Journal of Industrial and Management Optimization

JIMO, 22(5): 2410–2427.
DOI: 10.3934/jimo.2026088
Received: 11 December 2025
Revised: 16 March 2026
Accepted: 01 April 2026
Published: 16 April 2026

https://www.aimspress.com/journal/jimo

Research article

The green incentive wave: Dynamic optimization of referral rewards and green advertising in sustainable energy diffusion

Qi Chen*

School of Economics and Management, Shanghai Open University, Shanghai 200433, China

* **Correspondence:** Email: chenqireport@sou.edu.cn.

Abstract: This paper investigated optimal green advertising investment and referral reward strategies for sustainable energy product companies. We developed a dynamic optimization model that extends the Bass diffusion framework to simultaneously optimize referral rewards and green advertising investments throughout the product lifecycle. Our model integrates both marketing interventions and word-of-mouth effects, captured via a reduced-form Bass imitation mechanism, to analyze the diffusion of sustainable energy products. Applying optimal control theory, we derived analytical solutions and conducted sensitivity analyses across key parameters, including unit margin, referral efficiency, green advertising efficiency, spontaneous adoption rate, and word-of-mouth strength. Under our model parameterization, results revealed that the optimal referral reward strategy follows a U-shaped curve—initially high, decreasing during the middle phase, and gradually increasing in later stages—while optimal advertising investment decreases monotonically over time. Sensitivity analyses further demonstrate that (1) higher-margin sustainable technologies require greater referral rewards; (2) strong social network effects warrant simultaneous increases in both reward levels and advertising investments; and (3) technologies with strong intrinsic appeal require less marketing support. These findings provide theoretical guidance for sustainable energy companies seeking to optimize marketing strategies and offer insights for policymakers aiming to accelerate clean energy transitions through effective incentive mechanism design.

Keywords: sustainable energy adoption; referral rewards; green advertising; environmental sustainability; optimal control theory
Mathematics Subject Classification: 49K15, 90B60, 49N90

1. Introduction

As the primary solution to the global climate crisis, sustainable energy technologies are critical to achieving carbon neutrality goals [1]. The residential distributed energy market, including rooftop photovoltaic (PV) systems and smart energy management, is one of the main sectors requiring widespread technological adoption [2]. Meanwhile, environmental concerns are increasingly top of mind for consumers. It is estimated that a vast majority of households possess high pro-environmental behavioral intentions [3,4]. According to recent industry observations, consumer interest in reducing carbon footprints is a major factor in their energy system selection process [5].

Consumers' increased environmental awareness continues to drive the green technology mission of brand owners [3]. As originally demonstrated in the seminal economics of technological change, the diffusion of new technologies typically follows an S-shaped curve [6]. In addition, many large energy firms have committed to accelerating this transition with a common vision of achieving deep market penetration. However, several survey studies attest that it is hard for consumers to overcome initial inertia due to high upfront costs, long payback horizons, and information asymmetry [5,7]. Therefore, consumers mainly rely on firms' external marketing interventions, such as broad-reach green advertising, to initiate the purchase decision process in the early stages [8].

Green advertising is not always effective indefinitely, and the decay of advertising elasticity has flourished as a prominent issue in marketing practice [9]. Specifically, in the context of durable green technologies, relying purely on advertising refers to the behavior where a firm continuously pours budgets into mass media even when the innovator segment has saturated. Furthermore, recent analyses found that the marginal returns of pure advertising inherently diminish over the product lifecycle [10]. All of these reports suggest that treating green advertising as a static, standalone solution is an unsustainable phenomenon in the green technology market. Generally, in terms of marketing budget allocation, firms face a strategic trade-off. On the one hand, compared with relying solely on advertising, engaging the existing user base could save acquisition costs because social proof and peer trust are highly persuasive [11,12]. On the other hand, offering excessively large financial rewards to stimulate this network causes additional risks to the firm because it can erode product profitability. If the reward strategy is not carefully calibrated, the firm may experience a quantity–quality trade-off, as some opportunistic consumers may generate low-quality referrals that harm the firm's margins [13].

In addition to its own mass-market advertising efforts, a firm can also invite existing consumers to help accelerate technology diffusion. Referral reward programs (RRPs) have become popular among eco-conscious consumers in the residential energy industry, where firms incentivize consumers to use their own social networks to recommend products [14,15]. According to recent empirical evidence, spatial proximity to existing installations substantially increases neighborhood adoption probabilities [11,12]. Recently, strong evidence also emerged that customers acquired through referrals exhibit significantly higher downstream profitability and secondary referral rates compared to those acquired via advertising [14,16]. This means that, nowadays, a firm needs to make two different types of dynamic decisions regarding its product diffusion: (i) how much to invest in external green advertising, and (ii) how much to offer in internal peer-to-peer referral rewards over time. The interaction between the two makes the joint decision complex. For example, if a firm increases referral rewards, the internal organic diffusion accelerates, which the firm needs to take into account when making its advertising budget decisions. Also, because of the diminishing returns of advertising, consumer reliance on peer networks has grown. In response to this shift in the diffusion lifecycle, the firm should reconsider how to dynamically coordinate its marketing efforts in the market. Therefore, it is both relevant and important to conduct a systematic analysis of the two-way dynamic marketing decisions.

To deal with the abovementioned coordination problem associated with marketing channels, some existing literature relies on static econometric frameworks or conceptualizes these marketing tools in isolation [2,4,5]. While these studies provide a credible and reliable way to understand the efficacy of individual mechanisms, they fall short of providing exact mathematical guidance on how firms should strategically shift their investments across the product lifecycle.

In addition to static evaluations, optimal control theory has been increasingly adopted to address dynamic resource allocation. Specifically, to maximize profitability, researchers have applied continuous-time models to advertising paths and diffusion processes [9,10]. However, when a firm allows customers to engage in referral programs, both the firm's advertising and the consumers' word of mouth drive the diffusion simultaneously. Whereas existing optimal control models can optimize single marketing tools, their impact on simultaneous multi-channel coordination is less straightforward, thus generating unclear strategic implications for sustainable energy firms.

In this paper, we aim to provide insights into how a firm's green advertising interacts with its referral reward decisions in the sustainable energy industry. Specifically, our research questions are as follows:

Question 1. What is the firm's joint dynamic strategy for green advertising and referral rewards over the product lifecycle?

Question 2. If a firm adopts both channels, how do market characteristics (such as unit margin, word-of-mouth strength, and spontaneous adoption rate) impact these optimal trajectories?

