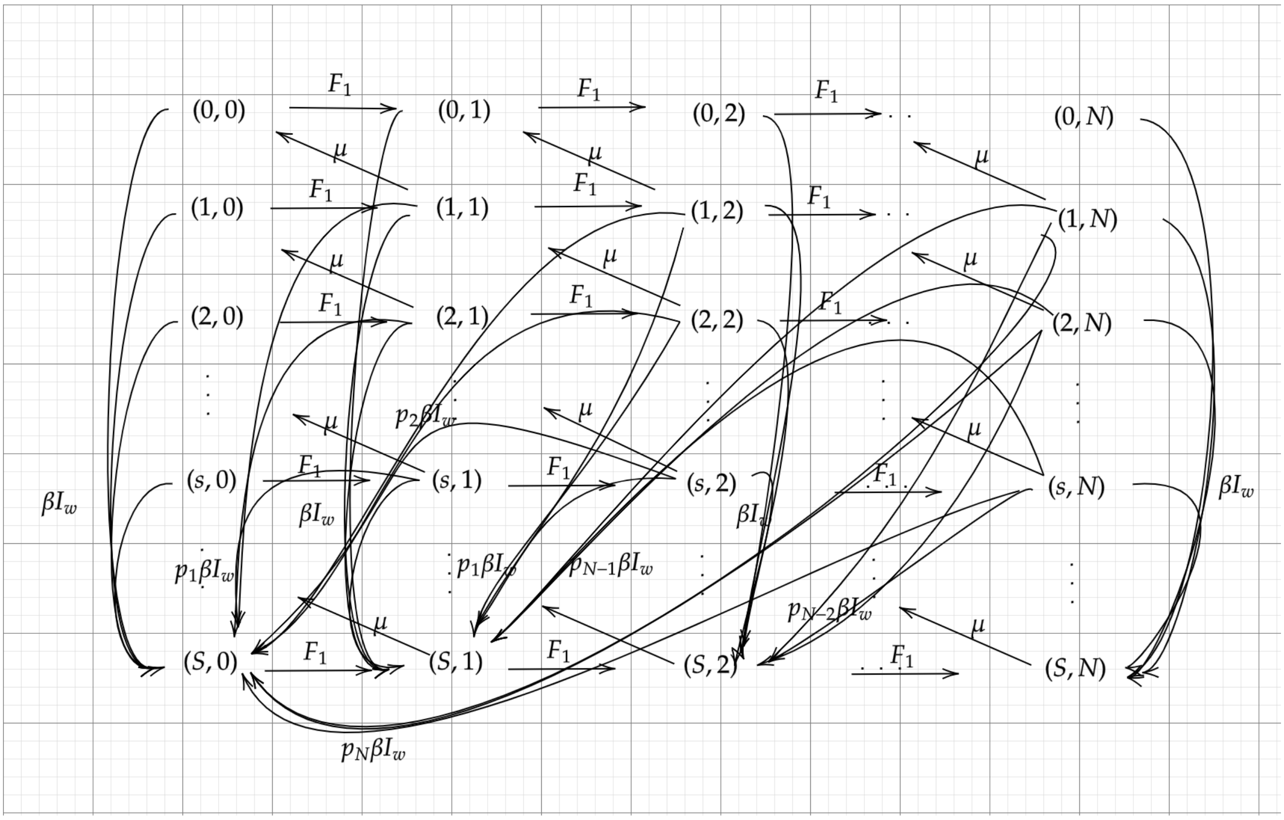

Figure 1. Graphical representation of these transitions.

The process's infinitesimal generator $\mathbb{V}$ is generated by

$$
\mathbb{V}=\begin{array}{c|cccccccccccc}
 & S & S-1 & S-2 & \dots & Q+1 & Q & \dots & s & s-1 & \dots & 1 & 0\\
\hline
S & A_2 & K_1 & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0}\\
S-1 & \mathbf{0} & A_2 & K_1 & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0}\\
S-2 & \mathbf{0} & \mathbf{0} & A_2 & \ddots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0}\\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
Q+1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & A_2 & K_1 & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0}\\
Q & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & A_2 & \ddots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0}\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots\\
s & B_1 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & A_1 & K_1 & \dots & \mathbf{0} & \mathbf{0}\\
s-1 & B_1 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & A_1 & \ddots & \mathbf{0} & \mathbf{0}\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots\\
1 & B_1 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & A_1 & K_1\\
0 & B_1 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & A_0
\end{array}
$$

where,

$$[K_1]_{(N+1)*(w)} = \begin{array}{cc} & \begin{array}{cccccc} 0 & 1 & 2 & \dots & N-1 & N \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \\ N-1 \\ N \end{array} & \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mu I_w & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mu I_w & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ddots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mu I_w & \mathbf{0} \end{pmatrix} \end{array},$$

and also

$$[B_1]_{(N+1)*(w)} = \begin{array}{cc} & \begin{array}{cccccc} 0 & 1 & 2 & \dots & N-1 & N \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \\ N-1 \\ N \end{array} & \begin{pmatrix} \beta I_w & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ p_1\beta I_w & (1-p_1)\beta I_w & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ p_2\beta I_w & p_1\beta I_w & (1-\sum_{i=1}^{2} p_i)\beta I_w & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ p_{N-1}\beta I_w & p_{N-2}\beta I_w & \dots & \ddots & (1-\sum_{i=1}^{N-1} p_i)\beta I_w & \mathbf{0} \\ p_N\beta I_w & p_{N-1}\beta I_w & \dots & \dots & p_1\beta I_w & (1-\sum_{i=1}^{N} p_i)\beta I_w \end{pmatrix} \end{array},$$

