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*Research article*

# Convex and sphere packing approaches to portfolio optimization

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**Abstract:** Classical mean–variance optimization yields exact efficient-frontier allocations for prescribed return or risk levels; however, such extremal solutions may be fragile under estimation error and mandate perturbations. This paper studies portfolio construction within the quadratic mean–variance framework from a feasibility and structural stability perspective by restricting attention to $\varepsilon$–approximate admissible allocations under return, variance, and relaxed budget constraints. Benchmark consistent feasibility restoration is formulated as a convex quadratic programming problem that computes the allocation closest to a reference portfolio while preserving the quadratic structure and guaranteeing existence and uniqueness of the admissible solution. To quantify interior stability, we introduce a sphere–packing–based geometric construction that inscribes the largest Euclidean ball within the linear return–budget region, yielding the Chebyshev center and an explicit robustness radius equal to the distance to the constraint boundary. The quadratic variance constraint is subsequently incorporated through analytic directional feasibility adjustment, ensuring membership in the intersection of linear and variance admissible sets without extremizing expected return or variance. Unlike ambiguity based or distributionally robust formulations, robustness is defined intrinsically through geometric interior maximization within the prescribed feasible region. Computational experiments calibrated to the investment mandate of the Sovereign Wealth Fund of Mongolia illustrate stable feasibility restoration and robustness under parameter perturbations, establishing sphere–packing–based interior maximization as a complementary geometric robustness principle within the classical mean–variance framework.

**Keywords:** sphere packing; convex optimization; robust portfolio optimization; mean–variance model
**Mathematics Subject Classification:** 90C25, 90C20, 52C17, 91G10, 91G80

## 1. Introduction

The mean–variance model of Markowitz [1], later formalized within modern investment science frameworks [2], forms the foundation of modern portfolio theory. In its classical formulation, portfolio selection reduces to a quadratic optimization problem whose solutions trace the efficient frontier of attainable risk–return combinations. While analytically elegant, practical implementation depends critically on accurate estimation of expected returns and covariance matrices. Empirical evidence shows that estimation errors, structural breaks, and regime shifts may generate unstable and highly concentrated portfolio weights, leading to poor out-of-sample performance [3, 4]. As a consequence, extremal efficient frontier allocations are often less desirable in institutional practice than portfolios that emphasize feasibility and structural stability under uncertainty.

Institutional asset management is typically governed by explicit mandates specifying admissible return targets, risk limits, and allocation constraints. Sovereign wealth funds, pension funds, and reserve managers frequently operate under minimum return requirements, maximum variance thresholds, and regulatory exposure bounds. Moreover, full capital deployment is not universally imposed; partial investment is often maintained to accommodate liquidity buffers and operational reserves. In practice, such mandate constraints are rarely enforced in a purely exact sense. Return and variance thresholds are typically implemented with tolerance bands reflecting estimation error, reporting conventions, and operational flexibility. This motivates an $\varepsilon$–approximate admissibility perspective, in which allocations are required to satisfy return, variance, and relaxed budget constraints up to a prescribed tolerance level $\varepsilon > 0$.

Within such mandate driven environments, portfolio construction commonly begins with a benchmark or strategic reference allocation. When return targets or risk limits are revised, the benchmark portfolio may fail to satisfy updated constraints. The practical problem therefore shifts from computing a new extremal efficient allocation to restoring feasibility possibly within an $\varepsilon$–relaxed admissible region while minimizing deviation from the reference portfolio and preserving structural stability within prescribed limits.

A related consideration concerns interior stability. Among admissible allocations, portfolios located near constraint boundaries may be highly sensitive to small perturbations in parameters or weights. Institutional investors often prefer allocations that remain sufficiently interior to the feasible region so as to tolerate moderate fluctuations without immediate violation of mandate limits. From the $\varepsilon$–admissibility viewpoint, this corresponds to maintaining a positive stability margin within the perturbed feasible set. These observations motivate two constructive problems within the classical mean–variance framework under $\varepsilon$–approximate admissibility:

1. benchmark–consistent feasibility restoration under return, variance (up to tolerance $\varepsilon$), and relaxed budget constraints;
2. identification of geometrically interior allocations within the linear–admissible region, followed by variance–consistent directional adjustment to ensure membership in the intersection of linear and variance–admissible sets within the prescribed tolerance.

Robustness in portfolio optimization has been extensively studied. Foundational contributions by Ben–Tal and Nemirovski [5] and El Ghaoui *et al.* [6] introduced uncertainty sets and conic reformulations to control worst-case risk. Bertsimas and Sim [7] proposed tractable uncertainty budget models, while Goldfarb and Iyengar [8] developed parametric robust portfolio formulations. Subsequent research incorporated ambiguity in return distributions and two-stage robust principles [9, 10], including $\alpha$-robust portfolio optimization under distributional uncertainty [11]. Broader treatments of robustness in active portfolio management appear in [12, 13].

