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Research article

Reset option pricing with predetermined levels for uncertain currency models

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Abstract: An option is a kind of financial derivative product with far-reaching influence, and its pricing problem is a high priority of quantitative research in modern finance. A reset option gives holders more choices, so it can bring the holder more profit opportunities. Therefore, this paper analyzes the reset option with predetermined levels on the basis of uncertainty theory. Assuming that the exchange rate is determined by an uncertain differential equation, namely the uncertain currency model, we study the reset option with predetermined levels in an uncertain financial market. Further, according to the principle of identical expected revenue between buyers and sellers, the corresponding pricing formulas of reset options with predetermined levels for an uncertain currency model are proposed. Finally, several numerical examples are designed to verify the rationality of the proposed formulas. The results show that the option price increases monotonically with the number of predetermined levels and the maturity time, which conforms to the actual financial market rules. This model provides a new theoretical method for the pricing of cross-border reset options and has important theoretical value and practical application potential in uncertain financial markets.

Keywords: option pricing; reset option; predetermined level; currency model; uncertain finance
Mathematics Subject Classification: 91G20, 91G80

1. Introduction

Investors can obtain certain benefits in the process of financial investment, but, at the same time, there are also various financial risks. As a financial derivative product, an option can help investors manage and hedge risks. It is not an obligation but the right to purchase or sell an agreed amount of underlying assets at the strike price before or on the maturity date. In order to reach a consensus between the buyers and sellers, the price of options has become a key factor. Reasonable option prices have become the research focus of financial institutions and relevant scholars. In 1900, the first strict mathematical description of the stock price trend was given by Bachelier [1], who established the modern theory system of option pricing. Since 1970s, the study of option pricing theory has made breakthrough progress with the rapid development of the option market. In 1973, a far-reaching option pricing model appeared: The Black-Scholes formula, established by Black and Scholes [2] and developed by Merton [3]. They gave the first complete option pricing model, which greatly promoted the pricing of multiple derivative financial instruments in the modern financial market. Scholars have made many extensions and modifications to the Black-Scholes option pricing model. For example, the jump-diffusion model was presented by Merton [4] in 1976, in which the jump process follows a Poisson distribution. A plain and intuitive option pricing model was suggested by Cox et al. [5] in 1979, namely the binomial model, which laid the foundation for numerical methods in option pricing and solved the problem for American option pricing. The G-K model, specifically aiming at the pricing of foreign exchange options, was put forward by Garman and Kohlhagen [6] in 1983. In addition, Boyle [7] first pioneered the use of the Monte Carlo method into option pricing in 1977, which is widely used in the complex derivatives pricing field.

As an effective method for risk hedging, options have developed rapidly. In order to attract investors' interest and better control risks, many new options have been developed. At the same time, the diversification of options also makes option pricing more difficult and complicated. A reset option is a new kind of exotic option [8], which is characterized by allowing the purchaser to reset the strike price before its expiration, specifically when the option is out of the money on designated reset dates within the reset time period, or when the spot price hits a predetermined level. In Jiang [9], reset options are classified into two types: One is a reset option with predetermined dates and the other is a reset option with predetermined levels. Numerous scholars have studied reset options in recent years. Gray and Whaley [10] priced the put warrants with periodic resets and introduced single-point reset put options. Further, they obtained an analytical formula for the price of reset put options with a single date [11]. Some scholars have extended the pricing of reset options to the case of multiple resets. Cheng and Zhang [12] defined the general option with n reset dates, compared it with other exotic options and deduced its pricing formula. Inspired by Cheng and Zhang, Jiang et al. [13] considered a kind of reset option with n predetermined levels, and its closed-form pricing formula is derived by the method of partial differential equations. On the basis of the results of previous investigations, Dai and Kwok [14] discussed the optimal reset strategy for the reset option and obtained the analytical pricing formulas for several specific reset options where the price of the underlying asset obeys the continuous diffusion model. Dai et al. [15, 16] derived an extension of the analytical model for multiple reset rights with optimal shouting policies. Xue et al. [17] further provided the explicit form of the pricing formula for an outside-reset option featuring multiple strike resets and reset dates. Konstandatos [18] examined the fair value valuation of executive stock reset options. Wang and Liu [19] investigated the pricing of vulnerable reset options under default risk and stochastic volatility jump-diffusion models.

Probability theory is an effective mathematical tool used to depict indeterminate phenomena. The law of large numbers determines that the function of the cumulative probability distribution has to be sufficiently close to the real frequency. In order to meet the condition of a cumulative probability distribution, we must conduct a large number of independent and repeatable experiments to obtain sufficient sample data. However, these experiments sometimes cannot be launched owing to the actual conditions and cost control. For instance, we cannot obtain the load-bearing data of bridges in use. Sometimes, the existing historical data cannot be used. For instance, we may have observed the passenger flow data of a shopping mall, but the previously collected data cannot be used under some special conditions, such as pandemic (e.g., COVID-19 in 2020), extreme weather (e.g., the Australian bush fires in 2019), etc. That is to say, some data in real life are unobtainable or unavailable. We have to acquire the confidence level of events estimated by experts in relevant fields according to their knowledge and experience. The estimates of human beings are often conservative. If we apply probability theory to describing their degree of belief, a counterintuitive result may be triggered [20]. To cope with the problem of insufficient historical data or even no historical data, considering all these situations, we have to seek for a new mathematical tool.

In 2007, Liu [21] found that uncertainty theory depended on the axioms of normality, duality, subadditivity, and the product. He introduced a series of basic concepts, including the uncertain variable, the uncertain measure, the uncertainty distribution, and the expected value and variance, etc. The uncertain variable characterizes uncertain quantities, the uncertain measure quantifies degrees of belief, and the uncertainty distribution serves as a direct descriptive tool for uncertain variables. Liu then proposed the expected value operator for ranking uncertain variables [21]. With some scholars' efforts, uncertainty theory has gradually matured and has now become a complete branch of mathematics. At present, uncertainty theory has been extensively adopted across multiple domains, including uncertain finance [22], uncertain risk assessment and reliability analyses [23, 24], and uncertain programming [25].

For describing uncertain variables that evolve over time, Liu [26] proposed the definition of an uncertain process. After that, some related concepts of the uncertain process were researched, such as the uncertainty distributions, independence, and operational law. An independent incremental process was proposed subsequently, featuring independent increments. Furthermore, Liu [27] put forward a stationary independent increment process with increments that are independent and identically distributed (i.i.d.) uncertain variables. A Liu process is produced if these i.i.d. uncertain variables follow the normal uncertainty distribution. Both uncertain calculus and the uncertain differential equation, which are driven by the Liu process, were developed by Liu [27]. The existence, uniqueness, and stability of solutions to uncertain differential equations were later confirmed by Chen and Liu [28]. Yao and Chen [29] initiated the Yao-Chen formula, a tool for transforming uncertain differential equations into ordinary differential equations.

The further improvement of uncertain differential equations has led to their widespread application, especially in the modern financial market, which is becoming increasingly sophisticated. Motivated by uncertain differential equations, Liu [27] introduced the uncertain stock model, and Liu et al. [30] constructed the uncertain currency model. Using these models of financial uncertainty, more and more related studies have been carried out. For example, Chen [31] priced American options, Liu [27] priced European options, and Sun and Chen [32] priced Asian options. In addition, the price of a zero-coupon bond was obtained by Jiao and Yao [33], and a model for the price of both the interest rate ceiling and the interest rate floor was presented by Zhang et al. [34]. In addition, in recent years, scholars have successively applied the uncertainty theory to the pricing of various derivatives, including the barrier swap option [35], the shout option [36], and the power option [37]. Li et al. [38] further extended the analysis to European option pricing under multifactor volatility models, thereby enriching the pricing system for uncertain financial derivatives.

The existing research on option pricing under uncertainty theory mainly focuses on traditional derivatives such as European options [27], American options [31], and barrier options [22], while there is no research on reset options with predetermined levels—a typical exotic option. At the same time, reset option pricing under the traditional probability framework (e.g., [13, 17]) relies on sufficient historical data, which are difficult to apply to cross-border financial markets with insufficient data. Therefore, this paper takes reset options with predetermined levels as the research object, constructs their pricing model under the uncertain currency model, and realizes the expansion of uncertainty theory in option type pricing.

Foreign exchange is a form of payment for international trade settlements. On the one hand, it can carry out international settlements and promote the development of international trade. On the other hand, it can also regulate the international flow of funds and serve as an international reserve. On the basis of the hypothesis that the exchange rate is determined by an uncertain differential equation, we will discuss the reset option and propose the corresponding pricing formulas in this paper. This paper adopts the fair pricing principle as the direct criterion for option pricing. That is, the equilibrium price of an option must equalize the expected net returns of both the buyer and the seller under an uncertain environment. As a classic pricing rule in the field of uncertain finance, it is worth noting that this principle differs fundamentally from the pricing logic centered directly on the no-arbitrage principle under the probability framework. The no-arbitrage principle is based on the core constraint that there is no trading strategy with zero cost, zero risk, and a positive return in the market, while the fair price principle is based on the pricing criterion that the expected net return of both option buyers and sellers is equal and zero. These two principles differ in their theoretical assumptions, applicable scenarios, and pricing objectives. Based on the research paradigm in the field of uncertain finance, this study selects the fair pricing principle as the basis for pricing, which is the core method used to depict market equilibrium in uncertain financial research.

The rest of the paper is arranged as follows: Section 2 introduces some fundamental concepts. Section 3 focuses on the general pricing formula for reset options with n predetermined levels, and provides a more specific pricing formula for the special case of $n = 1$, which is the pricing formula for reset options with a single predetermined level. Section 4 verifies the effectiveness and rationality of the pricing formula through using real data. Finally, Section 5 provides a concise conclusion that elucidates the differences between the proposed method and existing methods, as well as the innovative contribution of this study.

2. Preliminaries

This section will introduce some basic definitions and related properties that will be used in later derivations and proofs.

2.1. Uncertain variable

If Γ is a nonempty set, and $\mathcal{L}$ is a σ-algebra over Γ, then each element Λ in $\mathcal{L}$ is regarded as a measurable set. The two-tuple $(\Gamma, \mathcal{L})$ is used as the denotation of a measurable space. Next, an uncertain measure $\mathcal{M}$ is defined on $\mathcal{L}$ to show the degree of belief that the event Λ may happen.

Definition 2.1 ([21]). *Consider a measurable space* $(\Gamma, \mathcal{L})$. *A set function* $\mathcal{M}$ *is regarded as an uncertain measure if the three axioms listed below are satisfied.*

- *Normality:* $\mathcal{M}\{\Gamma\} = 1$.

- *Duality: Let* Λ *denote any one of the events, then* $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$.
- *Subadditivity: Let* $\Lambda_1, \Lambda_2, \cdots$ *denote any countable sequence of events, then*

$$\mathcal{M}\left\{\bigcup_{j=1}^{\infty} \Lambda_j\right\} \leq \sum_{j=1}^{\infty} \mathcal{M}\{\Lambda_j\}.$$

We say the triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ an uncertainty space. Liu [27] proposed the product axiom to perfect uncertainty theory.

- Product axiom: Suppose that $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ are uncertainty spaces ($k = 1, 2, \cdots$). Then the product uncertain measure $\mathcal{M}$ is an uncertain measure and satisfies the following equality:

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\}$$

  in which Λ_k represents respectively arbitrary events chosen from $\mathcal{L}_k$ ($k = 1, 2, \cdots$).