Further,

$$[A_0]_{(N+1)*(w)} = \begin{array}{cc} & \begin{array}{cccccc} 0 & 1 & 2 & \dots & N-1 & N \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \\ N-1 \\ N \end{array} & \begin{pmatrix} C_0 & F_1 & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & C_0 & F_1 & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & C_0 & \ddots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & F_1 & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_0 & F_1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_1 \end{pmatrix} \end{array},$$

$$C_0 = F_0 - \beta I_w \; ; C_1 = -\beta I_w,$$

and also

$$[A_1]_{(N+1)*(w)} = \begin{array}{cc} & \begin{array}{cccccc} 0 & 1 & 2 & \dots & N-1 & N \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \\ N-1 \\ N \end{array} & \begin{pmatrix} C_0 & F_1 & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & C_2 & F_1 & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & C_2 & \ddots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & F_1 & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_2 & F_1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_3 \end{pmatrix} \end{array},$$

$$C_2 = F_0 - (\mu + \beta)I_w \; ; C_3 = -(\mu + \beta)I_w,$$

with

$$[A_2]_{(N+1)*(w)} = \begin{array}{c|cccccc|} & 0 & 1 & 2 & \dots & N-1 & N \\ 0 & C_4 & F_1 & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ 1 & \mathbf{0} & C_5 & F_1 & \dots & \mathbf{0} & \mathbf{0} \\ 2 & \mathbf{0} & \mathbf{0} & C_5 & \ddots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \ddots & F_1 & \vdots \\ N-1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_5 & F_1 \\ N & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_6 \end{array},$$

$$C_4 = F_0\ ;\ C_5 = F_0 - \mu I_w\ ;\ C_6 = -\mu I_w.$$

3. Analysis in the steady state

The system is modeled as a continuous-time Markov process $\{(V_1(t), V_2(t), V_3(t)); t \geq 0\}$ defined on a finite state space K. It satisfies the Markov property and is represented by the generator matrix $\mathbb{V}$. The state space is finite and the transition structure ensures irreducible, aperiodic, and persistent non-null.

Thus, we have the following steady-state probability:

$$\mathbf{\Delta}^{(v_1,v_2,v_3)} = \lim_{t\to\infty} Pr[V_1(t) = v_1(t), V_2(t) = v_2(t), V_3(t) = v_3(t)|V_1(0), V_2(0), V_3(0)].$$

For notational convenience, define

$$\mathbf{\Delta} = [\mathbf{\Delta}^{(0)}, \mathbf{\Delta}^{(1)}, \mathbf{\Delta}^{(2)}, ..., \mathbf{\Delta}^{(S)}]$$

$$\mathbf{\Delta}^{(v_1)} = [\mathbf{\Delta}^{(v_1,0)}, \mathbf{\Delta}^{(v_1,1)}, \mathbf{\Delta}^{(v_1,2)}, ..., \mathbf{\Delta}^{(v_1,N)}], v_1 = 0, 1, 2, ..., S$$

$$\mathbf{\Delta}^{(v_1,v_2)} = [\mathbf{\Delta}^{(v_1,v_2,1)}, \mathbf{\Delta}^{(v_1,v_2,2)}, \mathbf{\Delta}^{(v_1,v_2,3)}, ..., \mathbf{\Delta}^{(v_1,v_2,w)}], v_1 = 0, 1, 2, ..., S;\ v_2 = 0, 1, 2, ..., N,$$

where $\mathbf{\Delta}^{(v_1,v_2,v_3)}$ denotes the steady state probability for the state (v_1, v_2, v_3) of the process, which exists and is given by

$$\mathbf{\Delta}\mathbb{V} = 0 \ \ and \ \ \mathbf{\Delta}e = 1, \tag{3.1}$$

where the equation represents the normalization condition.

The equation $\mathbf{\Delta}\mathbb{V} = 0$ yields the following set of equations:

$$\begin{aligned} \mathbf{\Delta}^{(c+1)}K_1 + \mathbf{\Delta}^{(c)}A_0 &= \mathbf{0},\ c = 0, && (3.2) \\ \mathbf{\Delta}^{(c+1)}K_1 + \mathbf{\Delta}^{(c)}A_1 &= \mathbf{0},\ c = 1, 2, ..., s, && (3.3) \\ \mathbf{\Delta}^{(c+1)}K_1 + \mathbf{\Delta}^{(c)}A_2 &= \mathbf{0}, && (3.4) \end{aligned}$$

$$\begin{aligned} & & & c = s+1, s+2, ..., S-1, \\ \mathbf{\Delta}^{(c)}A_2 + [\mathbf{\Delta}^{(c-Q)} + \mathbf{\Delta}^{(c-Q+1)} + \mathbf{\Delta}^{(c-Q+2)} + ... + \mathbf{\Delta}^{(c-Q+s)}]B_1 &=& \mathbf{0}, & \ c = S. \end{aligned} \tag{3.5}$$