Distributionally robust optimization further extended this literature through optimal transport and Wasserstein ambiguity sets [14, 15]. Multi-period and dynamic extensions of mean–variance models have been developed within stochastic control frameworks [16–19], while continuous time formulations under macroeconomic risks such as inflation have been analyzed in quadratic security market models [20]. Robust utility maximization under drift uncertainty has also been studied in continuous time settings [21]. Alternative robustness concepts include lower partial moment formulations [22], coherent risk measures such as conditional value-at-risk [23], and general robust convex programming frameworks [24].

In parallel, the financial machine learning literature has highlighted the instability of mean–variance allocations under estimation noise. López de Prado [25] demonstrates that small perturbations in return and covariance estimates may produce highly concentrated and unstable portfolios, motivating resampling, regularization, and cross validation techniques. These approaches primarily address statistical uncertainty in parameter estimation. By contrast, the present study focuses on the intrinsic geometric structure of the feasible allocation region itself. Rather than modifying estimation procedures or enlarging uncertainty sets, we characterize allocation stability through interior maximization within the prescribed $\varepsilon$–admissible constraint set. This geometric robustness perspective is orthogonal to, and potentially compatible with, machine learning–based return or covariance forecasting methods.

Although existing robust frameworks modify objective functions or enlarge uncertainty sets, comparatively limited attention has been devoted to the explicit geometry of the feasible allocation region within the classical mean–variance model under controlled feasibility relaxation. In particular, the constructive problem of generating portfolios that simultaneously satisfy prescribed return, variance, and relaxed budget constraints up to a specified tolerance $\varepsilon$—without extremizing expected return or variance, and that remain structurally interior to the admissible region, has not been systematically developed from a geometric design perspective. To the best of our knowledge, the explicit integration of Chebyshev center interior maximization with variance–consistent directional feasibility adjustment under relaxed budget constraints and $\varepsilon$–approximate admissibility has not been comprehensively examined within the classical mean–variance framework.

Motivated by this gap, we introduce two complementary portfolio design constructions.

**(i) Convex programming–based $\varepsilon$–feasibility restoration.** We formulate benchmark–consistent feasibility adjustment as a convex quadratic programming problem that computes the allocation closest to a given reference portfolio subject to return, variance (up to tolerance $\varepsilon$), and relaxed budget constraints. The formulation preserves the quadratic structure of the Markowitz model and guarantees existence and uniqueness of the admissible projection.

**(ii) Sphere packing–based geometric $\varepsilon$–interior maximization.** Instead of selecting an extremal efficient-frontier portfolio, this construction treats return, variance, and budget levels as feasibility thresholds and seeks an allocation that is maximally interior relative to the $\varepsilon$–admissible linear region.

In Stage 1, the largest Euclidean ball is inscribed in the linear region defined by return and budget constraints, yielding the Chebyshev center and an associated robustness radius. This radius provides a quantitative $\varepsilon$–stability margin equal to the minimal distance to the constraint boundary. In Stage 2, the quadratic variance constraint is incorporated through analytic directional feasibility adjustment, ensuring that the final allocation lies in the intersection of linear and variance admissible sets within the prescribed tolerance. This construction aligns with convex maximization interpretations of sphere packing problems [26] and provides an explicit geometric measure of interior stability.

The contribution of this study is threefold: (i) formalizing benchmark consistent $\varepsilon$–feasibility restoration within the classical mean–variance framework via convex quadratic programming; (ii) introducing sphere packing–based interior maximization to define an explicit geometric robustness radius as an $\varepsilon$–stability margin; and (iii) incorporating relaxed budget constraints reflecting institutional capital allocation practice.

The practical relevance of the proposed framework is illustrated through computational experiments conducted under the investment mandate of the Sovereign Wealth Fund of Mongolia. Using a representative multi-asset universe, we implement the convex programming and sphere packing constructions under prescribed return, variance (with tolerance $\varepsilon$), and relaxed budget constraints, emphasizing feasibility preservation and geometric interior stability rather than extremal risk–return optimization.

The remainder of the paper is organized as follows. Section 2 reviews classical Markowitz formulations. Section 3 presents the convex programming and sphere packing constructions. Section 4 reports computational experiments, and Section 5 concludes.

## 2. Markowitz optimization formulation

The classical mean–variance portfolio theory of Markowitz [1] models portfolio selection as a quadratic optimization problem balancing expected return and risk. Let $\mu = (r_1, r_2, \ldots, r_n)$ denote the vector of expected asset returns, let $C = (c_{ij})$ be the positive definite covariance matrix of returns, and let $x = (x_1, x_2, \ldots, x_n)$ denote the vector of portfolio weights satisfying $\sum_{i=1}^{n} x_i = 1$ and $x_i \geq 0$. The expected return and variance of a portfolio are given by

$$R(x) = \langle \mu, x \rangle, \qquad \sigma^2(x) = \langle Cx, x \rangle, \qquad C \succ 0.$$

Depending on the investor's objective, the mean–variance model admits several equivalent formulations.

