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Research article

Dynamic control of a multi-product firm's product and process innovation in a market with network externality

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Abstract: Despite the extensive literature on product and process innovation within firms, relatively little attention has been paid to how consumer reference perceptions regarding product differentiation dynamically influence the innovation strategies of multi-product firms in markets with network externality. This paper addressed this gap by developing a dynamic optimal control model to analyze the product and process innovation of a multi-product firm in the presence of reference effects about product differentiation and network externality. The key features of this study are as follows: (i) A multi-product firm offers horizontally differentiated products and engages in both product and process innovation; (ii) the network value depends on both network size and the degree of product substitutability; (iii) market demand incorporates price and, network size, as well as reference perceptions regarding product differentiation; and (iv) a special case was examined where network size evolves proportionally with market demand. The analytical results yielded several propositions and conclusions. The primary findings indicate that: (i) The saddle-point stability of the steady-state equilibrium, which is crucial for both firm decision-making and government regulation, hinges critically on the discount rate and the consumer memory parameters; (ii) product and process innovation efforts exhibit a complementary relationship; (iii) both types of innovation efforts increase with the proportional coefficient governing network expansion; and (iv) as market demand decreases, the firm intensifies its efforts in both product and process innovation. These findings contribute to the literature by clarifying the dynamic interactions among multi-product innovation strategies, consumer reference perceptions about the product differentiation, and network externality.

Keywords: network externality; product innovation; reference effects; process innovation
Mathematics Subject Classification: 49N90, 37N40

1. Introduction

Technological innovation plays a central role in firms' long-term development and industrial evolution [1]. Particularly, product and process innovation represent two fundamental dimensions of firms' innovative activities. Product innovation typically improves the intrinsic characteristics of the product [2, 3], while process innovation focuses on reducing production costs [4]. Because of their strategic importance, these two forms of innovation have attracted considerable attention in the economics and management literature. Pioneering research in the field of product and process innovation has been developed by notable scholars such as in references [5–8], among other important contributions. Utterback and Abernathy [5] and Athey and Schmutzler [6] explored the dynamic interplay between a firm's product and process innovation throughout the entire product life cycle. Klepper provided a model that underscored the disparities in a firm's innovation capabilities and the significance of firm size in the allocation of innovation returns to elucidate certain regularities [7]. Based on Klepper's work [7], Yin and Zuscovitch explored how firm strategy impacts its initiatives in both product and process innovation [8]. Bonanno and Haworth examined the decision-making between product and process innovation within the contexts of Bertrand and Cournot competition [9]. El Ouardighi and Tapiero investigated the optimal trajectory of a firm's product innovation efforts and analyzed the dynamic path for enhancing product quality [10]. Within the framework of a dynamic monopoly, Mantovani explored how product and process innovation can complement each other in a firm's strategic actions [11]. Saha discovered that a firm's process innovation is influenced by the quantity of products sold, whereas product innovation is affected by the willingness of buyers to pay for them, indicating that product innovation depends on the types of consumers purchasing the products [12].

More recently, a growing body of literature has examined firms' product and process innovation from various perspectives. For instance, Lambertini and Mantovani explored how a multi-product firm dynamically balances its efforts in product and process innovation over time [13, 14]. Subsequently, Lambertini and Orsini investigated the impact of market size on the strategic allocation of product and process innovation efforts [15]. Chenavaz formulated a dynamic model to identify the relationship between optimal pricing and product-process innovation efforts [16]. Lambertini et al. scrutinized the instability property of simultaneous product and process innovation activities [3]. Li delved into the dynamics of product and process innovation within a multi-product monopolistic framework [17]. On the basis of Li's work [17] , Li et al. explored the dynamic control of product and process innovation efforts under reference quality effects [18].

Despite extensive studies on product and process innovation, a notable gap remains in understanding how consumers' reference perceptions regarding product differentiation dynamically influence the innovation strategies of multi-product firms in markets with network externality. In many technology-intensive industries, consumers are repeatedly exposed to a firm's product portfolio and may gradually form reference perceptions regarding how differentiated the products within that portfolio should be over time [19]. For example, in telecommunications markets, service providers commonly offer multiple service plans that differ in pricing, data allowances, and bundled services. Under such circumstances, consumers may form expectations not only about the intrinsic quality of a product, but also about the degree of variety or differentiation within the firm's product line [20]. In this sense, horizontal differentiation can be viewed as a salient dimension of product richness that consumers repeatedly observe when comparing products offered by the same firm, such as service plans, software subscription tiers, or smartphone models. When the firm offers a level of differentiation consistent with consumers' prior expectations, consumers may experience a utility gain; otherwise, they may perceive insufficient variety or limited incremental value, which may reduce utility and alter purchasing decisions [21,22].

At the same time, network externality introduces an additional demand-side force. Network externality refers to the phenomenon that the utility derived by a consumer from using a product increases with the number of other users adopting the same product or participating in the same network [23]. This effect is especially relevant in telecommunications, information technology, transport, banking, and other high-tech and digital industries, especially in the era where the social interactions are costless [24,25]. In such settings, product and process innovation, network externality, and reference effects regarding horizontal differentiation are inherently interconnected. On the one hand, product innovation reduces substitutability among products and thus alleviates internal cannibalization within the firm's portfolio. On the other hand, process innovation lowers marginal production costs and improves production efficiency. Meanwhile, network externality increases the value perceived by consumers as market size expands, which may partly offset the demand pressure generated by within-portfolio substitutability. Consumers may also evaluate the degree of differentiation in the product line relative to a reference level formed through repeated exposure to the firm's portfolio structure. These interactions jointly suggest that product innovation, process innovation, network externality, and reference-dependent demand should be studied within a unified dynamic framework.

As such, this paper constructs a dynamic control model to investigate how a multi-product firm balances its efforts between product and process innovation, taking into account the influence of reference effects regarding product horizontal differentiation in a market characterized by network externality. The welfare implications, by assuming a benevolent regulator that seeks to maximize social welfare through optimal allocation of resources for product and process innovation, are also explored. This research builds on and extends the work of [17]. Li focused on a multi-product firm in a traditional market without network externality, engaging in product and process innovation driven by learning-by-doing effects [17]. In contrast, this paper examines a multi-product monopolist operating in a market with network externality, where reference perceptions about product differentiation are considered. The network value function depends on both network size and product substitutability, with reference perception serving as a state variable. Consumer demand is influenced by price, degree of product substitutability, reference perception, and network size. Moreover, this study considers a special case where network size is proportional to output level. Overall, our framework offers important insights into how firms adapt their innovation strategies in response to consumers' reference perceptions in markets with network externality.

The structure of the paper is organized as follows. Section 2 presents the basic model. Section 3 investigates the steady-state equilibrium when decisions are made by the firm. Section 4 explores the steady-state equilibrium under government regulation. Section 5 provides numerical experiments to compare the steady-state equilibria under both firm decision-making and government regulation. The managerial implications and economic intuitions are discussed in Section 6. Section 7 concludes the paper. Detailed proofs of all conclusions are available in the Appendix.

2. The baseline model

Consider a monopolistic market supplied by a firm producing n products, which are horizontally differentiated. We assume a *deterministic* operational environment (e.g., [14, 15]). Besides, strategic consumer behavior and interactions among firms are excluded from our analysis. The reasons for such formulations are twofold: Most crucially, the setting facilitates analytical tractability; in addition, such a framework helps us to isolate the impacts of network externality and consumers' reference perceptions regarding horizontal differentiation on firms' innovation decisions from secondary factors.

In general, process innovation is commonly characterized by reducing a firm's marginal production cost. Product innovation, on the other hand, can take various forms. For example, it can be defined as improving product quality [26] or as reducing product differentiation [13]. In this paper, we define product innovation as the reduction of horizontal differentiation among products and process innovation as the reduction of marginal production cost. Following the approaches of Lambertini and Mantovani [13] and Li [17], we use linear differential equations to model the dynamics of product and process innovation. This means that product substitutability $s(t) \in [0, 1]$ and marginal production cost $c(t)$ decrease over time $t \in [0, \infty)$ as a result of continuous efforts in product and process innovation, respectively.

$$\dot{s}(t) = -k(t) + \sigma s(t) \tag{2.1}$$

$$\dot{c}(t) = -h(t) + \delta c(t) \tag{2.2}$$

where $k(t)$ and $h(t)$ are control variables representing the efforts made by the firm to decrease product substitutability $s(t)$ and marginal production cost $c(t)$ over continuous time, respectively. Accordingly, we assume that the costs of these efforts are given by $\frac{\alpha}{2}k^2(t)$ and $\frac{\beta}{2}h^2(t)$, respectively [13], where α and β are positive constant parameters. In this paper, the analysis focuses on interior solutions, which is economically meaningful under specific parameter constellations [27]. Corner solutions may arise under extreme parameter configurations, but they are outside the benchmark setting considered in the present study.

Now, we introduce the concept of consumers' reference perceptions regarding the horizontal differentiation among all products offered by the firm. In this study, we consider a benchmark setting with *homogeneous* consumers. This assumption allows us to focus on the roles of network externality and consumers' reference perceptions regarding horizontal product differentiation, without introducing additional heterogeneity-driven channels in demand. The purpose of the model is not to capture all sources of market complexity, but to isolate the dynamic effects of these two demand-side forces on firms' product and process innovation decisions. Drawing on the work of Kopalle and Winer [28], the function of reference perception $r(t)$ can be characterized through an exponential smoothing process based on historical levels of substitutability degree, which indicates a decaying memory effect as shown below: $r(t) = e^{-\rho t}[r_0 + \rho \int_0^t e^{\rho z} s(z) dz]$. By differentiating the above function with respect to time t, we obtain:

$$\dot{r}(t) = \rho[s(t) - r(t)] \tag{2.3}$$

where r_0 stands for the initial reference perception regarding horizontal differentiation among products, and the parameter ρ indicates consumers' adjustment speed (or memory parameter). It is noteworthy that although the reference gap $s(t) - r(t)$ converges to 0 in the steady state under such adaptive expectation mechanisms, the reference effects are not totally independent of the steady-state outcomes. The reason is that the adjustment of $r(t)$ impacts the firm's optimal innovation incentives along the transition path, and these incentives in turn influence the steady-state levels of the state variables. In other words, even if the explicit gap term vanishes in the long run, the adjustment process influences the optimal dynamic innovation path and thereby influences the steady-state outcomes. To further illustrate this point, we also consider a benchmark model without reference effects. In this case, the dynamic adjustment mechanism associated with the reference state is absent, and the steady-state solution does not involve reference-related parameters. Comparing the two models highlights the role of the reference effects in shaping steady-state outcomes. The following remark is generated from the above discussions:

Remark 2.1. *Although the reference-gap term disappears in the steady state, the reference effects continue to influence long-term outcomes through cumulative effects over the transition path.*

The above formulations can be interpreted as a form of reference-dependent distortion in consumer evaluation: the current product-line structure is assessed not in isolation, but relative to an internally formed benchmark. Although direct empirical evidence on reference effects regarding horizontal differentiation is still limited, the broader literature provides closely related support. A large body of research has documented that consumers evaluate market offers relative to reference points rather than in purely absolute terms, especially in studies of reference-dependent preferences and reference price effects [20, 21]. In addition, empirical studies on product lines and assortment structure show that consumers are sensitive to within-firm differentiation, product variety, and the relative positioning of alternatives within the same portfolio (e.g., [29–31]). Taken together, these studies make it plausible that consumers may gradually form reference perceptions regarding product substitutability. Such effects are particularly important for multi-product firms, because product innovation may take the form of increasing horizontal differentiation and thereby reducing substitutability among products, which in turn mitigates internal cannibalization and reshapes the firm's dynamic innovation incentives.