After lengthy simplifications, the above equations, (except (3.5)), yield

$$\mathbf{\Delta}^{(c)} = \begin{cases} (-1)^{(Q+s)}\mathbf{\Delta}^{(S)}(K_1A_2^{-1})^{(Q-1)}(K_1A_1^{-1})^{(s)}(K_1A_0), & \text{c=0,} \\ (-1)^{(Q+s-c)}\mathbf{\Delta}^{(S)}(K_1A_2^{-1})^{(Q-1)}(K_1A_1^{-1})^{(s+1-c)}, & \text{c=1,2,...,s,} \\ (-1)^{(S-c)}\mathbf{\Delta}^{(S)}(K_1A_2^{-1})^{(S-c)}, & \text{c=s+1, s+2, ..., S-1,} \end{cases}$$

where $\mathbf{\Delta}^{(Q)}$ can be obtained by solving equations (3.5) and $\mathbf{\Delta}e = 1$.
Assume
$H_0 = (-1)^{(Q+s)}(K_1A_2^{-1})^{(Q-1)}(K_1A_1^{-1})^{(s)}(K_1A_0)$,

$$H_i = (-1)^{(Q+s-i)}(K_1A_2^{-1})^{(Q-1)}(K_1A_1^{-1})^{(s+1-i)}, i = 1, 2, ..., s.$$

Substitute $H_0, H_1, ..., H_s$ into equation (3.5):

$$\mathbf{\Delta}^{(0)}B_1 + \mathbf{\Delta}^{(1)}B_1 + \mathbf{\Delta}^{(2)}B_1 + \ldots + \mathbf{\Delta}^{(s)}B_1 + \mathbf{\Delta}^{(S)}A_2 = \mathbf{0},$$

$$\mathbf{\Delta}^{(S)}H_0B_1 + \mathbf{\Delta}^{(S)}H_1B_1 + \mathbf{\Delta}^{(S)}H_2B_1 + \ldots + \mathbf{\Delta}^{(S)}H_sB_1 + \mathbf{\Delta}^{(S)}A_2 = \mathbf{0},$$

$$\mathbf{\Delta}^{(S)}[H_0B_1 + H_1B_1 + H_2B_1 + ... + H_sB_1 + A_2] = \mathbf{0}. \tag{3.6}$$

Finally, applying the normalization condition, we get

$$\sum_{c=0}^{S} \mathbf{\Delta}^{(c)}e = 1,$$

$$[\mathbf{\Delta}^0 + \mathbf{\Delta}^1 + ... + \mathbf{\Delta}^S]e = 1,$$

$$\mathbf{\Delta}^{(S)}[H_0 + H_1 + ... + H_{(S-1)} + I]e = 1. \tag{3.7}$$

Solving (3.6) and (3.7) we will get $\mathbf{\Delta}^{(S)}$.

4. Assessment of expected system performance measures

To calculate the total cost (TC) rate, we compute a number of system performance metrics in its steady state in this section.

4.1. Expected inventory level [E_I]:

$$E_I = \sum_{v_1=1}^{S}\sum_{v_2=0}^{N} v_1\mathbf{\Delta}^{(v_1,v_2)}\mathbf{e}.$$

4.2. *Expected rate of reorder [E_R]:*

$$E_R = \sum_{v_2=1}^{N} \mu \boldsymbol{\Delta}^{(s+1,v_2)}\mathbf{e}.$$

4.3. *Expected number of clients in waiting platform [E_W]:*

$$E_W = \sum_{v_1=0}^{S}\sum_{v_2=1}^{N} v_2 \boldsymbol{\Delta}^{(v_1,v_2)}\mathbf{e}.$$

4.4. *Expected rate of client loss [E_L]:*

$$E_L = \sum_{v_1=0}^{S} \boldsymbol{\Delta}^{(v_1,N)} F_1 \mathbf{e}.$$

Formulation of the cost function

The following expenses are taken into account in order to calculate the total cost (TC) per unit time:

$$TC(S,s) = c_h E_I + c_{sc} E_R + c_w E_W + c_l E_L,$$

where

c_h : the cost of inventory holding per unit item,
c_{sc} : the setup cost per order,
c_w : client waiting costs in the waiting area per unit of time,
c_l : cost of a client lost per unit time.

5. Numerical experiments

We present several illustrative numerical examples that demonstrate the convexity of the MAP and the predicted cost rate.

1. **Hyper-exponential (HEX)**:

$$F_0 = \begin{bmatrix} -15 & 0 \\ 0 & -5 \end{bmatrix}; \quad F_1 = \begin{bmatrix} 13.5 & 1.5 \\ 4.5 & 0.5 \end{bmatrix}.$$

2. **Erlang (ER)**:

$$F_0 = \begin{bmatrix} -3 & 3 & 0 \\ 0 & -3 & 3 \\ 0 & 0 & -3 \end{bmatrix}; \quad F_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix}.$$

3. **Negative correlation (NC)**:

$$F_0 = \begin{bmatrix} -2.35 & 2.35 & 0 \\ 0 & -2.35 & 0 \\ 0 & 0 & -3.5 \end{bmatrix}; \quad F_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0.0235 & 0 & 2.3265 \\ 3.4650 & 0 & 0.0350 \end{bmatrix}.$$

4. **Positive correlation (PC)**:

$$F_0 = \begin{bmatrix} -2.35 & 2.35 & 0 \\ 0 & -2.35 & 0 \\ 0 & 0 & -3.5 \end{bmatrix}; \quad F_1 = \begin{bmatrix} 0 & 0 & 0 \\ 2.3265 & 0 & 0.0235 \\ 0.0350 & 0 & 3.4650 \end{bmatrix}.$$

The client process has a negative correlation (NC) (positive correlation (PC)) for arrival and coefficient of variance $C_{var} = 2\,\lambda\,\eta(-F_0)^{-1}e - 1$= 0.9342 (0.9342) and coefficient of correlation C_{cor} = ($\lambda\eta(-F_0)^{-1})\ F_1(-F_0)^{-1})e - 1\,/\,C_{var}$ = -0.25950 (0.25950) for an arrival rate $\lambda = 1$.