Theorem 2.1 ([21]). *The uncertainty measure* $\mathcal{M}$ *is a monotonically increasing set function, i.e., the inequality*

$$\mathcal{M}\{\Lambda_1\} \leq \mathcal{M}\{\Lambda_2\}$$

holds for arbitrary events Λ_1 *and* Λ_2 $(\Lambda_1 \subset \Lambda_2)$.

For ranking uncertain variables, Liu [21] introduced the definition of expected value, which is shown below.

Theorem 2.2 ([39]). *Let* η *represent an uncertain variable that has an uncertainty distribution* Φ*, assuming that the expected value of* η *exists. It yields*

$$E[\eta] = \int_{-\infty}^{+\infty} x \mathrm{d}\Phi(x).$$

If Φ *is regular, then*

$$E[\eta] = \int_0^1 \Phi^{-1}(\alpha) \mathrm{d}\alpha.$$

2.2. Uncertain process

The concept of an uncertain process was suggested by Liu [26] to model the dynamic uncertainty further.

Definition 2.2 ([40]). *The uncertainty distribution* $\Phi_s(y)$ *of an uncertain process* Y_s *is expressed as follows:*

$$\Phi_s(y) = \mathcal{M}\{Y_s \leq y\}$$

for an arbitrary number y *and an arbitrary time* s.

Definition 2.3 ([27]). *If the uncertain process* C_s *satisfies the three conditions shown below, then it is said to be a Liu process.*

- *Its initial value equals zero, i.e.,* $C_0 = 0$*, and nearly all its sample paths exhibit Lipschitz continuity.*
- C_s *exhibits independent and stationary increments.*
- *Every increment* $C_{t+s} - C_t$ *is a normal uncertain variable whose characteristics are an expected value of 0 and a variance of* s^2*. More precisely,* $C_{t+s} - C_t$ *has the following uncertainty distribution:*

$$\Phi(y) = \left(1 + \exp\left(\frac{-\pi y}{\sqrt{3}s}\right)\right)^{-1}, \quad y \in \Re.$$

2.3. Uncertain differential equation

Theorem 2.3 ([29]). *We consider the uncertain differential equation shown below:*

$$\mathrm{d}Y_s = f(s, Y_s)\mathrm{d}t + g(s, Y_s)\mathrm{d}C_s. \tag{2.1}$$

For an arbitrary $\alpha(0 < \alpha < 1)$*,* Y_s^α *relative to s is the solution of the corresponding equation*

$$\mathrm{d}Y_s^\alpha = f(s, Y_s^\alpha)\mathrm{d}s + |g(s, Y_s^\alpha)|\Phi^{-1}(\alpha)\mathrm{d}s$$

in which

$$\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}, \; 0 < \alpha < 1.$$

The deterministic function Y_s^α *is then the* α*-path of Equation* (2.1)*. Given that both functions* $f(s, y)$ *and* $g(s, y)$ *possess continuity, we can deduce the following results:*

$$\mathcal{M}\{Y_s \leq Y_s^\alpha, \; \forall s\} = \alpha,$$

$$\mathcal{M}\{Y_s > Y_s^\alpha, \; \forall s\} = 1 - \alpha.$$

Many of properties of the α-path have been deduced by Theorem 2.3. The inverse uncertainty distribution of Y_s was first demonstrated by Chen and Yao [29] and can be expressed as

$$\Phi^{-1}(\alpha) = Y_s^\alpha, \; 0 < \alpha < 1.$$

3. Reset option pricing model with predetermined levels

We discuss reset options with predetermined levels whose strike price can be reset many times at predetermined levels in this section. First, the general pricing framework for reset options with n predetermined levels is given.

The uncertain currency model was introduced by Liu et al. [30]

$$\begin{cases} \mathrm{d}X_t = \lambda X_t \mathrm{d}t \\ \mathrm{d}Y_t = \mu Y_t \mathrm{d}t \\ \mathrm{d}Z_t = \xi Z_t \mathrm{d}t + \sigma Z_t \mathrm{d}C_t \end{cases} \tag{3.1}$$

where X_t represents the domestic currency with the domestic interest rate λ; Y_t represents the foreign currency with the foreign interest rate μ; Z_t represents the exchange rate, which means that one unit of foreign currency can be exchanged for Z_t units of domestic currency at time t; ξ and σ denote the representation of drift and diffusion, respectively; and C_t represents a Liu process.

3.1. General pricing framework for reset options with n predetermined levels

A reset option with n predetermined levels is discussed, whose strike price can be reset many times at predetermined levels. Furthermore, their related pricing formulas are derived and formulated.

3.1.1. Reset call option with n predetermined levels

First, we examine the reset call option with n predetermined levels whose initial strike price is K_0, the maturity date is T, and the predetermined levels are $K_i (i = 1, 2, \ldots, n)(K_0 > K_1 > \ldots > K_n)$. Specifically, when the market price of the underlying asset surpasses the total of the strike price and the option premium, the option buyer can either transfer the call option to realize a profit or purchase the underlying asset at the predetermined price and quantity. In an uncertain financial market, the currency model for the reset call option with n predetermined levels is defined by Model (3.1).

If $\min_{0<t<T} Z_t$ satisfies

$$K_i < \min_{0<t<T} Z_t \leq K_{i-1},$$

then the strike price will be reset to K_{i-1}. That is to say

$$K_i = \begin{cases} K_0, & \text{if } K_1 < \min_{0<t<T} Z_t \leq K_0 \\ K_1, & \text{if } K_2 < \min_{0<t<T} Z_t \leq K_1 \\ \vdots & \\ K_n, & \text{if } \min_{0<t<T} Z_t \leq K_n \end{cases}. \tag{3.2}$$

Let f_{nl}^c represent the contract price in the domestic currency. According to the reset rule of the strike price K_i, the present value of the revenue possessed by the purchaser in the domestic currency is

$$(Z_T - K_i)^+,$$

where $(\cdot)^+$ denotes the non-negative part of the expression, i.e., $(x)^+ = \max(x, 0)$.

On the basis of the reset rule of the strike price, we derive the present value of revenue and net revenue for both the buyer and the seller of the option under the uncertain currency model. The net revenue at time 0 possessed by the buyer is

$$-f_{nl}^c + \exp(-\lambda T)(Z_T - K_i)^+,$$

where $\exp(-\lambda T)$ is the risk-free discount factor of the domestic currency, converting the expiration revenue to the present value at time 0. Similarly, the net revenue of the seller considers the discount factor $\exp(-\mu T)$ of the foreign currency and the exchange rate's conversion relationship. What the seller needs to pay at time 0 is $\exp(-\mu T)Z_0(1 - K_i/Z_T)^+$. The net revenue at time 0 possessed by the seller is

$$f_{nl}^c - \exp(-\mu T)Z_0(1 - K_i/Z_T)^+.$$

The contract price should ensure that the expected profits of both parties are equal, which implies that

$$-f_{nl}^c + E\left[\exp(-\lambda T)(Z_T - K_i)^+\right] = f_{nl}^c - E\left[\exp(-\mu T)Z_0(1 - K_i/Z_T)^+\right]. \tag{3.3}$$

Therefore, we further deduce the pricing equation of the reset call option with n predetermined levels.

Proposition 3.1. *Assuming that there is an reset call option with n predetermined levels whose maturity date is T, the initial strike price is K_0, and the predetermined levels are K_i. This reset call option price with n predetermined levels for Model (3.1) can be derived as*

$$f_{nl}^c = \frac{1}{2} E\left[\exp(-\lambda T)(Z_T - K_i)^+\right] + \frac{1}{2} E\left[\exp(-\mu T) Z_0 (1 - K_i / Z_T)^+\right]$$

where K_i satisfies Equation (3.2).

Proof. It can be immediately derived from Equation (3.3). □

Theorem 3.1. *Assuming that there is a reset call option with n predetermined levels whose maturity date is T, the initial strike price is K_0, and the predetermined levels are K_i. This reset call option price with n predetermined levels is controlled by the following equality:*

$$f_{nl}^c = \frac{1}{2}\int_0^1 \exp(-\lambda T)\left(Z_0 \exp\left(\xi T + \frac{\sqrt{3}\sigma T}{\pi}\ln\frac{\alpha}{1-\alpha}\right) - K_i\right)^+ \mathrm{d}\alpha + \frac{1}{2}\int_0^1 \exp(-\mu T)\left(Z_0 - K_i / \exp\left(\xi T + \frac{\sqrt{3}\sigma T}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\right)^+ \mathrm{d}\alpha$$

where K_i satisfies Equation (3.2).

Proof. To begin with, we check that the uncertain variable shown below

$$\exp(-\lambda T)(Z_T - K_i)^+$$

has the inverse uncertainty distribution, which may be expressed in the following form:

$$\phi^{-1}(\alpha) = \exp(-\lambda T)(Z_T^\alpha - K_i)^+$$

in which

$$Z_T^\alpha = Z_0 \exp\left(\xi T + \frac{\sqrt{3}\sigma T}{\pi}\ln\frac{\alpha}{1-\alpha}\right).$$

Noting that

$$\{\exp(-\lambda T)(Z_T - K_i)^+ \le \exp(-\lambda T)(Z_T^\alpha - K_i)^+\} \supseteq \{Z_T \le Z_T^\alpha\}$$

and

$$\{\exp(-\lambda T)(Z_T - K_i)^+ > \exp(-\lambda T)(Z_T^\alpha - K_i)^+\} \supseteq \{Z_T > Z_T^\alpha\},$$

we have

$$\begin{aligned}&\mathcal{M}\{\exp(-\lambda T)(Z_T - K_i)^+ \le \exp(-\lambda T)(Z_T^\alpha - K_i)^+\}\\ &\ge M\{Z_T \le Z_T^\alpha\}\\ &= \alpha\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}&\mathcal{M}\{\exp(-\lambda T)(Z_T - K_i)^+ > \exp(-\lambda T)(Z_T^\alpha - K_i)^+\}\\ &\ge M\{Z_T > Z_T^\alpha\}\\ &= 1-\alpha\end{aligned} \tag{3.5}$$

due to Theorem 2.1 and Theorem 2.3.

Furthermore, from the duality axiom, it can be seen that

$$\begin{aligned}&\mathcal{M}\{\exp(-\lambda T)(Z_T - K_i)^+ \le \exp(-\lambda T)(Z_T^\alpha - K_i)^+\} +\\ &\mathcal{M}\{\exp(-\lambda T)(Z_T - K_i)^+ > \exp(-\lambda T)(Z_T^\alpha - K_i)^+\} = 1.\end{aligned} \tag{3.6}$$

It follows from Equations (3.4)–(3.6) that

$$\mathcal{M}\{\exp(-\lambda T)(Z_T - K_i)^+ \le \exp(-\lambda T)(Z_T^\alpha - K_i)^+\} = \alpha.$$

Therefore, the uncertain variable

$$\exp(-\lambda T)(Z_T - K_i)^+$$

has the inverse uncertainty distribution which may be expressed in the following form:

$$\phi^{-1}(\alpha) = \exp(-\lambda T)(Z_T^\alpha - K_i)^+.$$

Therefore, we can obtain

$$f_{nl}^{c1} = \int_0^1 \exp(-\lambda T)(Z_T^\alpha - K_i)^+ \mathrm{d}\alpha$$

from Theorem 2.2.