In the following, we aim to construct the network value function $F(t)$, which we assume depends on product substitutability $s(t)$ and product variety i's network size $Q_i(t)$ in a separable multiplicative form between the state variable $s(t)$ and exogenous variable $Q_i(t)$, i.e., $F(t) = G(s(t))f(Q_i(t))$. Here, the functions $G(s(t))$ and $f(Q_i(t))$ represent the impacts of product substitutability $s(t)$ and product variety i's network size $Q_i(t)$ on network value $F(t)$, respectively. This network value function $F(t)$ indicates that the firm affects network value directly by controlling product substitutability $s(t)$ through efforts for product innovation and indirectly through network size $Q_i(t)$. Further, we assume that the effect of product substitutability $s(t)$ on function $G(s(t))$ is linearly decreasing, expressed as $G(s(t)) = \upsilon - \nu(s(t))$, where the parameter $\upsilon \geq 0$ denotes the fixed intensity of network externality, and the function $\nu(s(t))$ indicates the variable intensity of network externality. Similar to Jing [32], we assume that the variable intensity of network externality is given by $\nu(s(t)) = \vartheta s(t)$, where the parameter $\vartheta \geq 0$ captures the unit variable intensity of network externality, measuring how sensitive product substitutability $s(t)$ is to changes in the variable intensity of network externality $\nu(s(t))$. Additionally, we assume that the effect of the network size $Q_i(t)$ on function $f(Q_i(t))$ is linearly increasing, expressed as $f(Q_i(t)) = \varphi Q_i(t)$, where the parameter $\varphi \geq 0$ denotes the effect of network size $Q_i(t)$ on network value $F(t)$. Therefore, considering both network size $Q_i(t)$ and product substitutability $s(t)$, the network value function $F(t)$ can be written as:

$$F(t) = \varphi[\upsilon - \vartheta s(t)]Q_i(t) \tag{2.4}$$

The above network value function (2.4) captures how the firm's product innovation and the network size jointly influence the network value in the market. Furthermore, we obtain from network value function (2.4) that the network value is negatively affected by product substitutability, i.e., $\frac{\partial F(t)}{\partial s(t)} = -\varphi\vartheta Q_i(t) < 0$, and positively affected by network size, i.e., $\frac{\partial F(t)}{\partial Q_i(t)} = \varphi[\upsilon - \vartheta s(t)] > 0$. Note, when $\vartheta = 0$ (i.e., $G(s(t)) = \upsilon$), users of different product variety enjoy the same level of network value. Notably, in the present model, network externality operates through market size rather than through direct technological compatibility across products. Product innovation influences demand structures by changing substitutability within the firm's portfolio so as to weaken internal cannibalization within the firm's product portfolio, whereas network externality impacts demand structures by changing the value generated by market participation. These are two distinct channels, and their coexistence is particularly relevant in high-tech and digital industries [33].

Having described network value function $F(t)$ and consumers' reference perception $r(t)$, we next investigate firm's inverse demand function with network externality. We first assume that the firm sells differentiated products with output $q_i(t)$ at price $p_i(t)$ for the product variety i. Additionally, we assume that consumers exhibit a preference for product variety and tend to purchase all products. At the same time, consumers hold the belief that products are, to varying degrees, interchangeable. Then, when the firm supplies $n > 1$ products in a market without network externality, the inverse demand function for the product variety i can be written as

$$p_i(t) = a - a_1 q_i(t) - s(t) \sum_{j \neq i} q_j(t) \tag{2.5}$$

where parameter a denotes the reservation price, and the parameter a_1 stands for the influence coefficient of product variety i on price $p_i(t)$. Note that, in demand function (2.5), $s(t) = 1$ indicates that the products are completely homogeneous; while $s(t) = 0$ denotes that the products are independent, which means that the higher the product substitutability $s(t)$, the lower the degree of differentiation.

Since consumers' reference perceptions regarding horizontal differentiation among products are formed by using information from various sources over time [34], it plays an important role in shaping consumers' demand structure [35]. As such, following the framework of Hardie et al. [36], we assume that the consumers' inverse demand is a function of the difference between the current degree of substitutability $s(t)$ and reference perception $r(t)$. Accordingly, we can modify the inverse demand function (2.5) as follows:

$$p_i(t) = a - a_1 q_i(t) - \mu[s(t) - r(t)] - s(t) \sum_{j \neq i} q_j(t) \tag{2.6}$$

where μ is a positive parameter, and the item $\mu[s(t) - r(t)]$ denotes the impact of the difference between the current degree of substitutability $s(t)$ and reference perception $r(t)$ on the price of product i. Further, it is straightforward to see from the expression $\mu[s(t) - r(t)]$ that when the realized degree of substitutability exceeds the consumers' reference level, the products are perceived as insufficiently differentiated, which depresses the willingness to pay and induces the firm to lower prices. Conversely, when the realized degree of substitutability is below the reference level, the product portfolio is perceived as more differentiated than expected, which raises the willingness to pay and supports higher prices. Here, the linear specifications are adopted in the inverse demand function (2.6) and in the innovation-related dynamics (2.1)–(2.2). This follows common practices in the theoretical literature on dynamic innovation and network industries (e.g., [15, 17] and the references therein), which provide a transparent benchmark that facilitates analytical characterization of the qualitative conclusions of interest.

Now, we further consider the effect of the network value function $F(t)$ on inverse demand function (2.6). According to Naskar and Pal [37] and Zhao and Ni [38], the inverse demand is a function of network value. Accordingly, we assume the influence of network value on the inverse demand function is linear, and using the network value function (2.4), we can modify the inverse demand function (2.6) as follows:

$$p_i(t) = a - a_1 q_i(t) + a_2\varphi[\upsilon - \vartheta s(t)]Q_i(t) - \mu[s(t) - r(t)] - s(t)\sum_{j\neq i} q_j(t) \tag{2.7}$$

where the parameter a_2 represents the influence coefficient of network value $F(t)$ on price $p_i(t)$. Notably, from (2.7), one can see that although the model adopts an additively separable structure at the production level, the products are not fully independent in terms of inverse demand. Specifically, the inverse demand of each product depends not only on its own output, but also on the outputs of the other products supplied by the same firm. This intrinsic structural property of the model captures internal cannibalization within the product portfolio. The strength of this cannibalization is governed by the degree of horizontal differentiation among products: higher substitutability strengthens within-firm demand competition, whereas product innovation reduces substitutability and thereby alleviates internal cannibalization [13]. To guarantee the validity of analytical analysis, we assume that $a_1 > a_2\phi m(\upsilon - \vartheta s)$, reflecting the observation that the influence of network effects on price is secondary to that of actual sales [39].

Then, at continuous time $t \in [0, +\infty)$, the firm's profit function $\pi(t)$ is given by:

$$\begin{aligned}\pi(t) = \sum_{i=1}^{n} \{&a - a_1 q_i(t) + a_2\varphi[\upsilon - \vartheta s(t)]Q_i(t) - \mu[s(t) - r(t)] - s(t)\sum_{j\neq i} q_j(t) \\ &- c(t)\}q_i(t) - \tfrac{\alpha}{2}k^2(t) - \tfrac{\beta}{2}h^2(t)\end{aligned} \tag{2.8}$$

We also assume that the firm has *complete* information about the key model parameters, including the intensity of network effects, the adjustment speed of consumers' reference perceptions, and the proportional coefficient linking network expansion to demand. In practice, these parameters may be informed by user-growth data, consumer surveys, historical sales records, and demand estimation procedures [30, 40]. In the subsequent section, we will conduct equilibrium analysis under both firm decision-making and governmental regulation.

3. Equilibrium analysis under firm decision-making

3.1. The optimal conditions and characteristics

Under firm decision-making, the firm's objective is to determine the optimal levels of effort for product and process innovation, as well as the output level over continuous time $t \in [0, +\infty)$, in order to maximize the discounted profit flow Π. This optimization is subject to the differential equations (2.1)–(2.3), which describe the dynamics of product substitutability $s(t)$, marginal production cost $c(t)$, and reference perception $r(t)$. Specifically, the firm seeks to solve the following optimal control problem:

$$\Pi = \max_{q_i(t),k(t),h(t)} \int_0^{+\infty} e^{-\tau t}\{\sum_{i=1}^{n} [a - a_1 q_i(t) + a_2\varphi(\upsilon - \vartheta s(t))Q_i(t) - \mu(s(t) - r(t)) - s(t)\sum_{j\neq i} q_j(t) - c(t)]q_i(t) - \frac{\alpha}{2}k^2(t) - \frac{\beta}{2}h^2(t)\}dt \tag{3.1}$$

$$s.t.\begin{cases} \dot{s}(t) = -k(t) + \sigma s(t) \\ \dot{c}(t) = -h(t) + \delta c(t) \\ \dot{r}(t) = \rho[s(t) - r(t)] \end{cases}$$

where τ is the discount factor and $s(0) = s_0$, $c(0) = c_0$, and $r(0) = r_0$ are the initial conditions of the state variables representing product substitutability $s(t)$, marginal production cost $c(t)$, and reference perception $r(t)$, respectively. Accordingly, the current-value Hamiltonian function H for optimization problem (3.1) is expressed as

$$H = \sum_{i=1}^{n} \{a - a_1 q_i(t) + a_2\varphi[\upsilon - \vartheta s(t)]Q_i(t) - \mu[s(t) - r(t)] - s(t)\sum_{j\neq i} q_j(t) - c(t)\}q_i(t) - \frac{\alpha}{2}k^2(t) - \frac{\beta}{2}h^2(t) + \lambda_1(t)[-k(t) + \sigma s(t)] + \lambda_2(t)[-h(t) + \delta c(t)] + \lambda_3(t)\rho[s(t) - r(t)] \tag{3.2}$$