**Problem M1 (Minimum-variance portfolio; [1]).**

Determine the portfolio with minimum variance under full investment. This problem identifies the lower bound of the efficient frontier.

$$\begin{aligned} \text{minimize} \quad & \langle Cx, x \rangle \\ \text{subject to} \quad & \sum_{i=1}^{n} x_i = 1. \end{aligned} \tag{2.1}$$

**Problem M2 (Minimum variance with target return; [1]).**

Minimize portfolio variance subject to achieving a required return level $r_0$.

$$\begin{aligned} \text{minimize} \quad & \frac{1}{2}\langle Cx, x\rangle, \\ \text{subject to} \quad & \langle \mu, x\rangle \geq r_0, \\ & \sum_{i=1}^{n} x_i = 1. \end{aligned} \tag{2.2}$$

**Problem M3 (Mean–variance utility maximization; [2]).**
Maximize a quadratic utility function balancing expected return and variance.

$$\begin{aligned} \text{maximize} \quad & \langle \mu, x\rangle - \frac{1}{2}\langle Cx, x\rangle, \\ \text{subject to} \quad & \sum_{i=1}^{n} x_i = 1. \end{aligned} \tag{2.3}$$

**Problem M4 (Return maximization under risk constraint; [1]).**
Maximize expected return subject to an upper bound on allowable variance.

$$\begin{aligned} \text{maximize} \quad & \langle \mu, x\rangle, \\ \text{subject to} \quad & \langle Cx, x\rangle \leq \sigma_0^2, \\ & \sum_{i=1}^{n} x_i = 1. \end{aligned} \tag{2.4}$$

These formulations characterize exact efficient frontier portfolios under the classical full investment constraint $\sum_{j=1}^{n} x_j = 1$. In practice, however, institutional investors frequently operate under mandate constraints and benchmark allocations that require feasibility restoration rather than complete re-optimization of an extremal objective. Moreover, stability considerations may favor allocations that remain sufficiently interior to the admissible region so as to tolerate moderate perturbations without immediate violation of mandate limits.

Section 3 adopts the relaxed budget constraint $\langle \mathbf{1}, x\rangle \leq 1$ to allow partial investment (liquidity buffers), while Section 2 reviews the classical fully invested frontier $\langle \mathbf{1}, x\rangle = 1$. The next section introduces two complementary constructions: convex programming–based feasibility restoration and sphere packing–based geometric interior maximization that address these practical considerations while preserving the quadratic structure of the mean–variance framework.

## 3. Convex and sphere packing approaches to portfolio optimization

We consider $\varepsilon$–approximate formulations that permit controlled tolerances in the constraints and reinterpret the portfolio selection problem from two complementary perspectives: convex programming and sphere packing approach [27].
Let $\varepsilon > 0$ denote a small tolerance parameter. The approximate feasible region can be written as

$$D = \{x \in \mathbb{R}^n : \langle Cx, x\rangle \leq \sigma^2 + \varepsilon,\ \langle \mu, x\rangle \geq \mu_0,\ \sum_{j=1}^{n} x_j \leq 1\}. \tag{3.1}$$

Equivalently, one may find a feasible $x$ satisfying

$$\begin{aligned} &\langle Cx, x\rangle \leq \sigma^2 + \varepsilon, \\ &\langle \mu, x\rangle \geq \mu_0, \\ &\sum_{j=1}^{n} x_j \leq 1, \end{aligned} \tag{3.2}$$

where $C \geq 0$ denotes the covariance matrix of asset returns in the quadratic variance constraint of Markowitz [1]. The inequality budget constraint allows partial investment, reflecting the practical possibility of holding liquidity buffers rather than imposing full investment at all times. The quantity $\sigma^2$ is defined as

$$\sigma^2 = \min\left\{\langle Cx, x\rangle : \sum_{j=1}^{n} x_j = 1,\ x \geq 0\right\},$$

i.e., the minimum achievable portfolio variance under the classical full investment constraint. The minimization defining $\sigma^2$ is performed under the classical full investment constraint $\sum_{j=1}^{n} x_j = 1$. Thus, $\sigma^2$ represents the theoretical minimum-variance benchmark associated with the fully invested portfolio. The tolerance parameter $\varepsilon > 0$ allows controlled deviation from this lower bound while preserving consistency with the classical mean–variance framework.