A brief overview of our perspective model numerical examples and how the parameters ($\mu = 11.2$, $\beta = 8.1$, $\lambda = 1$, N=10) and cost values ($C_h = 0.05, C_{sc} = 4, C_w = 2.1, C_l = 0.0062$) are listed below in the results and discussions section.

The numerical parameter values considered in this study are assumed for computational illustration and were varied systematically through a trial-and-error approach. The purpose of this experimentation is to investigate the structural behavior of the proposed model and to verify important properties such as the convexity and stability of the total cost function TC(S, s). The selected ranges of S and s are centered around the optimal region in order to demonstrate the existence of a unique minimum. Although the parameters are not calibrated from a specific dataset, they are chosen to reflect realistic operational magnitudes commonly observed in service inventory systems, including e-commerce platforms and processing-based service environments.

5.1. *Results and discussions*

- In the case of a hyper-exponential distribution, we discuss the behavior of the total cost of two variables, TC(S, s). We note that each row and column has the minimum possible value for TC. The lowest feasible TC rate is indicated by the **bold** values in each row and the underlined values in each column. At the same time, the function TC(S, s) specifies the value of the least cost rate by both bold and underlined values. At $S = 20$ and $s = 4$, the optimal total cost value TC(S, s) = 12.217673. The function TC(S, s) is convex, as shown in **Table 1 and Figure 2**.
  To explain the motivation behind our simulation settings, we selected parameter values that reflect realistic operating conditions in service inventory systems facing demand uncertainty. In particular, under the hyper-exponential arrival structure, we examine the behaviour of the total cost function TC(S, s) to represent highly variable demand patterns often observed in practice, such as in e-commerce fulfilment centers and spare parts management systems.
- **Table 2 and Table 3** clearly show that the modified (s, S) policy outperforms the normal (s, S) policy in terms of cost efficiency. For every combination of μ and β, the expected total cost under the modified policy is consistently lower than that of the normal policy. This comparison highlights that adopting the modified (s, S) policy effectively reduces the overall expected system cost.

To clarify the real world motivation behind the simulation settings, we compare the proposed modified (s, S) policy with the normal (s, S) policy under varying service rates μ and replenishment rates β, which reflect practical conditions in e-commerce, manufacturing, and spare parts systems.

- **Figure 3** displays how the optimal expected total cost responds, where S $\in$ [17, 23] and s $\in$ [1, 7], together with reorder rates β =7.1, 8.1, 9.1. The illustration shows that the interaction between these parameters has a notable influence on the system's overall cost behavior. As the reorder rate β increases, the optimal cost TC(S, s) consistently declines, with the corresponding values observed as 12.245176, 12.217673, and 12.194109.
- **Figure 4** presents how the optimal expected total cost varies with changes, where S $\in$ [17, 23] and s $\in$ [1, 7], together with service rates μ =11, 11.5, 12. The illustration shows that the interaction between these parameters has a notable influence on the system's overall cost behavior. As the service rate μ increases, the optimal cost TC(S, s) consistently declines, with the corresponding values observed as 12.338781, 12.033865, and 11.724631.
- **Figure 5** illustrates the impact of the optimal expected total cost, where S $\in$ [17, 23] and s $\in$ [1, 7], together with setup cost rates C_{sc} =3.5, 4, 4.5. The illustration shows that the interaction between these parameters has a notable influence on the system's overall cost behavior. As the setup cost rate C_{sc} increases, the optimal cost TC(S, s) consistently rises, with the corresponding values observed as 11.976742, 12.217673, and 12.431981.
- **Figure 6** shows how modifications affect the ideal estimated total cost, where S $\in$ [17, 23] and s $\in$ [1, 7], together with holding cost rates C_h =0.04, 0.06, 0.08. The illustration shows that the interaction between these parameters has a notable influence on the system's overall cost behavior. As the holding cost rate C_h increases, the optimal cost TC(S, s) consistently rises, with the corresponding values observed as 12.106303, 12.324572, and 12.520949.
- **I**n Figures 7 and **8**, the results show that both the service rate μ and the waiting platform (WP) capacity strongly influence the expected number of clients in the waiting platform and the level of client loss. An increase in the service rate reduces the expected number of clients and expected client loss in WP, whereas if the WP capacity allows more clients to accumulate, then there is an increase in the expected number of clients and expected client loss in WP.
- **Figure 9** shows how the expected reorder level is influenced by both the service rate μ and the reorder rate β. As the service rate μ increases, the server completes services more quickly, leading to a rise in the expected reorder level. Likewise, when the reorder rate β increases, the system replenishes more frequently, which also results in a higher expected reorder level.
- **Table 4** illustrates how the expected total cost (TC) is affected by both the service rate μ and the reorder rate β. As the service rate μ and the reorder rate β increases, the expected total cost decreases, reflecting the improved overall performance of the system.
- **Table 5** demonstrates how variations in the ordering rate β and the WP capacity affect the system's estimated total cost. When the reorder rate β increases, replenishment occurs more rapidly, reducing shortages and lowering the total cost. In contrast, raising the WP capacity allows more clients to wait in the system, which increases the associated holding and waiting fees. As a result, a larger WP capacity translates to a higher projected overall cost.
- **Table 7** illustrates how the expected inventory level responds to variations in the reorder rate β and the service rate μ. When the reorder rate β increases, inventory is replenished more frequently, allowing the system to raise the expected inventory level. In contrast, as the service rate μ

increases, items are delivered more rapidly, which reduces the overall inventory level in the system. Consequently, this lowers the expected inventory level.

