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Research article

Digital innovation and product innovation in a monopoly exhibiting network externality: A dynamic analysis

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Abstract: With the continuous iteration of information technology, the level of digitalization has become a key factor in product development for a monopolist, especially in markets with network externality. In this paper, we took a dynamic control approach to investigate a monopolist's digital innovation and product innovation in a market characterized by network externality. The major features of this work are: (i) A monopolist conducted digital innovation to improve the level of digitalization and product innovation to enhance product quality; (ii) the demand function consisted of price, product quality, digitalization level, and network size; and (iii) the network value function depended on network size and demand. The major findings are as follows: (i) Regardless of the monopolist decision-making or government regulation, the saddle-point stability of steady-state equilibrium depends on the depreciation rate of product quality and the digitalization level; (ii) price increases with product quality, the level of digitalization, and network size; and (iii) there is always a complementary relationship between the effort for digital innovation and that for product innovation near the steady-state equilibrium. Our findings serve as a valuable addition to the literature on monopolistic digital innovation and product innovation, particularly in the context of network effects.

Keywords: digital innovation; product innovation; network externality
Mathematics Subject Classification: 49N90, 37N40

1. Introduction

In the era of global digital transformation, the rapid evolution of digital technologies has reshaped the competitive landscape of industries and the operational logic of organizations. Traditional business models, which rely on linear processes and offline interactions, are increasingly challenged by dynamic market demands and emerging customer expectations. Digital innovation refers to the use of digital technologies to transform or enhance business processes, organizational culture, and customer experiences in response to evolving market conditions and customer needs [1–4]. It is not just about inventing new technology; it is about using existing and emerging digital tools in novel ways to generate new value. In essence, it is the application of digital technology and new ideas to solve old problems in better ways or to create new opportunities.

In today's rapidly evolving business landscape, digital innovation has emerged as a transformative force, revolutionizing traditional methods and driving competitive differentiation [5, 6]. Through the strategic adoption and application of digital technologies, firms can optimize operations, develop innovative products and services, enhance customer engagement, and better respond to evolving market demands [7, 8]. Digital innovation encompasses concepts such as digital transformation and digitization, which are closely intertwined. It relies on advanced technologies like artificial intelligence, big data analytics, and cloud computing to identify new opportunities and fuel growth within systems and tools. The concept of digital transformation has gained significant attention as firms strive to stay competitive in the digital age. Understanding the dynamics of this transformation becomes crucial for firms. Researchers have explored the impact of digital transformation on firm performance [9] and enterprise innovation [10], highlighting the mediating role of total factor productivity and innovation outputs. Moreover, in [11], the researchers investigate the impact of non-R&D subsidies on digital product innovation among Chinese manufacturing firms and explore the conditions that influence this relationship. They propose that non-R&D subsidies have a positive effect on a firm's digital product innovation. On the other hand, the researchers in [12] explore how digital innovation transforms industries and reshapes business practices. They emphasize the importance of network externalities and reduced switching costs in creating customer lock-in, and highlight the role of digital transformation in creating customer loyalty and reducing the likelihood of customers switching to competitors.

In this study, we present a dynamic analysis of the monopolist's digitalization level, considering the effects of product innovation and network externality. Based on the Nerlove–Arrow framework, we extend studies by bringing product quality, digital innovation, and innovation effort into one dynamic setting [13, 14]. In the model, firms invest in product and digital innovation to improve quality and digitalization. The size of the user network grows over time with demand, and network effects make digitalization more valuable by reinforcing user adoption. As a result, demand depends on product quality, digitalization, and network size. We then study the equilibrium under firm decision-making and government regulation, focusing on the steady state and its saddle-point stability.

By investigating the intricate relationship between digital innovation, pricing strategies, and product innovation, this research contributes to a deeper understanding of how firms can navigate the digital landscape to achieve sustainable growth and competitive advantage. Our findings of this study will provide valuable insights for businesses in formulating effective strategies for digital innovation. Additionally, our analysis of government regulation will inform policymakers on the potential impact of regulatory measures in the digital era.

The paper is structured in the following manner: In Section 2, we discuss the Basic Model, followed by Section 3, where we explore the monopolist's decision behavior. In Section 4, we focus on government regulation, while in Section 5, we present the numerical analysis. Theoretical and managerial implications are discussed in Section 6, and we conclude with Section 7, providing the concluding remarks.

2. The basic model

In this paper, we study a monopolist that chooses its price as well as its efforts in digital innovation (which raises the level of digitalization) and product innovation (which improves product quality) for a single product in a dynamic market with network externalities over continuous time $t \in [0, \infty)$. Following [13, 15–20], we model product quality $q(t)$ as a state variable that evolves over time $t \in [0, \infty)$. Economically, this captures the idea that product innovation is a cumulative process: Current investment improves quality, while depreciation reflects technological obsolescence and changing consumer expectations. Under this framework, we specify the dynamic evolution of product quality through a differential equation that balances innovation investment and depreciation, enabling us to analyze the firm's optimal decisions and the resulting equilibrium dynamics in economically meaningful terms.

$$\dot{q}(t) = k(t) - \sigma_1 q(t) \tag{2.1}$$

where $k(t)$ represents the monopolist's effort for product innovation at time $t \in [0, \infty)$, and parameter $\sigma_1 > 0$ denotes the depreciation rate of product quality. Besides, the effort cost borne by the monopolist will be assumed quadratic $\frac{\beta_1}{2}k^2(t)$, where β_1 is the positive parameter [18, 21–24]. The cost of product innovation increases at an increasing rate. In simple terms, the more a firm tries to improve product quality, the harder and more expensive it becomes to achieve further improvements.