### *3.1. Convex programming formulation*

The reference portfolio $y$ is assumed to be given exogenously, for example as a benchmark or mandate derived allocation that may become infeasible under updated risk or return constraints. We view the convex formulation as projecting a reference portfolio $y \in \mathbb{R}^n$ onto the feasible region:

$$\begin{aligned} \text{minimize} \quad & u(x) = \|x - y\|^2 \\ \text{subject to} \quad & \langle Cx, x\rangle \leq \sigma^2 + \varepsilon, \\ & \langle \mu, x\rangle \geq \mu_0, \\ & \langle \mathbf{1}, x\rangle \leq 1, \\ & x \geq 0. \end{aligned} \tag{3.3}$$

It is clear that problem (3.3) is a convex quadratic program. The objective is strictly convex, so the minimizer is unique whenever the feasible region is non–empty.

### *3.2. Sphere packing approach*

Following the sphere packing approach developed in [26], we analyze the robustness of a feasible region by inscribing the largest possible Euclidean ball within it. Let $B(x_0, r)$ denote a Euclidean ball with center $x_0 \in \mathbb{R}^n$ and radius $r \geq 0$:

$$B(x_0, r) = \{\, x \in \mathbb{R}^n : \|x - x_0\| \leq r \,\}, \tag{3.4}$$

where $\|\cdot\|$ is the Euclidean norm. The $n$-dimensional volume of $B(x_0, r)$ is

$$V(B(x_0, r)) = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)} r^n, \tag{3.5}$$

where $\Gamma(\cdot)$ denotes Euler's gamma function.
Consider the polyhedral set

$$\overline{D} = \{x \in \mathbb{R}^n : \langle a_i, x\rangle \le b_i,\ i = 1, \dots, m\}, \tag{3.6}$$

where $a_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$, $i = 1, \dots, m$. For a point $x \in \mathbb{R}^n$ and radius $r \ge 0$, define the closed ball

$$B(x, r) = \{z \in \mathbb{R}^n : \|z - x\| \le r\}. \tag{3.7}$$

The inclusion condition $B(x, r) \subset \overline{D}$ can be written in the following equivalent form (see, e.g., Enkhbat [26]):

$$B(x, r) \subset \overline{D} \iff \langle a_i, x\rangle + r\|a_i\| \le b_i,\ i = 1, \dots, m. \tag{3.8}$$

**Geometric interpretation of the robustness radius.** Let $\overline{D} \subset \mathbb{R}^n$ be the nonempty closed convex polyhedral set defined in (3.6). The optimal radius $r^*$ obtained from the linear program associated with (3.8) defines the largest Euclidean ball $B(x^*, r^*) \subset \overline{D}$. Consequently,

$$r^* = \operatorname{dist}(x^*, \partial\overline{D}),$$

where $\partial\overline{D}$ denotes the boundary of $\overline{D}$.

Thus, for any perturbation $\Delta$ satisfying $\|\Delta\| \le r^*$, the perturbed allocation $x^* + \Delta$ remains feasible. The quantity $r^*$ therefore provides a geometric measure of allocation robustness with respect to portfolio weight perturbations under the Euclidean norm.

Thus, the one sphere packing problem in the polyhedral set $D$ reduces to linear programming:

$$\begin{aligned} \text{maximize} \quad & r \\ \text{subject to} \quad & \langle a_i, x\rangle + r\|a_i\| \le b_i, \quad i = 1, \dots, m, \\ & r \ge 0. \end{aligned} \tag{3.9}$$

The optimal solution $(x^*, r^*)$ gives the center $x^*$ and radius $r^*$ of the largest ball that can be inscribed in the polyhedral set $D$.
Now we apply sphere packing approach to portfolio optimization.
Consider the system

$$\begin{aligned} & \langle Cx, x\rangle \le \sigma^2 + \varepsilon, \\ & \langle \mu, x\rangle \ge \mu_0, \\ & \sum_{j=1}^{n} x_j \le 1, \quad x \ge 0. \end{aligned} \tag{3.10}$$

In practice, rather than solving the extremal formulation of Problem (2.2), we seek a portfolio that satisfies prescribed return, variance, and relaxed budget thresholds without minimizing variance or maximizing expected return. To construct such a feasible allocation, we decompose the admissible region into two convex sets.

For this purpose, define two convex sets:

$$D_1 = \Big\{x : \langle \mu, x\rangle \geq \mu_0,\ \sum_{j=1}^{n} x_j \leq 1,\ 0 \leq x_j \leq 1\Big\}, \qquad D_2 = \Big\{x : \langle Cx, x\rangle \leq \sigma^2 + \varepsilon\Big\}.$$

To find a feasible solution $x^0 \in D_1 \cap D_2$, we first determine a feasible point in $D_1$ by solving the following linear program:

$$\begin{aligned} \text{maximize} \quad & r \\ \text{subject to} \quad & \langle \mu, x\rangle - r\|\mu\| \geq \mu_0, \\ & \sum_{j=1}^{n} x_j + r\sqrt{n} \leq 1, \\ & x_j + r \leq 1, \quad j = 1, \ldots, n, \\ & x_j \geq 0, \quad r \geq 0. \end{aligned} \tag{3.11}$$

The optimal pair $(x^0, r^*)$ provides the center $x^0$ and radius $r^*$ of the largest sphere contained in $D_1$. Since $D_1$ is a bounded polyhedral convex set, problem (3.11) computes its Chebyshev center, i.e., the unique point that maximizes the minimum Euclidean distance to all supporting hyperplanes defining $D_1$. Consequently, $r^*$ equals the maximal uniform perturbation radius such that every point in the ball $B(x^0, r^*)$ remains feasible with respect to the linear constraints in $D_1$.