Next, we check that the uncertain variable shown below

$$\exp(-\mu T)I_{sl}(1 - K_i/Z_T)^+$$

has the inverse uncertainty distribution, which may be expressed in the following form:

$$\phi^{-1}(\alpha) = \exp(-\mu T)1 - K_i/Z_T^\alpha)^+$$

in which

$$Z_T^\alpha = Z_0 \exp\left(\xi T + \frac{\sqrt{3}\sigma T}{\pi} \ln \frac{\alpha}{1-\alpha}\right).$$

Noting the fact that

$$\{\exp(-\mu T)(1 - K_i/Z_T)^+ \le \exp(-\mu T)(1 - K_i/Z_T^\alpha)^+\}$$

$$\supseteq \{Z_T \le Z_T^\alpha\}$$

and

$$\begin{aligned}&\{\exp(-\mu T)(1 - K_i/Z_T)^+ > \exp(-\mu T)(1 - K_i/Z_T^\alpha)^+\}\\&\supseteq \{Z_T > Z_T^\alpha\},\end{aligned}$$

we have

$$\begin{aligned}&\mathcal{M}\{\exp(-\mu T)(1 - K_i/Z_T)^+ \le \exp(-\mu T)(1 - K_i/Z_T^\alpha)^+\}\\&\ge M\{Z_T \le Z_T^\alpha\}\\&= \alpha\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}&\mathcal{M}\{\exp(-\mu T)(1 - K_i/Z_T)^+ > \exp(-\mu T)(1 - K_i/Z_T^\alpha)^+\}\\&\ge M\{Z_T > Z_T^\alpha\}\\&= 1 - \alpha\end{aligned} \tag{3.8}$$

due to Theorem 2.1 and Theorem 2.3.

Moreover, from the duality axiom, it can be seen that

$$\begin{aligned}&\mathcal{M}\{\exp(-\mu T)(1 - K_i/Z_T)^+ \le \exp(-\mu T)(1 - K_i/Z_T^\alpha)^+\} +\\&\mathcal{M}\{\exp(-\mu T)(1 - K_i/Z_T)^+ > \exp(-\mu T)(1 - K_i/Z_T^\alpha)^+\} = 1.\end{aligned} \tag{3.9}$$

It follows from Equations (3.7)–(3.9) that

$$\mathcal{M}\{\exp(-\mu T)(1 - K_i/Z_T)^+ \le \exp(-\mu T)(1 - K_i/Z_T^\alpha)^+\} = \alpha.$$

Therefore, the uncertain variable

$$\exp(-\mu T)(1 - K_i/Z_T)^+$$

has the inverse uncertainty distribution, which may be expressed in the following form:

$$\phi^{-1}(\alpha) = \exp(-\mu T)(1 - K_i/Z_T^\alpha)^+.$$

Therefore, the equality

$$f_{nl}^{c2} = \int_0^1 \exp(-\mu T)(1 - K_i/Z_T^\alpha)^+ \mathrm{d}\alpha$$

holds due to Theorem 2.2.

Consequently, the reset call option pricing formula with n predetermined levels is

$$\begin{aligned} f_{nl}^{c} &= \frac{1}{2}\int_0^1 \exp(-\lambda T)(Z_T^{\alpha} - K_i)^{+}\mathrm{d}\alpha \\ &+ \frac{1}{2}\int_0^1 \exp(-\mu T)Z_0(1 - K_i/Z_T^{\alpha})^{+}\mathrm{d}\alpha \\ &= \frac{1}{2}\int_0^1 \exp(-\lambda T)\left(Z_0 \exp\left(\xi T + \frac{\sqrt{3}\sigma T}{\pi}\ln\frac{\alpha}{1-\alpha}\right) - K_i\right)^{+}\mathrm{d}\alpha \\ &+ \frac{1}{2}\int_0^1 \exp(-\mu T)\left(Z_0 - K_i/\exp\left(\xi T + \frac{\sqrt{3}\sigma T}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\right)^{+}\mathrm{d}\alpha \end{aligned}$$

where K_i satisfies Equation (3.2).

□

Example 3.1. The parameters of Model (3.1) are set as $\lambda = 0.03$, $\mu = 0.025$, $\xi = 0.04$, and $\sigma = 0.03$. Suppose that $Z_0 = 5$, $K_0 = 6$, $T = 10$, $K_1 = 4$, and $K_2 = 3$. Then $f_{nl}^{c} = 2.908$ according to Theorem 3.1.

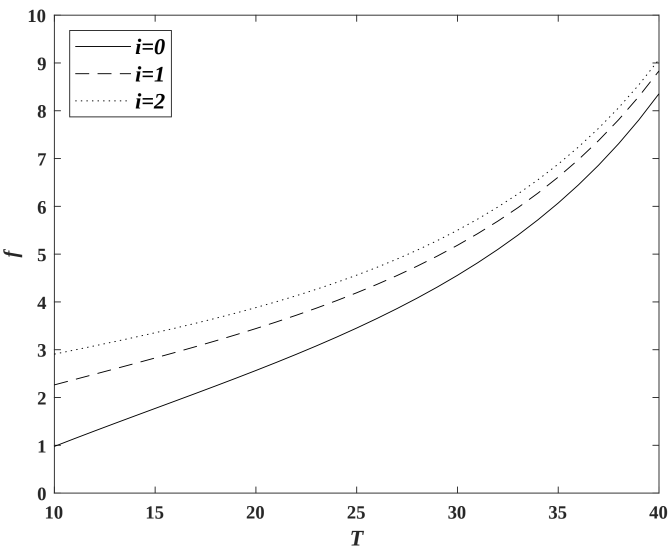

Figure 1. Price comparison of reset call options with different reset times.

In Figure 1, when the other parameters remain unchanged, we observe that f_{nl}^{c} is monotonously increasing with respect to T and i. Because it has more profit opportunities, it conforms to reality. More predetermined levels mean a more detailed interval division of the exchange rate. For call options, the holder can reset the strike price to a lower level (e.g., K_2 instead of K_1, $K_1 > K_2$) when $\min_{0<t<T} Z_t$ falls into a lower interval, locking in a lower purchase cost. In uncertain financial markets, the exchange rate has inherent volatility. More predetermined levels provide the holder with more "protection thresholds." When the exchange rate fluctuates unfavorably (such as a significant drop in Z_t), the holder can reset the strike price to reduce losses; when it fluctuates favorably, the holder can abandon the reset to obtain excess returns. This one-way protection is continuously strengthened with an increase in i.

3.1.2. Reset put options with n predetermined levels

We also study the reset put option with n predetermined levels whose strike price is K_0, the maturity date is T, and the predetermined levels are $K_i(i = 1, 2, \ldots, n)(K_0 < K_1 < \ldots < K_n)$, i.e., when the market price of the underlying asset falls below the total of the strike price and the option premium, the option buyer can either transfer the put option to realize a profit or sell the underlying asset at the predetermined price and quantity. In an uncertain financial market, the currency model for the reset put option with n predetermined levels is defined by Model (3.1).

If $\max_{0<t<T} Z_t$ satisfies

$$K_i > \max_{0<t<T} Z_t \geq K_{i-1},$$

then the strike price will be reset to K_{i-1}. That is to say

$$K_i = \begin{cases} K_0, & \text{if } K_1 > \max_{0<t<T} Z_t \\ K_1, & \text{if } K_2 > \max_{0<t<T} Z_t \geq K_1 \\ \vdots & \\ K_n, & \text{if } \max_{0<t<T} Z_t \geq K_n \end{cases}. \tag{3.10}$$

Let f_{nl}^p represent the contract price in the domestic currency. According to the reset rule of the strike price K_i, the present value of the revenue possessed by the buyer in the domestic currency, which is

$$(K_i - Z_T)^+,$$

where $(\cdot)^+$ denotes the non-negative part of the expression, i.e., $(x)^+ = \max(x, 0)$.

According to the reset rule of the strike price, we derive the present value of revenue and net revenue for both the buyer and the seller of the option under the uncertain currency model. The net revenue at time 0 possessed by the buyer is

$$-f_{nl}^p + \exp(-\lambda T)(K_i - Z_T)^+,$$

where $\exp(-\lambda T)$ is the risk-free discount factor of the domestic currency, converting the expiration revenue to the present value at time 0. Similarly, the net revenue of the seller considers the discount factor $\exp(-\mu T)$ of the foreign currency and the exchange rate's conversion relationship. What the seller then needs to pay at time 0 is $\exp(-\mu T)Z_0(K_i/Z_T - 1)^+$. The net revenue at time 0 possessed by the seller is

$$f_{nl}^p - \exp(-\mu T)Z_0(K_i/Z_T - 1)^+.$$

The price of this contract should ensure that the expected profits of both parties are equal, which implies that

$$-f_{nl}^p + E\left[\exp(-\lambda T)(K_i - Z_T)^+\right] = f_{nl}^p - E\left[\exp(-\mu T)Z_0(K_i/Z_T - 1)^+\right]. \tag{3.11}$$

Therefore, we further deduce the pricing equation of the reset put option with n predetermined levels.

Proposition 3.2. *Assuming that there is a reset put option with n predetermined levels whose maturity date is T, the initial strike price is K_0, and the predetermined levels are K_i. This reset put option price with n predetermined levels for Model (3.1) can be derived as*

$$f_{nl}^{p} = \frac{1}{2}E\left[\exp(-\lambda T)(K_i - Z_T)^+\right] + \frac{1}{2}E\left[\exp(-\mu T)Z_0(K_i/Z_T - 1)^+\right]$$

where K_i satisfies Equation (3.10).

Proof. This can be immediately derived from Equation (3.11). □

Theorem 3.2. *Assuming that there is a reset put option with n predetermined levels whose maturity date is T, the initial strike price is K_0, and the predetermined levels are K_i. This reset put option price with n predetermined levels is controlled by the following equality:*

$$f_{nl}^{p} = \frac{1}{2}\int_0^1 \exp(-\lambda T)\left(K_i - Z_0 \exp\left(\xi T + \frac{\sqrt{3}\sigma T}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\right)^+ \mathrm{d}\alpha + \frac{1}{2}\int_0^1 \exp(-\mu T)\left(K_i/\exp\left(\xi T + \frac{\sqrt{3}\sigma T}{\pi}\ln\frac{\alpha}{1-\alpha}\right) - Z_0\right)^+ \mathrm{d}\alpha$$

where K_i satisfies Equation (3.10).