where $\lambda_i(t)$ $(i = 1, 2, 3)$ are co-state variables representing the shadow prices of the state variables $s(t)$, $c(t)$, and $r(t)$, respectively. Specifically, the costate variable associated with product substitutability reflects the marginal value to the firm of relaxing or tightening within-portfolio cannibalization through product differentiation, the costate variable associated with marginal production cost measures the marginal value of cost reduction through process innovation, and the costate variable associated with the reference state captures the marginal value of changes in consumers' reference perceptions regarding horizontal differentiation. Then, the current-value Hamiltonian can be interpreted as the sum of the firm's instantaneous net profit and the intertemporal values generated by changes in the three state variables [41]. The first-order conditions and the co-state equations derived from the Hamiltonian function (3.2) are presented as follows:

$$\frac{\partial H}{\partial q_i(t)} = \sum_{i=1}^{n} \{a - 2a_1 q_i(t) + a_2\varphi[\upsilon - \vartheta s(t)]Q_i(t) - \mu[s(t) - r(t)] - s(t)\sum_{j\neq i} q_j(t) - c(t)\} = 0 \tag{3.3}$$

$$\frac{\partial H}{\partial k(t)} = -\alpha k(t) - \lambda_1(t) = 0 \tag{3.4}$$

$$\frac{\partial H}{\partial h(t)} = -\beta h(t) - \lambda_2(t) = 0 \tag{3.5}$$

$$\dot{\lambda}_1(t) = \tau\lambda_1(t) - \frac{\partial H}{\partial s(t)} = (\tau - \sigma)\lambda_1(t) + \sum_{i=1}^{n} q_i(t)[\sum_{j\neq i} q_j(t) + a_2\varphi\vartheta Q_i(t) + \mu] - \rho\lambda_3(t) \tag{3.6}$$

$$\dot{\lambda}_2(t) = \tau\lambda_2(t) - \frac{\partial H}{\partial c(t)} = (\tau - \delta)\lambda_2(t) + \sum_{i=1}^{n} q_i(t) \tag{3.7}$$

$$\dot{\lambda}_3(t) = \tau\lambda_3(t) - \frac{\partial H}{\partial r(t)} = (\tau+\rho)\lambda_3(t) - \mu\sum_{i=1}^{n} q_i(t) \tag{3.8}$$

where the corresponding transversal conditions are given by $\lim_{t\to\infty}\lambda_1(t)s(t)e^{-\tau t} = 0$, $\lim_{t\to\infty}\lambda_2(t)c(t)e^{-\tau t} = 0$, and $\lim_{t\to\infty}\lambda_3(t)r(t)e^{-\tau t} = 0$, respectively. It should be noted that, to ensure the existence of a steady-state equilibrium in the economic sense for the optimization problem (3.1), following a similar approach to that of Li [17], we assume that $\tau - \sigma > 0$ and $\tau - \delta > 0$. Moreover, these optimal conditions and characteristics provide crucial guidance on how the firm should allocate its resources for product and process innovation, and how to determine the optimal output level to maximize its discounted profit flow in a market with network externality and reference effects.

Notably, the influence of reference-related parameters on steady-state outcomes can be understood through their impact on the co-state dynamics (3.6) and (3.8). Specifically, the parameters affect the shadow price associated with product innovation, which reflects the marginal value of innovation effort for differentiating products. By altering this marginal value, μ and ρ influence the firm's optimal investment path in product innovation. Similarly, the reference-related parameters affect the shadow price associated with the reference level of horizontal differentiation, which captures the marginal value of the reference state. Through this channel, μ and ρ shape the monopolist's intertemporal investment decisions. Through their impacts on the dynamic adjustment paths of optimal decisions, these parameters generate cumulative effects over time. Consequently, they influence the steady-state levels as stated in Remark 2.1.

Now, we introduce symmetry assumptions: $p_i(t) = p_j(t) = p(t)$, $q_i(t) = q_j(t) = q(t)$, and $Q_i(t) = Q_j(t) = Q(t)$. Then, solving first-order condition (3.3) w.r.t. $q_i(t)$, and using symmetry assumptions, one can obtain the following output condition:

$$q(t) = \frac{a_2\varphi[\upsilon - \vartheta s(t)]Q(t) - \mu[s(t) - r(t)] - c(t) + a}{2a_1 + (n-1)s(t)} \tag{3.9}$$

Since $2a_1 + (n-1)s(t) > 0$, and considering the fact that $q(t) \geq 0$, then from output condition (3.9), we have $a_2\varphi[\upsilon - \vartheta s(t)]Q(t) - \mu[s(t) - r(t)] - c(t) + a \geq 0$. In addition, we assume that the firm is always rational, that is, $a - c(t) > 0$. To facilitate the application of these results in the subsequent analysis, we summarize the above findings as the following Condition 3.1:

Condition 3.1. *For the output level to be non-negative, the following condition must be satisfied for the admissible parameters:* $a_2\varphi[\upsilon - \vartheta s(t)]Q(t) - \mu[s(t) - r(t)] - c(t) + a \geq 0$.

Next, we explore how both types of effort influence the output level $q(t)$. By taking the partial derivatives of the output condition (3.9) w.r.t. the efforts $k(t)$ and $h(t)$, respectively, we arrive at the following Conclusion 3.2 (Proof: See Appendix 1):

Conclusion 3.2. *Under the firm's decision-making framework, the following results are obtained: (i)* $\frac{\partial q(t)}{\partial k(t)} > 0$*; (ii)* $\frac{\partial q(t)}{\partial h(t)} > 0$.

Conclusion 3.2 indicates that under the firm's decision-making process, both efforts have a direct impact on the output level, with the output increasing in response to an increase in either effort. The underlying logic is that efforts in either product innovation or process innovation can enhance the firm's profitability. Specifically, for Conclusion 3.2(i): Lower product substitutability enhances consumer demand for the output. By reducing substitutability, the firm attracts consumers who are less willing to pay for highly differentiated products. This broadens the consumer base and increases overall demand for the output level. For Conclusion 3.2(ii): A lower marginal production cost directly improves the firm's profit margin. With reduced costs, the firm can expand its output level while maintaining profitability. This cost advantage supports higher output levels to meet increased demand.

Now, substituting the output level (3.9) into the inverse demand function (2.7), and using symmetry assumptions, yields the following pricing condition:

$$p(t) = \frac{a_1\{a_2\varphi[\upsilon - \vartheta s(t)]Q(t) - \mu[s(t) - r(t)] + a\} + [a_1 + (n-1)s(t)]c(t)}{2a_1 + (n-1)s(t)} \tag{3.10}$$

By differentiating the pricing condition (3.10) w.r.t. $k(t)$ and $h(t)$, respectively, we arrive at the following Conclusion 3.3 (Proof: See Appendix 2):

Conclusion 3.3. *Under the firm's decision-making framework, the following holds true: (i)* $\frac{\partial p(t)}{\partial k(t)} > 0$*; (ii)* $\frac{\partial p(t)}{\partial h(t)} < 0$.

Conclusion 3.3 reveals that while both types of effort impact the price level, their effects are in opposite directions. Specifically, the price level rises with increased effort in product innovation and falls with increased effort in process innovation. To elaborate, for Conclusion 3.3(i): The increase in price due to product innovation is intuitive. Lower product substitutability, achieved through product innovation, enhances the firm's market power and profit margin, leading to a higher price level. For Conclusion 3.3(ii): The decrease in price due to process innovation is equally clear. Lower marginal production costs, achieved through process innovation, improve the firm's profit margin while reducing production costs. This cost reduction allows the firm to lower the price level while maintaining profitability.

The shadow price, which indicates the maximum price a manager would pay for an extra unit of a limited resource, serves as a critical tool for decision-makers by providing key insights into the problems being analyzed. Therefore, we next investigate the effects of both efforts on shadow prices $\lambda_1(t)$, $\lambda_2(t)$, and $\lambda_3(t)$, respectively. By solving the differential equations (3.6)–(3.8) w.r.t. $\lambda_1(t)$, $\lambda_2(t)$, and $\lambda_3(t)$, together with transversality conditions $\lim_{t\to\infty} \lambda_1(t)s(t)e^{-\tau t} = 0$, $\lim_{t\to\infty} \lambda_2(t)c(t)e^{-\tau t} = 0$, and $\lim_{t\to\infty} \lambda_3(t)r(t)e^{-\tau t} = 0$, as well as the symmetry assumptions, output level (3.9), and Condition 3.1, respectively, the shadow prices can be expressed in terms of the parameters and state variables of the optimization problem, that is to say,

$$\begin{aligned} \lambda_1(t) = -\{&\tfrac{n\{a_2\varphi[\upsilon-\vartheta s(t)]Q(t)-\mu[s(t)-r(t)]-c(t)+a\}}{(\tau-\sigma)[2a_1+(n-1)s(t)]}\} \\ &\times \{\tfrac{(n-1)\{a_2\varphi[\upsilon-\vartheta s(t)]Q(t)-\mu[s(t)-r(t)]-c(t)+a\}}{2a_1+(n-1)s(t)} \\ &+ \tfrac{a_2\varphi\vartheta(\tau+\rho)Q(t)+\tau\mu}{\tau+\rho}\} \le 0 \end{aligned} \tag{3.11}$$

$$\lambda_2(t) = -\frac{n\{a_2\varphi[\upsilon - \vartheta s(t)]Q(t) - \mu[s(t) - r(t)] - c(t) + a\}}{(\tau - \delta)[2a_1 + (n-1)s(t)]} \le 0 \tag{3.12}$$

$$\lambda_3(t) = \frac{n\mu\{a_2\varphi[\upsilon - \vartheta s(t)]Q(t) - \mu[s(t) - r(t)] - c(t) + a\}}{(\tau + \rho)[2a_1 + (n-1)s(t)]} \ge 0 \tag{3.13}$$

From equations (3.11) and (3.12), we observe that the shadow prices $\lambda_1(t)$ and $\lambda_2(t)$ are negative. Consequently, the values $\lambda_1(t)\dot{s}(t) = \lambda_1(t)[-k(t) + \sigma s(t)]$ and $\lambda_2(t)\dot{c}(t) = \lambda_2(t)[-h(t) + \delta c(t)]$ are positive (in a mathematical sense, the multiplication of two negative numbers yields a positive result), which clearly indicates that the firm's dynamic profit margin tends to expand as both product substitutability and marginal production cost decline, respectively. Additionally, from expression (3.13), it can be observed that the shadow price $\lambda_3(t)$ is positive. Therefore, the value $\lambda_3(t)\dot{r}(t) = \lambda_3(t)\rho[s(t) - r(t)]$ is positive (negative) if and only if (iff) $s(t) - r(t) \geq (<) \, 0$, which implies that the firm's dynamic profit margin rises (falls) with an increase in the reference perception iff $s(t) - r(t) \geq (<) \, 0$.