Next, we determine $x^h$ such that $x^h \in D_2$. Construct points $x^h = x^0 + \alpha h$ such that $x^h \in D_2$ for some $h \in \mathbb{R}^n$.

Compute

$$\langle Cx^0, x^0\rangle + 2\alpha \langle Cx^0, h\rangle + \alpha^2 \langle Ch, h\rangle \leq \sigma^2 + \varepsilon.$$

Denote by

$$a = \langle Ch, h\rangle, \qquad b = 2\langle Cx^0, h\rangle, \qquad c = \langle Cx^0, x^0\rangle - \sigma^2 - \varepsilon.$$

Then, $\alpha$ must satisfy $a\alpha^2 + b\alpha + c \leq 0$, whose discriminant $\Delta = b^2 - 4ac > 0$ ensures the feasibility. The corresponding roots are

$$\alpha_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \qquad \alpha_1 < \alpha_2.$$

Hence, $\alpha$ is feasible whenever $\alpha_1 \leq \alpha \leq \alpha_2$, and for $x = x(\alpha)$, we have

$$x_i^0 + \alpha_1 h_i \leq x_i \leq x_i^0 + \alpha_2 h_i, \qquad i = 1, \ldots, n.$$

No post-hoc normalization or rescaling is applied; feasibility with respect to the quadratic constraint is ensured directly through the interval condition on $\alpha$. For a fixed direction $h$, any $\alpha \in [\alpha_1, \alpha_2]$ yields $x(\alpha) = x^0 + \alpha h$, satisfying the quadratic constraint defining $D_2$.

Feasible portfolios are therefore obtained by intersecting the directional interval $[\alpha_1, \alpha_2]$ (ensuring $D_2$-feasibility) with the additional requirement that $x(\alpha) \in D_1$. The overall feasible region of the sphere packing portfolio problem remains $D_1 \cap D_2$.

The optimal $r^*$ from (3.11) measures the maximal uniform perturbation radius with respect to the linear constraint set $D_1$, i.e., the largest Euclidean distance from $x^0$ to the supporting hyperplanes defining $D_1$. This robustness measure does not incorporate the quadratic variance constraint $D_2$, which is subsequently enforced through the directional construction. Accordingly, $x^0$ represents the Chebyshev center of $D_1$ and serves as a geometrically stable reference allocation relative to the linear admissible region.

In practice, the direction vector $h$ is not predetermined but is selected from admissible directions, for example through Monte Carlo sampling or structured search procedures within $D_1 \cap D_2$, to reflect desired patterns of portfolio adjustment while preserving feasibility.

## 4. Computational experiments

To illustrate the proposed convex programming and sphere packing framework, we consider a four asset universe aligned with the investment categories of the Sovereign Wealth Fund of Mongolia. The assets correspond to the mandate defined allocation segments:

- 0–5 Year U.S. Treasury: fixed-income portfolio,
- S&P 500 index: equity portfolio,
- 3-Month LIBOR: deposit portfolio,
- Federal Reserve overnight rate: liquidity portfolio.

Monthly return data covering January 2000 to September 2025 were used to construct the empirical inputs of the model. The mean return vector $\mu$ and covariance matrix $C$ were estimated from monthly return observations and subsequently annualized by multiplying the sample monthly estimates by 12. Tables 1 and 2 report the resulting statistical inputs employed in the portfolio analysis.

**Table 1.** Estimated expected returns for the four-asset portfolio.

| **Asset class** | **Expected return (%)** | **Vector component ($\mu_j$)** |
|---|---|---|
| 0–5 Year U.S. Treasury | 2.73 | 0.0273 |
| S&P 500 Index | 7.22 | 0.0722 |
| 3-Month LIBOR | 2.27 | 0.0227 |
| Federal Reserve Overnight Rate | 1.92 | 0.0192 |

**Table 2.** Estimated covariance matrix of annualized asset returns.

| **Covariance** | **0–5 Year UST** | **S&P 500** | **LIBOR 3M** | **Fed overnight** |
|---|---|---|---|---|
| 0–5 Year U.S. Treasury | 0.00030 | -0.00052 | 0.00003 | 0.00003 |
| S&P 500 | -0.00052 | 0.02303 | -0.00006 | -0.00004 |
| LIBOR 3M | 0.00003 | -0.00006 | 0.00003 | 0.00003 |
| Fed Overnight | 0.00003 | -0.00004 | 0.00003 | 0.00004 |