- **Tables 6** and **8** show the expected inventory level, reorder level, number of clients in WP, and client loss whenever μ and β increase under hyper-exponential, Erlang, negative, and positive correlation.
- **Table 9** explores the influence of increasing setup and holding costs on system performance. As both cost components grow, the optimal projected total cost also increases, indicating the extra financial burden associated with more expensive inventory maintenance and replenishment activities.
- **Table 10** displays an expected overall cost for various cost parameter configurations. The findings make it possible to compare how changes in the underlying cost components affect the total cost of the system. **Table 11** highlights the repercussions of major performance measurements, demonstrating how changes in system parameters affect inventory levels, client behaviour, and overall operational efficiency.

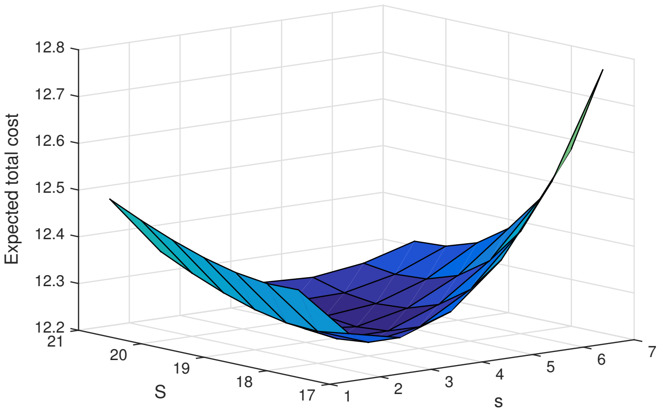

Figure 2. Impact of S ∈ [17, 24] and s ∈ [1, 7] on the optimal expected total cost value.

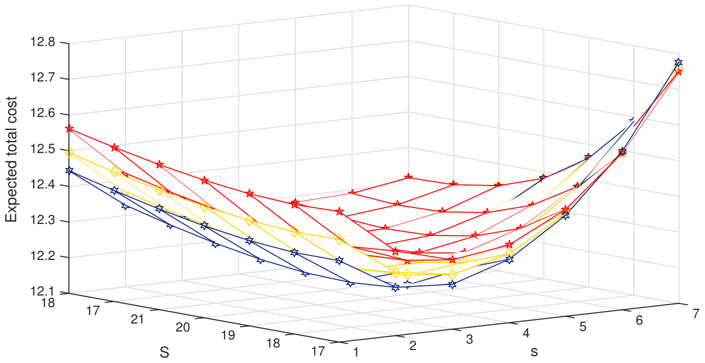

Figure 3. Effect of S ∈ [17, 23], s ∈ [1, 7], and β =7.1, 8.1, 9.1 on the optimal expected total cost value.

Table 1. Optimal expected total cost on (S vs s).

S/s	1	2	3	4	5	6	7
17	12.387593	12.277844	**12.253541**	12.292941	12.387955	12.540770	12.763000
18	12.386569	12.268309	**12.228997**	12.246138	12.309796	12.418939	12.579942
19	12.395817	12.272137	12.222154	**12.223053**	12.263758	12.341160	12.458282
20	12.412912	12.286085	12.228626	**12.217673**	12.241464	12.295535	12.380707
21	12.436007	12.307713	12.245151	**12.225589**	12.236879	12.273669	12.335304
22	12.463686	12.335169	12.269278	**12.243531**	12.245589	12.269521	12.313673
23	12.494852	12.367032	12.299148	12.269039	**12.264312**	12.278667	12.309764
24	12.528654	12.402207	12.333338	12.300248	**12.290584**	12.297821	12.319150

Table 2. Expected total cost on (μ vs β) using a modified (s,S) policy.

μ/β	2.5	2.6	2.7	2.8	2.9
4.6	18.966210	18.958048	18.950564	18.943683	18.937341
4.7	18.884108	18.875446	18.867501	18.860194	18.853455
4.8	18.800908	18.791730	18.783308	18.775560	18.768412
4.9	18.716619	18.706909	18.697996	18.689792	18.682222
5	18.631254	18.620996	18.611576	18.602902	18.594897

Table 3. Expected total cost on (μ vs β) using a normal (s,S) policy.

μ/β	2.5	2.6	2.7	2.8	2.9
4.6	20.254454	20.253712	20.253204	20.252893	20.252749
4.7	20.211423	20.210563	20.209951	20.209552	20.209332
4.8	20.167836	20.166852	20.166134	20.165643	20.165344
4.9	20.123697	20.122586	20.121759	20.121172	20.120790
5	20.079013	20.077772	20.076831	20.076146	20.075679

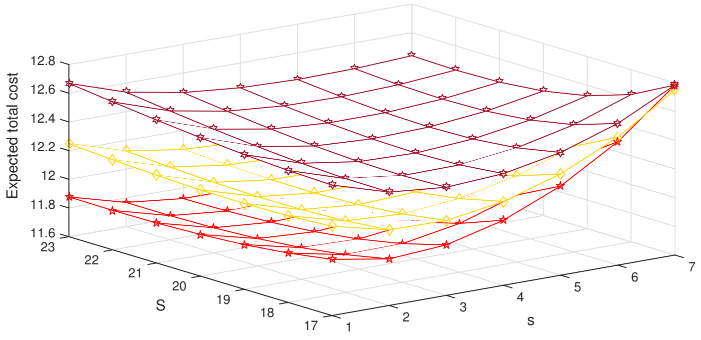

Figure 4. Effect of S ∈ [17, 23], s ∈ [1, 7], and μ =11, 11.5, 12 on the optimal expected total cost value.