Proof. To begin with, we check that the uncertain variable shown below

$$\exp(-\lambda T)(K_i - Z_T)^+$$

possesses the inverse uncertainty distribution

$$\phi^{-1}(\alpha) = \exp(-\lambda T)(K_i - Z_T^{1-\alpha})^+$$

in which

$$Z_T^{1-\alpha} = Z_0 \exp\left(\xi T + \frac{\sqrt{3}\sigma T}{\pi}\ln\frac{1-\alpha}{\alpha}\right).$$

Noting that

$$\left\{\exp(-\lambda T)(K_i - Z_T)^+ \le \exp(-\lambda T)(K_i - Z_T^{1-\alpha})^+\right\} \supseteq \left\{Z_T \ge Z_T^{1-\alpha}\right\}$$

and

$$\left\{\exp(-\lambda T)(K_i - Z_T)^+ > \exp(-\lambda T)(K_i - Z_T^{1-\alpha})^+\right\} \supseteq \left\{Z_T < Z_T^{1-\alpha}\right\},$$

we obtain

$$\begin{aligned}&\mathcal{M}\left\{\exp(-\lambda T)(K_i - Z_T)^+ \le \exp(-\lambda T)(K_i - Z_T^{1-\alpha})^+\right\}\\&\ge M\left\{Z_T \ge Z_T^{1-\alpha}\right\}\\&= \alpha\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}&\mathcal{M}\left\{\exp(-\lambda T)(K_i - Z_T)^+ > \exp(-\lambda T)(K_i - Z_T^{1-\alpha})^+\right\}\\&\ge M\left\{Z_T < Z_T^{1-\alpha}\right\}\\&= 1-\alpha\end{aligned} \tag{3.13}$$

due to Theorem 2.1 and Theorem 2.3.

Moreover, from the duality axiom, it can be seen that

$$\begin{aligned}&\mathcal{M}\left\{\exp(-\lambda T)(K_i - Z_T)^+ \le \exp(-\lambda T)(K_i - Z_T^{1-\alpha})^+\right\} +\\&\mathcal{M}\left\{\exp(-\lambda T)(K_i - Z_T)^+ > \exp(-\lambda T)(K_i - Z_T^{1-\alpha})^+\right\} = 1.\end{aligned} \tag{3.14}$$

It follows from Equations (3.12)–(3.14) that

$$\mathcal{M}\left\{\exp(-\lambda T)(K_i - Z_T)^+ \le \exp(-\lambda T)(K_i - Z_T^{1-\alpha})^+\right\} = \alpha.$$

So the uncertain variable

$$\exp(-\lambda T)(K_i - Z_T)^+$$

has the inverse uncertainty distribution, which may be expressed in the following form:

$$\phi^{-1}(\alpha) = \exp(-\lambda T)(K_i - Z_T^{1-\alpha})^+.$$

Therefore, we can get

$$f_{nl}^{p1} = \int_0^1 \exp(-\lambda T)(K_i - Z_T^{1-\alpha})^+ \mathrm{d}\alpha$$

from Theorem 2.2.

Next, we check that the uncertain variable shown below

$$\exp(-\mu T)(K_i/Z_T - 1)^+$$

has the inverse uncertainty distribution, which may be expressed in the following form:

$$\phi^{-1}(\alpha) = \exp(-\mu T)(K_i/Z_T^{1-\alpha} - 1)^+$$

in which

$$Z_T^{1-\alpha} = Z_0 \exp\left(\xi T + \frac{\sqrt{3}\sigma T}{\pi} \ln \frac{1-\alpha}{\alpha}\right).$$

Noting the fact that

$$\left\{\exp(-\mu T)(K_i/Z_T - 1)^+ \le \exp(-\mu T)(K_i/Z_T^{1-\alpha} - 1)^+\right\}$$

$$\supseteq \left\{ Z_T \geq Z_T^{1-\alpha} \right\}$$

and

$$\left\{ \exp(-\mu T)(K_i/Z_T - 1)^+ > \exp(-\mu T)(K_i/Z_T^{1-\alpha} - 1)^+ \right\}$$
$$\supseteq \left\{ Z_T < Z_T^{1-\alpha} \right\},$$

we have

$$\begin{aligned} &\mathcal{M}\left\{ \exp(-\mu T)(K_i/Z_T - 1)^+ \leq \exp(-\mu T)(K_i/Z_T^{1-\alpha} - 1)^+ \right\} \\ &\geq M\left\{ Z_T \geq Z_T^{1-\alpha} \right\} \\ &= \alpha \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} &\mathcal{M}\left\{ \exp(-\mu T)(K_i/Z_T - 1)^+ > \exp(-\mu T)(K_i/Z_T^{1-\alpha} - 1)^+ \right\} \\ &\geq M\left\{ Z_T < Z_T^{1-\alpha} \right\} \\ &= 1 - \alpha \end{aligned} \tag{3.16}$$

due to Theorem 2.1 and Theorem 2.3.

Moreover, from the duality axiom, it can be seen that

$$\begin{aligned} &\mathcal{M}\left\{ \exp(-\mu T)(K_i/Z_T - 1)^+ \leq \exp(-\mu T)(K_i/Z_T^{1-\alpha} - 1)^+ \right\} + \\ &\mathcal{M}\left\{ \exp(-\mu T)(K_i/Z_T - 1)^+ > \exp(-\mu T)(K_i/Z_T^{1-\alpha} - 1)^+ \right\} = 1. \end{aligned} \tag{3.17}$$

We can deduce

$$\mathcal{M}\left\{ \exp(-\mu T)(K_i/Z_T - 1)^+ \leq \exp(-\mu T)(K_i/Z_T^{1-\alpha} - 1)^+ \right\} = \alpha$$

from Equations (3.15)–(3.17).

Therefore, the uncertain variable

$$\exp(-\mu T)(K_i/Z_T - 1)^+$$

has the inverse uncertainty distribution, which may be expressed in the following form:

$$\phi^{-1}(\alpha) = \exp(-\mu T)(K_i/Z_T^{1-\alpha} - 1)^+.$$

Therefore, the equality

$$f_{nl}^{p2} = \int_0^1 \exp(-\mu T)(K_i/Z_T^{1-\alpha} - 1)^+ \mathrm{d}\alpha$$

holds, due to Theorem 2.2.

Consequently, the reset put option pricing formula with n predetermined levels is

$$
\begin{aligned}
f_{nl}^{p} &= \frac{1}{2}\int_0^1 \exp(-\lambda T)(K_i - Z_T^{1-\alpha})^+ \mathrm{d}\alpha \\
&+ \frac{1}{2}\int_0^1 \exp(-\mu T) Z_0 (K_i / Z_T^{1-\alpha} - 1)^+ \mathrm{d}\alpha \\
&= \frac{1}{2}\int_0^1 \exp(-\lambda T)\left(K_i - Z_0 \exp\left(\xi T + \frac{\sqrt{3}\sigma T}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\right)^+ \mathrm{d}\alpha \\
&+ \frac{1}{2}\int_0^1 \exp(-\mu T)\left(K_i / \exp\left(\xi T + \frac{\sqrt{3}\sigma T}{\pi}\ln\frac{\alpha}{1-\alpha}\right) - Z_0\right)^+ \mathrm{d}\alpha
\end{aligned}
$$

where K_i satisfies Equation (3.10).

□

Example 3.2. The parameters of Model (3.1) are set as $\lambda = 0.03$, $\mu = 0.025$, $\xi = -0.04$, and $\sigma = 0.03$. Suppose that $Z_0 = 5$, $K_0 = 4$, $T = 10$, $K_1 = 6$, and $K_2 = 7$. Then $f_{nl}^{p} = 3.602$ according to Theorem 3.2.

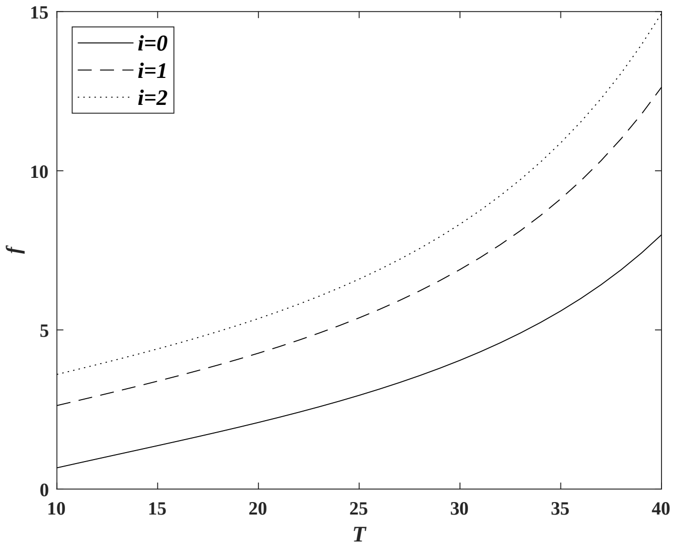

Figure 2. Price comparison of reset put options with different reset times.

In Figure 2, when the other parameters remain unchanged, we observe that f_{nl}^{p} is monotonously increasing with respect to T and i. It has more profit opportunities, which conforms to reality. More predetermined levels mean a more detailed interval division of the maximum exchange rate $\max_{0<t<T} Z_t$. For put options, the holder benefits from a higher strike price, and more predetermined levels (e.g., $K_0 < K_1 < \ldots < K_n$) allow the holder to reset the strike price to a higher level when $\max_{0<t<T} Z_t$ falls into a higher interval, locking in a higher sales revenue. In uncertain financial markets, the exchange rate has inherent volatility. More predetermined levels provide the holder with more "protection thresholds." When the exchange rate fluctuates unfavorably (such as a significant increase in Z_t), the holder can reset the strike price to reduce losses; when it fluctuates favorably, the

holder can abandon the reset option to obtain excess returns. This one-way protection is continuously strengthened with an increase in i.

3.2. Special case: Reset options with a single predetermined level

In fact, the pricing model for reset options with a single predetermined level is a special case when the pricing model for the reset option with n predetermined levels is $n = 1$. In this subsection, we validated the rationality of the pricing formulas for reset options with n predetermined levels and provided a more specific expression for the pricing formulas for reset options with a single predetermined level.

3.2.1. Reset call options with a single predetermined level

For the convenience of subsequent derivations, we define an indicator function to characterize the reset condition of the strike price as following

$$I_{sl} = \begin{cases} 1, & \text{if } K_1 > \min\limits_{0<t<T} Z_t \\ 0, & \text{otherwise} \end{cases}$$

where K_1 is the predetermined level and $\min\limits_{0<t<T} Z_t$ represents the minimum value of the underlying exchange rate Z_t during the period $(0, T)$. On the basis of this indicator function, we further clarify the reset rule of the strike price for the reset call option with a single predetermined level.

First, we study the reset call option with a single predetermined level whose strike price is K, the maturity date is T and the predetermined level is K_1; i.e., when the market price of the underlying asset surpasses the total of the strike price and the option premium, the option buyer can either transfer the call option to realize a profit or purchase the underlying asset at the predetermined price and quantity. In an uncertain financial market, the currency model for the reset call option with a single predetermined level is defined by Model (3.1). If $\min\limits_{0<t<T} Z_t$ is lower than K_1, then it will be reset to K_1; otherwise, the initial strike price K will be maintained. Let $\overline{K}$ be the final strike price, which is

$$\overline{K} = \begin{cases} K_1, & \text{if } K_1 > \min\limits_{0<t<T} Z_t \\ K, & \text{otherwise} \end{cases}.$$

Let f_{sl}^c represent the contract price in the domestic currency. According to the reset rule of the strike price $\overline{K}$, the present value of the revenue possessed by the buyer in the domestic currency is

$$(Z_T - \overline{K})^+,$$

where $(\cdot)^+$ denotes the non-negative part of the expression, i.e., $(x)^+ = \max(x, 0)$.