We next consider the dynamics of digital innovation. In economic terms, digital innovation can be understood as a cumulative process through which firms adopt and integrate digital technologies to improve efficiency, enhance customer experience, and differentiate their products. Unlike one-time investments, digitalization builds over time and requires continuous effort to adapt to technological progress and changing market conditions. In this context, digital innovation raises the effective quality and attractiveness of the product by improving functionality, usability, and connectivity. It also strengthens network effects, as a higher level of digitalization makes it easier for users to interact and derive value from a larger network. We therefore model digitalization as a state variable that evolves dynamically over time. Its evolution reflects a balance between firms' investment efforts, which increase the level of digitalization, and depreciation, which captures technological obsolescence and the need for ongoing upgrades. This specification captures the economic reality that maintaining and improving digital capabilities requires sustained investment rather than one-off decisions. Based on these considerations, the dynamics of digitalization level $\theta(t)$ are described as follows:

$$\dot{\theta}(t) = h(t) - \sigma_2 \theta(t) \tag{2.2}$$

where $h(t)$ stands for the monopolist's effort for digital innovation at time $t \in [0, \infty)$, and parameter $\sigma_2 > 0$ denotes the depreciation rate of the digitalization level. Similar to the effort cost borne by the monopolist for product innovation, we also assume that the effort cost borne by the monopolist for digital innovation is given by $\frac{\beta_2}{2}h^2(t)$, where β_2 is the positive parameter.

Now, we consider the evolution of network size. In markets with network externalities, consumers' decisions depend not only on current product attributes but also on their expectations about how the user base will evolve [25]. If consumers anticipate that the network will expand, they are more likely to adopt the product, whereas expectations of decline may discourage participation or induce exit. From an economic perspective, changes in network size reflect the balance between consumer entry and exit. We assume that the number of consumers $R(t)$ entering the network is positively related to market demand $D(t)$. This captures the idea that stronger demand signals higher perceived value, which attracts new users into the system. For simplicity, we adopt a linear relationship between consumer inflow and demand, $R(t) = \alpha D(t)$, where proportionality parameter $\alpha \geq 0$ measures the sensitivity of entry to market conditions. Following [26], network size is therefore modeled as a dynamic state variable whose evolution is driven by this demand-induced entry. This formulation reflects the economic mechanism through which market demand translates into network expansion over time. Then, the dynamic equation of network size $Q(t)$ can be written as follows:

$$\dot{Q}(t) = \alpha D(t) - \delta Q(t) \tag{2.3}$$

where coefficient $\delta \geq 0$ represents the rate at which consumers exit the network system. Under steady-state condition $\dot{Q}(t) = 0$, it follows that $Q(t) = \frac{\alpha}{\delta} D(t)$, indicating a linear proportional relationship between network size and demand. When $\alpha = \delta$, meaning that the entry rate of consumers equals the exit rate, we obtain $D(t) = Q(t)$. In addition, studies such as [13, 27, 28] treat $Q(t)$ as an exogenous variable.

Having described the evolution of network size, we now turn to the specification of consumer demand. In economic terms, demand is shaped by price and non-price factors, including product quality, digitalization, and the value generated by the user network. We assume that demand $D(t)$ depends on product price $p(t)$, product quality $q(t)$, digitalization level $\theta(t)$, and network size $Q(t)$. Price negatively affects demand through the standard willingness-to-pay channel. In contrast, higher product quality and a greater level of digitalization increase demand by improving product performance, functionality, and user experience. Network size further enhances demand through network externalities, as a larger user base raises the value of adoption for each consumer. For tractability, we adopt an additively separable specification. This captures the idea that each factor independently contributes to demand, while keeping the model analytically manageable. Although more complex interactions could be considered, this formulation enables us to isolate and interpret the role of each determinant [17, 29]. In addition, we assume that there is no inventory, so that all output is sold in each period. This reflects markets where production closely matches demand, such as digital goods or made-to-order products, and ensures that output is directly tied to realized demand. Therefore, the demand function can be written as follows:

$$D(t) = a - b_1 p(t) + b_2 q(t) + b_3 \theta(t) + b_4 Q(t) \tag{2.4}$$

in which a represents the basic market potential, and b_1, b_2, b_3, and b_4 denote the effects of the price, product quality, digitalization level, and network size on the demand function, respectively, where a, b_1, b_2, b_3, and b_4 are the positive parameters.

Substituting the demand function (2.4) into (2.3), we can derive the dynamic state of the network size as follows:

$$\dot{Q}(t) = \alpha[a - b_1 p(t) + b_2 q(t) + b_3 \theta(t)] - (\delta - \alpha b_4) Q(t) \tag{2.5}$$

Without loss of generality, we assume that the marginal production cost of the monopolist is equal to c. Thus, at continuous time $t \in [0, \infty)$, the monopolist's net profit $\pi(t)$ is given by:

$$\pi(t) = [p(t) - c][a - b_1 p(t) + b_2 q(t) + b_3 \theta(t) + b_4 Q(t)] - \frac{\beta_1}{2} k^2(t) - \frac{\beta_2}{2} h^2(t) \tag{2.6}$$

In the next section, we conduct equilibrium analysis under firm decision-making and government regulation, respectively.

3. The monopolist's decision behavior

3.1. The general properties

In this setting, the objective of the monopolist is to choose the price $p(t)$ and the efforts $k(t)$ $h(t)$ over continuous time $t \in [0, \infty)$ to maximize the discounted present value of net profit Π under the constraints of dynamic equations (2.1), (2.2), and (2.5), namely

$$\begin{aligned} &\Pi = \max_{p,k,h} \int_0^{\infty} e^{-\rho t}\{[p(t) - c][a - b_1 p(t) + b_2 q(t) + b_3\theta(t) + b_4 Q(t)] - \frac{\beta_1}{2}k^2(t) - \frac{\beta_2}{2}h^2(t)\}dt \\ &s.t. \begin{cases} \dot{q}(t) = k(t) - \sigma_1 q(t) \\ \dot{\theta}(t) = h(t) - \sigma_2\theta(t) \\ \dot{Q}(t) = \alpha[a - b_1 p(t) + b_2 q(t) + b_3\theta(t)] - (\delta - \alpha b_4)Q(t) \end{cases} \end{aligned} \tag{3.1}$$

where ρ represents the discount rate, and the initial conditions are $q(0) = q_0$, $\theta(0) = \theta_0$, and $Q(0) = Q_0$.