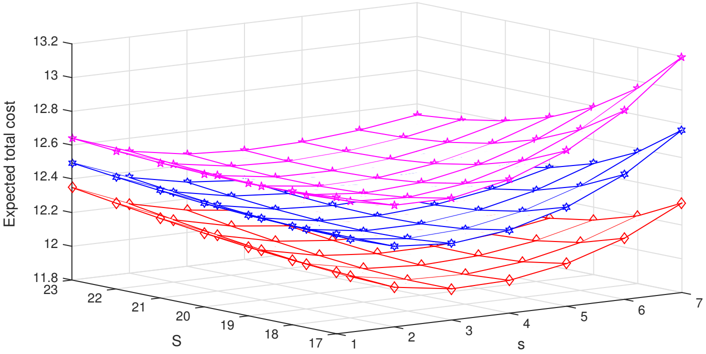

Figure 5. Effect of S ∈ [17, 23], s ∈ [1, 7], and C_{sc} =3.5, 4, 4.5 on the optimal expected total cost value.

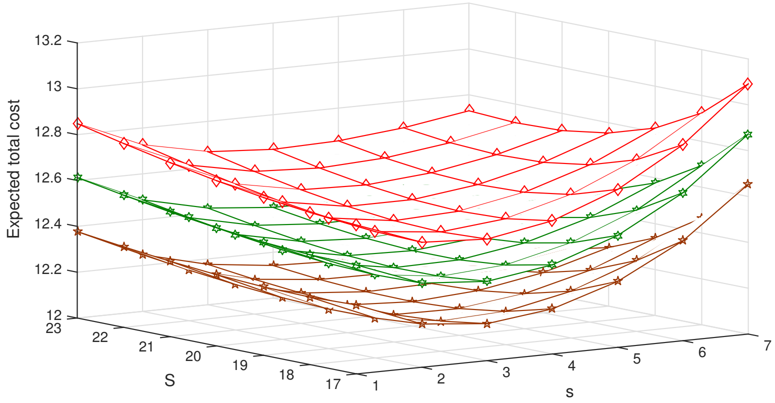

Figure 6. Effect of S ∈ [17, 23], s ∈ [1, 7], and C_h =0.04, 0.06, 0.08 on the optimal expected total cost value.

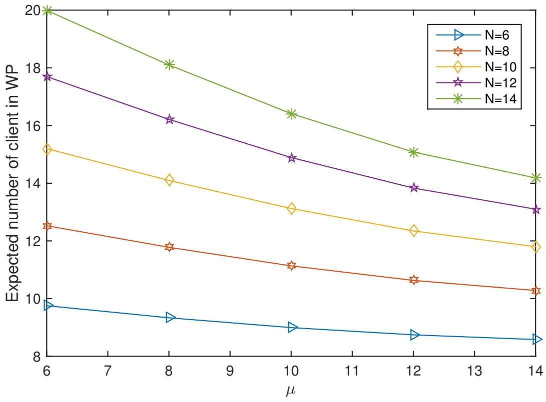

Figure 7. Expected number of clients in the waiting platform on (μ vs N).

Table 4. Expected total cost on (μ vs β).

μ/β	7	7.5	8	8.5	9
10	13.157107	13.139914	13.125851	13.114236	13.104553
10.5	12.740420	12.723373	12.709573	12.698298	12.689007
11	12.377780	12.361348	12.348220	12.337645	12.329064
11.5	12.072618	12.057263	12.045208	12.035683	12.028120
12	11.826299	11.812457	11.801849	11.793702	11.787443

Table 5. Expected total cost on (β vs N).

β/N	7	8	9	10	11	12	13
6.5	9.660797	10.576215	11.448443	12.269094	13.032020	13.734629	14.377484
7	9.653674	10.564682	11.432485	12.248697	13.007177	13.705360	14.343864
7.5	9.648673	10.555999	11.420115	12.232642	12.987442	13.681978	14.316909
8	9.645237	10.549454	11.410464	12.219895	12.971616	13.663108	14.295068
8.5	9.642962	10.544529	11.402896	12.209699	12.958813	13.647738	14.277199
9	9.641552	10.540840	11.396938	12.201489	12.948375	13.635111	14.262449
9.5	9.640790	10.538102	11.392233	12.194839	12.939803	13.624656	14.250172

Table 6. Impact of β on system measures with various MAP representations.

Client arrivals	β	Ei	Er	Ew	El
MAP with HEX	7	11.090940	0.442392	4.721967	1.363545
	7.5	11.112034	0.445260	4.708383	1.353944
	8	11.132839	0.447778	4.697045	1.346377
	8.5	11.153072	0.450005	4.687484	1.340340
	9	11.172566	0.451988	4.679347	1.335470
MAP with ER	7	11.379321	0.617147	0.090091	0.000001
	7.5	11.386688	0.617324	0.090094	0.000001
	8	11.393218	0.617472	0.090097	0.000001
	8.5	11.399048	0.617597	0.090099	0.000001
	9	11.404285	0.617703	0.090101	0.000001
MAP with NC	7	11.447987	0.965257	0.180930	0.000003
	7.5	11.458099	0.965723	0.180933	0.000003
	8	11.466921	0.966080	0.180936	0.000003
	8.5	11.474675	0.966351	0.180939	0.000003
	9	11.481537	0.966554	0.180941	0.000003
MAP with PC	7	11.371347	0.890702	0.192533	0.000004
	7.5	11.381184	0.891678	0.192511	0.000004
	8	11.389877	0.892532	0.192496	0.000004
	8.5	11.397613	0.893285	0.192485	0.000004
	9	11.404543	0.893954	0.192476	0.000004

Table 7. Expected inventory level on (β vs μ).