On the basis of the reset rule of the strike price, we derive the present value of revenue and net revenue for both the buyer and the seller of the option under the uncertain currency model. The net revenue at time 0 possessed by the buyer is

$$-f_{sl}^c + \exp(-\lambda T)(Z_T - \overline{K})^+,$$

where $\exp(-\lambda T)$ is the risk-free discount factor of the domestic currency, converting the expiration revenue to the present value at time 0. Similarly, the net revenue of the seller considers the discount factor $\exp(-\mu T)$ of the foreign currency and the exchange rate's conversion relationship. What the seller needs to pay at time 0 is $\exp(-\mu T)Z_0(1-\overline{K}/Z_T)^+$. The net revenue at time 0 possessed by the seller is

$$f_{sl}^c - \exp(-\mu T)Z_0(1-\overline{K}/Z_T)^+.$$

The contract price should ensure that the expected profits of both parties are equal, which implies that

$$-f_{sl}^c + E\left[\exp(-\lambda T)(Z_T-\overline{K})^+\right] = f_{sl}^c - E\left[\exp(-\mu T)Z_0(1-\overline{K}/Z_T)^+\right]. \tag{3.18}$$

Therefore, we also deduce the pricing equation of the reset call option with a single predetermined level.

Proposition 3.3. *Consider a reset call option with a single predetermined level whose maturity date is T, the strike price is K, and the predetermined level is K_1. This reset call option price with a single predetermined level for Model (3.1) can be derived as follows*

$$\begin{aligned} f_{sl}^c =& \frac{1}{2}E\left[\exp(-\lambda T)I_{sl}(Z_T-K_1)^+\right] \\ &+ \frac{1}{2}E\left[\exp(-\mu T)I_{sl}Z_0(1-K_1/Z_T)^+\right] \\ &+ \frac{1}{2}E\left[\exp(-\lambda T)(1-I_{sl})(Z_T-K)^+\right] \\ &+ \frac{1}{2}E\left[\exp(-\mu T)(1-I_{sl})Z_0(1-K/Z_T)^+\right]. \end{aligned}$$

Proof. It can be immediately derived from Equation (3.18). □

Theorem 3.3. *Consider a reset call option with a single predetermined level whose maturity date is T, the strike price is K, and the predetermined level is K_1. This reset call option price with a single predetermined level is controlled by the equality shown below*

$$\begin{aligned} f_{sl}^c =& \frac{1}{2}\int_0^1 \exp(-\lambda T)I_{sl}\left(Z_0\exp\left(\xi T+\frac{\sqrt{3}\sigma T}{\pi}\ln\frac{\alpha}{1-\alpha}\right)-K_1\right)^+ \mathrm{d}\alpha \\ &+ \frac{1}{2}\int_0^1 \exp(-\mu T)I_{sl}\left(Z_0-K_1/\exp\left(\xi T+\frac{\sqrt{3}\sigma T}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\right)^+ \mathrm{d}\alpha \\ &+ \frac{1}{2}\int_0^1 \exp(-\lambda T)(1-I_{sl})\left(Z_0\exp\left(\xi T+\frac{\sqrt{3}\sigma T}{\pi}\ln\frac{\alpha}{1-\alpha}\right)-K\right)^+ \mathrm{d}\alpha \\ &+ \frac{1}{2}\int_0^1 \exp(-\mu T)(1-I_{sl})\left(Z_0-K/\exp\left(\xi T+\frac{\sqrt{3}\sigma T}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\right)^+ \mathrm{d}\alpha. \end{aligned}$$

Proof. To begin with, we check that the uncertain variable shown below

$$\exp(-\lambda T)I_{sl}(Z_T-K_1)^+$$

has the inverse uncertainty distribution

$$\phi^{-1}(\alpha) = \exp(-\lambda T)I_{sl}(Z_T^\alpha-K_1)^+$$

in which

$$Z_T^\alpha = Z_0 \exp\left(\xi T + \frac{\sqrt{3}\sigma T}{\pi} \ln \frac{\alpha}{1-\alpha}\right).$$

Noting that

$$\begin{aligned}&\{\exp(-\lambda T)I_{sl}(Z_T - K_1)^+ \leq \exp(-\lambda T)I_{sl}(Z_T^\alpha - K_1)^+\}\\ &\supseteq \{Z_T \leq Z_T^\alpha\}\end{aligned}$$

and

$$\begin{aligned}&\{\exp(-\lambda T)I_{sl}(Z_T - K_1)^+ > \exp(-\lambda T)I_{sl}(Z_T^\alpha - K_1)^+\}\\ &\supseteq \{Z_T > Z_T^\alpha\},\end{aligned}$$

we have

$$\begin{aligned}&\mathcal{M}\{\exp(-\lambda T)I_{sl}(Z_T - K_1)^+ \leq \exp(-\lambda T)I_{sl}(Z_T^\alpha - K_1)^+\}\\ &\geq M\{Z_T \leq Z_T^\alpha\}\\ &= \alpha\end{aligned} \tag{3.19}$$

and

$$\begin{aligned}&\mathcal{M}\{\exp(-\lambda T)I_{sl}(Z_T - K_1)^+ > \exp(-\lambda T)I_{sl}(Z_T^\alpha - K_1)^+\}\\ &\geq M\{Z_T > Z_T^\alpha\}\\ &= 1-\alpha\end{aligned} \tag{3.20}$$

due to Theorem 2.1 and Theorem 2.3.

Moreover, from the duality axiom, it can be seen that

$$\begin{aligned}&\mathcal{M}\{\exp(-\lambda T)I_{sl}(Z_T - K_1)^+ \leq \exp(-\lambda T)I_{sl}(Z_T^\alpha - K_1)^+\} +\\ &\mathcal{M}\{\exp(-\lambda T)I_{sl}(Z_T - K_1)^+ > \exp(-\lambda T)I_{sl}(Z_T^\alpha - K_1)^+\} = 1.\end{aligned} \tag{3.21}$$

We can deduce

$$\mathcal{M}\{\exp(-\lambda T)I_{sl}(Z_T - K_1)^+ \leq \exp(-\lambda T)I_{sl}(Z_T^\alpha - K_1)^+\} = \alpha$$

from Equations (3.19)–(3.21).

Therefore, the uncertain variable

$$\exp(-\lambda T)I_{sl}(Z_T - K_1)^+$$

has the inverse uncertainty distribution which may be expressed in the following form:

$$\phi^{-1}(\alpha) = \exp(-\lambda T)I_{sl}(Z_T^\alpha - K_1)^+.$$

Therefore, it can be seen that

$$f_{sl}^{c1} = \int_0^1 \exp(-\lambda T) I_{sl}(Z_T^\alpha - K_1)^+ \mathrm{d}\alpha$$

from Theorem 2.2.

Next, we check that the uncertain variable shown below

$$\exp(-\mu T) I_{sl}(1 - K_1/Z_T)^+$$

has the inverse uncertainty distribution, which may be expressed in the following form:

$$\phi^{-1}(\alpha) = \exp(-\mu T) I_{sl}(1 - K_1/Z_T^\alpha)^+$$

in which

$$Z_T^\alpha = Z_0 \exp\left(\xi T + \frac{\sqrt{3}\sigma T}{\pi} \ln \frac{\alpha}{1-\alpha}\right).$$

Noting the fact that

$$\begin{aligned}&\{\exp(-\mu T) I_{sl}(1 - K_1/Z_T)^+ \le \exp(-\mu T) I_{sl}(1 - K_1/Z_T^\alpha)^+\}\\ &\supseteq \{Z_T \le Z_T^\alpha\}\end{aligned}$$

and

$$\begin{aligned}&\{\exp(-\mu T) I_{sl}(1 - K_1/Z_T)^+ > \exp(-\mu T) I_{sl}(1 - K_1/Z_T^\alpha)^+\}\\ &\supseteq \{Z_T > Z_T^\alpha\},\end{aligned}$$

we have

$$\begin{aligned}&\mathcal{M}\{\exp(-\mu T) I_{sl}(1 - K_1/Z_T)^+ \le \exp(-\mu T) I_{sl}(1 - K_1/Z_T^\alpha)^+\}\\ &\ge M\{Z_T \le Z_T^\alpha\}\\ &= \alpha\end{aligned} \tag{3.22}$$

and

$$\begin{aligned}&\mathcal{M}\{\exp(-\mu T) I_{sl}(1 - K_1/Z_T)^+ > \exp(-\mu T) I_{sl}(1 - K_1/Z_T^\alpha)^+\}\\ &\ge M\{Z_T > Z_T^\alpha\}\\ &= 1 - \alpha\end{aligned} \tag{3.23}$$

due to Theorem 2.1 and Theorem 2.3.

Moreover, from the duality axiom, it can be seen that

$$\begin{aligned}&\mathcal{M}\{\exp(-\mu T) I_{sl}(1 - K_1/Z_T)^+ \le \exp(-\mu T) I_{sl}(1 - K_1/Z_T^\alpha)^+\} +\\ &\mathcal{M}\{\exp(-\mu T) I_{sl}(1 - K_1/Z_T)^+ > \exp(-\mu T) I_{sl}(1 - K_1/Z_T^\alpha)^+\} = 1.\end{aligned} \tag{3.24}$$

We can deduce

$$\mathcal{M}\{\exp(-\mu T)I_{sl}(1-K_1/Z_T)^+ \le \exp(-\mu T)I_{sl}(1-K_1/Z_T^\alpha)^+\} = \alpha$$

from Equations (3.22)–(3.24).

Therefore, the uncertain variable shown below

$$\exp(-\mu T)I_{sl}(1-K_1/Z_T)^+$$

has the inverse uncertainty distribution, which may be expressed in the following form:

$$\phi^{-1}(\alpha) = \exp(-\mu T)I_{sl}(1-K_1/Z_T^\alpha)^+.$$

Therefore the equality

$$f_{sl}^{c2} = \int_0^1 \exp(-\mu T)I_{sl}(1-K_1/Z_T^\alpha)^+ \mathrm{d}\alpha$$

holds due to Theorem 2.2.

Similarly, we can prove that the uncertain variable shown below

$$\exp(-\lambda T)(1-I_{sl})(Z_T-K)^+$$

has the inverse uncertainty distribution, which may be expressed in the following form:

$$\phi^{-1}(\alpha) = \exp(-\lambda T)(1-I_{sl})(Z_T^\alpha-K)^+,$$

and the uncertain variable shown below

$$\exp(-\mu T)(1-I_{sl})(1-K/Z_T)^+$$

has the inverse uncertainty distribution which may be expressed in the following form:

$$\phi^{-1}(\alpha) = \exp(-\mu T)(1-I_{sl})(1-K/Z_T^\alpha)^+.$$

Consequently, the reset call option pricing formula with a single predetermined level is

$$\begin{aligned}
f_{sl}^c &= \frac{1}{2}\int_0^1 \exp(-\lambda T)I_{sl}(Z_T^\alpha-K_1)^+ \mathrm{d}\alpha \\
&\quad + \frac{1}{2}\int_0^1 \exp(-\mu T)I_{sl}Z_0(1-K_1/Z_T^\alpha)^+ \mathrm{d}\alpha \\
&\quad + \frac{1}{2}\int_0^1 \exp(-\lambda T)(1-I_{sl})(Z_T^\alpha-K)^+ \mathrm{d}\alpha \\
&\quad + \frac{1}{2}\int_0^1 \exp(-\mu T)(1-I_{sl})Z_0(1-K/Z_T^\alpha)^+ \mathrm{d}\alpha \\
&= \frac{1}{2}\int_0^1 \exp(-\lambda T)I_{sl}\left(Z_0\exp\left(\xi T+\frac{\sqrt{3}\sigma T}{\pi}\ln\frac{\alpha}{1-\alpha}\right)-K_1\right)^+ \mathrm{d}\alpha \\
&\quad + \frac{1}{2}\int_0^1 \exp(-\mu T)I_{sl}\left(Z_0-K_1/\exp\left(\xi T+\frac{\sqrt{3}\sigma T}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\right)^+ \mathrm{d}\alpha \\
&\quad + \frac{1}{2}\int_0^1 \exp(-\lambda T)(1-I_{sl})\left(Z_0\exp\left(\xi T+\frac{\sqrt{3}\sigma T}{\pi}\ln\frac{\alpha}{1-\alpha}\right)-K\right)^+ \mathrm{d}\alpha \\
&\quad + \frac{1}{2}\int_0^1 \exp(-\mu T)(1-I_{sl})\left(Z_0-K/\exp\left(\xi T+\frac{\sqrt{3}\sigma T}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\right)^+ \mathrm{d}\alpha.
\end{aligned}$$

□

Example 3.3. The parameters of Model (3.1) are set as $\lambda = 0.03$, $\mu = 0.025$, $\xi = 0.04$, and $\sigma = 0.03$. Suppose that $Z_0 = 5$, $K = 6$, $T = 10$, and $K_1 = 4$. Then $f_{sl}^c = 2.264$ according to Theorem 3.3.