To address these questions, we build a dynamic optimal control model to study a firm's advertising intensity, referral reward size, and their implications on technology diffusion. In the base model, there is a firm making and selling a durable green product in a market consisting of potential adopters. The firm can choose to invest in mass-media green advertising. In addition, the firm can also select the size of the referral reward to incentivize organic word of mouth. We extend the classic Bass diffusion framework [17] by mathematically embedding these two controllable interventions within the continuous-time state equation. Finally, we conduct systematic sensitivity analyses to examine how varying market conditions impact optimal policy paths.

We summarize our main findings below. First, we find that the optimal green advertising investment should follow a monotonically decreasing pattern over time. Whereas this result aligns with conventional wisdom that advertising is heavily needed early on, our analysis analytically characterizes a novel trajectory for the second channel: the optimal referral reward strategy follows a distinct U-shaped curve—initially high, decreasing during the middle phase, and rising again in later stages. Such a result occurs because the late-stage rebound is predominantly driven by internal network characteristics (e.g., word-of-mouth strength and referral efficiency) rather than external market factors. Furthermore, when a firm adjusts its strategy based on product profitability, we find that high-margin technologies warrant more generous referral rewards. Whereas policymakers and managers traditionally rely on static marketing budgets, our results suggest that firms should exercise dynamic resource allocation, strategically pivoting from external acquisition to internal network activation. In particular, instead of solely concentrating on flat-rate subsidies, governments and policymakers could also consider dynamic subsidy designs that theoretically complement these optimal corporate strategies.

2. Literature review

This section reviews three key areas of literature: consumer behavior in sustainable energy adoption, the role of social networks in technology diffusion, and the foundational Bass diffusion and optimal control models that underpin our analytical framework.

2.1. Consumer behavior and marketing interventions in sustainable energy adoption

A complex interaction of both internal social forces and external marketing interventions determines the diffusion of sustainable energy technologies, i.e., residential photovoltaic (PV) systems. The technical and economic viability of systems does not ensure actual adoption, a phenomenon known as the sustainability adoption gap [1,2,5]. The solution to this gap is to ensure that firms strategically use targeted marketing tools to overcome the well-established behavioral friction [7].

Traditional green advertising is an important mechanism of external influence. With the high capital requirements, long payback periods, and information asymmetry that come with long-lasting green technologies, broad coverage advertising becomes key to building market awareness [8]. Empirical research indicates that well-defined marketing messages that describe the positive environmental and financial benefits will greatly facilitate consumers' initial engagement in the adoption process [8,18]. But as the market and the innovator segment mature, the marginal returns of pure advertising inevitably decline throughout the product lifecycle [9], requiring a strategic shift toward internal influence mechanisms.

Word of mouth (WOM) and peer effects are the most powerful sources of internal influence for highly visible, high-involvement products such as rooftop PV. Empirical research has found that residential proximity to current installations is a strong predictor of neighborhood adoption, as proximity to installations provides social proof and trust [7,11,12,19]. In an effort to actively leverage this organic WOM, companies are more likely to use referral reward programs [15]. Although the reward size must be well-tuned to balance acquisition costs and the profitability of new customers [13], it is difficult to doubt the strategic merit of referrals. Referred customers have much higher downstream engagement and a greater likelihood of making additional referrals than those from the traditional advertising process [14,16].

Although there are strong empirical observations on the effectiveness of advertising and referrals, a methodological gap must be critically addressed. The available literature is based on inert econometric models or conceptualizes these marketing instruments independently of each other and does not consider their dynamic processes over time [2,5,9,20]. The core methodological innovation of this paper is the simultaneous dynamic optimal control of both green advertising and referral rewards in continuous time. By embedding these two controllable interventions into a time-dependent extension of the Bass diffusion model, our analysis yields exact policy trajectories without relying on numerical approximations. In particular, we analytically characterize a novel U-shaped optimal policy for referral rewards alongside a monotonically decreasing advertising strategy, under explicit market conditions.

2.2. The role of organic word of mouth and referral mechanisms

Social networks are also very important for the transmission of information and for reducing uncertainty in the spread of durable green technologies [7]. External advertising, though critical in creating initial market awareness, is only secondary in the preliminary adoption of such advantages, as internal forces within social structures take over subsequently [11]. In the modern context of sustainable energy, residential photovoltaic systems, which are highly visible technologies, have significant spatial and social spillovers. The empirical data indicate that organic word of mouth and proximity to existing installations significantly affect adoption by neighboring households [12]. The reason is that reductions in perceived risk and the creation of robust social rules regarding household energy practices occur when others are viewed [7]. Moreover, one of the long-term effects of the strategic positioning of early adopters is the direction of subsequent diffusion, underscoring the role of early seeding effects in diffusion [21,22]. It is this diffusion, which is organic peer-to-peer behavior that is not based on the incentives of a firm but rather on the foundational model of social reinforcement, that is modeled as the baseline word-of-mouth effect as part of our extended Bass framework.

To accelerate this organic diffusion, companies increasingly introduce referral reward programs [14,16]. Compared with conventional advertising via mass media, a referral program leverages the existing trust and quality information flows that strong social ties foster to ease the transition into the market [23,24]. Experimentation in the field indicates that incentives generated through financial means are an effective stimulus for eliciting consumer self-interest, and peer recommendations are a strong motivating force [18]. According to recent large-scale marketing research, customers obtained via referrals are more profitable and, more importantly, exhibit a much greater propensity to address and make subsequent referrals than those obtained via advertising [14,16]. These incentives, however, need to be carefully financially balanced depending on their design. Rewards that are too high may undermine product profitability, result in a quantity–quality trade-off, and lure opportunistic customers, thereby compromising the quality of recommendations overall [13,25]. This internal pressure in our model is simply characterized by a manageable variable, the referral reward, and the associated parameter, referral efficiency, which would theoretically isolate the effect of financial incentives on speeding up the adoption curve.

From a strategic perspective, the core managerial challenge lies in coordinating these marketing channels over time [26,27]. Recent empirical brand research points out that marketing efforts, which include green advertising, not only create instant sales but also the necessary secondary impacts as a result of the initial awareness creation, which subsequently creates community word of mouth [26]. This synergy is specifically important as initial advertising creates the required brand awareness that will later lead to peer suggestions being more convincing and induce long-term brand performance [26]. When selling high-involvement products with a high cost, a successful temporal sequencing usually implies the use of advertising to create preliminary momentum and attract influential first adopters [21,22] and then dominance of referrals to enhance the extent of penetration. Despite these mechanisms being strongly empirically validated, the optimal dynamic course for integrating them remains elusive. This paper uses a consistent-time optimal control model to mathematically compute the optimal dynamic allocation of resources between external advertising and internal referral rewards across the entire lifecycle of the technology by merging these validated behavioral insights.