Furthermore, by differentiating the aforementioned expressions (3.11)–(3.13) w.r.t. $k(t)$ and $h(t)$, respectively, we arrive at the following Conclusion 3.4 (Proof: See Appendix 3):

Conclusion 3.4. *Under the firm's decision-making framework, it follows that: (i)* $\frac{\partial \lambda_1(t)}{\partial k(t)} < 0$, $\frac{\partial \lambda_2(t)}{\partial k(t)} < 0$, *and* $\frac{\partial \lambda_3(t)}{\partial k(t)} > 0$; *(ii)* $\frac{\partial \lambda_1(t)}{\partial h(t)} < 0$, $\frac{\partial \lambda_2(t)}{\partial h(t)} < 0$, *and* $\frac{\partial \lambda_3(t)}{\partial h(t)} > 0$.

Conclusion 3.4 demonstrates that the shadow prices associated with product substitutability and marginal production cost each decrease as the respective efforts increase. This inverse relationship implies that the dynamic profit margin expands with higher levels of these efforts. Additionally, the shadow price of reference perception rises with increased efforts, reinforcing the notion that the dynamic profit margin also benefits from elevated effort levels.

Now, we examine the dynamic equations governing the efforts for product and process innovation. By utilizing equations (3.4)–(3.8), incorporating the output level specified in equation (3.9), invoking the symmetry assumptions, and rearranging the terms, we derive the following nonlinear dynamic differential equations for the efforts $k(t)$ and $h(t)$, respectively:

$$\begin{aligned}\dot{k}(t) = (\tau - \sigma)k(t) - &\frac{n\{a_2\varphi[\upsilon - \vartheta s(t)]Q(t) - \mu[s(t) - r(t)] - c(t) + a\}}{\alpha[2a_1 + (n-1)s(t)]} \\ &\times \left\{\frac{(n-1)[a_2\varphi(\upsilon - \vartheta s(t))Q(t) - \mu(s(t) - r(t)) - c(t) + a]}{2a_1 + (n-1)s(t)} + \frac{a_2\varphi\vartheta(\tau+\rho)Q(t) + \mu\tau}{\tau+\rho}\right\}\end{aligned} \tag{3.14}$$

$$\dot{h}(t) = (\tau - \delta)h(t) - \frac{n\{a_2\varphi[\upsilon - \vartheta s(t)]Q(t) - \mu[s(t) - r(t)] - c(t) + a\}}{\beta[2a_1 + (n-1)s(t)]} \tag{3.15}$$

From expressions (3.14) and (3.15), it can be observed that both dynamic differential equations involve the state variables $s(t)$, $c(t)$, and $r(t)$. This confirms the interdependent relationship between the two efforts. Moreover, the presence of the product variety range n and the network size $Q(t)$ in both equations indicates that these factors will have a sustained impact on the dynamics of the two efforts.

Next, we examine how changes in the state variables influence the rates of change of the two efforts. The following Conclusion 3.5 can be easily obtained (Proof: See Appendix 4):

Conclusion 3.5. *In the context of firm decision-making, the following holds true: (i)* $\frac{\partial \dot{k}(t)}{\partial s(t)} > 0$, $\frac{\partial \dot{k}(t)}{\partial c(t)} > 0$, *and* $\frac{\partial \dot{k}(t)}{\partial r(t)} < 0$; *(ii)* $\frac{\partial \dot{h}(t)}{\partial s(t)} > 0$, $\frac{\partial \dot{h}(t)}{\partial c(t)} > 0$, *and* $\frac{\partial \dot{h}(t)}{\partial r(t)} < 0$.

From Conclusion 3.5, it is observed that the rates of change of both efforts increase as product substitutability $s(t)$ and marginal production cost $c(t)$ increase, respectively. Conversely, the rates of change of both efforts decrease with an increase in reference perception. Furthermore, Conclusion 3.5 reveals that $\frac{\partial \dot{k}(t)/\partial s(t)}{\partial \dot{h}(t)/\partial s(t)} > 0$, $\frac{\partial \dot{k}(t)/\partial c(t)}{\partial \dot{h}(t)/\partial c(t)} > 0$, and $\frac{\partial \dot{k}(t)/\partial r(t)}{\partial \dot{h}(t)/\partial r(t)} > 0$, indicating that when there are variations in the state variables $s(t)$, $c(t)$, and $r(t)$, there is always a complementary relationship between the rates of change of both efforts, respectively.

Additionally, by taking the derivative of the dynamic differential equations (3.14) and (3.15) w.r.t. n, we arrive at the following Conclusion 3.6 (Proof: See Appendix 5):

Conclusion 3.6. *Under the firm's decision-making process, when $s(t) \geq (<)\, 2a_1$, we derive the following: (i) $\frac{\partial \dot{k}(t)}{\partial n} \geq (<)\, 0$; (ii) $\frac{\partial \dot{h}(t)}{\partial n} \geq (<)\, 0$.*

The significance of Conclusion 3.6 is that the influence of the range of product variety on the rates of change of both efforts depends on the parameter a_1 and product substitutability $s(t)$. Specifically, the rates of change of these two efforts increase (decrease) with the increase of the range of product variety iff $s(t) \geq (<) 2a_1$. Furthermore, from Conclusion 3.6, we can see that $\frac{\partial \dot{k}(t)/\partial n}{\partial \dot{h}(t)/\partial n} > 0$, which indicates that when there is variation in the range of product variety n, there is always a complementary relationship between the rates of changes of these two efforts.

Further, we investigate the interaction between the two efforts. Since $\frac{\partial \dot{k}(t)}{\partial h(t)} = \frac{\partial \dot{k}(t)}{\partial c(t)} \frac{\partial c(t)}{\partial h(t)}$, $\frac{\partial \dot{h}(t)}{\partial k(t)} = \frac{\partial \dot{h}(t)}{\partial s(t)} \frac{\partial s(t)}{\partial k(t)}$. In the proof of Conclusion 3.2 (see Appendix 1), we have obtained $\frac{\partial s(t)}{\partial k(t)} < 0$ and $\frac{\partial c(t)}{\partial h(t)} < 0$. Furthermore, in Conclusion 3.5, we have obtained $\frac{\partial \dot{k}(t)}{\partial c(t)} > 0$ and $\frac{\partial \dot{h}(t)}{\partial s(t)} > 0$.

Therefore, one can derive the following insights: (i) $\frac{\partial \dot{k}(t)}{\partial h(t)} < 0$; (ii) $\frac{\partial \dot{h}(t)}{\partial k(t)} < 0$. These findings highlight an interactive relationship between the two efforts at any given continuous time $t \in [0, +\infty)$, namely, (i) the rate of change of effort for product innovation decreases as the effort for process innovation increases; (ii) the rate of change of effort for process innovation diminishes as the effort for product innovation rises. These findings are consistent with prior research, including the studies by Lambertini and Orsini [15] and Li [27].

The primary objective of this paper is to investigate whether the two efforts result in a stable steady-state equilibrium. In the following subsection, we delve into the characteristics and stability properties of the steady-state equilibrium.

3.2. Steady-state equilibrium

To begin analyzing the effects of changes in state variables on control variables, we first impose the steady-state condition $\dot{k}(t) = \dot{h}(t) = 0$ to characterize the steady-state efforts in terms of the state variables and parameters, and we apply the steady-state condition $\dot{r}(t) = 0$, under which we obtain $s(t) = r(t)$. From this, we can derive the following:

$$k(s,c) = \frac{n\{a_2\varphi[\upsilon - \vartheta s(t)]Q(t) - c(t) + a\}}{\alpha(\tau-\sigma)[2a_1+(n-1)s(t)]}\Big\{\frac{(n-1)[a_2\varphi(\upsilon-\vartheta s(t))Q(t)-c(t)+a]}{2a_1+(n-1)s(t)} + \frac{a_2\varphi\vartheta(\tau+\rho)Q(t)+\mu\tau}{\tau+\rho}\Big\} \tag{3.16}$$

$$h(s,c) = \frac{n\{a_2\varphi[\upsilon - \vartheta s(t)]Q(t) - c(t) + a\}}{\beta(\tau-\delta)[2a_1 + (n-1)s(t)]} \tag{3.17}$$

By taking the derivatives of equations (3.16) and (3.17) w.r.t. the state variables $s(t)$ and $c(t)$, respectively, and incorporating Condition 3.1, we arrive at the following Conclusion 3.7 (Proof: See Appendix 6).

Conclusion 3.7. *Under the firm's decision-making framework, the following results are obtained: (i) $\frac{\partial k(s,c)}{\partial s} < 0$, $\frac{\partial k(s,c)}{\partial c} < 0$; (ii) $\frac{\partial h(s,c)}{\partial s} < 0$, $\frac{\partial h(s,c)}{\partial c} < 0$.*

Conclusion 3.7 indicates that the steady-state efforts for product and process innovation decline as product substitutability s and marginal production cost c increase, respectively. Moreover, Conclusion 3.7 reveals that $\frac{\partial k(s,c)/\partial s}{\partial h(s,c)/\partial s} > 0$ and $\frac{\partial k(s,c)/\partial c}{\partial h(s,c)/\partial c} > 0$, suggesting that when there are changes in product substitutability s and marginal production cost c, the steady-state efforts $k(s,c)$ and $h(s,c)$ consistently exhibit a complementary relationship.

Naturally, one might wonder whether the firm focuses more on product innovation or process innovation, and what conditions lead the firm to prioritize one over the other. By analyzing equations (3.16) and (3.17), we can determine the relationship between the efforts allocated to product and process innovation: $k(s,c) = \frac{\beta(\tau-\delta)}{\alpha(\tau-\sigma)}h(s,c)[\frac{(n-1)\beta(\tau-\delta)h(s,c)}{n} + \frac{a_2\varphi\vartheta(\tau+\rho)Q(t)+\mu\tau}{\tau+\rho}]$. Let us consider $k(s,c)$ as a function of $h(s,c)$, and by differentiating the derived equation w.r.t. $h(s,c)$, we obtain $\frac{\partial k(s,c)}{\partial h(s,c)} = \frac{\beta(\tau-\delta)}{\alpha(\tau-\sigma)}[\frac{2(n-1)\beta(\tau-\delta)}{n}h(s,c) + \frac{a_2\varphi\vartheta(\tau+\rho)Q(t)+\mu\tau}{\tau+\rho}] > 0$, which indicates that $k(s,c)$ increases with the increase of $h(s,c)$. In other words, the efforts $k(s,c)$ and $h(s,c)$ consistently exhibit a complementary relationship.