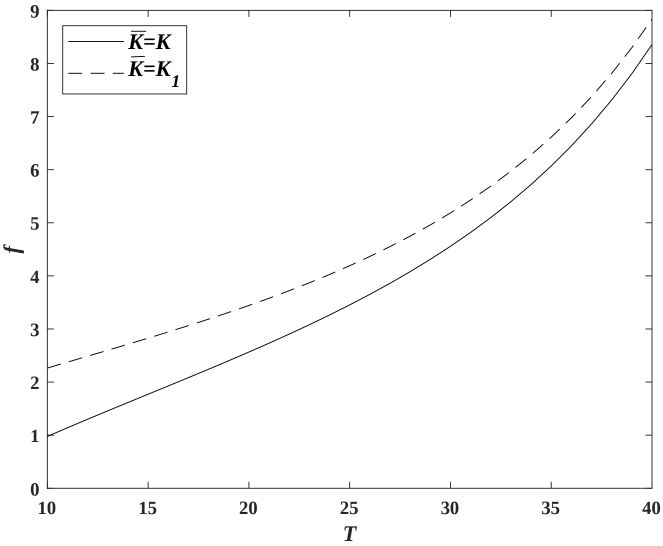

Figure 3. Price comparison of European options and reset options.

In Figure 3, when the other parameters remain unchanged, we observe that f_{sl}^c is monotonously increasing with respect to T. Moreover, we find that the price of reset options is significantly higher than that of European options, because it has more profit opportunities, which conforms to reality.

3.2.2. Reset put options with a single predetermined level

For another thing, we study the reset put option with a single predetermined level whose strike price is K, the maturity date is T, and the predetermined level is K_1; i.e., when the market price of the underlying asset falls below the total of the strike price and the option premium, the option buyer can either transfer the put option to realize a profit or sell the underlying asset at the predetermined price and quantity. In an uncertain financial market, the currency model for the reset put option with a single predetermined level is defined by Model (3.1).

For reset put options, the holder benefits from a higher strike price. Therefore, if $\max_{0<t<T} Z_t$ is higher than K_1, then it will be reset to K_1; otherwise, the initial strike price K will be maintained. Let $\overline{K}$ be the final strike price, which is

$$\overline{K} = \begin{cases} K_1, & \text{if } K_1 < \max\limits_{0<t<T} Z_t \\ K, & \text{otherwise} \end{cases}.$$

Let f_{sl}^p represent the contract price in the domestic currency. According to the reset rule of the strike price $\overline{K}$, the present value of the revenue possessed by the buyer in the domestic currency, which is

$$(\overline{K} - Z_T)^+,$$

where $(\cdot)^+$ denotes the non-negative part of the expression, i.e., $(x)^+ = \max(x, 0)$.

On the basis of the reset rule of the strike price, we derive the present value of revenue and net revenue for both the buyer and the seller of the option under the uncertain currency model. The net revenue at time 0 possessed by the buyer is

$$-f_{sl}^p + \exp(-\lambda T)(\overline{K} - Z_T)^+,$$

where $\exp(-\lambda T)$ is the risk-free discount factor of the domestic currency, converting the expiration revenue to the present value at time 0. Similarly, the net revenue of the seller considers the discount factor $\exp(-\mu T)$ of the foreign currency and the exchange rate's conversion relationship. What the seller needs to pay at time 0 is $\exp(-\mu T)Z_0(\overline{K}/Z_T - 1)^+$. The net revenue at time 0 possessed by the seller is

$$f_{sl}^p - \exp(-\mu T)Z_0(\overline{K}/Z_T - 1)^+.$$

The price of this contract should ensure that the expected profits of both parties are equal, which implies that

$$-f_{sl}^p + E\left[\exp(-\lambda T)(\overline{K} - Z_T)^+\right] = f_{sl}^p - E\left[\exp(-\mu T)Z_0(\overline{K}/Z_T - 1)^+\right]. \tag{3.25}$$

For the convenience, we define an indicator function to characterize the reset condition of the strike price as follows

$$I_{sl} = \begin{cases} 1, & \text{if } K_1 < \max\limits_{0<t<T} Z_t \\ 0, & \text{otherwise} \end{cases}.$$

Therefore, we also deduce the pricing equation of the reset put option with a single predetermined level.

Proposition 3.4. *Assuming that there is a reset put option with a single predetermined level whose maturity date is T, the strike price is K, and the predetermined level is K_1. This reset put option price with a single predetermined level for Model (3.1) can be derived as follows:*

$$\begin{aligned} f_{sl}^p =& \frac{1}{2}E\left[\exp(-\lambda T)I_{sl}(K_1 - Z_T)^+\right] \\ &+ \frac{1}{2}E\left[\exp(-\mu T)I_{sl}Z_0(K_1/Z_T - 1)^+\right] \\ &+ \frac{1}{2}E\left[\exp(-\lambda T)(1 - I_{sl})(K - Z_T)^+\right] \\ &+ \frac{1}{2}E\left[\exp(-\mu T)(1 - I_{sl})Z_0(K/Z_T - 1)^+\right]. \end{aligned}$$

Proof. It can be immediately derived from Equation (3.25). □

Theorem 3.4. *Assuming that there is a reset put option with a single predetermined level whose maturity date is T, the strike price is K, and the predetermined level is K_1. This reset put option price with a single predetermined level is controlled by the equality shown below*

$$f_{sl}^p = \frac{1}{2}\int_0^1 \exp(-\lambda T)I_{sl}\left(K_1 - Z_0\exp\left(\xi T + \frac{\sqrt{3}\sigma T}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\right)^+ \mathrm{d}\alpha$$

$$+\frac{1}{2}\int_0^1 \exp(-\mu T)I_{sl}\left(K_1/\exp\left(\xi T+\frac{\sqrt{3}\sigma T}{\pi}\ln\frac{\alpha}{1-\alpha}\right)-Z_0\right)^+ \mathrm{d}\alpha$$
$$+\frac{1}{2}\int_0^1 \exp(-\lambda T)(1-I_{sl})\left(K-Z_0\exp\left(\xi T+\frac{\sqrt{3}\sigma T}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\right)^+ \mathrm{d}\alpha$$
$$+\frac{1}{2}\int_0^1 \exp(-\mu T)(1-I_{sl})\left(K/\exp\left(\xi T+\frac{\sqrt{3}\sigma T}{\pi}\ln\frac{\alpha}{1-\alpha}\right)-Z_0\right)^+ \mathrm{d}\alpha.$$

Proof. To begin with, we check that the uncertain variable shown below

$$\exp(-\lambda T)I_{sl}(K_1-Z_T)^+$$

has the inverse uncertainty distribution

$$\phi^{-1}(\alpha)=\exp(-\lambda T)I_{sl}(K_1-Z_T^{1-\alpha})^+$$

in which

$$Z_T^{1-\alpha}=Z_0\exp\left(\xi T+\frac{\sqrt{3}\sigma T}{\pi}\ln\frac{1-\alpha}{\alpha}\right).$$

Noting the fact that

$$\begin{aligned}&\left\{\exp(-\lambda T)I_{sl}(K_1-Z_T)^+\le \exp(-\lambda T)I_{sl}(K_1-Z_T^{1-\alpha})^+\right\}\\ &\supseteq\left\{Z_T\ge Z_T^{1-\alpha}\right\}\end{aligned}$$

and

$$\begin{aligned}&\left\{\exp(-\lambda T)I_{sl}(K_1-Z_T)^+> \exp(-\lambda T)I_{sl}(K_1-Z_T^{1-\alpha})^+\right\}\\ &\supseteq\left\{Z_T< Z_T^{1-\alpha}\right\},\end{aligned}$$

it can be seen that

$$\begin{aligned}&\mathcal{M}\left\{\exp(-\lambda T)I_{sl}(K_1-Z_T)^+\le \exp(-\lambda T)I_{sl}(K_1-Z_T^{1-\alpha})^+\right\}\\ &\ge M\left\{Z_T\ge Z_T^{1-\alpha}\right\}\\ &=\alpha\end{aligned}\tag{3.26}$$

and

$$\begin{aligned}&\mathcal{M}\left\{\exp(-\lambda T)I_{sl}(K_1-Z_T)^+> \exp(-\lambda T)I_{sl}(K_1-Z_T^{1-\alpha})^+\right\}\\ &\ge M\left\{Z_T< Z_T^{1-\alpha}\right\}\\ &=1-\alpha\end{aligned}\tag{3.27}$$

due to Theorem 2.1 and Theorem 2.3.

Moreover, from the duality axiom, it can be seen that

$$\begin{aligned}&\mathcal{M}\left\{\exp(-\lambda T)I_{sl}(K_1 - Z_T)^+ \le \exp(-\lambda T)I_{sl}(K_1 - Z_T^{1-\alpha})^+\right\} + \\ &\mathcal{M}\left\{\exp(-\lambda T)I_{sl}(K_1 - Z_T)^+ > \exp(-\lambda T)I_{sl}(K_1 - Z_T^{1-\alpha})^+\right\} = 1.\end{aligned} \tag{3.28}$$

It follows from Equations (3.26)–(3.28) that

$$\mathcal{M}\left\{\exp(-\lambda T)I_{sl}(K_1 - Z_T)^+ \le \exp(-\lambda T)I_{sl}(K_1 - Z_T^{1-\alpha})^+\right\} = \alpha.$$

Therefore, the uncertain variable

$$\exp(-\lambda T)I_{sl}(K_1 - Z_T)^+$$

has the inverse uncertainty distribution, which may be expressed in the following form:

$$\phi^{-1}(\alpha) = \exp(-\lambda T)I_{sl}(K_1 - Z_T^{1-\alpha})^+.$$

Therefore, it can be seen

$$f_{sl}^{p1} = \int_0^1 \exp(-\lambda T)I_{sl}(K_1 - Z_T^{1-\alpha})^+ \mathrm{d}\alpha$$

from Theorem 2.2.