β/μ	7	7.5	8	8.5	9	9.5
6.5	11.091872	11.081879	11.074514	11.069530	11.066632	11.065489
7	11.117531	11.107385	11.099682	11.094211	11.090715	11.088895
7.5	11.141374	11.131304	11.123508	11.117805	11.113965	11.111719
8	11.163454	11.153619	11.145906	11.140155	11.136160	11.133675
8.5	11.183868	11.174378	11.166871	11.161207	11.157197	11.154615
9	11.202734	11.193662	11.186449	11.180968	11.177046	11.174473
9.5	11.220175	11.211568	11.204709	11.199481	11.195723	11.193237

Table 8. Impact of μ on system measures with various MAP representations.

client arrivals	μ	Ei	Er	Ew	El
MAP with HEX	10	11.136661	0.267548	5.468542	1.998179
	10.5	11.136325	0.336727	5.139460	1.700439
	11	11.136668	0.414715	4.819668	1.439597
	11.5	11.137437	0.500891	4.511982	1.213211
	12	11.138406	0.594427	4.218619	1.018418
MAP with ER	10	11.396033	0.615492	0.101204	0.000003
	10.5	11.395310	0.616417	0.096255	0.000001
	11	11.394671	0.617212	0.091774	0.000002
	11.5	11.394103	0.617899	0.087696	0.000002
	12	11.393595	0.618497	0.083970	0.000003
MAP with NC	10	11.471538	0.948854	0.206250	0.000002
	10.5	11.470197	0.956603	0.194888	0.000001
	11	11.468997	0.963556	0.184714	0.000002
	11.5	11.467916	0.969827	0.175552	0.000003
	12	11.466935	0.975510	0.167256	0.000002
MAP with PC	10	11.397077	0.868220	0.222526	0.000011
	10.5	11.394578	0.879122	0.208926	0.000007
	11	11.392331	0.888997	0.196914	0.000005
	11.5	11.390303	0.897980	0.186227	0.000003
	12	11.388466	0.906185	0.176655	0.000002

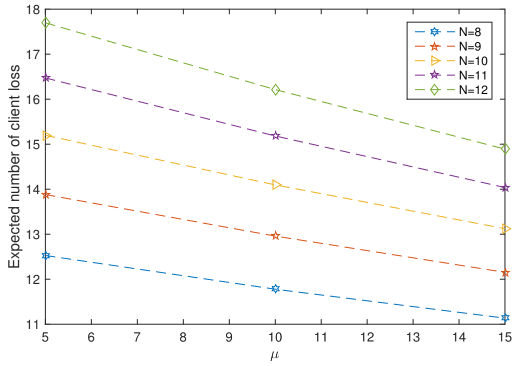

Figure 8. Expected number of client loss on (μ vs N).

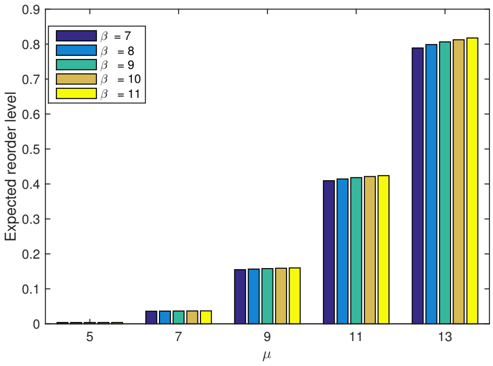

Figure 9. Expected reorder level on (μ vs β).

Table 9. Expected total cost on (C_h vs C_{sc})

C_h/C_{sc}	3 S s TC(S, s)	3.5 S s TC(S, s)	4 S s TC(S, s)	4.5 S s TC(S, s)	5 S s TC(S, s)
0.01	20 6 11.274718	22 6 11.523364	24 6 11.735398	24 5 11.924384	24 5 12.102432
0.03	18 5 11.500613	20 5 11.764932	21 4 11.992645	22 4 12.191512	24 4 12.369344
0.05	17 4 11.695658	18 4 11.976742	20 4 12.217673	21 4 12.431981	22 3 12.624423
0.07	17 4 11.887742	17 3 12.169372	18 3 12.423631	20 3 12.649600	21 3 12.851878
0.09	17 4 12.079826	17 3 12.353800	18 3 12.618266	19 3 12.855404	21 4 12.431981

Table 10. Expected total cost values for different cost values.