Further, by imposing stationarity on equations (2.1) and (2.2), then equations (3.16) and (3.17) are transformed into:

$$k(c) = -\frac{B}{3A} + \sqrt[3]{-\frac{N}{2} + \sqrt{(\frac{N}{2})^2 + (\frac{M}{3})^3}} + \sqrt[3]{-\frac{N}{2} - \sqrt{(\frac{N}{2})^2 + (\frac{M}{3})^3}} \tag{3.18}$$

$$h(s) = \frac{n\delta\{a + a_2\varphi[\upsilon - \vartheta s(t)]Q(t)\}}{\delta\beta(\tau-\delta)[2a_1 + (n-1)s(t)] + n} \tag{3.19}$$

where $M = \frac{3AC-B^2}{3A^2}$, $N = \frac{2B^3-9ABC+27A^2D}{27A^2}$, in which $A = \alpha(\tau - \sigma)(\tau + \rho)(n - 1)^2$, $B = (n - 1)[4\alpha a_1\sigma(\tau - \sigma)(\tau + \rho) + na_2\varphi\vartheta\mu\tau Q(t)]$, $C = 4\alpha(\tau-\sigma)(\tau+\rho)(a_1\sigma)^2 + 2na_2\varphi\vartheta(n-1)(\tau+\rho)[a\sigma + a_2\varphi\upsilon\sigma Q(t) - \sigma c(t)]Q(t) + [a_2\varphi\vartheta(\tau+\rho)Q(t)+\mu\tau]\{2a_1\sigma a_2\varphi\vartheta nQ(t) - n(n-1)[a\sigma + a_2\varphi\upsilon\sigma Q(t) - \sigma c(t)]\}$, $D = 2a_1\sigma[a_2\varphi\vartheta(\tau+\rho)Q(t) + \mu\tau]\}[a\sigma + a_2\varphi\upsilon\sigma Q(t) - \sigma c(t)] - n\{(n-1)(\tau+\rho)[a\sigma + a_2\varphi\upsilon\sigma Q(t) - \sigma c(t)]$.

We observe that the structure of expression (3.18) is highly complex, making it challenging to analyze the effects of product variety n and marginal production cost $c(t)$ on the effort for product innovation. To address this, we consider a hypothetical example. To validate the model through numerical methods, we draw on the parameter values utilized in prior research, such as [13, 17, 42, 43]. The specific values are detailed in Table 1 below.

Table 1. Baseline numerical values used in the simulation exercises.

τ	μ	α	β	σ	δ	ρ	a	a_1	a_2	φ	υ	ϑ	Q
0.2	0.16	0.7	0.8	0.08	0.05	0.2	0.3	0.4	0.3	0.15	0.12	0.3	10

Figure 1 illustrates the path of the steady-state effort $k(c,n)$ as a function of the marginal production cost c and the range of product variety n by using the values provided in Table 1.

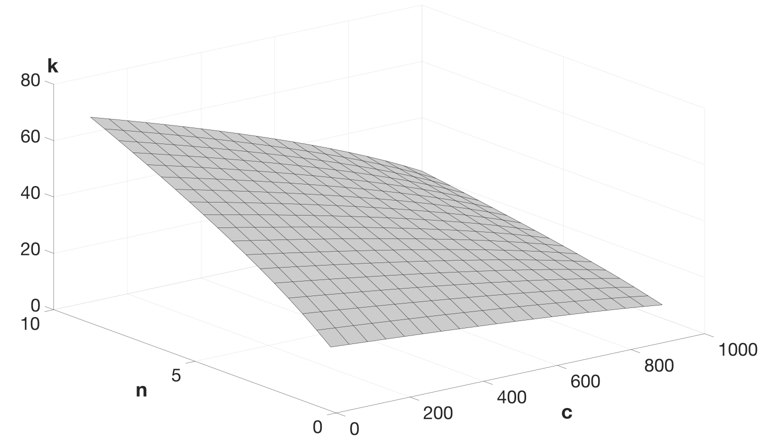

Figure 1. The candidate steady-state effort $k(c, n)$.

As depicted in Figure 1, for permissible numerical values of the parameters, the steady-state effort allocated to product innovation grows with an increase in the range of product variety and also increases with a rise in marginal production cost.

Additionally, by taking the derivative of expression (3.19) w.r.t. $s(t)$ and n, we arrive at the following: $\frac{\partial h(s)}{\partial s} = -\frac{n\delta\{a_2\varphi[2a_1\vartheta\delta\beta(\tau-\delta)+\upsilon\delta\beta(\tau-\delta)(n-1)+n\vartheta]Q(t)+a\delta\beta(\tau-\delta)(n-1)\}}{\{\delta\beta(\tau-\delta)[2a_1+(n-1)s(t)]+n\}^2} < 0$ and $\frac{\partial h(s)}{\partial n} = \frac{\beta(\tau-\delta)\delta^2\{a+a_2\varphi[\upsilon-\vartheta s(t)]Q(t)\}[2a_1-s(t)]}{\{\delta\beta(\tau-\delta)[2a_1+(n-1)s(t)]+n\}^2}$, from which one can obtain that $\frac{\partial h(s)}{\partial n} \geq (<)\ 0$ if and only if $s(t) \leq (>)\ 2a_1$. This suggests that the steady-state effort for process innovation diminishes as product substitutability rises, while it rises (or falls) with an increase in the range of product variety, contingent upon $s(t) \leq (>)\ 2a_1$.

The main goal of this section is to explore the steady-state equilibrium and its stability characteristics. To accomplish this, we integrate the dynamic differential equations (2.1)–(2.3) with equations (3.14)–(3.15) to obtain the following dynamic control system:

$$\begin{cases} \dot{k}(t) = (\tau-\sigma)k(t) - \frac{n\{a_2\varphi[\upsilon-\vartheta s(t)]Q(t)-\mu[s(t)-r(t)]-c(t)+a\}}{\alpha[2a_1+(n-1)s(t)]} \\ \qquad \times\{\frac{(n-1)[a_2\varphi(\upsilon-\vartheta s(t))Q(t)-\mu(s(t)-r(t))-c(t)+a]}{2a_1+(n-1)s(t)} + \frac{a_2\varphi\vartheta(\tau+\rho)Q(t)+\mu\tau}{\tau+\rho}\} \\ \dot{h}(t) = (\tau-\delta)h(t) - \frac{n\{a_2\varphi[\upsilon-\vartheta s(t)]Q(t)-\mu[s(t)-r(t)]-c(t)+a\}}{\beta[2a_1+(n-1)s(t)]} \\ \dot{s}(t) = -k(t) + \sigma s(t) \\ \dot{c}(t) = -h(t) + \delta c(t) \\ \dot{r}(t) = \rho[s(t) - r(t)] \end{cases} \tag{3.20}$$

Under the steady-state condition $\dot{k}(t) = \dot{h}(t) = \dot{s}(t) = \dot{c}(t) = \dot{r}(t) = 0$, solving the system (3.20) and denoting the steady-state equilibrium with the superscript "f", we arrive at following Proposition 3.8 (Proof: See Appendix 7):

Proposition 3.8. *Under firm decision-making, the dynamic control system* (3.20) *possesses a unique saddle-point equilibrium* $\{k^f, h^f, s^f, c^f, r^f\}$ *iff* $\rho < 2\tau$ *holds. Conversely, the steady-state equilibrium* $\{k^f, h^f, s^f, c^f, r^f\}$ *is unstable in the saddle-point sense if* $\rho > 2\tau$*, where* $k^f = \sigma s^f$*,* $h^f = \delta c^f$*,* $c^f = \frac{n[a+a_2\varphi(\upsilon-\vartheta s^f)Q]}{\beta(\tau-\delta)\delta[2a_1+(n-1)s^f]+n}$*,* $s^f = r^f = \{-\frac{Y}{2} + [(\frac{Y}{2})^2 + (\frac{Z}{3})^3]^{\frac{1}{2}}\}^{\frac{1}{3}} + \{-\frac{Y}{2} - [(\frac{Y}{2})^2 + (\frac{Z}{3})^3]^{\frac{1}{2}}\}^{\frac{1}{3}} - \frac{K_2}{3K_1}$*, in which* Y*,* Z*,* K_1*, and* K_2 *are presented in Appendix 7.*

The economic significance of Proposition 3.8 is that the memory parameter ρ and discount factor τ directly affect the stability of the steady-state equilibrium $\{k^f, h^f, s^f, c^f, r^f\}$. Specifically, the steady-state equilibrium $\{k^f, h^f, s^f, c^f, r^f\}$ is stable (unstable) in the saddle-point sense iff $\rho < (>)\ 2\tau$. Since ρ represents the consumers' memory parameter, a larger value of ρ indicates a shorter memory of the degree of product substitutability among consumers, or lower loyalty to the product. To ensure the system has a unique saddle-point equilibrium, the firm must increase the memory parameter ρ through strategies such as advertising, ensuring that $\rho < 2\tau$ is met.

In the next subsection, we investigate a particular case where the network scale is directly proportional to consumers' demand over continuous time $t \in [0, +\infty)$.