Accordingly, the Hamiltonian function of the dynamic control model (3.1) can be expressed as:

$$\begin{aligned} H = &[p(t) - c][a - b_1 p(t) + b_2 q(t) + b_3\theta(t) + b_4 Q(t)] - \frac{\beta_1}{2}k^2(t) - \frac{\beta_2}{2}h^2(t) + \lambda_1(t)[k(t) - \sigma_1 q(t)] \\ &+ \lambda_2(t)[h(t) - \sigma_2\theta(t)] + \lambda_3(t)\{\alpha[a - b_1 p(t) + b_2 q(t) + b_3\theta(t)] - (\delta - \alpha b_4)Q(t)\} \end{aligned} \tag{3.2}$$

where $\lambda_1(t)$, $\lambda_2(t)$, and $\lambda_3(t)$ denote the shadow prices (co-state variables) corresponding to the state variables $q(t)$, $\theta(t)$, and $Q(t)$, respectively.

The first-order conditions and the corresponding dynamic co-state equations for the Hamiltonian function (3.2) are, respectively:

$$\frac{\partial H}{\partial p(t)} = a - b_1[2p(t) - c] + b_2 q(t) + b_3\theta(t) + b_4 Q(t) - \alpha b_1\lambda_3(t) = 0 \tag{3.3}$$

$$\frac{\partial H}{\partial k(t)} = \lambda_1(t) - \beta_1 k(t) = 0 \tag{3.4}$$

$$\frac{\partial H}{\partial h(t)} = \lambda_2(t) - \beta_2 h(t) = 0 \tag{3.5}$$

$$\dot{\lambda}_1(t) = \rho\lambda_1(t) - \frac{\partial H}{\partial q(t)} = (\rho + \sigma_1)\lambda_1(t) - b_2[p(t) - c] - \alpha b_2\lambda_3(t) \tag{3.6}$$

$$\dot{\lambda}_2(t) = \rho\lambda_2(t) - \frac{\partial H}{\partial \theta(t)} = (\rho + \sigma_2)\lambda_2(t) - b_3[p(t) - c] - \alpha b_3\lambda_3(t) \tag{3.7}$$

$$\dot{\lambda}_3(t) = \rho\lambda_3(t) - \frac{\partial H}{\partial Q(t)} = (\rho + \delta - \alpha b_4)\lambda_3(t) - b_4[p(t) - c] \tag{3.8}$$

where the associated transversality conditions are $\lim_{t\to\infty}\lambda_1(t)q(t)e^{-\rho t} = 0$, $\lim_{t\to\infty}\lambda_2(t)\theta(t)e^{-\rho t} = 0$, and $\lim_{t\to\infty}\lambda_3(t)Q(t)e^{-\rho t} = 0$, respectively. In essence, the above transversality conditions pin down the terminal values of the state and/or co-state variables. Furthermore, to guarantee the existence of a steady-state equilibrium for the dynamic system (3.1), we follow [30] and introduce the following Condition 1:

Condition 1. *For the dynamic control model (3.1), given acceptable parameters, we assume that* $\rho + \delta - \alpha b_4 > 0$.

It is important to note that the above assumption may appear restrictive. However, it is necessary to ensure analytical tractability, as relaxing it could lead to an unbounded expansion of network size, thereby preventing the dynamic control system from converging to a stable steady-state equilibrium. Moreover, although some results do not explicitly rely on this assumption, the existence of an equilibrium solution fundamentally requires that the objective function in the dynamic control model (3.1) remains bounded when the monopolist actively engages in product innovation and digital innovation.

We first investigate the shadow prices (or co-state variables) $\lambda_1(t)$, $\lambda_2(t)$, and $\lambda_3(t)$ associated with the state equations $q(t)$, $\theta(t)$, and $Q(t)$, respectively. Solving dynamic co-state equations (3.6)-(3.8) with respect to co-state variables $\lambda_1(t)$, $\lambda_2(t)$, and $\lambda_3(t)$, and using transversality conditions $\lim_{t\to\infty}\lambda_1(t)q(t)e^{-\rho t} = 0$, $\lim_{t\to\infty}\lambda_2(t)\theta(t)e^{-\rho t} = 0$, and $\lim_{t\to\infty}\lambda_3(t)Q(t)e^{-\rho t} = 0$, respectively, one can obtain that

$$\lambda_1(t) = \frac{b_2(\rho + \delta)[p(t) - c]}{(\rho + \sigma_1)(\rho + \delta - \alpha b_4)} \tag{3.9}$$

$$\lambda_2(t) = \frac{b_3(\rho + \delta)[p(t) - c]}{(\rho + \sigma_2)(\rho + \delta - \alpha b_4)} \tag{3.10}$$

$$\lambda_3(t) = \frac{b_4[p(t) - c]}{\rho + \delta - \alpha b_4} \tag{3.11}$$

Since we have $\rho + \delta - \alpha b_4 > 0$ (see Condition 1), then from equations (3.9)–(3.11), we can obtain positive (or buying) shadow prices, i.e., $\lambda_i(t) \geq 0$, $i = 1, 2, 3$. Note that a shadow price is an estimated or implicit value assigned to a good or resource for which no official market price exists. It is not a price you pay at a market; it is a theoretical value used for internal decision-making. Fundamentally, the shadow price represents the opportunity cost of a constraint. It answers the question: “How much would the value of our objective (e.g., profit, social welfare) increase if we had one more unit of this constrained resource?”. Furthermore, a shadow price reflects the scarcity of the goods or factors, and its theoretical basis is marginal utility. The marginal benefit of each incremental unit of goods or factors is its shadow price. This reflects the relationship between the scarcity and value of goods or factors. A positive shadow price indicates that relaxing a constraint by one additional unit will improve the value of the objective function. In simpler terms, it means that if you could have a little more of a limited resource (like more raw materials, more machine hours, or more budget), it would enable you to achieve a better outcome (like higher profit, lower cost, etc.). A negative shadow price indicates that an increase of one unit in the right-hand side of a constraint reduces the value of the objective function. Conversely, a decrease in the constraint, corresponding to a reduction in the availability of the resource, leads to an improvement in the objective function. A shadow price of zero implies that the constraint is non-binding, indicating the presence of surplus resources, such that an increase in their availability has no effect on the objective function value.