The government mandate specifies admissible asset categories but does not prescribe explicit numerical targets for expected return or volatility. Accordingly, the admissible portfolio region is defined through the feasibility thresholds introduced in Section 3:

- return requirement: $\langle \mu, x \rangle \geq u_0$,
- variance cap: $\langle Cx, x \rangle \leq \sigma_{\max}^2 + \varepsilon$,
- relaxed budget constraint: $\langle \mathbf{1}, x \rangle \leq 1$.

The feasibility parameters $(u_0, \sigma_{\max})$ are calibrated internally from the investable asset universe. Let $(R_{\max}, \sigma_{\max}^R)$ denote the expected return and standard deviation of the maximum-return portfolio, and let $(R_{\min}, \sigma_{\min})$ denote those of the minimum-variance portfolio. The feasibility thresholds are selected as interior midpoints between these two extremal portfolios. This calibration identifies a moderate region of the attainable risk–return set and is used solely to illustrate the feasibility–based construction methods, without extremizing either expected return or variance.

Under the baseline calibration,

$$u_0 = 0.0476, \qquad \sigma_{\max} = 0.0788, \qquad \varepsilon = 10^{-4}.$$

All portfolios reported below satisfy the corresponding feasibility conditions defined above.

### 4.1. *Convex approach to portfolio optimization*

To construct a locally stable, feasible portfolio, we first formulate a convex quadratic programming problem. Specifically, a given reference allocation is adjusted by minimizing its deviation subject to the admissible return, variance, and relaxed budget constraints introduced in Section 3.1. This leads to the strictly convex quadratic program (3.3), which guarantees existence and uniqueness of the solution whenever the feasible set is non–empty. The resulting allocation restores feasibility while remaining close to the benchmark and is positioned stably within the prescribed constraint region.

**Objective function and constraint setup.** The convex quadratic program minimizes the squared Euclidean distance $\|x - y\|^2$ from a reference portfolio $y$, selected as the maximum-return allocation. This formulation determines the minimal adjustment required to restore feasibility under the prescribed return and variance constraints. The target return $\mu_0$ is chosen as the midpoint between the returns of the maximum-return and minimum-variance portfolios, while the admissible risk level $\sigma$ is defined as the average of their corresponding standard deviations. These choices place the feasibility thresholds in an interior region between the two extremal benchmark portfolios, thereby avoiding boundary effects. Table 3 reports the optimal objective value together with the active constraints at the solution.

**Table 3.** Objective value and constraints of the convex programming portfolio.

| Component | Description | Value |
|---|---|---|
| Objective value | $\min \|x - y\|^2$ (squared distance) | 0.2868 |
| Return threshold | $u_0$ in $\langle \mu, x \rangle \geq u_0$ | 4.76% |
| Risk cap | $\sigma_{\max}$ in $\sqrt{\langle Cx, x \rangle} \leq \sigma_{\max}$ | 7.88% |
| Budget constraint (relaxed) | $\langle \mathbf{1}, x \rangle \leq 1$ | ≤ 100% |

**Optimization results.** The convex quadratic program was implemented in MATLAB to obtain the numerical solution. Tables 4 and 5 compares the resulting convex programming portfolio with several benchmark allocations, including the maximum-return portfolio, the minimum-variance portfolio, the equal-weight portfolio, and the classical maximum Sharpe (Markowitz) portfolio. The convex solution represents the unique Euclidean projection of the reference portfolio onto the convex feasible region defined by the calibrated return threshold, variance cap, and relaxed budget constraint. Since the objective function is strictly convex and the feasible region is convex (provided the covariance matrix is positive semi-definite), the projection solution exists and is unique whenever the feasible set is non–empty.

**Table 4.** Comparison of portfolio allocations across benchmark strategies.

| Asset / Portfolio | Max Return | Min Risk | Equal Weight | Markowitz | Convex (min $\|x - y\|^2$) |
|---|---|---|---|---|---|
| 0–5 Year U.S. Treasury | 0.00% | 1.36% | 25.00% | 20.23% | 17.27% |
| S&P 500 | 100.00% | 0.42% | 25.00% | 2.70% | 52.31% |
| LIBOR 3M | 0.00% | 98.22% | 25.00% | 77.06% | 13.18% |
| Fed Overnight | 0.00% | 0.00% | 25.00% | 0.00% | 11.08% |
| **Total weight** | 100% | 100% | 100% | 100% | 93.8% |