Next, we check that the uncertain variable shown below

$$\exp(-\mu T)I_{sl}(K_1/Z_T - 1)^+$$

has the inverse uncertainty distribution, which may be expressed in the following form:

$$\phi^{-1}(\alpha) = \exp(-\mu T)I_{sl}(K_1/Z_T^{1-\alpha} - 1)^+$$

in which

$$Z_T^{1-\alpha} = Z_0 \exp\left(\xi T + \frac{\sqrt{3}\sigma T}{\pi} \ln \frac{1-\alpha}{\alpha}\right).$$

Noting that

$$\begin{aligned}&\left\{\exp(-\mu T)I_{sl}(K_1/Z_T - 1)^+ \le \exp(-\mu T)I_{sl}(K_1/Z_T^{1-\alpha} - 1)^+\right\} \\ &\supseteq \left\{Z_T \ge Z_T^{1-\alpha}\right\}\end{aligned}$$

and

$$\begin{aligned}&\left\{\exp(-\mu T)I_{sl}(K_1/Z_T - 1)^+ > \exp(-\mu T)I_{sl}(K_1/Z_T^{1-\alpha} - 1)^+\right\} \\ &\supseteq \left\{Z_T < Z_T^{1-\alpha}\right\},\end{aligned}$$

we have

$$\begin{aligned}&\mathcal{M}\left\{\exp(-\mu T)I_{sl}(K_1/Z_T-1)^+ \le \exp(-\mu T)I_{sl}(K_1/Z_T^{1-\alpha}-1)^+\right\}\\&\ge M\left\{Z_T \ge Z_T^{1-\alpha}\right\}\\&=\alpha\end{aligned} \tag{3.29}$$

and

$$\begin{aligned}&\mathcal{M}\left\{\exp(-\mu T)I_{sl}(K_1/Z_T-1)^+ > \exp(-\mu T)I_{sl}(K_1/Z_T^{1-\alpha}-1)^+\right\}\\&\ge M\left\{Z_T < Z_T^{1-\alpha}\right\}\\&=1-\alpha\end{aligned} \tag{3.30}$$

due to Theorem 2.1 and Theorem 2.3.

Moreover, from the duality axiom, it can be seen that

$$\begin{aligned}&\mathcal{M}\left\{\exp(-\mu T)I_{sl}(K_1/Z_T-1)^+ \le \exp(-\mu T)I_{sl}(K_1/Z_T^{1-\alpha}-1)^+\right\}+\\&\mathcal{M}\left\{\exp(-\mu T)I_{sl}(K_1/Z_T-1)^+ > \exp(-\mu T)I_{sl}(K_1/Z_T^{1-\alpha}-1)^+\right\}=1.\end{aligned} \tag{3.31}$$

We can deduce

$$\mathcal{M}\left\{\exp(-\mu T)I_{sl}(K_1/Z_T-1)^+ \le \exp(-\mu T)I_{sl}(K_1/Z_T^{1-\alpha}-1)^+\right\}=\alpha$$

from Equations (3.29)–(3.31).

Therefore, the uncertain variable

$$\exp(-\mu T)I_{sl}(K_1/Z_T-1)^+$$

has the inverse uncertainty distribution, which may be expressed in the following form:

$$\phi^{-1}(\alpha)=\exp(-\mu T)I_{sl}(K_1/Z_T^{1-\alpha}-1)^+.$$

Therefore, the equality

$$f_{sl}^{p2}=\int_0^1 \exp(-\mu T)I_{sl}(K_1/Z_T^{1-\alpha}-1)^+\mathrm{d}\alpha$$

holds due to Theorem 2.2.

Similarly, we can prove that the uncertain variable shown below

$$\exp(-\lambda T)(1-I_{sl})(K-Z_T)^+$$

has the inverse uncertainty distribution, which may be expressed in the following form:

$$\phi^{-1}(\alpha)=\exp(-\lambda T)(1-I_{sl})(K-Z_T^{1-\alpha})^+,$$

and the uncertain variable shown below

$$\exp(-\mu T)(1-I_{sl})(K/Z_T-1)^+$$

possesses the inverse uncertainty distribution, which may be expressed in the following form:

$$\phi^{-1}(\alpha) = \exp(-\mu T)(1 - I_{sl})(K/Z_T^{1-\alpha} - 1)^+.$$

Consequently, the reset put option pricing formula with a single predetermined level is

$$\begin{aligned}
f_{sl}^p = & \frac{1}{2}\int_0^1 \exp(-\lambda T)I_{sl}(K_1 - Z_T^{1-\alpha})^+ \mathrm{d}\alpha \\
& + \frac{1}{2}\int_0^1 \exp(-\mu T)I_{sl}Z_0(K_1/Z_T^{1-\alpha} - 1)^+ \mathrm{d}\alpha \\
& + \frac{1}{2}\int_0^1 \exp(-\lambda T)(1 - I_{sl})(K - Z_T^{1-\alpha})^+ \mathrm{d}\alpha \\
& + \frac{1}{2}\int_0^1 \exp(-\mu T)(1 - I_{sl})Z_0(K/Z_T^{1-\alpha} - 1)^+ \mathrm{d}\alpha \\
= & \frac{1}{2}\int_0^1 \exp(-\lambda T)I_{sl}\left(K_1 - Z_0\exp\left(\xi T + \frac{\sqrt{3}\sigma T}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\right)^+ \mathrm{d}\alpha \\
& + \frac{1}{2}\int_0^1 \exp(-\mu T)I_{sl}\left(K_1/\exp\left(\xi T + \frac{\sqrt{3}\sigma T}{\pi}\ln\frac{\alpha}{1-\alpha}\right) - Z_0\right)^+ \mathrm{d}\alpha \\
& + \frac{1}{2}\int_0^1 \exp(-\lambda T)(1 - I_{sl})\left(K - Z_0\exp\left(\xi T + \frac{\sqrt{3}\sigma T}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\right)^+ \mathrm{d}\alpha \\
& + \frac{1}{2}\int_0^1 \exp(-\mu T)(1 - I_{sl})\left(K/\exp\left(\xi T + \frac{\sqrt{3}\sigma T}{\pi}\ln\frac{\alpha}{1-\alpha}\right) - Z_0\right)^+ \mathrm{d}\alpha.
\end{aligned}$$

□

Example 3.4. The parameters of Model (3.1) are set as $\lambda = 0.03$, $\mu = 0.025$, $\xi = -0.04$, and $\sigma = 0.03$. Suppose that $Z_0 = 5$, $K = 4$, $T = 10$, and $K_1 = 6$. Then $f_{sl}^p = 2.624$ according to Theorem 3.4.

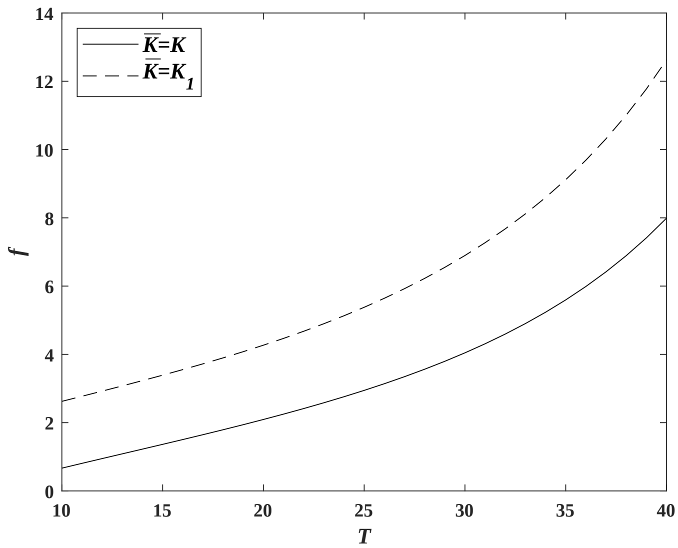

Figure 4. Price comparison of European options and reset options.

In Figure 4, when the other parameters remain unchanged, we observe that f_{sl}^{p} is monotonously increasing with respect to T. Moreover, we see that the price of the reset option is significantly higher than that of the European option because it has more profit opportunities, which conforms to reality.

Compared with reset options with a single predetermined level, reset options with n predetermined levels exhibit distinct unique properties and substantially higher computational complexity due to their multithreshold and multi-level reset mechanisms. The single predetermined level model only requires a binary judgment (reset triggered or not) to determine the final strike price, whereas the model with n predetermined levels relies on tiered inequalities $K_0 > K_1 > \ldots > K_n$ for the call options and $K_0 < K_1 < \ldots < K_n$ for the put options to conduct multi-interval threshold judgments. The corresponding predetermined level K_i is assigned according to the specific interval into which $\min_{0<t<T} Z_t$ (for call options) or $\max_{0<t<T} Z_t$ (for put options) falls. Such a hierarchical reset mechanism is more consistent with the complex design requirements of practical option products in real financial markets. It enables investors to benefit from more refined reset alternatives, rather than being restricted to the model with a single reset scheme offered by the single predetermined level. Furthermore, the pricing formula of the model with n predetermined levels is considerably more complex in terms of its integral solution than that of the model with a single predetermined level. The integral in the model with a single predetermined level only needs to be partitioned according to the binary condition of a single indicator function. By contrast, the integral in the model with n predetermined levels needs to be split according to a multi-interval condition. That is, the integration is performed separately over each interval corresponding to a predetermined level K_i and then summed. As n increases, the number of integral intervals rises accordingly, leading to an increase in the complexity of solving the pricing formula.

4. Real data analysis

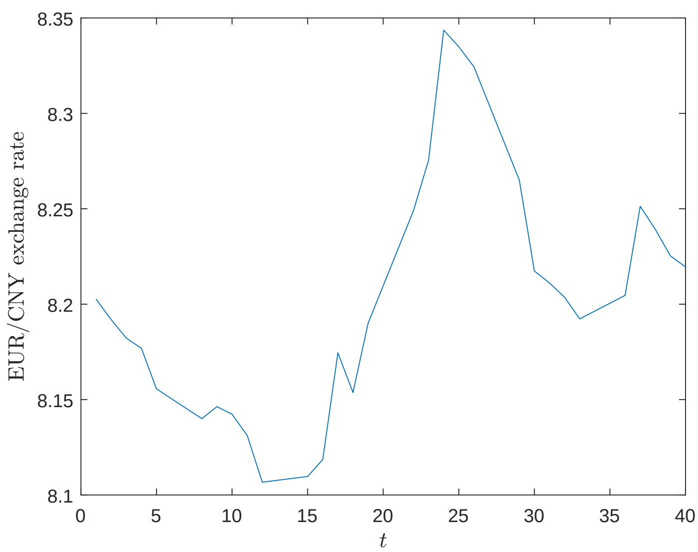

Figure 5. EUR-CNY exchange rate from January 5, 2026 to February 13, 2026.