Now, we analyze the monopolist's pricing condition. From Condition 1, we have $2(\rho+\delta)-\alpha b_4 > 0$. Further, from equation (3.3) and using equation (3.11), one can obtain the following pricing condition:

$$p(t) = \frac{(\rho+\delta-\alpha b_4)[a+b_2 q(t)+b_3\theta(t)+b_4 Q(t)]+b_1(\rho+\delta)c}{b_1[2(\rho+\delta)-\alpha b_4]} \tag{3.12}$$

From pricing condition (3.12), we always have (i) $\frac{\partial p(t)}{\partial q(t)} > 0$, (ii) $\frac{\partial p(t)}{\partial \theta(t)} > 0$, and (iii) $\frac{\partial p(t)}{\partial Q(t)} > 0$, which state that the price increases with the increase of the product quality $q(t)$, digitization level $\theta(t)$, and network size $Q(t)$, respectively. We can summarize these conclusions in the following Proposition 1.

Proposition 1. *At continuous time $t \in [0,\infty)$, we have (i) $\frac{\partial p(t)}{\partial q(t)} > 0$, (ii) $\frac{\partial p(t)}{\partial \theta(t)} > 0$, and (iii) $\frac{\partial p(t)}{\partial Q(t)} > 0$.*

Proposition 1 reveals that: (i) Higher Production Costs: Superior quality often requires more expensive raw materials, more skilled labor, stricter quality control processes, and longer manufacturing times. These increased costs are naturally passed on to the consumer in the form of a higher price. Enhanced Perceived Value: A high-quality product offers better performance, durability, reliability, and user experience. Customers recognize this added utility and are generally willing to pay more for a product that lasts longer, works better, or satisfies their needs more completely. For example, a handcrafted leather bag commands a higher price than a mass-produced synthetic one because buyers value the superior materials and craftsmanship. (ii) Significant R&D Investment: Developing advanced digital features (e.g., sophisticated software, AI algorithms, seamless user interfaces, and cloud infrastructure) requires a substantial upfront investment in research, development, and engineering. This high fixed cost must be recouped through sales. Ongoing Value & Updates: A highly digital product is often not a static purchase. It can receive continuous software updates, new features, and cloud-based services that improve it over time. This creates an ongoing value proposition, effectively making the product better long after the initial sale, which supports a higher price point. Superior Functionality and Convenience: Digitization often translates to greater efficiency, automation, data analytics, and connectivity. These features solve complex problems or save the user time and effort, which they are willing to pay a premium for. For instance, a "smart" thermostat that learns your schedule and saves energy is more valuable than a simple programmable one. (iii) Increased Value for Every User: A network effect occurs when a product or service becomes more valuable as more people use it. Premium Pricing Power: As the network grows and becomes more valuable to each user, they become increasingly "locked-in" due to the high cost of switching to a competitor with a smaller network. This gives the company immense pricing power. They can charge higher fees for access to this valuable network (e.g., through subscriptions, transaction fees, or premium tiers) because the value the user receives is so great. Natural Monopoly Tendency: In many cases, markets with strong network effects tend toward a winner-takes-most dynamic. The largest network becomes the most valuable, making it difficult for competitors to arise. This dominant position enables the leading company to set prices that reflect the unique value of its massive user base.

In summary, the relationship between price and these three factors is a function of value creation and cost: Quality increases value through superior physical attributes and performance. Digitization increases value through intelligent features, services, and convenience. Network Size increases value through connectivity and ecosystem effects. A firm invests more to achieve these enhancements (increasing cost) and deliver significantly more value to the customer. The customer, in turn, demonstrates their willingness to receive this greater value by accepting a higher price. This is a fundamental principle of value-based pricing.

Substituting the pricing condition (3.12) into (2.4), we can obtain the following customers' demand function:

$$D(t) = \frac{(\rho + \delta)[a + b_2 q(t) + b_3\theta(t) + b_4 Q(t) - b_1 c]}{2(\rho + \delta) - \alpha b_4} \tag{3.13}$$

From pricing condition (3.12), we can obtain the monopolist's marginal gross profit $p(t) - c = \frac{(\rho+\delta-\alpha b_4)[a+b_2 q(t)+b_3\theta(t)+b_4 Q(t)-b_1 c]}{b_1[2(\rho+\delta)-\alpha b_4]}$. Then, one can obtain the Conclusion 1 as follows.

Conclusion 1. *For admissible parameters, we always have* $\frac{D(t)}{p(t)-c} = \frac{b_1(\rho+\delta)}{\rho+\delta-\alpha b_4}$.

Conclusion 1 characterizes the relationship between the monopolist's marginal gross profit, $p(t) - c$, and customer demand, $D(t)$, under admissible parameter conditions. The derived proportionality implies a stable and systematic linkage between these two variables. In particular, Condition 1 ensures that this relationship is positive, indicating that higher marginal gross profit is associated with stronger demand. From a managerial perspective, this result provides useful guidance for pricing decisions. Rather than relying on simplified intuitions about the profit–demand trade-off, the identified proportional relationship offers a more structured basis for decision-making. By accounting for the underlying parameter conditions, the monopolist can better anticipate how changes in pricing or cost structures affect demand, thereby improving the effectiveness of pricing strategies and enhancing profitability.

Substituting the pricing condition (3.13) into (2.3), we can obtain the dynamic state of the network size as follows:

$$\dot{Q}(t) = \frac{\alpha(\rho + \delta)[a + b_2 q(t) + b_3\theta(t) - b_1 c] - [2\delta(\rho + \delta) - \alpha b_4(\rho + 2\delta)]Q(t)}{[2(\rho + \delta) - \alpha b_4]} \tag{3.14}$$

Thus, under steady-state $\dot{Q}(t) = 0$, we have

$$Q(t) = \frac{\alpha(\rho + \delta)[a + b_2 q(t) + b_3\theta(t) - b_1 c]}{2\delta(\rho + \delta) - \alpha b_4(2\delta + \rho)} \tag{3.15}$$

Then differentiating equation (3.15) with respect to (w.r.t.) the product quality $q(t)$ and digitization level $\theta(t)$, respectively, we can obtain the following Proposition 2.

Proposition 2. *At continuous time* $t \in [0, \infty)$, *we have (i)* $\frac{\partial Q(t)}{\partial q(t)} > (\leq) 0$ *if and only if* $b_4 < (\geq) \frac{\alpha(2\delta+\rho)}{2\delta(\rho+\delta)}$; *and (ii)* $\frac{\partial Q(t)}{\partial \theta(t)} > 0$ *if and only if* $b_4 < (\geq) \frac{\alpha(2\delta+\rho)}{2\delta(\rho+\delta)}$.