3.3. *A special case under firm decision-making*

In this subsection, adopting the methodology of Hurkens and López [44], we concentrate on a specific scenario where we assume that the network size is directly proportional to consumers' demand over continuous time $t \in [0, +\infty)$, i.e., $q_i(t) = Q_i(t)/m_i$, where m_i denotes the proportional coefficient between network size $Q_i(t)$ and demand $q_i(t)$. For simplicity, we introduce the following symmetry assumption: $m_i = m_j = m$. Substituting $q_i(t) = Q_i(t)/m$ into equation (2.7) and rearranging the terms, we arrive at the following inverse demand function:

$$p_i(t) = a - a_1 q_i(t) + a_2\varphi m[\upsilon - \vartheta s(t)]q_i(t) - \mu[s(t) - r(t)] - s(t)\sum_{j\neq i} q_j(t) \tag{3.21}$$

Under the assumption $q_i(t) = Q_i(t)/m$, the optimization problem of the multi-product firm can be formulated as follows:

$$\begin{aligned}\Pi = \max_{q_i(t),k(t),h(t)} \int_0^{+\infty} e^{-\tau t}\{&\sum_{i=1}^{n} [a - a_1 q_i(t) + a_2\varphi m(\upsilon - \vartheta s(t))q_i(t) \\ &- \mu(s(t) - r(t)) - s(t)\sum_{j\neq i} q_j(t) - c(t)]q_i(t) \\ &- \tfrac{\alpha}{2}k^2(t) - \tfrac{\beta}{2}h^2(t)\}dt \\ s.t. \begin{cases} \dot{s}(t) = -k(t) + \sigma s(t) \\ \dot{c}(t) = -h(t) + \delta c(t) \\ \dot{r}(t) = \rho[s(t) - r(t)] \end{cases}\end{aligned} \tag{3.22}$$

where τ is the discount factor and $s(0) = s_0$, $c(0) = c_0$, $r(0) = r_0$ are the initial conditions. The current-value Hamiltonian function for the optimization problem (3.22) is expressed as

$$\begin{aligned}H = \sum_{i=1}^{n} &\{a - a_1 q_i(t) + a_2\varphi m[\upsilon - \vartheta s(t)]q_i(t) - \mu[s(t) - r(t)] \\ &- s(t)\sum_{j\neq i} q_j(t) - c(t)\}q_i(t) - \tfrac{\alpha}{2}k^2(t) - \tfrac{\beta}{2}h^2(t) \\ &+ \lambda'_1(t)[-k(t) + \sigma s(t)] + \lambda'_2(t)[-h(t) + \delta c(t)] + \lambda'_3(t)\rho[s(t) - r(t)]\end{aligned} \tag{3.23}$$

where $\lambda'_i(t)$ $(i = 1, 2, 3)$ are the co-state variables that correspond to the shadow prices of the state variables $s(t)$, $c(t)$, and $r(t)$, respectively. The first-order conditions and co-state equations derived from the Hamiltonian function (3.23) are presented as follows:

$$\frac{\partial H}{\partial q_i(t)} = \sum_{i=1}^{n} \{a - 2a_1 q_i(t) + 2a_2\varphi m[\upsilon - \vartheta s(t)]q_i(t) - \mu[s(t) - r(t)] \tag{3.24}$$

$$- s(t)\sum_{j\neq i} q_j(t) - c(t)\} = 0$$

$$\frac{\partial H}{\partial k(t)} = -\alpha k(t) - \lambda_1(t) = 0 \tag{3.25}$$

$$\frac{\partial H}{\partial h(t)} = -\beta h(t) - \lambda_2(t) = 0 \tag{3.26}$$

$$\dot{\lambda}_1'(t) = \tau\lambda_1(t) - \frac{\partial H}{\partial s(t)} \tag{3.27}$$

$$= (\tau - \sigma)\lambda_1'(t) + \sum_{i=1}^{n} q_i(t)[\sum_{j\neq i} q_j(t) + a_2\varphi\vartheta m q_i(t) + \mu] - \rho\lambda_3'(t)$$

$$\dot{\lambda}_2'(t) = \tau\lambda_2'(t) - \frac{\partial H}{\partial c(t)} \tag{3.28}$$

$$= (\tau - \delta)\lambda_2'(t) + \sum_{i=1}^{n} q_i(t)$$

$$\dot{\lambda}_3'(t) = \tau\lambda_3'(t) - \frac{\partial H}{\partial r(t)} \tag{3.29}$$

$$= (\tau + \rho)\lambda_3'(t) - \mu\sum_{i=1}^{n} q_i(t)$$

where the transversality conditions are given by $\lim_{t\to\infty}\lambda_1'(t)s(t)e^{-\tau t} = 0$, $\lim_{t\to\infty}\lambda_2'(t)c(t)e^{-\tau t} = 0$, and $\lim_{t\to\infty}\lambda_3'(t)r(t)e^{-\tau t} = 0$, respectively. It should be noted that, to ensure the existence of a steady-state equilibrium, we still assume $\tau - \sigma > 0$ and $\tau - \delta > 0$.

We now solve equation (3.24) w.r.t. $q(t)$, and by employing symmetry assumptions, we obtain the following output level:

$$q(t) = \frac{a - \mu[s(t) - r(t)] - c(t)}{2(a_1 - a_2\varphi m\upsilon) + (2a_2\varphi m\vartheta + n - 1)s(t)} \tag{3.30}$$

Since $2(a_1 - a_2\varphi m\upsilon) + (2a_2\varphi m\vartheta + n - 1)s(t) > 0$ (see Condition 3.10 below), and given that $q(t) \geq 0$, we derive that $a - \mu[s(t) - r(t)] - c(t) \geq 0$. We summarize this result as the following Condition 3.9:

Condition 3.9. *For admissible parameters, when* $q(t) = Q(t)/m$, *the condition* $a - \mu[s(t) - r(t)] - c(t) \geq 0$ *must be satisfied to ensure that the output level remains non-negative.*

Substituting the output level (3.30) into the inverse demand function (3.21) and applying symmetry assumptions, one can derive the following pricing strategy:

$$p(t) = \frac{[a_1 - a_2\varphi m\upsilon + a_2\varphi m\vartheta s(t)]\{a - \mu[s(t) - r(t)] + c(t)\} + (n-1)s(t)c(t)}{2(a_1 - a_2\varphi m\upsilon) + (2a_2\varphi m\vartheta + n - 1)s(t)} \tag{3.31}$$

Differentiating expressions (3.30) and (3.31) w.r.t. m and applying Condition 3.9 yields $\frac{\partial q(t)}{\partial m} \propto 2a_2\varphi[\upsilon - \vartheta s(t)]\{a - \mu[s(t) - r(t)] - c(t)\} \geq 0$ and $\frac{\partial p(t)}{\partial m} \propto a_2\varphi[\upsilon - \vartheta s(t)]s(t)[4a_2\varphi m\vartheta + 2(n-1)c(t) + n - 1]\{a - \mu[s(t) - r(t)] + c(t)\} \geq 0$. This indicates that, when $q(t) = Q(t)/m$, the output level $q(t)$ and price level $p(t)$ increase with the proportional coefficient m. Furthermore, we have $\frac{\partial q(t)/\partial m}{\partial p(t)/\partial m} > 0$, which implies that, when $q(t) = Q(t)/m$, a variation in proportional coefficient m results in a complementary relationship between the output level $q(t)$ and price level $p(t)$.

We now solve the differential equation (3.29) w.r.t. $\lambda_3'(t)$. By incorporating the transversality condition $\lim_{t\to\infty} \lambda_3'(t) r(t) e^{-\tau t} = 0$, the symmetry assumptions, and the output level (3.30), we derive the following shadow price, formulated in terms of the state variables and parameters of the model:

$$\lambda_3'(t) = \frac{n\mu\{a - \mu[s(t) - r(t)] - c(t)\}}{(\tau + \rho)[2(a_1 - a_2\varphi m\upsilon) + (2a_2\varphi m\vartheta + n - 1)s(t)]} \tag{3.32}$$

Similar to the analysis in subsection 3.1, we derive the following dynamics equations for these two efforts, i.e.,

$$\begin{aligned} \dot{k}(t) = (\tau - \sigma)k(t) - \{\tfrac{n[a-\mu(s(t)-r(t))-c(t)]}{\alpha[2(a_1-a_2\varphi m\upsilon)+(2a_2\varphi m\vartheta+n-1)s(t)]}\} \\ \times \{\tfrac{(a_2\varphi\vartheta m+n-1)[a-\mu(s(t)-r(t))-c(t)]}{2(a_1-a_2\varphi m\upsilon)+(2a_2\varphi m\vartheta+n-1)s(t)} + \tfrac{\mu\tau}{\tau+\rho}\} \end{aligned} \tag{3.33}$$

$$\dot{h}(t) = (\tau - \delta)h(t) - \frac{n\{a - \mu[s(t) - r(t)] - c(t)\}}{\beta[2(a_1 - a_2\varphi m\upsilon) + (2a_2\varphi m\vartheta + n - 1)s(t)]} \tag{3.34}$$

Additionally, under the steady-state condition $\dot{r}(t) = 0$, we have $s(t) = r(t)$. As a result, the steady-state efforts $k(s, c)$ and $h(s, c)$ can be explicitly represented in terms of the state variables and model parameters, as shown below:

$$\begin{aligned} k(s, c) = \{\tfrac{n[a-c(t)]}{\alpha(\tau-\sigma)[2(a_1-a_2\varphi m\upsilon)+(2a_2\varphi m\vartheta+n-1)s(t)]}\} \\ \times\{\tfrac{(a_2\varphi\vartheta m+n-1)[a-c(t)]}{2(a_1-a_2\varphi m\upsilon)+(2a_2\varphi m\vartheta+n-1)s(t)} + \tfrac{\mu\tau}{\tau+\rho}\} \end{aligned} \tag{3.35}$$

$$h(s, c) = \frac{n[a - c(t)]}{\beta(\tau - \delta)[2(a_1 - a_2\varphi m\upsilon) + (2a_2\varphi m\vartheta + n - 1)s(t)]} \tag{3.36}$$

Taking into account that $h(s, c) > 0$ and applying Condition 3.1(ii), we derive $2(a_1 - a_2\varphi m\upsilon) + (2a_2\varphi m\vartheta + n - 1)s(t) > 0$. We summarize this result as the following Condition 3.10:

Condition 3.10. *For admissible parameters, when $q(t) = Q(t)/m$, the condition $2(a_1 - a_2\varphi m\upsilon) + (2a_2\varphi m\vartheta + n - 1)s(t) > 0$ must be satisfied to ensure that the effort level remains non-negative.*

From equations (3.35) and (3.36), we can derive $k(s, c) = \frac{\beta(\tau-\delta)}{\alpha(\tau-\sigma)} h(s, c)[\frac{\mu\tau}{\tau+\rho} + \frac{\beta(a_2\varphi\vartheta m+n-1)(\tau-\delta)}{n} h(s, c)]$. Now, by treating $k(s, c)$ as a function of $h(s, c)$, and differentiating the above equation w.r.t. $h(s, c)$, we obtain $\frac{\partial k(s,c)}{\partial h(s,c)} = \frac{\beta(\tau-\delta)}{\alpha(\tau-\sigma)}[\frac{\mu\tau}{\tau+\rho} + \frac{2\beta(a_2\varphi\vartheta m+n-1)(\tau-\delta)}{n} h(s, c)] > 0$. This suggests that when the network size is proportional to consumers' demand, the two efforts $k(s, c)$ and $h(s, c)$ consistently exhibit a complementary relationship.