2.3. Bass diffusion and optimal control in sustainable consumption

The technology adoption model assumes that the diffusion of innovations over time is usually S-shaped, as first observed by Mansfield [6]. Expanding on this time-tested economic concept, the Bass diffusion model offers a strong mathematical basis for operationalizing the S-shaped curve by breaking down market drivers into external drivers, such as mass-media advertising, and internal drivers, such as word of mouth [17]. Implementing such a binary structure for sustainable consumption is highly promising for understanding the sustainability adoption gap [1,5,28]. Early market entry is important for launching the market, given the high upfront costs and information asymmetry associated with residential photovoltaic systems and other long-lasting green technologies [4,5,8,28]. At the same time, the visibility of these technologies is higher, which increases the peripheral role of internal peer diffusion and spatial spillovers in subsequent phases [11,12].

The normative approach to management does not simply ask whether to observe such an S-shaped diffusion but rather to utilize resources to increase the speed at which it occurs. The best methodological toolkit for this challenge is the optimal control theory. Recent literature has effectively incorporated optimal control into advertising tracks and diffusion processes, and it has been confirmed that time-varying dynamic interventions are strictly better than constant-rate budgets [9,10]. Moreover, dynamic programming and optimal control have demonstrated significant potential for optimizing engineering and economic decisions in the sustainable energy sector [29,30]. Nonetheless, the current diffusion models used in the sustainable energy marketing context are mostly concerned with assessing the efficiency of individual marketing devices or consider marketing aspects as fixed parameters [4,5,28]. This creates a significant missing link in the dynamical coordination of multi-channel dynamics, particularly with respect to the decays of advertising returns over time [10] and the dynamic capture of downstream customer value at the referral program [14,16].

In this respect, the main contribution of this paper is the joint dynamic optimal control of both green advertising and referral rewards in continuous time. Our optimal-control problem, based on combining these two manageable interventions into a longer Bass state equation, will endogenously construct the exact, time-varying policy paths without making any chargeable approximations. In particular, in this strident, continuous-time framework, we analytically define a new U-shaped optimal policy that involves referral rewards and a monotonically declining advertising policy. This method is a straightforward way to bridge the existing gap between unchanging empirical results and dynamic firm strategy, where the specific strategic direction the firm must change regarding its marketing investments is to real-time network activation rather than the acquisition of external customers across the entire product lifecycle.

3. Model

To examine the optimal dynamic marketing strategy for a sustainable energy company, we use a model based on Bass's classic diffusion framework and then extend it using optimal control theory. In this section, the development of the model is discussed in a systematic way. We start with the standard Bass model. Then, we specify our extensions to add advertising and referral rewards and the resulting final-state equation. Finally, we formulate the firm's profit-maximization problem and identify the main propositions that define the ideal strategies.

3.1. The foundational Bass diffusion model

The Bass diffusion model [17] provides the underlying principles for adopting new products. It assumes that there are two types of driving forces in adoption: innovators adopt independently of social influence, and imitators adopt in response to word of mouth from former adopters. The rate of change of the cumulative adoption fraction, $q(t)$, is depicted by the classic model as follows:

$$q'(t) = [p + s \cdot q(t)][1 - q(t)]. \tag{3.1}$$

Here, p is the coefficient of innovation (representing external influence), and s is the coefficient of imitation (representing internal, word-of-mouth influence). The term $[1 - q(t)]$ represents the remaining market potential at time t.

3.2. Model extension: incorporating advertising and referral rewards

Although the Bass model captures the essence of diffusion dynamics, it fails to explicitly account for a firm's marketing activities. Our framework is based on previous studies that have combined marketing variables into diffusion models [17] and more recent studies that indicate the importance of the peer effect in adopting sustainable energy services [11,31]. We use these models to simultaneously explain both green advertising and referral rewards.

We model these instruments as direct influences on the diffusion parameters. Specifically:

Green advertising: We assume that advertising intensity, denoted by $g(t)$, enhances the external influence on potential adopters. We define a baseline spontaneous adoption rate, parameter η, representing the proportion of environmentally conscious consumers who adopt without influence from advertising or referrals. The coefficient σ measures green advertising efficiency, with $\sigma g(t)$ capturing the proportion of new adopters attributable to advertising.

Referral rewards: We assume that referral rewards, denoted by $r(t)$, amplify the internal, word-of-mouth effect. The parameter τ represents the baseline word-of-mouth effect—the organic spread of environmental awareness within communities independent of marketing interventions. The coefficient θ measures referral reward efficiency, with $\theta r(t)$ representing the per-adopter successful referral probability under reward level $r(t)$; higher rewards correspond to higher referral success rates.

By substituting these extended parameters back into the Bass framework, we arrive at our modified state equation for the evolution of the cumulative adoption rate $q(t)$ over a finite time horizon T:

$$q'(t) = [\eta + \sigma g(t) + (\tau + \theta r(t))q(t)][1 - q(t)], q(0) = 0. \tag{3.2}$$

Here, $q'(t)$ is the instantaneous adoption rate (the derivative of the cumulative adoption rate). The initial condition $q(0) = 0$ indicates zero adoption at the beginning of the promotion period. This instantaneous adoption rate derives from four distinct sources: (i) spontaneous adoption driven by environmental awareness, $\eta(1 - q(t))$; (ii) advertising-induced adoption, $\sigma g(t)(1 - q(t))$; (iii) adoption from organic word-of-mouth effects, $\tau q(t)(1 - q(t))$; and (iv) adoption from incentivized referrals,

$\theta r(t)q(t)(1-q(t))$, with $\theta r(t)$ representing the per-adopter successful referral probability under reward level $r(t)$. In our calibrated market parameters, the endogenous optimal reward ensures that the mathematical probability constraint $\theta r(t) \le 1$ is strictly satisfied at all times.

3.3. The firm's optimization problem

The sustainable energy company aims to maximize expected profit over the entire promotion period, with the objective function:

$$J = \max \int_0^T \left\{ \mu q'(t) - \theta r(t)^2 q(t)[1-q(t)] - \frac{cg(t)^2}{2} \right\} dt, \\ s.t.\ q'(t) = [\eta + \sigma g(t) + (\tau + \theta r(t))q(t)][1-q(t)] \tag{3.3}$$

where μ represents unit margin, $\mu q'(t)$ is the revenue at time t, $\frac{cg(t)^2}{2}$ captures advertising costs with cost coefficient c, and $\theta r(t)^2 q(t)[1-q(t)]$ represents total referral reward expenditures (since each successful referral costs $r(t)$, and there are $\theta r(t)q(t)[1-q(t)]$ such referrals). The decision variables are advertising intensity $g(t)$ and referral reward level $r(t)$.