Proposition 2 reflects a nonlinear and regime-dependent effect of product quality and digitization on network expansion. When the effect of network size on demand is weak, user adoption is primarily driven by intrinsic product attributes. Improvements in product quality enhance user experience, reduce usage friction, and increase satisfaction, thereby lowering churn rates and encouraging positive word-of-mouth. These mechanisms facilitate user acquisition and retention, leading to an expansion of network size. Similarly, higher levels of digitization improve accessibility, functionality, and scalability. Digital technologies reduce barriers to entry, enable efficient user matching through algorithms, and enhance platform integration, all of which increase the attractiveness of the product and accelerate network growth. In contrast, when the effect of network size on demand is strong, the marginal contribution of additional users depends heavily on the network scale. In such a setting, improvements in product quality and digitization may generate diminishing or negative marginal effects on network expansion.

Higher quality and more advanced digital features can raise user expectations and intensify competition within the network, thereby reducing marginal adoption incentives. Moreover, as the network becomes large, congestion, coordination frictions, or strategic interactions among users may emerge, weakening the positive impact of further product improvements. As a result, network size may decline despite increases in quality and digitization.

Proposition 2 yields several important managerial insights. First, firms should align their innovation strategies with the strength of network externalities. When network effects are weak, investments in product quality and digitization directly promote user acquisition and retention. When network effects are strong, firms should shift from product improvement to network management, focusing on user interaction, platform governance, and congestion mitigation. Second, product quality is a necessary but not sufficient condition for sustained network growth. Although it enhances trust, reduces churn, and supports word-of-mouth diffusion, its marginal contribution diminishes as network externalities strengthen. Overreliance on quality improvements may therefore lead to suboptimal outcomes. Third, digitization exhibits a dual effect. While it improves accessibility, scalability, and functionality in early stages, excessive or misaligned digital investment in mature networks may increase complexity, raise user expectations, and intensify competition, thereby hindering growth.

We next examine the dynamic control equations governing digital and product innovation efforts. By differentiating the first-order conditions (3.4)–(3.5) w.r.t. time t, and incorporating the co-state dynamic equations (3.6)–(3.8), the pricing condition (3.12), and the expressions for the co-state variables (3.9)–(3.11), we can rearrange terms to obtain the following results, respectively,

$$\dot{k}(t) = (\rho + \sigma_1)k(t) - \frac{b_2(\rho + \delta)[a + b_2 q(t) + b_3\theta(t) + b_4 Q(t) - b_1 c]}{b_1\beta_1[2(\rho + \delta) - \alpha b_4]} \tag{3.16}$$

$$\dot{h}(t) = (\rho + \sigma_2)h(t) - \frac{b_3(\rho + \delta)[a + b_2 q(t) + b_3\theta(t) + b_4 Q(t) - b_1 c]}{b_1\beta_2[2(\rho + \delta) - \alpha b_4]} \tag{3.17}$$

From dynamic equations (3.16)–(3.17), we have the following Proposition 3 (Proof: See Appendix A):

Proposition 3. *At continuous time $t \in [0, \infty)$, we have (i) $\frac{\partial \dot{k}(t)}{\partial h(t)} < 0$; (ii) $\frac{\partial \dot{h}(t)}{\partial k(t)} < 0$.*

Proposition 3 highlights the negative impact that product innovation efforts and digital innovation efforts have on each other's rates of change. For example, when the rate of change in product innovation efforts is positive, $\dot{k}(t) > 0$, indicating an increase in product innovation over time, increasing digital innovation efforts can slow the rate of product innovation growth. A company focusing heavily on product design and features may experience faster innovation in the short term. However, shifting resources toward digital innovation, such as developing new software or digital platforms, may divert attention and slow product development.

This interdependence between product and digital innovation efforts creates a trade-off in resource allocation and innovation strategies. While focusing on product innovation can drive short-term growth, increasing digital innovation efforts can slow product innovation. Managers must carefully balance these efforts to avoid hindering progress in one area by overemphasizing the other. It is essential to assess the company's market position and long-term strategy before reallocating resources between product and digital innovation. If rapid product innovation is a priority, managers should carefully consider the timing and scale of digital innovation investments to ensure they do not undermine product development.

This insight underscores the importance of strategic planning and prioritization. In fast-evolving markets driven by technological advancements, managers must balance incremental product improvements with disruptive digital innovations, adjusting strategies to maintain overall innovation momentum.

In the following subsection, we examine the local properties of firms' digital and product innovation decisions around the steady-state equilibrium and characterize the saddle-point stability of the steady-state equilibrium.

3.2. Steady-state equilibrium and saddle-point property

First, we impose the stationarity conditions on the control variables, i.e., $\dot{k}(t) = \dot{h}(t) = 0$, to derive the steady-state levels of product innovation effort $k^m(q, \theta, Q)$ and digital innovation effort $h^m(q, \theta, Q)$ as functions of the state variables and model parameters. Specifically, we obtain

$$k^m(q, \theta, Q) = \frac{b_2(\rho + \delta)[a + b_2 q(t) + b_3\theta(t) + b_4 Q(t) - b_1 c]}{b_1\beta_1(\rho + \sigma_1)[2(\rho + \delta) - \alpha b_4]} \tag{3.18}$$

$$h^m(q, \theta, Q) = \frac{b_3(\rho + \delta)[a + b_2 q(t) + b_3\theta(t) + b_4 Q(t) - b_1 c]}{b_1\beta_2(\rho + \sigma_2)[2(\rho + \delta) - \alpha b_4]} \tag{3.19}$$

In what follows, we investigate complementarity or substitutability between the efforts $k^m(q, \theta, Q)$ and $h^m(q, \theta, Q)$. It is easy to verify Proposition 4 as follows (Proof: See Appendix B):

Proposition 4. *For state variables $q(t)$, $\theta(t)$, and $Q(t)$, the following relationship always exists* $\frac{\partial k^m(q,\theta,Q)}{\partial q(t)} / \frac{\partial h^m(q,\theta,Q)}{\partial q(t)} = \frac{\partial k^m(q,\theta,Q)}{\partial \theta(t)} / \frac{\partial h^m(q,\theta,Q)}{\partial \theta(t)} = \frac{\partial k^m(q,\theta,Q)}{\partial Q(t)} / \frac{\partial h^m(q,\theta,Q)}{\partial Q(t)} > 0.$

Proposition 4 elucidates the complex yet consistent relationship between the steady-state efforts allocated for product innovation, $k^m(q, \theta, Q)$, and those for digital innovation, $h^m(q, \theta, Q)$, relative to the state variables $q(t)$, $\theta(t)$, and $Q(t)$. Specifically, the proposition asserts that the ratio of the marginal effects of these state variables on $k^m(q, \theta, Q)$ and $h^m(q, \theta, Q)$ is non-negative.