Now, differentiating the above expressions (3.35) and (3.36) w.r.t. m, we arrive at the following Conclusion 3.11 (Proof: See Appendix 8).

Conclusion 3.11. *Under the firm's decision-making approach, when $q(t) = Q(t)/m$, the following conclusions can be drawn: (i) $\frac{\partial k(s,c)}{\partial m} > 0$; (ii) $\frac{\partial h(s,c)}{\partial m} > 0$.*

Conclusion 3.11 indicates that the proportional coefficient m affects the changes in the steady-state efforts $k(s, c)$ and $h(s, c)$. Specifically, $k(s, c)$ and $h(s, c)$ increase with the proportional coefficient m. Furthermore, from Conclusion 3.11, we observe that $\frac{\partial k(s,c)/\partial m}{\partial h(s,c)/\partial m} > 0$. This suggests that when there is a variation in the proportional coefficient m, a complementary relationship exists between the steady-state efforts $k(s,c)$ and $h(s,c)$. Additionally, since $q(t) = Q(t)/m$, as m increases, $q(t)$ decreases. In other words, the economic significance of Conclusion 3.11 is that as the proportional coefficient increases, leading to a decrease in market demand, the firm will increase its efforts in product and process innovation.

Subsequently, by integrating the dynamic state equations (2.1)–(2.3) with the dynamic control equations (3.33)–(3.34), we derive the following dynamic control system:

$$\begin{cases} \dot{k}(t) = (\tau - \sigma)k(t) - \{\frac{n[a-\mu(s(t)-r(t))-c(t)]}{\alpha[2(a_1-a_2\varphi m\upsilon)+(2a_2\varphi m\vartheta+n-1)s(t)]}\} \\ \qquad \times \{\frac{(a_2\varphi\vartheta m+n-1)[a-\mu(s(t)-r(t))-c(t)]}{2(a_1-a_2\varphi m\upsilon)+(2a_2\varphi m\vartheta+n-1)s(t)} + \frac{\mu\tau}{\tau+\rho}\} \\ \dot{h}(t) = (\tau - \delta)h(t) - \frac{n\{a-\mu[s(t)-r(t)]-c(t)\}}{\beta[2(a_1-a_2\varphi m\upsilon)+(2a_2\varphi m\vartheta+n-1)s(t)]} \\ \dot{s}(t) = -k(t) + \sigma s(t) \\ \dot{c}(t) = -h(t) + \delta c(t) \\ \dot{r}(t) = \rho[s(t) - r(t)] \end{cases} \tag{3.37}$$

Upon solving the dynamic control system (3.37) under the steady-state condition $\dot{k}(t) = \dot{h}(t) = \dot{s}(t) = \dot{c}(t) = \dot{r}(t) = 0$ and denoting the steady-state equilibrium with the superscript "f", we derive the following Proposition 3.12 (Proof: See Appendix 9):

Proposition 3.12. *Under firm decision-making, when $q(t) = Q(t)/m$, the steady-state equilibrium $\{k^{f'}, h^{f'}, s^{f'}, c^{f'}, r^{f'}\}$ is stable (unstable) in the saddle-point sense iff $\rho < (>)\ 2\tau$, where $k^{f'} = \sigma s^{f'}$, $h^{f'} = \delta c^{f'}$, $s^{f'} = r^{f'} = \frac{an-[2\beta\delta(\tau-\delta)(a_1-a_2\varphi m\upsilon)+n]c^{f'}}{\beta\delta(\tau-\delta)(2a_2\varphi m\vartheta+n-1)c^{f'}}$, $c^{f'} = \{-\frac{V}{2}+[(\frac{V}{2})^2+(\frac{U}{3})^3]^{\frac{1}{2}}\}^{\frac{1}{3}}+\{-\frac{V}{2}-[(\frac{V}{2})^2+(\frac{U}{3})^3]^{\frac{1}{2}}\}^{\frac{1}{3}}-\frac{W_2}{3W_1}$, in which U, V, W_1, and W_2 are presented in Appendix 9.*

The economic significance of Proposition 3.12 mirrors that of Proposition 3.8. To further complement our analysis, we proceed to conduct an equilibrium analysis under government regulation in continuous time $t \in [0, +\infty)$ in the subsequent section.

4. The government regulation

4.1. Steady-state equilibrium

In this section, we consider a scenario where a benevolent government is tasked with optimizing both types of efforts to maximize the discounted flow of social welfare over continuous time $t \in [0, +\infty)$. Concurrently, the output level remains under the control of the firm, as defined in equation (3.9). It is worth noting that the approach to analyzing the (in)efficiency of a firm's efforts in product and process innovation is consistent with the analysis in previous studies (e.g., [15, 17, 27, 45]). We now define the social welfare function as additively separable into two components: the firm's profit $\pi(t)$ and the consumer surplus $cs(t)$. Specifically, the social welfare function is expressed as $sw(t) = \pi(t) + cs(t)$, where $\pi(t)$ is derived from the profit function (2.8). Under symmetry assumptions, the profit $\pi(t)$ can be formulated as follows:

$$\begin{aligned} \pi(t) = n\{a + a_2\varphi[\upsilon - \vartheta s(t)]Q(t) - \mu[s(t) - r(t)] - [a_1 + (n-1)s(t)]q(t) - \\ c(t)\}q(t) - \tfrac{\alpha}{2}k^2(t) - \tfrac{\beta}{2}h^2(t) \end{aligned} \tag{4.1}$$

Moreover, under the assumptions of symmetry, the consumer surplus $cs(t)$ can be formulated as

$$cs(t) = \int_0^{q(t)} \{a - a_1 z(t) + a_2\varphi[\upsilon - \vartheta s(t)]Q(t) - \mu[s(t) - r(t)] -$$

$$(n-1)s(t)z(t)\}dz(t) \tag{4.2}$$
$$= \{a + a_2\varphi[\upsilon - \vartheta s(t)]Q(t) - \mu[s(t) - r(t)]\}q(t) - \frac{1}{2}[a_1 + (n-1)s(t)]q^2(t)$$

Thus, the social welfare function $sw(t)$ is specified as

$$\begin{aligned} sw(t) = (n+1)\{a + a_2\varphi[\upsilon - \vartheta s(t)]Q(t) - \mu[s(t) - r(t)]\}q(t) - nc(t)q(t) \\ - \tfrac{(2n+1)}{2}[a_1 + (n-1)s(t)]q^2(t) - \tfrac{\alpha}{2}k^2(t) - \tfrac{\beta}{2}h^2(t) \end{aligned} \tag{4.3}$$

Given the output level specified in equation (3.9), i.e., $q(t) = \frac{a_2\varphi[\upsilon-\vartheta s(t)]Q(t)-\mu[s(t)-r(t)]-c(t)+a}{2a_1+(n-1)s(t)}$, the government's objective is to ascertain the optimal levels of effort $k(t)$ and $h(t)$ to maximize the discounted flow of social welfare SW. Specifically, this can be formulated as

$$\begin{aligned} SW = \max_{k(t),h(t)} \int_0^{+\infty} e^{-\tau t}\{(n+1)[a + a_2\varphi(\upsilon - \vartheta s(t))Q(t) - \mu(s(t) - r(t))] \\ \times q(t) - nc(t)q(t) - \tfrac{(2n+1)}{2}[a_1 + (n-1)s(t)] \\ \times q^2(t) - \tfrac{\alpha}{2}k^2(t) - \tfrac{\beta}{2}h^2(t)\}dt \\ s.t. \begin{cases} \dot{s}(t) = -k(t) + \sigma s(t) \\ \dot{c}(t) = -h(t) + \delta c(t) \\ \dot{r}(t) = \rho[s(t) - r(t)] \end{cases} \end{aligned} \tag{4.4}$$

where $s(0) = s_0$, $c(0) = c_0$, and $r(0) = r_0$ are the initial conditions, respectively. Using the superscript "g" to represent the steady-state equilibrium, we arrive at the following Proposition 4.1 (Proof: See Appendix 10):

Proposition 4.1. *Under government regulation, the steady-state equilibrium* $\{k^g, h^g, s^g, c^g, r^g\}$ *is stable (unstable) in the saddle-point sense iff* $\rho < (>)\ 2\tau$, *in which* $k^g = \sigma s^g$, $h^g = \delta c^g$, $r^g = s^g = \{-\frac{M}{2} + [(\frac{M}{2})^2 + (\frac{N}{3})^3]^{\frac{1}{2}}\}^{\frac{1}{3}} + \{-\frac{M}{2} - [(\frac{M}{2})^2 + (\frac{N}{3})^3]^{\frac{1}{2}}\}^{\frac{1}{3}} - \frac{F_2}{3F_1}$, $c^g = \frac{n[a+a_2\varphi(\upsilon-\vartheta s^g)Q]}{\{\beta(\tau-\delta)\delta[2a_1+(n-1)s^g]+n\}}$, *where* M, N, F_1, *and* F_2 *are presented in Appendix 10.*

4.2. *A special case under government regulation*

When $q(t) = Q(t)/m$, the inverse demand function is given by equation (3.21). Under symmetry assumptions, the profit function $\pi(t)$ can be expressed as

$$\begin{aligned} \pi(t) = n\{a - \mu[s(t) - r(t)] - c(t)\}q(t) - n\{a_1 - a_2\varphi m[\upsilon - \vartheta s(t)] + (n-1) \\ \times s(t)\}q^2(t) - \tfrac{\alpha}{2}k^2(t) - \tfrac{\beta}{2}h^2(t) \end{aligned} \tag{4.5}$$

Additionally, under symmetry assumptions, the consumer surplus $cs(t)$ is specified by

$$cs(t) = \int_0^{q(t)} \{a - a_1 z(t) + a_2\varphi m[\upsilon - \vartheta s(t)]z(t) - \mu[s(t) - r(t)] - (n-1)s(t)z(t)\}dz(t) \tag{4.6}$$
$$= \{a - \mu[s(t) - r(t)]\}q(t) - \frac{1}{2}\{a_1 - a_2\varphi m[\upsilon - \vartheta s(t)] + (n-1)s(t)\}q^2(t)$$

Thus, the social welfare function $sw(t)$ can be expressed as

$$\begin{aligned} sw(t) = (n+1)\{a - \mu[s(t) - r(t)]\}q(t) - \tfrac{(2n+1)}{2}\{a_1 - a_2\varphi m[\upsilon - \vartheta s(t)] + \\ (n-1)s(t)\}q^2(t) - nc(t)q(t) - \tfrac{\alpha}{2}k^2(t) - \tfrac{\beta}{2}h^2(t) \end{aligned} \tag{4.7}$$