Following optimal control theory, we construct the Hamiltonian function:

$$H(q,r,g,\lambda) = (\mu+\lambda)[\eta + \sigma g + (\tau + \theta r)q](1-q) - \frac{cg^2}{2} - \theta r^2 q(1-q), \tag{3.4}$$

where $\lambda = \lambda(t)$ denotes the costate variable associated with the adoption rate, and the optimal solution (g^*, r^*) must satisfy $(g^*, r^*) = \max_{g\ge 0, r\geqslant 0} H(q,r,g,\lambda)$. According to the maximum principle, the costate variable must satisfy $\lambda' = -\frac{\partial H}{\partial q}$, with boundary condition $\lambda(T) = 0$. By analyzing the necessary conditions, we derive the following propositions. (The detailed proofs for all propositions are provided in the Appendix.)

Proposition 1. *The optimal control strategy* (g^*, r^*) *for the sustainable energy company is given by* $g^*(t) = \sigma(\mu + \lambda(t))(1-q(t))/c$, $r^*(t) = (\mu + \lambda(t))/2$. *The corresponding optimal cumulative adoption rate is* $q^*(t) = \dfrac{4c(\tau-\eta) + (\mu+\lambda(t))\left(c\theta - 4\sigma^2\right)}{8c\tau + (\mu+\lambda(t))\left(2c\theta - 4\sigma^2\right)}$.

Based on the conclusion of Proposition 1, we further derive the following propositions.

Proposition 2. *The sustainable energy company's optimal advertising strategy* $g^*(t)$ *satisfies* $g'(t) \le 0$*, indicating that advertising investment decreases monotonically over time.*

The result of Proposition 2 has important managerial implications. During the early stages of sustainable energy technology diffusion, companies should allocate substantial resources to advertising to build market awareness and establish an initial adoption base. This early advertising not only directly drives adoption but also lays the foundation for subsequent word-of-mouth dissemination. As the adoption base expands over time, advertising's marginal effectiveness diminishes, warranting a gradual reduction in advertising investment. At later stages, companies should increasingly leverage the more cost-effective social network dissemination and word-of-mouth effects to drive further adoption.

Proposition 3. *There exists a point* $t^* \in [0,T)$ *such that the sustainable energy company's optimal referral reward* $r(t)$ *decreases over time when* $t \in [0,t^*)$ *(i.e.,* $r'(t) \le 0$*), and increases over time when* $t \in [t^*,T)$ *(i.e.,* $r'(t) \ge 0$*).*

Proposition 3 reveals that the optimal referral reward strategy follows a U-shaped pattern over time. This trend has grave strategic consequences for the diffusion of sustainable energy technology. In the initial promotion stage, firms ought to utilize high advertising using a free referral offer, where advertising takes most of the center stage. With a wider and wider adoption base, and as the technology becomes more visible, the ratio of advertising to what it reaps is high; hence, it should be reduced further in terms of advertising investments. Once all requirements have been met after the critical time point t^*, the promotion strategy should switch to word of mouth via social networks. In this stage, the companies are encouraged to gradually offer referral rewards as they aim to build an adopter base into a network of transmission, in essence, pushing the remaining potential adopters into adoption. The specified strategy of reducing and later increasing referral rewards is economically reasonable and consistent with the specifics of sustainable energy technology diffusion. Once a critical mass of followers is reached, the weight of social learning and word of mouth becomes more influential, and the appropriate strategy to gain benefits is to offer referral rewards.

4. Numerical analysis

To explain the theoretical propositions generated in Section 3 and discuss their managerial implications, we perform a numerical analysis. In this paper, we develop a theoretical optimal control model; this means we calibrate our analysis without econometric calculations using a given dataset. There are three principles that determine our approach. First, we provide specifications of all the variables and parameters required by the model. Second, we define a baseless situation that is adjusted to simulate an average diffusion path of long-duration green technologies (i.e., low take-off, acceleration, and saturation). Third, we perform comparative statics and robustness checks by varying key parameters within ranges similar to those in the literature. Table 1 explains the baseline values and their reasons based on recent empirical findings and the traditional theory of diffusion [8,16,12,32].

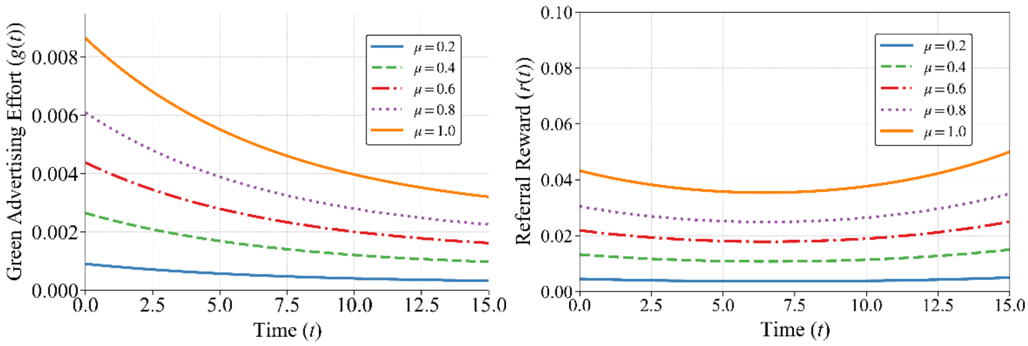
Table 1. Baseline parameter values.

Parameter	Baseline	Justification/recent source
Unit margin	$\mu = 0.2$	Normalized baseline for profits; not empirically pinned—used for calibration with robustness checks around this value (modeling convention).
Referral effectiveness	$\theta = 0.1$	Referral-acquired customers exhibit stronger downstream referral activity, validating a meaningful referral channel [14,16].
Advertising effectiveness	$\sigma = 0.1$	Advertising/messages raise initiation for solar adoption [8,18].
Spontaneous adoption	$\eta = 0.01$	Small innovator share aligns with classic diffusion patterns [32].
Organic word of mouth	$\tau = 0.15$	Material peer/neighborhood spillovers documented for rooftop solar power [11,12].
Advertising cost coefficient	$c = 1$	Quadratic cost normalization typical in control problems; varied in sensitivity (modeling convention).
Planning horizon	$T = 15$	Finite horizon covering a full diffusion wave (weeks or quarters).

We analyze how key parameters affect optimal referral reward and advertising strategies, yielding important managerial insights.