		C_h=0.01					C_h=0.03				
	C_w	1	2	3	4	5	1	2	3	4	5
C_l	0.001	6.824820	11.519823	16.214826	20.909829	25.604832	7.047558	11.742562	16.437565	21.132568	25.827571
	0.003	6.827510	11.522513	16.217516	20.912519	25.607522	7.050249	11.745252	16.440255	21.135258	25.830261
	0.005	6.830200	11.525203	16.220206	20.915209	25.610212	7.052939	11.747942	16.442945	21.137948	25.832951
	0.007	6.832890	11.527893	16.222896	20.917899	25.612903	7.055629	11.750632	16.445635	21.140638	25.835641
	0.009	6.835580	11.530583	16.225586	20.920590	25.615593	7.058319	11.753322	16.448325	21.143328	25.838331
	0.011	6.838270	11.533273	16.228277	20.923280	25.618283	7.061009	11.756012	16.451015	21.146018	25.841022
		C_h=0.05					C_h=0.07				
C_l	0.001	7.270297	11.965300	16.660303	21.355307	26.050310	7.493036	12.188039	16.883042	21.578045	26.273049
	0.003	7.272987	11.967990	16.662994	21.357997	26.053000	7.495726	12.190729	16.885732	21.580735	26.275739
	0.005	7.275677	11.970681	16.665684	21.360687	26.055690	7.498416	12.193419	16.888422	21.583426	26.278429
	0.007	7.278368	11.973371	16.668374	21.363377	26.058380	7.501106	12.196109	16.891113	21.586116	26.281119
	0.009	7.281058	11.976061	16.671064	21.366067	26.061070	7.503796	12.198800	16.893803	21.588806	26.283809
	0.011	7.283748	11.978751	16.673754	21.368757	26.063760	7.506487	12.201490	16.896493	21.591496	26.286499
		C_h=0.09					C_h=0.11				
C_l	0.001	7.715775	12.410778	17.105781	21.800784	26.495787	7.938513	12.633517	17.328520	22.023523	26.718526
	0.003	7.718465	12.413468	17.108471	21.803474	26.498477	7.941204	12.636207	17.331210	22.026213	26.721216
	0.005	7.721155	12.416158	17.111161	21.806164	26.501168	7.943894	12.638897	17.333900	22.028903	26.723906
	0.007	7.723845	12.418848	17.113851	21.808854	26.503858	7.946584	12.641587	17.336590	22.031593	26.726596
	0.009	7.726535	12.421538	17.116541	21.811545	26.506548	7.949274	12.644277	17.339280	22.034283	26.729287
	0.011	7.729225	12.424228	17.119232	21.814235	26.509238	7.951964	12.646967	17.341970	22.036973	26.731977

Table 11. Ramification of some measures.

	S=18				S=19			
s	Ei	Er	Ew	El	Ei	Er	Ew	El
2	9.419832	0.445194	4.765644	1.401514	9.929137	0.410264	4.821767	1.437631
3	9.731727	0.488440	4.657352	1.324362	10.241796	0.447156	4.720465	1.364820
4	10.115484	0.538792	4.560661	1.259280	10.626382	0.489759	4.630769	1.303859
5	10.541342	0.598356	4.467557	1.199020	11.053083	0.539675	4.545080	1.247795
6	10.991742	0.669872	4.372763	1.139344	11.504353	0.598946	4.458459	1.192581
7	11.456256	0.757036	4.272526	1.077638	11.969806	0.670266	4.367470	1.135734
	S=20				S=21			
2	10.438221	0.380167	4.873517	1.471120	10.947087	0.354042	4.921371	1.502241
3	10.751594	0.411884	4.778483	1.402279	11.261136	0.381505	4.831983	1.437037
4	11.136938	0.448245	4.695003	1.345060	11.647183	0.412784	4.754057	1.383225
5	11.564393	0.490487	4.615848	1.292786	12.075325	0.448846	4.680693	1.334381
6	12.016418	0.540161	4.536369	1.241584	12.528021	0.490888	4.607495	1.286793
7	12.482659	0.599270	4.453392	1.189095	12.994942	0.540427	4.531511	1.238220
	S=22				S=23			
2	11.455738	0.331209	4.965740	1.531220	11.964174	0.311127	5.006979	1.558257
3	11.770432	0.355145	4.881460	1.469356	12.279487	0.332117	4.927337	1.499466
4	12.157140	0.382247	4.808519	1.418651	12.666822	0.355757	4.858890	1.451602
5	12.585917	0.413280	4.740315	1.372918	13.096194	0.382657	4.795305	1.408700
6	13.039221	0.449177	4.672674	1.328596	13.550060	0.413553	4.732607	1.367334
7	13.506743	0.491107	4.602833	1.283546	14.018126	0.449358	4.668195	1.325461

6. Conclusion and suggestions

This work introduces a new queueing-inventory system that incorporates a CCP into an extended (s, S) structure, complete with MAP arrivals and a finite-waiting platform. The selected special offer mechanism at reorder instants captures client behavior and reshapes inventory service interactions. Steady-state analysis using a finite generator matrix enables precise cost evaluation across holding, replenishment, processing, waiting, and special offer components, culminating in the determination of an optimal strategy. By connecting CCP to the reorder instant, each replenishment moment becomes a strategic control point, minimizing congestion, stabilizing demand, and improving effective capacity. As the ordering rate increases, these benefits become more obvious, resulting in a more flexible and efficient operational mechanism than (s, S) models.

Future research may extend the proposed queueing-inventory system by incorporating item perishability to reflect time sensitive products and by introducing multiple servers to analyze more complex service dynamics under the client choice mechanism. The model can also be enhanced by considering client impatience behaviors such as balking or reneging, which would provide a more realistic assessment of system performance.

Notations and abbreviations

$[A]_{ij}$: (i,j)-th element of the matrix A
0 : Zero matrix
I : Identity matrix of an appropriate order.
e : Column vector of 1's with appropriate order.
$\sum_{i=k}^{j} p_i = p_k + p_{k+1} + \ldots + p_j$
MAP : Markovian arrival process
Q = **(S - s) > s + 1)**
CCP : Client Choice Policy

Author contributions

K. Lawrence: Conceptualization, data curation, resources, and formal analysis; N. Anbazhagan: Conceptualization, methodology, formal analysis, writing—original draft preparation; S. Amutha: Investigation, validation, writing review and editing, and project administration; Tran Son Hai: Software, visualization and Validation; Woong Cho: Data curation, resources, supervision, project administration, funding acquisition; Gyanendra Prasad Joshi: Validation, visualization, supervision, writing-review and editing.

Use of Generative-AI tools declaration

The authors declare that no generative AI tools were used in the preparation of this manuscript.

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Conflict of interest

The authors declare no conflicts of interest.

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