Given the output level specified in equation (3.30), i.e., $q(t) = \frac{a-\mu[s(t)-r(t)]-c(t)}{2(a_1-a_2\varphi m\upsilon)+(2a_2\varphi m\vartheta+n-1)s(t)}$, the government seeks to identify the optimal efforts $k(t)$ and $h(t)$ to maximize the discounted flow of social welfare SW. Specifically, this objective can be formulated as follows:

$$\begin{aligned} SW = \max_{k(t),h(t)} \int_0^{+\infty} & e^{-\tau t}\{(n+1)[a-\mu(s(t)-r(t))]q(t) - \tfrac{(2n+1)}{2}[a_1 \\ & - a_2\varphi m(\upsilon - \vartheta s(t)) + (n-1)s(t)]q^2(t) \\ & - nc(t)q(t) - \tfrac{\alpha}{2}k^2(t) - \tfrac{\beta}{2}h^2(t)\}dt \\ s.t. & \begin{cases} \dot{s}(t) = -k(t) + \sigma s(t) \\ \dot{c}(t) = -h(t) + \delta c(t) \\ \dot{r}(t) = \rho[s(t) - r(t)] \end{cases} \end{aligned} \tag{4.8}$$

where $s(0) = s_0$, $c(0) = c_0$, and $r(0) = r_0$ are the initial conditions. We will use the superscript "g" to indicate the steady-state equilibrium. This leads us to the following Proposition 4.2 (Proof: See Appendix 11):

Proposition 4.2. *Under government regulation, when $q(t) = Q(t)/m$, the steady-state equilibrium $\{k^{g'}, h^{g'}, s^{g'}, c^{g'}, r^{g'}\}$ is stable (unstable) in the saddle-point sense iff $\rho < (>)\ 2\tau$, where $k^{g'} = \sigma s^{g'}$, $h^{g'} = \delta c^{g'}$, $s^{g'} = \frac{(n+1)(\tau-\delta)\mu\tau\delta}{\sigma\alpha(\tau-\sigma)(\tau+\rho)}c^{g'} + r^{g'}\frac{(2n+1)[a_2\varphi m\vartheta+(n-1)](\tau-\delta)^2\delta^2}{2\sigma\alpha(\tau-\sigma)}(c^{g'})^2$, $c^{g'} = \{-\frac{E}{2} + [(\frac{E}{2})^2 + (\frac{R}{3})^3]^{\frac{1}{2}}\}^{\frac{1}{3}} + \{-\frac{E}{2} - [(\frac{E}{2})^2 + (\frac{R}{3})^3]^{\frac{1}{2}}\}^{\frac{1}{3}} - \frac{V_2}{3V_1}$, in which E, R, V_1, and V_2 are presented in Appendix 11.*

Next, we provide numerical examples to compare the steady-state equilibrium under firm decision-making with that under government regulation.

5. Numerical examples

Due to the complexity of the steady-state equilibria $\{k^f, h^f, s^f, c^f, r^f\}$, $\{k^g, h^g, s^g, c^g, r^g\}$, $\{k^{f'}, h^{f'}, s^{f'}, c^{f'}, r^{f'}\}$, and $\{k^{g'}, h^{g'}, s^{g'}, c^{g'}, r^{g'}\}$, under both firm decision-making and government regulation, it is challenging to analytically determine whether the efforts under government regulation are higher or lower than those under firm decision-making. However, a comparative analysis can be conducted using numerical examples. The baseline parameterization can be interpreted as representing technology-intensive markets, such as telecommunication or smart device markets, where firms offer multiple differentiated products in a network environment. In such markets, consumer utility depends on network size, while product innovation affects the degree of differentiation within the firm's portfolio [46–48].

By setting the proportional coefficient and utilizing the parameter values provided in Table 1, we obtain the following results (under steady-state equilibrium, we have $s^f = r^f$, $s^g = r^g$, $s^{f'} = r^{f'}$, and $s^{g'} = r^{g'}$):

(i) $\{k^f, h^f, s^f, c^f\} = \{3.42, 4.04, 0.42, 2.35\}$, $\{k^g, h^g, s^g, c^g\} = \{3.78, 4.89, 0.37, 2.12\}$, and then we obtain $\{\Delta k^{f-g}, \Delta h^{f-g}, \Delta s^{f-g}, \Delta c^{f-g}\} = \{k^f, h^f, s^f, c^f\} - \{k^g, h^g, s^g, c^g\} = \{-0.36, -0.85, 0.05, 0.23\}$;

(ii) $\{k^{f'}, h^{f'}, s^{f'}, c^{f'}\} = \{4.17, 5.32, 0.39, 2.89\}$, $\{k^{g'}, h^{g'}, s^{g'}, c^{g'}\} = \{5.22, 5.89, 0.27, 2.12\}$, and then we have $\{\Delta k^{f-g}, \Delta h^{f-g}, \Delta s^{f-g}, \Delta c^{f-g}\} = \{k^{f'}, h^{f'}, s^{f'}, c^{f'}\} - \{k^{g'}, h^{g'}, s^{g'}, c^{g'}\} = \{-1.06, -0.57, 0.12, 0.77\}$.

Based on the results from (i) and (ii), it can be observed that for admissible parameter values, the equilibrium efforts are greater under government regulation compared to those under firm decision-making. To assess the robustness of the numerical findings, we further vary the key parameters related to network externality intensity and consumers' adjustment speed. The qualitative conclusions remain unchanged across these alternative configurations (see Figures. A.1–A.2 in Appendix 12).

6. Discussions

Social-welfare implications

The comparison between profit-seeking and socially optimal outcomes indicates that both product and process innovation efforts are subject to under-investment problems. This result is broadly consistent with the classical literature showing that firms tend to under-provide innovation efforts, which generates more consumer-side benefits over private profits [15, 49]. In our model, decreasing horizontal differentiation among products or reducing marginal production cost mitigates internal cannibalization and improves product-line structure, whereas the firm does not fully internalize the social-welfare gains associated with more efficient market structure and stronger demand performance. This finding differs from earlier studies showing that under-investment arises in only one innovation dimension, while the other remains socially efficient [4]. In contrast, our results indicate that, under the present framework where product innovation reduces product substitutability and process innovation lowers production costs, both types of innovation suffer from under-investment. Therefore, the under-investment problem is not confined to demand-side innovation alone. This result extends the existing literature by showing that multidimensional under-investment may emerge even in a setting centered on reducing substitutability and production costs. Practically, this result suggests that targeted policy support for both product and process innovation, such as innovation subsidies, tax credits, or support for portfolio redesign, may be justified in network-related industries [18].

Stability property of the innovation system

The stability analysis identifies the conditions under which the firm's dynamic decisions ultimately converge to a steady state. This is relevant for practitioners because firms operating in technology-intensive markets often need to evaluate whether their innovation policies will reach a predictable long-term outcome. Our results show that, under the stated conditions, the system admits a stable steady-state equilibrium. This means that the joint evolution of product differentiation, production cost, and consumers' reference perceptions may not generate unstable dynamics, even in the presence of network-related demand effects.

Product and process innovation relationship

The relationship between product and process innovation has received mixed conclusions in the literature, partly because product innovation is often defined as improving vertical quality [49]. In contrast, the present paper defines product innovation as reducing substitutability among products within a multi-product firm's portfolio, thereby weakening internal cannibalization. Under this setting, we find that product and process innovation tend to be complementary. This result has practical relevance for multi-product firms. For example, a telecommunications provider redesigning service plans to make them more differentiated may also have stronger incentives to invest in process improvements that reduce service delivery costs. Additionally, we show that when demand weakens, firms may increase both product and process innovation efforts rather than relying on only one dimension of innovative activities.

Network expansion as a complement for innovation activities

We also conclude that both types of innovation efforts increase with the proportional coefficient governing network expansion. This reflects a complementary relationship between innovative activities and network expansion: when the market environment more strongly turns demand into network growth, firms have greater incentives to invest in both product differentiation and cost reduction in order to support the expansion process. Intuitively, network expansion and innovation may be viewed as substitutes, because network expansion tends to generate path dependence. Once a firm has accumulated a sufficiently large user base, it can rely more heavily on the self-reinforcing effect of the network, which reduces the need for further innovation [50]. However, our finding is contrary to this intuition, which provides managerial insights for technology-intensive and network-related markets. For example, in telecommunications markets, stronger user-base growth increases the payoff from differentiating service and from reducing marginal delivery costs, because both forms of innovation support a larger and more valuable network.

7. Conclusions and the managerial implications

In this paper, we develop a dynamic control model to examine the product and process innovation strategies of a multi-product firm operating in a market with network externality, while incorporating reference effects regarding horizontal differentiation among products. The main features of our framework are as follows: (i) a multi-product firm offers differentiated products in a dynamic environment with network externality and invests in both product and process innovation, taking into account consumers' reference perceptions of horizontal differentiation; (ii) the network value depends on both network size and the degree of product substitutability; (iii) market demand is influenced by price, product substitutability, reference perception, and network size; and (iv) we explore a special case in which network size is proportional to market demand.

Our research findings are presented in the form of conclusions and propositions. The main results can be outlined as follows: (i) the stability properties of the system depend on key parameters such as the discount rate and the consumers' memory parameter under both monopolist decision-making and government regulation. In particular, the steady-state equilibrium exhibits saddle-point stability under certain parameter conditions, including cases where $\rho < 2\tau$; (ii) our analysis suggests that product and process innovation efforts tend to exhibit a complementary relationship within the considered framework; (iii) the price level is positively associated with product innovation effort and negatively associated with process innovation effort; (iv) the impact of product variety on innovation efforts depends on model parameters such as a_1 and the degree of product substitutability $s(t)$, and may vary across different parameter regions; (v) in the special case where the demand is proportional to the network size, the steady-state levels of product and process innovation are generally increasing in the proportional coefficient m. Moreover, as the proportional coefficient increases and market demand declines, firms may have incentives to intensify both product and process innovation efforts.

Overall, our analysis provides a theoretical framework for understanding how product innovation, process innovation, network externality, and reference effects jointly shape firms' dynamic decisions in a multi-product setting. Future research may extend this framework by incorporating additional behavioral factors, such as reference price effects, or by considering more general market structures.

Author contributions

Huiquan Li: Conceptualization and writing original draft; Huiquan Li and Lijia Ge: Programming; Lijia Ge and Ran Jiang: Validation and revising; Ran Jiang: Methodology and supervision.

Use of Generative-AI tools declaration

In the preparation of this work, the authors used Generative AI tools such as ChatGPT to assist in improving the clarity of the language. All AI-assisted content was carefully reviewed and revised by the authors, who take full responsibility for the final version of the manuscript.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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