4.1. Unit margin sensitivity (μ)

Figure 1 illustrates the temporal paths of optimal referral rewards and the green advertising strategy. The optimal advertising strategy will always show a monotonically declining trend: the investment will be high at the beginning and decline over time, in line with Proposition 2. The unit margin, however, largely affects the optimal reward to use regarding the referral. The U-shaped curve of referral rewards at high-margin levels is unique: high at the beginning, declining in the middle, and gradually increasing once a specific critical point of time t^*. On the contrary, this curve is much flatter at lower levels of the margin. It means there is a profitability threshold: to achieve large returns from increasing returns to scale due to network effects, it is only cost-efficient to substantially increase referral rewards in the later stages, when the product generates a sufficiently high marginal profit. These findings provide evidence of a subtle managerial perspective. In the case of high-margin sustainable energy technologies, the initial consideration for companies is heavy investment in green advertising and large referral rewards for early adopters to build market presence. Both investments ought to be minimized at mid-course to maximize the cost. Once at the turning point t^*, when the product's profit margin is also very large, companies should progressively increase referral rewards to concentrate the adopter pool as transmission nodes and convert the increasing returns of network transformation into profitable adoptions.



(a) green advertising effort (b) optimal referral rewards

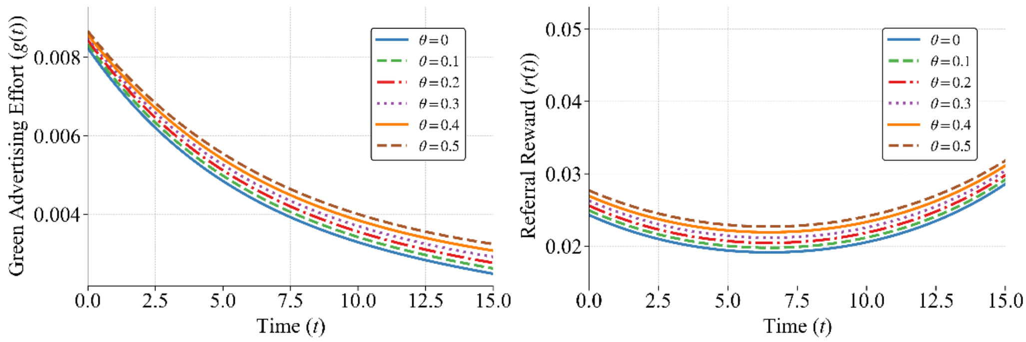
Figure 1. Impact of unit margin on optimal strategies.

4.2. Referral reward efficiency (θ)

Figure 2 confirms that referral rewards are U-shaped, as provided in proposition 3, whereas advertising decreases as time passes. This follows the study by Arbatskaya and Konishi [33], which found that firms that have adopted referral rewards tend to lower their advertising rates. As clearly demonstrated in Figure 2(b), the entire referral reward curve rises without altering its unique U-shaped pattern as referral efficiency increases. Mathematically, the costate variable $\lambda(t)$ is endogenously driven by θ, which explains why higher efficiency dynamically shifts the optimal reward trajectory upward, raising the overall budget ceiling to reflect a higher shadow value for new adopters.

In a realistic business situation, increasing the efficiency of referrals improves the quality of the target market's social network and its users' trust in it. In particular, an increased referral efficiency would mean that (1) the firm has been able to reduce the stress of users to make a recommendation (e.g., through the use of easily shareable one-click sharing solutions); and (2) the product is viralizing in high-degree-of-trust, close-knit communities where word of mouth will be much more effective. In the case of sustainable energy companies, however, low referral efficiency can signal negative word of mouth about the technology, rendering referrals useless. El Ouardighi et al. [34] observed that when technology quality is low and negative word of mouth prevails, companies should not increase advertising but focus on quality improvement, then resume advertising after quality is enhanced.

This offers the following management implications: Sustainable energy technology firms need to focus on improving referral efficiency, as this helps minimize friction in sharing, and market social network properties in the target markets ought to be evaluated before promotion campaigns are initiated. The overall referral reward budget in socially connected and high-trust communities with high environmental awareness can be used to effectively harness social network effects. When the market does not yet have a reputation for technology or the consumer has a low level of confidence, the company must initially ensure that its products deliver satisfactory performance and environmental benefits before shifting to referral rewards and continued increases in advertising investment.

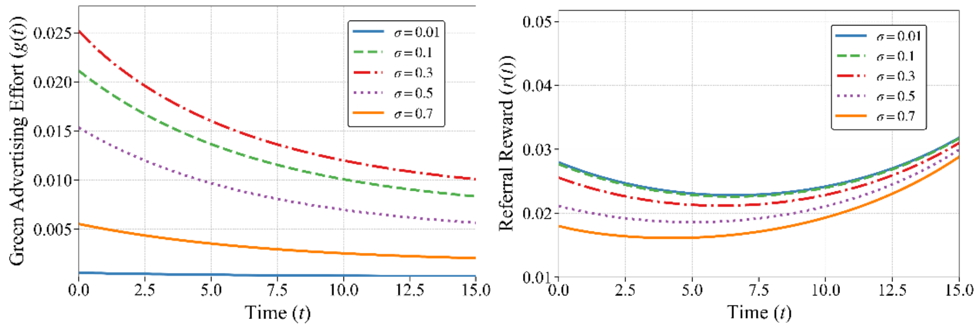


(a) green advertising effort (b) optimal referral rewards

Figure 2. The impact of referral reward efficiency on optimal strategies.

4.3. Green advertising efficiency (σ)

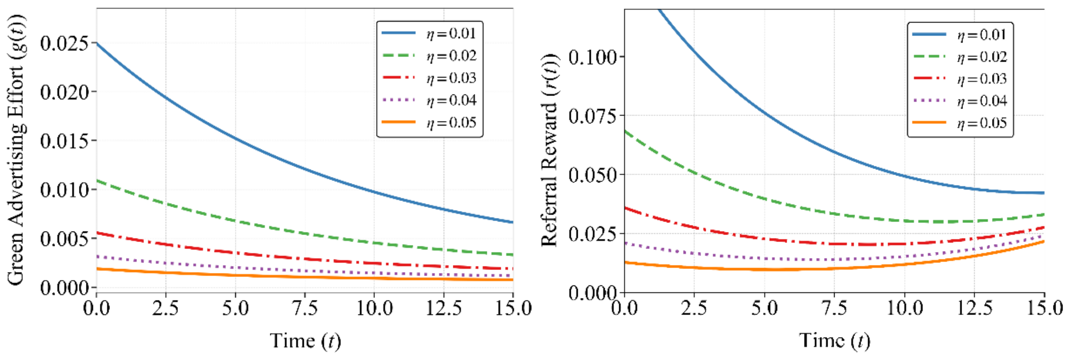
Figure 3 reveals an inverse relationship between green advertising efficiency and optimal referral rewards for sustainable energy enterprises. In particular, the best referral reward decreases with increasing green advertising efficiency, and vice versa. The trend indicates that these two strategic aspects are substitutes but not complements. Interestingly, advertising efficacy and optimal advertising commitment do not follow a linear relationship. Moderate advertising efficiencies ($\sigma = 0.3, \sigma = 0.1$, $\sigma = 0.5$) correspond to higher advertising investments, while both very high ($\sigma = 0.7$) and very low ($\sigma = 0.01$) advertising efficiencies result in lower optimal advertising investments by sustainable energy enterprises. These results have the following management implications: Sustainable energy technology companies should tailor their marketing mix to specific market environments. Green advertising effectiveness may be lower in markets where consumers are less convinced that advertising claims are environmental (e.g., in areas where consumers are skeptical of the truth of the claims). In such markets, higher referral reward budgets will make up for it. Firms in markets highly responsive to advertising are advised to use advertising placement strategies to reduce frequency and regain visibility. Notably, extremely high advertising efficiency does not always justify increased investment, as it is often associated with higher unit costs. Sustainable energy firms are encouraged to regularly assess the real effects of green advertising in their target markets to ensure an appropriate cost-benefit relationship.