Propositions 3 and 4 indicate that, although product innovation and digital innovation are complementary near the equilibrium, their relationship becomes asymmetric as investment increases. In particular, when product innovation effort rises, digital innovation effort also increases, but at a diminishing rate.

Another objective we have for this section is to examine the stability of the equilibrium and assess whether the dynamical system exhibits saddle-point behavior. Accordingly, combining (2.1), (2.2), (3.14), and (3.16)–(3.17) yields the following dynamic control system:

$$\begin{cases} \dot{k}(t) = (\rho + \sigma_1)k(t) - \frac{b_2(\rho+\delta)[a+b_2 q(t)+b_3\theta(t)+b_4 Q(t)-b_1 c]}{b_1\beta_1[2(\rho+\delta)-\alpha b_4]} \\ \dot{h}(t) = (\rho + \sigma_2)h(t) - \frac{b_3(\rho+\delta)[a+b_2 q(t)+b_3\theta(t)+b_4 Q(t)-b_1 c]}{b_1\beta_2[2(\rho+\delta)-\alpha b_4]} \\ \dot{q}(t) = k(t) - \sigma_1 q(t) \\ \dot{\theta}(t) = h(t) - \sigma_2\theta(t) \\ \dot{Q}(t) = \frac{\alpha(\rho+\delta)[a+b_2 q(t)+b_3\theta(t)-b_1 c]-[2\delta(\rho+\delta)-\alpha b_4(2\delta+\rho)]Q(t)}{2(\rho+\delta)-\alpha b_4} \end{cases} \tag{3.20}$$

Solving system (3.20) under stationarity conditions $\dot{k}(t) = \dot{h}(t) = \dot{q}(t) = \dot{\theta}(t) = \dot{Q}(t) = 0$, we have the Proposition 5 as follows (Proof: See Appendix C):

Proposition 5. *The steady-state equilibrium $\{k^m, h^m, q^m, \theta^m, Q^m\}$ is a unique saddle-point equilibrium if and only if $\sigma_1 + \sigma_2 > \bar{\sigma}$, where the expressions of $\{k^m, h^m, q^m, \theta^m, Q^m\}$ and $\bar{\sigma}$ are given in Appendix C.*

In the next section, to complement the analysis, we conduct equilibrium analysis under government regulation over continuous time $t \in [0, \infty)$.

4. The government regulation

In this section, we assume that a benevolent government selects both effort levels to maximize the discounted stream of social welfare over continuous time t, while the price remains determined by the monopolist according to equation (3.12). It is worth noting that our approach to evaluating the efficiency or inefficiency of firms' product and process innovation efforts follows the methodology used in dynamic models of product quality, as developed in [13,30,31], among others. Now, we define the social welfare function as additively separable with respect to two elements: The profit $\pi(t)$ and the consumer surplus $cs(t)$, that is, $sw(t) = \pi(t) + cs(t)$, where $\pi(t)$ is given by the profit function (2.6). Then, using pricing condition (3.12), we have

$$\pi(t) = \frac{(\rho+\delta)(\rho+\delta-\alpha b_4)}{b_1[2(\rho+\delta)-\alpha b_4]^2}[a + b_2 q(t) + b_3\theta(t) + b_4 Q(t) - b_1 c]^2 - \frac{\beta_1}{2}k^2(t) - \frac{\beta_2}{2}h^2(t) \tag{4.1}$$

while the consumers' surplus $cs(t)$ writes $cs(t) = \int_{p(t)}^{\frac{a+b_2q(t)+b_3\theta(t)+b_4Q(t)}{b_1}} \{a - b_1 z(t) + b_2 q(t) + b_3\theta(t) + b_4 Q(t)\}dz(t)$, then using pricing condition (3.12), we have

$$cs(t) = \frac{(\rho+\delta)^2}{2b_1[2(\rho+\delta)-\alpha b_4]^2}[a + b_2 q(t) + b_3\theta(t) + b_4 Q(t) - b_1 c]^2 \tag{4.2}$$

Thus, the social welfare function $sw(t)$ is given by

$$sw(t) = \frac{(\rho+\delta)[3(\rho+\delta)-2\alpha b_4][a + b_2 q(t) + b_3\theta(t) + b_4 Q(t) - b_1 c]^2}{2b_1[2(\rho+\delta)-\alpha b_4]^2} - \frac{\beta_1}{2}k^2(t) - \frac{\beta_2}{2}h^2(t) \tag{4.3}$$

Thus, the government's objective is to find optimal efforts $k(t)$ and $h(t)$ such that the discounted social welfare flow SW is maximized under the constraints of dynamic equations (2.1), (2.2), and (3.14), namely,

$$\begin{aligned} SW = \max_{k,h} \int_0^\infty e^{-\rho t}\{&\frac{(\rho+\delta)[3(\rho+\delta)-2\alpha b_4][a + b_2 q(t) + b_3\theta(t) + b_4 Q(t) - b_1 c]^2}{2b_1[2(\rho+\delta)-\alpha b_4]^2} \\ &- \frac{\beta_1}{2}k^2(t) - \frac{\beta_2}{2}h^2(t)\}dt \\ s.t. &\begin{cases} \dot{q}(t) = k(t) - \sigma_1 q(t) \\ \dot{\theta}(t) = h(t) - \sigma_2\theta(t) \\ \dot{Q}(t) = \dfrac{\alpha(\rho+\delta)[a + b_2 q(t) + b_3\theta(t) - b_1 c] - [2\delta(\rho+\delta) - \alpha b_4(2\delta+\rho)]Q(t)}{2(\rho+\delta)-\alpha b_4} \end{cases} \end{aligned} \tag{4.4}$$

where ρ represents the discount rate, and the initial conditions are $q(0) = q_0$, $\theta(0) = \theta_0$, and $Q(0) = Q_0$, respectively. We use superscript "g" to represent steady-state equilibrium, and we have the following Proposition 6 (Proof: See Appendix D):