In this section, we will validate our proposed pricing method through actual data analysis. We have considered the EUR-CNY exchange rate reported by China Monetary Network* from January 5, 2026 to February 13, 2026, which is shown in Figure 5, where $t = 1$ represents January 5, 2026; $t = 2$ represents January 6, 2026; and so on. Assume that Z_t indicates uncertain processes that represent the EUR-CNY exchange rate according to the uncertain differential equation

$$\mathrm{d}Z_t = \xi Z_t \mathrm{d}t + \sigma Z_t \mathrm{d}C_t, \tag{4.1}$$

where ξ and σ are unknown parameters that need to be estimated. First, we utilize the existing observational data to estimate these unknown parameters. For this purpose, we adopted the maximum likelihood estimation method introduced by Liu and Liu [41, 42]. Given the EUR-CNY exchange rates $z_{t_1}, z_{t_2}, \ldots, z_{t_{40}}$ observed at time points $t_1, t_2, \ldots, t_{40}$, we obtain the updated form of Equation (4.1) for each exponent i, where $2 \le i \le 40$

$$\begin{cases} \mathrm{d}Z_t = \xi Z_t \mathrm{d}t + \sigma Z_t \mathrm{d}C_t \\ Z_{t_{i-1}} = z_{t_{i-1}}, \end{cases}$$

and obtain the uncertainty distribution of Z_{t_i} as follows

$$\Phi_{t_i}(x) = \left(1 + \exp\left(\frac{\pi(\ln z_{t_{i-1}} + \xi(t_i - t_{i-1}) - \ln x)}{\sqrt{3}\sigma(t_i - t_{i-1})}\right)\right)^{-1}.$$

It follows from Liu and Liu [43] that the ith residual is

$$\varepsilon_i = \left(1 + \exp\left(\frac{\pi(\ln z_{t_{i-1}} + \xi(t_i - t_{i-1}) - \ln z_{t_i})}{\sqrt{3}\sigma(t_i - t_{i-1})}\right)\right)^{-1}.$$

By the maximum likelihood estimation method introduced by Liu and Liu [41, 42], we estimated the parameters as

$$\xi = 0.0023, \ \sigma = 0.0029.$$

Therefore, the uncertain EUR-CNY exchange rate model can be derived as follows:

$$\mathrm{d}Z_t = 0.0023 Z_t \mathrm{d}t + 0.0029 Z_t \mathrm{d}C_t. \tag{4.2}$$

Before pricing the reset options based on Model (4.2), we need to determine whether the fitted model adequately describes the corresponding observational data. To address this issue, the uncertainty hypothesis testing method was introduced by Ye and Liu [44]. We test whether the linear uncertainty distribution $\mathcal{L}(0, 1)$ conforms to these 29 residuals $\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_{40}$. As shown in Figure 6, given a confidence level $\alpha = 0.05$, the rejection region for the test is

$$W = \{(y_2, y_3, \cdots, y_{40}) : \text{there are at least 2 indices } i\text{'s with } 2 \le i \le 40 \text{ such that } y_i < 0.025 \text{ or } y_i > 0.975\}$$

given that $\lceil \alpha \times 29 \rceil = 2$. Since only $\varepsilon_{30} \notin [0.025, 0.975]$, we have

$$(\varepsilon_2, \varepsilon_3, \cdots, \varepsilon_{40}) \notin W.$$

*https://www.chinamoney.com.cn/chinese/bkccpr/

Therefore, Model (4.2) fits the EUR-CNY exchange rates $z_{t_1}, z_{t_2}, \ldots, z_{t_{40}}$ very well.

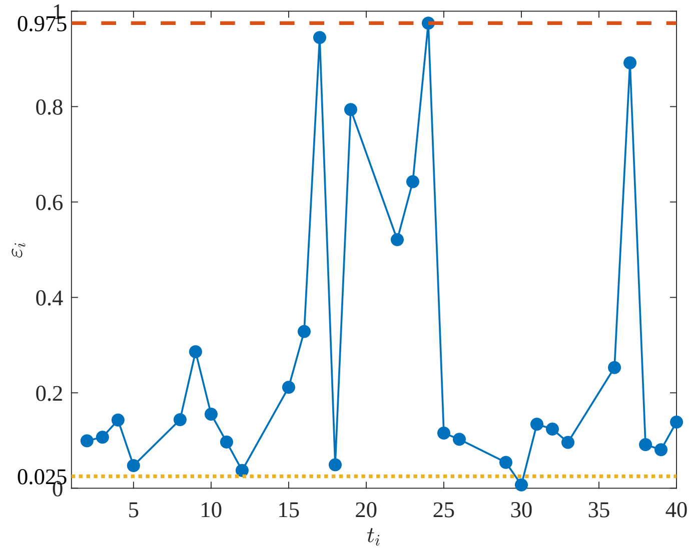

Figure 6. Residual plot of Model (4.2).

Assume that the domestic riskless interest rate is $\lambda = 0.013$, and the foreign riskless interest rate is $\mu = 0.020$. Next, we evaluate the fair value of the reset option with n predetermined levels on the basis of the estimated Model (4.2) with the initial value $Z_0 = 8.2195$.

Example 4.1. For the reset call option with n predetermined levels, let the strike prices be $K_0 = 8.5$, $K_1 = 8.3$, and $K_2 = 8.2$. Suppose that $T = 10$, and then $f_{nl}^c = 0.198$ according to Theorem 3.1. For the predetermined levels $i = 0, 1, 2$, keeping the other parameters constant, the relationship between the fair price and the maturity date T is shown in Figure 7.

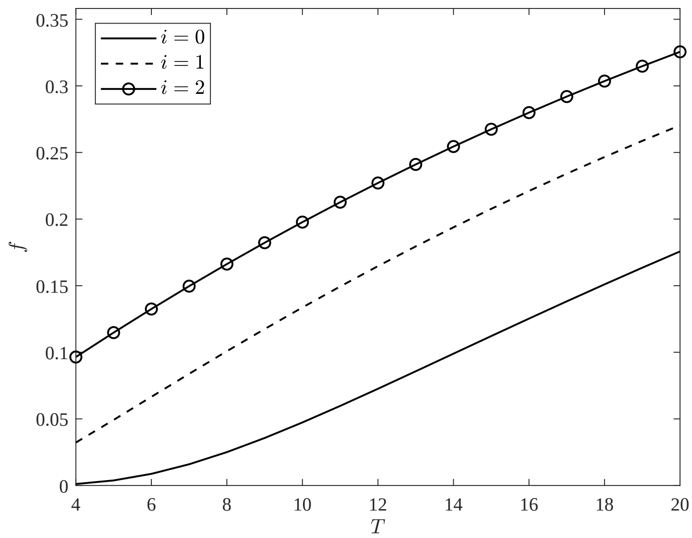

Figure 7. Price comparison of reset call options with different reset times.

In Figure 7, when the other parameters remain unchanged, we observe that f_{nl}^{c} is monotonously increasing with respect to T and i.

Example 4.2. For the reset put option with n predetermined levels, let the strike prices be $K_0 = 7.6$, $K_1 = 7.9$, and $K_2 = 8.1$. If we suppose that $T = 10$, then $f_{nl}^{c} = 0.010$ according to Theorem 3.2. For the predetermined levels $i = 0, 1, 2$, keeping the other parameters constant, the relationship between the fair price and the maturity date T is shown in Figure 8.

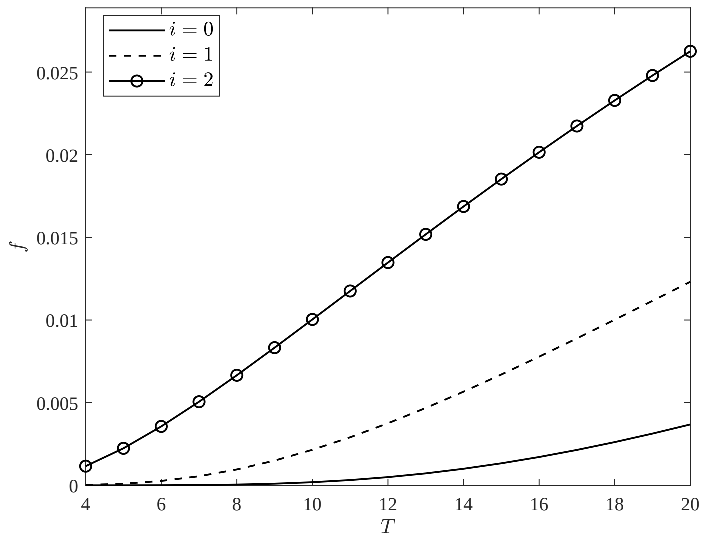

Figure 8. Price comparison of reset put options with different reset times.

In Figure 8, when the other parameters remain unchanged, we observe that f_{nl}^{p} is monotonously increasing with respect to T and i.

5. Conclusions

Building upon the analysis of reset options for currency models in an uncertain financial market, this paper proposes the pricing formulas for four types of reset option, namely, the reset call option with a single predetermined level, the reset put option with a single predetermined level, the reset call option with n predetermined levels and the reset put option with n predetermined levels. Finally, the rationality and feasibility of the formulas are verified by examples and comparisons. Numerical results indicate that the price difference between reset options and European options is significant, and the price of reset options monotonically increases with the number of preset levels and the extension of the maturity date. Finally, we also verified the this conclusion through real data.

Existing reset option pricing methods (e.g., [10, 17]) are based on probability theory, which requires sufficient historical data to fit the asset price distribution. However, in cross-border financial markets, the data may be insufficient or unavailable (e.g., under emergencies or policy changes), leading to invalidation of the probability models. This paper adopts uncertainty theory, which quantifies degrees of belief through expert evaluations, making it applicable to data-scarce scenarios. Meanwhile, although uncertainty theory has been applied to option pricing problems, studies under uncertainty theory (e.g., [22, 27, 31]) mainly focus on traditional options (European, American and barrier options), and no research has been conducted on reset options with predetermined levels. This paper extends the application boundary of uncertainty theory to reset options, enriching the research category of uncertain finance.

This is the first time that uncertainty theory has been combined with reset option pricing to construct a unified pricing framework for reset options with predetermined levels under the uncertain currency model. This framework breaks through the limitations of traditional probability theory in data-scarce scenarios and provides a new theoretical tool for cross-border derivative pricing. We derive the pricing formulas of reset options and extract the general model for n predetermined levels. The general model can be simplified to the single-level model by setting $n = 1$, showing strong generality.

This paper constructs the pricing model of reset options under the uncertain currency model. However, this paper has several limitations that need to be addressed in future research. The model assumes that the domestic and foreign risk-free interest rates λ and μ are constant, ignoring interest rate volatility. In practice, interest rate fluctuations affect the exchange rate and further impact option prices, especially in cross-border markets with frequent monetary policy adjustments. Furthermore, the uncertain currency model only considers continuous fluctuations driven by the Liu process, without incorporating jump terms. However, the exchange rate often experiences sudden jumps due to policy changes, geopolitical events, or economic crises, which may lead to deviations in the pricing results.

Given these limitations, future research can be carried out on the following aspects. A study could introduce an uncertain currency model with floating interest rates, setting λ and μ as uncertain processes to study the impact of interest rate volatility on pricing. Meanwhile, we could construct an uncertain currency model with jump terms to depict discontinuous exchange rate fluctuations and derive the corresponding reset option pricing formula. Moreover, with the groundwork laid by this paper, future research can further investigate reset options with a single predetermined date as well as reset options with n predetermined dates. In addition, the corresponding pricing formulas can be proposed.

Author contributions

Rong Gao: Conceptualization, Methodology, Writing-original draft, Supervision, Funding acquisition, Writing-review and editing. **Deguo Yang**: Formal analysis, Writing-review and editing. **Kaixiang Liu**: Writing-original draft.

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Conflict of interest

The authors declare that they have no conflict of interest.

Use of Generative-AI tools declaration

The authors declare that they did not utilize any artificial intelligence (AI) tools in the creation of this article.

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