(a) green advertising effort (b) optimal referral rewards

Figure 3. The impact of green advertising efficiency on optimal strategies.

4.4. Spontaneous adoption rate (η)



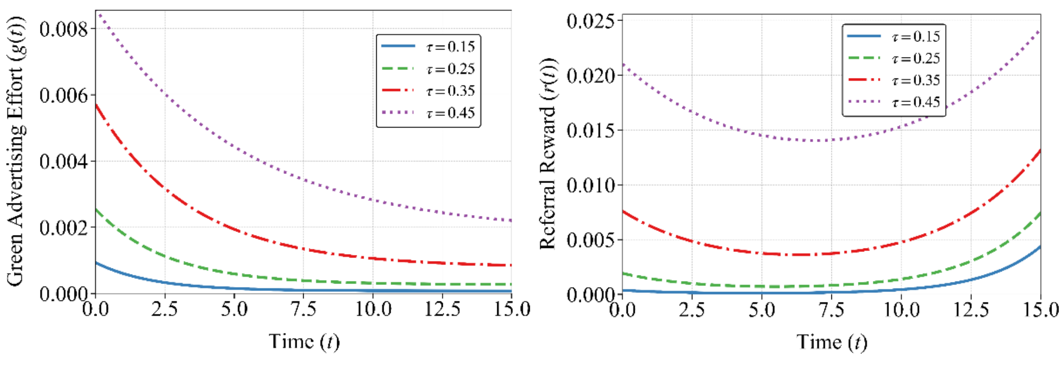
(a) green advertising effort (b) optimal referral rewards

Figure 4. The impact of spontaneous adoption rate on optimal strategies.

Figure 4 shows that optimal referral rewards and advertising investments decline with rising spontaneous adoption rates. Based on a classical theory by Mansfield [6], technological change, the implementation of innovation, is usually characterized by an S-shaped process of change with time. An increased spontaneous adoption rate would mean the technology would face much less resistance and friction during the early takeoff phase of this S-curve. The latter finding is particularly important in the context of sustainable energy markets, where spontaneous adoption directly indicates consumers' environmental awareness and their intrinsic positive attitude toward the technology. Systems with high spontaneous adoption rates (e.g., proven value and high brand recognition of established solar brands) can exist without extensive marketing support to sustain the initial adoption cascade. Otherwise, technologies with low spontaneous adoption rates (e.g., new but untested energy storage solutions) need to be strongly rewarded in a referral program and advertised to offset their low intrinsic attractiveness. This provides the following insights for management: Sustainable energy technology companies ought to align their strategies with the target market's awareness of environmental issues and orientation toward the technology. In environmental markets (e.g., eco-friendly communities or cities), resources need to focus on product functionality and customer experience, not on widespread marketing. In markets where environmental awareness is still low, companies must enhance educational advertisement and referral rewards in order to make consumers appreciate the value of technology in the long-term.

4.5. *Word-of-mouth effect (τ)*

Figure 5 shows that optimal referral and advertising investments increase with the strength of word of mouth. This is in line with the observation by Brown and Reingen [35] that information flows and influence can be more easily conducted through strong social relationships than through weak ties. The empirical study by Graziano and Gillingham [31] also revealed that proximity to existing solar installations was an important factor influencing households' likelihood of adopting them. The information about referral processes will spread more effectively and with greater impact in closely knit communities and will be more successful at increasing referral rates. Moreover, marketing memory can be used to amplify the effect of word of mouth by raising brand awareness, which in turn increases potential adopters' receptiveness to their peers' recommendations. Simultaneously increasing referral rewards and advertising has a synergistic effect in communities with highly dense social networks, accelerating the healthy adoption of technologies.



(a) green advertising effort (b) optimal referral rewards

Figure 5. The impact of word of mouth on optimal strategies.

The management insights from this are as follows: Sustainable energy technology companies should focus on communities with strong social networks, e.g., a highly populated residential neighborhood, a professional community, or a group of environmental organizations. The synergistic benefits of optimized referral rewards and targeted advertising are much more effective in these high-influence settings than in dispersed promotion strategies. Firms should further come up with community-based models of adoption, including the neighborhood group buying or installation-sharing programs, to enhance word-of-mouth impacts.

4.6. Summary of sensitivity dynamics

The sensitivity analyses in Sections 4.1–4.5 reveal how different parameters shape the optimal referral reward trajectory established in Proposition 3. Internal network characteristics and product profitability amplify the U-shaped curvature, making the late-stage rebound increasingly pronounced. A higher unit margin (μ) makes the U-shape visibly more distinct by enabling the firm to profitably reinvest in referral rewards at later stages (Section 4.1). A higher referral efficiency (θ) shifts the entire trajectory upward while preserving and accentuating the U-shaped pattern (Section 4.2). A stronger word-of-mouth effect (τ) simultaneously elevates both referral rewards and advertising investments, reinforcing the late-stage rebound through synergistic network effects (Section 4.5).

External market variables exhibit differentiated effects on the U-shaped pattern. A higher green advertising efficiency (σ) acts as a substitute for referral rewards, reducing the overall reward level, but preserves the U-shaped curvature across all values of σ (Section 4.3). In contrast, a higher spontaneous adoption rate (η) not only lowers the overall investment level but also substantially flattens the U-shaped curvature, as technologies with strong intrinsic appeal face less friction throughout the diffusion process, reducing the need for late-stage network reactivation (Section 4.4). In summary, the practical significance of the U-shaped referral reward strategy is most pronounced when product profitability is high and internal network effects are strong, and is attenuated primarily when the technology possesses strong intrinsic adoption appeal (high η) rather than when advertising channels are more efficient.