Proposition 6. *Under government regulation, the steady-state equilibrium* $\{k^g, h^g, q^g, \theta^g, Q^g\}$ *is a unique saddle-point equilibrium if and only if* $\sigma_1 + \sigma_2 > \bar{\sigma}$, *where the expressions of* $\{k^g, h^g, q^g, \theta^g, Q^g\}$ *are given in Appendix D.*

Propositions 5 and 6 show that, under both monopolist decision-making and government regulation, the system converges to a saddle-point steady state only when the combined depreciation rates of product quality and digitalization exceed a critical threshold. This condition is essential for dynamic stability. Corollaries 5 and 6 therefore provide useful guidance for maintaining stability in the presence of market fluctuations. From a managerial perspective, firms should pay close attention to the depreciation of product quality and digital capabilities. Very low depreciation may lead to excessive accumulation and destabilize the system. This suggests that innovation should be continuous but disciplined. Firms need to update products and digital features regularly, while aligning innovation intensity with market demand to avoid inefficiencies. From a policy perspective, regulators should foster an environment that supports stable technological evolution. Policies can promote timely upgrading and renewal through standards, innovation incentives, and appropriate depreciation mechanisms. Furthermore, prolonging the use of outdated technologies should be avoided, as it may hinder adjustment and reduce system stability. Overall, effective policy should balance innovation promotion with the need for a stable market trajectory.

5. Numerical analysis

In this section, we provide numerical solutions of systems (3.1) and (4.4) to analyze evolutionary paths of the decision variables and state variables over time. We apply numerical examples with the help of MATLAB. The forward–backward sweep method is a widely used numerical technique for solving optimal control problems [32, 33]. The relevant parameter values are given in Table 1. Note that to numerically prove the validity of the system and obtain the appropriate parameter values, references are cited from [13,24,30,34]. The parameter values are calibrated to capture the fundamental characteristics of innovative monopolists operating in markets with network externalities, in line with other studies [13, 24, 30, 34]. In addition, the calibration ensures that all relevant variables remain nonnegative, as reflected in the baseline parameter values reported in Table 1.

Table 1. The parameters used in diagrammatic analysis.

a	b_1	b_2	b_3	b_4	α	σ_1	σ_2
10	0.3	0.15	0.1	0.12	0.2	0.4	0.35
β_1	β_2	ρ	δ	c	q_0	θ_0	Q_0
0.3	0.35	0.4	0.3	1	3	5	2

The numerical solutions of systems (3.1) and (4.4) are illustrated in Figures 1–2, respectively, where the solid line denotes the numerical solutions under monopolist decision-making, while the dotted line denotes the numerical solutions under government regulation.

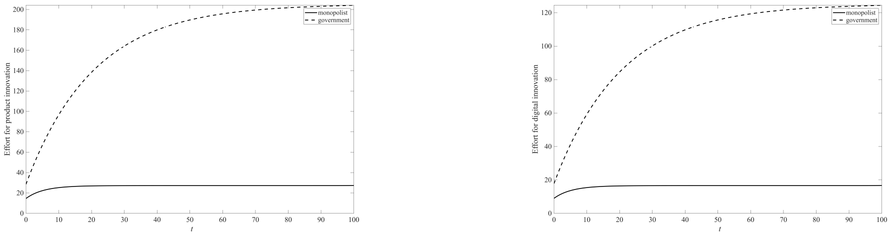

Figure 1. The evolutionary paths of the efforts for product innovation and digital innovation over time.

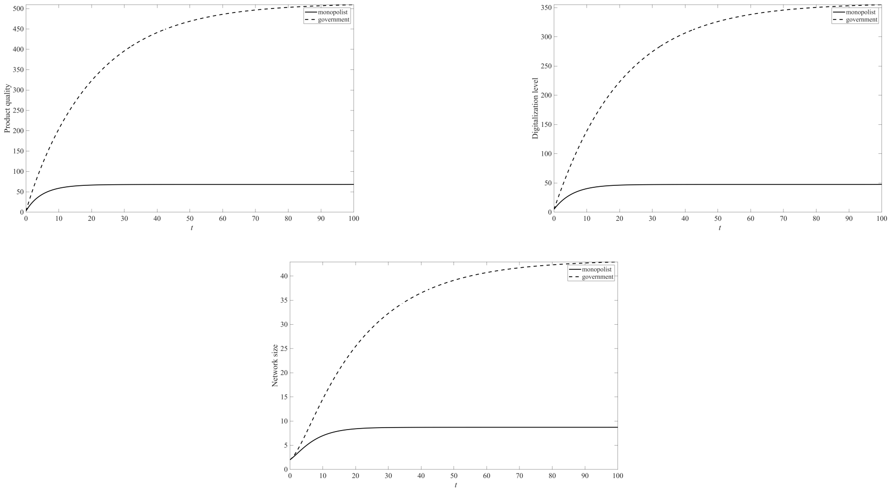

Figure 2. The evolutionary paths of product quality, digitalization level, and network size over time.

From Figure 1, we find that the increase in product and digital innovation efforts over time suggests a dynamic environment where firms are continually adapting to technological advancements and market demands. This can be attributed to factors such as competition, consumer preferences, and the rapid pace of technological change. Moreover, under monopolist decision-making, the monopolist may prioritize profit maximization over innovation, leading to lower investments in both product and digital innovations. The monopolist may focus on short-term gains rather than long-term innovation strategies, potentially stifling overall market growth. In contrast, under government regulation, the government aims to maximize societal welfare, which often involves promoting higher levels of innovation. This approach can lead to greater investments in research and development, as the government considers the broader impacts on society, including consumer surplus. This means that monopolies can lead to suboptimal outcomes due to a lack of competition, justifying the need for intervention by the government to enhance innovation. From a management perspective, the findings suggest that firms should adopt a long-term view of innovation, considering not just immediate profits but also the potential for sustainable growth through innovation. Collaboration between firms and policymakers can enhance innovation outcomes, as aligning business interests with societal goals can lead to more effective resource allocation.