**Table 5.** Portfolio performance measures.

| Portfolio | Max return | Min risk | Equal weight | Markowitz | Convex min $\|x - y\|^2$ |
|---|---|---|---|---|---|
| Portfolio return (%) | 7.22 | 2.30 | 3.54 | 2.50 | 4.76 |
| Portfolio risk (%) | 15.18 | 0.58 | 3.74 | 0.72 | 7.88 |
| Sharpe ratio | 0.35 | 0.65 | 0.43 | 0.81 | 0.36 |
| Risk-free rate (%) | 1.92 | 1.92 | 1.92 | 1.92 | 1.92 |

### *4.2. Sphere packing approach to portfolio optimization*

We now implement the sphere packing framework developed in Section 3.2. The admissible region is decomposed into two components:

- the linear return–budget region $D_1$, and
- the quadratic variance region $D_2$.

The portfolio construction proceeds in two stages.

#### 4.2.1. Stage 1: Determining the robustness radius in $D_1$

In the first stage, the sphere packing linear program (3.11) is solved to determine the largest inscribed Euclidean ball within the return–budget region $D_1$. This yields the stable portfolio center $x^0$ and the associated robustness radius $r^*$:

$$x^0 = (0,\ 0.783236,\ 0,\ 0), \qquad r^* = 0.108382.$$

Here, $x^0$ is the Chebyshev center of $D_1$, and $r^*$ represents the maximal uniform perturbation radius such that all points within distance $r^*$ from $x^0$ remain feasible with respect to the linear constraints defining $D_1$.

By construction, Stage 1 incorporates only the linear return and budget constraints; the quadratic variance constraint is enforced separately in Stage 2. The corresponding linear constraints used in the sphere packing formulation are summarized in Table 6.

**Table 6.** Stage 1 constraints used in the sphere packing linear program.

| Constraint | Specification |
|---|---|
| Return threshold | $\langle \mu, x\rangle - r\|\mu\| \geq 0.0476$ |
| Budget constraint | $\sum_{j=1}^{n} x_j + r\sqrt{n} \leq 1$ |
| Box constraint | $x_j + r \leq 1$ |
| Non-negativity | $x \geq 0,\ r \geq 0$ |

#### 4.2.2. Stage 2: Directional feasibility with respect to $D_2$

In the second stage, the center $x^0$ obtained from the linear admissible region $D_1$ is directionally adjusted toward the variance feasible region $D_2$ along rays of the form

$$x(\alpha) = x^0 + \alpha h,$$

where $h \in \mathbb{R}^n$ is a candidate direction, and $\alpha \in \mathbb{R}$ is a scalar step size.

Substituting $x(\alpha)$ into the quadratic variance constraint defining $D_2$,

$$\langle Cx, x\rangle \leq \sigma^2_{\max} + \varepsilon,$$

yields a quadratic inequality in $\alpha$ of the form derived in Section 3.2. If the discriminant is non-negative, the admissible parameters form a closed interval $[\alpha_1, \alpha_2]$. By construction, the unmodified portfolio $x^0 + \alpha h$ satisfies the variance constraint if and only if $\alpha \in [\alpha_1, \alpha_2]$.

For numerical conditioning, directions $h$ may be rescaled, for example, so that $\langle Ch, h\rangle = 1$. Such normalization merely reparameterizes $\alpha$ and does not alter the feasible set or the quadratic boundary of $D_2$.

**Monte Carlo directional search.**

To obtain feasible portfolios in $D_1 \cap D_2$, a Monte Carlo search over candidate directions $h$ is performed. Directions are generated using both stochastic and economically motivated constructions, including Dirichlet sampling, Gaussian perturbations, return aligned directions ($h \propto \mu$), variance reducing directions ($h \propto -Cx^0$), and $C^{-1}\mu$ directions.

For each direction, the admissible interval $[\alpha_1, \alpha_2]$ is computed analytically from

$$\langle C(x^0 + \alpha h),\ x^0 + \alpha h\rangle \leq \sigma^2_{\max}.$$

If no real solution exists or $\alpha_2 \leq 0$, the direction is discarded. Otherwise, the initial step size is set to $\alpha = \alpha_2$, corresponding to the analytic intersection of the ray $x^0 + \alpha h$ with the quadratic boundary of $D_2$.

The tentative portfolio

$$\tilde{x}(\alpha) = x^0 + \alpha h$$

is then constructed, and non–negativity is enforced componentwise:

$$x(\alpha) = [\,\tilde{x}(\alpha)\,]_+.$$

While the interval $[\alpha_1, \alpha_2]$ guarantees variance feasibility for the unmodified portfolio $\tilde{x}(\alpha)$, componentwise clipping modifies the portfolio vector. Accordingly, the variance, return, and budget constraints defining $D_1 \cap D_2$ are explicitly re–evaluated.