5. Conclusions

This paper develops and analyzes a dynamic optimal control model to determine the optimal advertising and referral reward strategies for a sustainable energy company over a finite time horizon. By extending the Bass diffusion framework, we derive the optimal temporal paths for these two key marketing instruments and examine how they are influenced by market characteristics. Our findings offer significant theoretical contributions and practical implications for firms and policymakers seeking to accelerate the adoption of green technologies.

5.1. Summary of key findings

Our analysis yields two primary findings regarding optimal strategy. First, the investment in advertising follows an optimal, monotonically decreasing path, implying a front-loaded model that should be used to build market awareness initially. Second, when unit margins are high and internal network effects are strong, the optimum referral reward policy must incur a U-shaped curve: high at a low level to jump start the market, falling during the growth phase, and then increasing to reactivate the existing adopter base and unexploited market potential. We have also learned some valuable lessons in our sensitivity analyses: (1) High-margin technologies should have bigger referral rewards; (2) the impact of word of mouth is stronger, and the investment in advertising and referral rewards should be increased; (3) technologies that come with high intrinsic characteristics (i.e., high environmental benefits) have less need for marketing support.

5.2. Managerial and policy implications

Our findings provide actionable guidance for both managers and policymakers seeking to address the sustainability adoption gap outlined in the introduction. To managers, the findings demand a change in the traditional marketing budgets to the dynamic allocation of resources. The curve reward structure and declining advertising curves indicate a premise of committing resources to both channels upon launch, and moving resources to referral rewards as the market matures and new network effects are established. The product should also be optimally fitted in this marketing mix; in this regard, it is our analysis that superior unit profitability with high-margin technologies best suits deploying generous referral rewards aimed at maximizing the high profitability levels of the product, whilst innovative technologies with low initial appeal best suit increased levels of advertising and referral rewards to overcome consumer inertia. Moreover, promotional activities should be prioritized for high-influence communities with strong social networks, as our model shows the greatest synergy between advertising and referral actions in these communities.

From a policy perspective, while our model explicitly optimizes firm-level profitability rather than social welfare or government budget constraints, we can offer a speculative extension regarding public interventions. These findings qualitatively suggest that governments might enhance their impact by moving beyond flat-rate subsidies toward dynamic subsidy design. For example, public subsidies may be designed to be at their maximum at the initial introduction of a technology and at the final stages, when it is theoretically complementary to optimal corporate strategies and may even enhance the efficiency of public funds. Also, policymakers can contribute significantly to increasing the presence of information spillovers by subsidizing community workshops, peer-to-peer sharing services, or neighborhood group-purchasing arrangements to enhance the effectiveness of referral strategies and to hasten the clean energy transition.

5.3. Limitations and future research

This study has several limitations that open avenues for future research. First, while our results demonstrate strong internal validity under the specified mathematical framework, they are derived under the assumptions of a monopoly environment and a homogeneous market. In reality, strategic reactions from competitors and consumer heterogeneity could compress the profit margins associated with referral rewards. Therefore, extending the model to a competitive setting to analyze strategic interactions between firms would be a highly valuable contribution. Furthermore, future research could explicitly incorporate consumer heterogeneity to explore differentiated marketing strategies for various customer segments. Second, we model the effects of advertising and referrals as instantaneous. Incorporating time-lag effects or memory effects in advertising could yield further insights into long-term budget allocation. Third, our framework captures word of mouth via a reduced-form Bass imitation mechanism. Extending the model to incorporate explicit network structures—such as complex contagion dynamics or heterogeneous network topologies—represents a promising direction for future research. Finally, our model could be enriched by incorporating additional behavioral factors, such as prospect theory's loss aversion or reference points, to further examine how psychological biases influence the effectiveness of referral rewards.

Author contributions

Qi Chen: Conceptualization, software, formal analysis, investigation, resources, data curation, writing—original draft preparation, supervision, project administration.

Use of Generative-AI tools declaration

No Artificial Intelligence (AI) tools were used in the creation of this article.

Data availability statement

All relevant data are within the paper.

Acknowledgments

We thank the editor-in-chief and anonymous reviewers whose insights enhanced this manuscript's quality.

This research received funding through grants from the National Natural Science Foundation of China (72304208), the Shanghai Open University Center for Research on Digital Management and Service Innovation (yjzx2403), and the Shanghai Municipal Educa System Union (2025GHL19).

Conflicts of interest

The authors certify that they have no affiliations with or involvement in any organization or entity with any financial interest or non-financial interest in the subject matter or materials discussed in this manuscript.

References

1. G. Konzen, R. Best, N. J. de Castro, The energy injustice of household solar energy: A systematic review of distributional disparities in residential rooftop solar adoption, *Energy Res. Soc. Sci.*, **111** (2024), 103473. https://doi.org/10.1016/j.erss.2024.103473
2. Z. Wang, M. L. Arlt, C. Zanocco, A. Majumdar, R. Rajagopal, DeepSolar++: Understanding residential solar adoption trajectories with computer vision and technology diffusion models, *Joule*, **6** (2022), 2611–2625. https://doi.org/10.1016/j.joule.2022.09.011
3. A. Ghosh, V. K. S. Prasad, Evaluating the influence of environmental factors on household solar PV pro-environmental behavioral intentions: A meta-analysis review, *Renew. Sustain. Energy Rev.*, **190** (2024), 114047. https://doi.org/10.1016/j.rser.2023.114047
4. E. Schulte, F. Scheller, D. Sloot, T. Bruckner, A meta-analysis of residential PV adoption: The important role of perceived benefits, intentions and antecedents in solar energy acceptance, *Energy Res. Soc. Sci.*, **84** (2022), 102339. https://doi.org/10.1016/j.erss.2021.102339
5. S. R. Shakeel, H. Yousaf, M. Irfan, A. Rajala, Solar PV adoption at household level: Insights based on a systematic literature review, *Energy Strategy Rev.*, **50** (2023), 101178. https://doi.org/10.1016/j.esr.2023.101178