Figure 2 depicts that firms operating under monopolistic conditions may prioritize short-term profits and may lack the incentive to invest in improving product quality, digitalization, and network size. This can result in stagnation and lower overall market performance. In contrast, the government focuses on maximizing societal welfare, which often leads to higher investments in quality improvements, digital technologies, and expanding networks. This approach encourages a more holistic view of business success, considering long-term benefits for consumers and society. The differences in outcomes between monopolistic and government regulation scenarios highlight the impact of market structure on innovation. Monopolies may lead to inefficiencies and underinvestment in critical areas. Moreover, investments in product and digital innovation often yield positive externalities, benefiting not just the firm but also consumers and the economy. The government is more likely to recognize and promote these investments. Therefore, firms should adopt a long-term perspective on product and digital innovation, recognizing that investments in these areas can lead to sustained competitive advantage. Focusing on consumer needs and preferences can drive improvements in product quality and digitalization efforts, leading to better market performance. We have checked these findings under various parameter values and the results are robust (Appendix E).

6. Theoretical and managerial implications

The exploration of monopolistic strategies in pricing, product innovation, and digital innovation, as outlined in the series of conclusions and propositions, provides valuable insights into the dynamic interactions among these elements. This analysis has significant theoretical and managerial implications that are relevant for academic discourse and practical application in economics and operations management.

6.1. Theoretical implications

Proposition 1 shows that higher product quality, greater digitalization, and a larger network size all raise consumers' willingness to pay, enabling firms to charge higher prices. Although improvements in quality and digitalization increase costs, they simultaneously generate additional value. Moreover, network externalities reinforce pricing power by strengthening user dependence. Taken together, this result reflects a value-based pricing mechanism in a dynamic environment with network effects.

Proposition 2 highlights that the influence of product quality and digitalization on network growth is neither linear nor uniform. When network effects are relatively weak, improvements in these dimensions encourage user adoption and support network expansion. By contrast, under strong network effects, their marginal contribution may decline or turn negative due to congestion, heightened expectations, or strategic interactions among users. This suggests that the role of innovation in driving demand depends critically on the underlying strength of network externalities.

Propositions 3 and 4 emphasize the dynamic interdependence between product innovation and digital innovation. Around the steady-state equilibrium, the two are complementary, as increases in one tend to be accompanied by increases in the other. However, this relationship is not symmetric: As investment in product innovation rises, digital innovation also increases, but at a decreasing rate, reflecting limited resources. Overall, these findings point to a jointly determined and nonlinear relationship between different types of innovation.

6.2. *Managerial implications*

From a managerial perspective, these findings provide more actionable guidance when interpreted in the context of different monopolistic industries, such as Internet platforms and high-end manufacturing firms. In platform-based monopolies characterized by strong direct network externalities (e.g., digital platforms), Propositions 3 and 4 suggest that product innovation and digital innovation are interdependent but constrained by limited resources. Managers should therefore avoid excessive concentration of resources in either dimension. In the early stages of platform development, greater emphasis can be placed on digital innovation (e.g., algorithm optimization, platform infrastructure, and user interaction design) to accelerate network expansion. However, as the network matures, overinvestment in digital features may yield diminishing returns. At this stage, shifting attention toward improving product quality and user experience (e.g., service reliability, content quality, and trust mechanisms) becomes more effective for sustaining user engagement.

In contrast, for high-end manufacturing monopolies (e.g., firms producing advanced equipment or premium goods), product innovation typically plays a more central role in value creation. Nevertheless, digitalization (e.g., smart manufacturing, data analytics, and digital services) increasingly complements product innovation. Propositions 3 and 4 imply that while both types of innovation should be pursued jointly, managers must recognize that the marginal gains from digital investment may decline as product innovation intensifies. Therefore, firms should adopt a coordinated investment strategy, ensuring that digital transformation supports rather than crowds out core product development. More broadly, these results highlight the importance of dynamic resource allocation. Managers should adjust the balance between product and digital innovation according to the firm's lifecycle stage, the strength of network externalities, and market conditions. Rather than pursuing uniform investment strategies, firms should adopt flexible and state-dependent innovation policies to sustain long-term performance.

In summary, the results suggest that effective innovation management in monopolistic markets requires a context-specific balance between product and digital innovation. Firms should tailor their resource allocation to industry characteristics and network dynamics, ensuring that innovation investments are coordinated and adaptable over time.

7. Concluding remarks

By enhancing their digitalization level, monopolists can improve product functionality and user experience while boosting operational efficiency and market competitiveness. This trend is encouraging monopolists to prioritize digital features in their innovations to better address the evolving needs and expectations of consumers. In this paper, we present a dynamic analysis of a monopolist's digital innovation and product innovation in a monopoly exhibiting network externality. Our analysis provides insight into the monopolist's behavior, including the stability of the steady-state equilibrium, the impact of product quality and digitalization level on profitability, the relationship between efforts for product innovation and digital innovation, and the interplay between pricing conditions and customer demand. Our findings indicate that a positive relationship exists between price and product quality, digitalization level, and network size. There is a complementary relationship between the efforts for product innovation and digital innovation near the steady-state equilibrium.

For the analysis, we focus on digital and product innovation for a single product in a monopoly market with network externalities. In future studies, researchers could extend this framework to a multi-product setting, enabling a systematic examination of how firms allocate resources between digital and product innovation across a product portfolio. Furthermore, the demand function assumes additive separability among price, quality, digitalization level, and network size, which may overlook potential interaction effects in real-world markets. Moreover, researchers could extend the analysis by considering more general demand specifications, particularly those incorporating complementarities between digitalization and network effects, to better capture the complexity of firms' investment decisions in digital and product innovation.

Author contributions

Shengbiao Ma: Conceptualization, methodology and writing original draft; Huiquan Li and Ran Jiang: Validation and revising.

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Conflict of interest

No potential conflict of interest was reported by the author.

Use of Generative-AI tools declaration

In the preparation of this work, the authors used Generative AI tools such as ChatGPT to assist in improving the clarity of the language. All AI-assisted content was carefully reviewed and revised by the authors, who take full responsibility for the final version of the manuscript.

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