6. E. Mansfield, Technical change and the rate of imitation, *Econometrica*, **29** (1961), 741–766. https://doi.org/10.2307/1911817
7. K. S. Wolske, K. T. Gillingham, P. W. Schultz, Peer influence on household energy behaviours, *Nat. Energy*, **5** (2020), 202–212. https://doi.org/10.1038/s41560-019-0541-9
8. D. Bär, S. Feuerriegel, T. Li, M. Weinmann, Message framing to promote solar panels, *Nat. Commun.*, **14** (2023), 7187. https://doi.org/10.1038/s41467-023-42904-0
9. G. Feichtinger, D. Grass, R. F. Hartl, P. M. Kort, A. Seidl, The digital economy and advertising diffusion models: Critical mass and the stalling equilibrium, *Eur. J. Oper. Res.*, **318** (2024), 966–978. https://doi.org/10.1016/j.ejor.2024.05.043
10. R. Guo, Z. Jiang, Optimal dynamic advertising policy considering consumer ad fatigue, *Decis. Support Syst.*, **187** (2024), 114323. https://doi.org/10.1016/j.dss.2024.114323
11. Y. Min, Spatial dynamics of low-carbon transitions: Peer effects and disadvantaged communities in solar energy, electric vehicle, and heat pump adoption in the United States, *Energy Res. Soc. Sci.*, **121** (2025), 103981. https://doi.org/10.1016/j.erss.2025.103981
12. E. O'Shaughnessy, A. Grayson, G. Barbose, The role of peer influence in rooftop solar adoption inequity in the United States, *Energy Econ.*, **127** (2023), 107009. https://doi.org/10.1016/j.eneco.2023.107009
13. H. M. Wolters, C. Schulze, K. Gedenk, Referral reward size and new customer profitability, *Mark. Sci.*, **39** (2020), 1166–1180. https://doi.org/10.1287/mksc.2020.1242
14. D. Kuang, B. Ma, X. Li, Acquiring customers via referral reward programs versus advertising: Who is more likely to provide future referrals? *J. Retail. Consum. Serv.*, **81** (2024), 103982. https://doi.org/10.1016/j.jretconser.2024.103982
15. L. Wang, Z. Chen, The effect of incentive structure on referral: The determining role of self-construal, *J. Acad. Mark. Sci.*, **50** (2022), 1091–1110. https://doi.org/10.1007/s11747-022-00852-z
16. R. Gershon, Z. Jiang, Referral contagion: Downstream benefits of customer referrals, *J. Mark. Res.*, **62** (2025), 97–116. https://doi.org/10.1177/00222437241257886
17. F. M. Bass, A new product growth model for consumer durables, *Manage. Sci.*, **15** (1969), 215–227. https://doi.org/10.1287/mnsc.15.5.215
18. B. Bollinger, K. T. Gillingham, M. Ovaere, Field experimental evidence shows that self-interest attracts more sunlight, *Proc. Natl. Acad. Sci. U.S.A.*, **117** (2020), 20503–20510. https://doi.org/10.1073/pnas.2004428117
19. J. L. Barnes, A. S. Krishen, A. Chan, Passive and active peer effects in the spatial diffusion of residential solar panels: A case study of the Las Vegas Valley, *J. Clean. Prod.*, **363** (2022), 132634. https://doi.org/10.1016/j.jclepro.2022.132634
20. Q. Chen, Y. Duan, Impact of information disclosure on global supply chain greenwashing: Is more information transparency always better? *Transp. Res. Part E Logist. Transp. Rev.*, **178** (2023), 103288. https://doi.org/10.1016/j.tre.2023.103288
21. S. Malaekeh, S. Castellanos, Residential solar photovoltaic seeding effects and their unequal impact on diffusion across racial and ethnic groups in the U.S., *Energy Policy*, **206** (2025), 114782. https://doi.org/10.1016/j.enpol.2025.114782
22. F. Cui, L. Wang, X. R. Luo, X. Cui, Influentials, early adopters, or random targets? Optimal seeding strategies under vertical differentiations, *Decis. Support Syst.*, **183** (2024), 114263. https://doi.org/10.1016/j.dss.2024.114263

23. C. Song, T. Wang, H. Lee, M. Y. Hu, The moderating role of perceived social risk in bank credit card referral programs, *Int. J. Bank Mark.*, **38** (2020), 1601–1616. https://doi.org/10.1108/IJBM-05-2020-0291
24. M. Zhan, M. Huang, A. Li, Y. Yang, The role of impulsive behaviour and meta-perception in referral reward programs, *J. Retail. Consum. Serv.*, **75** (2023), 103496. https://doi.org/10.1016/j.jretconser.2023.103496
25. H. Chen, S. Hao, Dollar or percentage? The effect of reward presentation on referral likelihood, *Asia Pac. J. Mark. Logist.*, **36** (2024), 1504–1519.https://doi.org/10.1108/APJML-07-2023-0678
26. S. C. Bahadir, K. Pauwels, When does word of mouth versus marketing drive brand performance most? *J. Mark. Anal.*, **12** (2024), 473–491. https://doi.org/10.1057/s41270-023-00267-x
27. R. Wang, Q. Wang, W. Y. K. Chiang, Optimal promotional mix and pricing when faced with uncertain product value, *Eur. J. Oper. Res.*, **313** (2024), 637–651. https://doi.org/10.1016/j.ejor.2023.08.042
28. P. K. Kopalle, J. Burkhardt, K. Gillingham, L. S. Grewal, N. Ordabayeva, Delivering affordable clean energy to consumers, *J. Acad. Mark. Sci.*, **52** (2024), 1452–1474. https://doi.org/10.1007/s11747-024-01003-2
29. W. H. Chen, F. You, Sustainable energy management and control for decarbonization of complex multi-zone buildings with renewable solar and geothermal energies using machine learning, robust optimization, and predictive control, *Appl. Energy*, **372** (2024), 123802. https://doi.org/10.1016/j.apenergy.2024.123802
30. T. Diller, A. Soppelsa, H. Nagpal, R. Fedrizzi, G. Henze, A dynamic programming based method for optimal control of a cascaded heat pump system with thermal energy storage, *Optim. Eng.*, **25** (2024), 229–251. https://doi.org/10.1007/s11081-023-09853-5
31. M. Graziano, K. Gillingham, Spatial patterns of solar photovoltaic system adoption: the influence of neighbors and the built environment, *J. Econ. Geogr.*, **15** (2015), 815–839. https://doi.org/10.1093/jeg/lbu036
32. R. Peres, E. Muller, V. Mahajan, Innovation diffusion and new product growth models: A critical review and research directions, *Int. J. Res. Mark.*, **27** (2010), 91–106. https://doi.org/10.1016/j.ijresmar.2009.12.012
33. M. Arbatskaya, H. Konishi, Consumer referrals, *Int. J. Ind. Organ.*, **48** (2016), 34–58. https://doi.org/10.1016/j.ijindorg.2016.06.001
34. F. El Ouardighi, G. Feichtinger, D. Grass, R. F. Hartl, P. M. Kort, Advertising and qualitydependent word-of-mouth in a contagion sales model, *J. Optim. Theory Appl.*, **170** (2016), 323–342. https://doi.org/10.1007/s10957-015-0855-0
35. J. J. Brown, P. H. Reingen, Social ties and word-of-mouth referral behavior, *J. Consum. Res.*, **14** (1987), 350–362. https://doi.org/10.1086/209118