If any constraint is violated, the step size $\alpha$ is reduced via backtracking and the clipped portfolio recomputed until full feasibility is restored. This produces a verified step size

$$\alpha_{\text{used}} \in (0, \alpha_2],$$

and a verified feasible portfolio

$$x_{\text{ver}} = [\, x^0 + \alpha_{\text{used}} h \,]_+, \qquad x_{\text{ver}} \in D_1 \cap D_2.$$

Among all verified feasible portfolios generated by the directional search, the final allocation is selected via

$$x^* \in \arg\max\{\langle \mu, x\rangle : x \text{ verified feasible}\}.$$

In the baseline implementation $K = 1$, so no aggregation across multiple candidates is performed.

**Proposition (Correctness of Stage 2 construction).**

Let

$$D_2 = \left\{ x \in \mathbb{R}^n : \langle Cx, x\rangle \le \sigma^2_{\max} + \varepsilon \right\}$$

denote the quadratic variance admissible region. Let $x^0 \in D_1$ denote the Chebyshev center of the linear admissible region, and let $h \in \mathbb{R}^n$ satisfy $\langle Ch, h\rangle > 0$. Suppose that substitution of

$$x(\alpha) = x^0 + \alpha h$$

into the quadratic constraint

$$\langle Cx, x\rangle \le \sigma^2_{\max} + \varepsilon$$

yields a quadratic inequality whose admissible set of parameters is a non-empty closed interval $[\alpha_1, \alpha_2]$. Then:

*(i)* For every $\alpha \in [\alpha_1, \alpha_2]$, the unmodified portfolio $x^0 + \alpha h$ satisfies $x^0 + \alpha h \in D_2$.

*(ii)* If componentwise clipping is applied and feasibility with respect to the return, budget, and variance constraints is enforced via the backtracking procedure of Algorithm 1, then whenever the algorithm terminates with $\alpha > \delta$, the resulting portfolio $x_{\text{ver}}$ satisfies

$$x_{\text{ver}} \in D_1 \cap D_2.$$

**Proof.** Substitution of $x(\alpha) = x^0 + \alpha h$ into the quadratic form $\langle Cx, x\rangle$ produces a scalar quadratic inequality in $\alpha$. Whenever the discriminant is non-negative, the admissible parameters form a closed interval $[\alpha_1, \alpha_2]$, which establishes part (i).

For part (ii), componentwise clipping modifies the ray $x^0 + \alpha h$, so Algorithm 1 explicitly re-evaluates the return, budget, and variance constraints after clipping. If any constraint is violated, the step size $\alpha$ is reduced via backtracking until either feasibility is restored or $\alpha \le \delta$. By construction, whenever the algorithm accepts a portfolio (i.e., $\alpha > \delta$ at termination), all defining inequalities of $D_1$ and $D_2$ are satisfied. Hence $x_{\text{ver}} \in D_1 \cap D_2$.

*Remark.* Algorithm 1 implements a heuristic Monte Carlo directional search; however, by construction, every accepted portfolio is explicitly verified to satisfy the defining constraints of $D_1 \cap D_2$, thereby ensuring feasibility preservation at each step.

**Algorithm 1** Verified Monte Carlo directional search within $D_1 \cap D_2$

**Require:** Covariance matrix $C$, center $x^0$, variance cap $\sigma^2_{\max}$, variance tolerance $\varepsilon > 0$, return threshold $u_0$, mean vector $\mu$, number of trials $N$, direction styles $\mathcal{S}$, backtracking factor $\beta \in (0,1)$, backtracking tolerance $\delta > 0$.

**for** each direction style $s \in \mathcal{S}$ **do**
  **for** $k = 1$ to $N/|\mathcal{S}|$ **do**
    Generate candidate direction $h^{(k)}$.
    Optionally rescale $h^{(k)}$ so that $\langle Ch^{(k)}, h^{(k)} \rangle = 1$ {rescales $\alpha$ only; improves conditioning}
    Compute $(\alpha_1, \alpha_2)$ from

$$\langle C(x^0 + \alpha h^{(k)}),\ x^0 + \alpha h^{(k)} \rangle \leq \sigma^2_{\max} + \varepsilon.$$

    **if** no real solution or $\alpha_2 \leq 0$ **then**
      Discard direction.
    **else**
      Set $\alpha \leftarrow \alpha_2$.
      Compute clipped portfolio $x \leftarrow [x^0 + \alpha h^{(k)}]_+$.
      **while** ($\langle \mathbf{1}, x \rangle > 1$ **or** $\langle \mu, x \rangle < u_0$ **or** $\langle Cx, x \rangle > \sigma^2_{\max} + \varepsilon$) **and** $\alpha > \delta$ **do**
        $\alpha \leftarrow \beta\alpha$
        $x \leftarrow [x^0 + \alpha h^{(k)}]_+$.
      **end while**
      **if** $\alpha > \delta$ **then**
        Set $\alpha_{\text{used}} \leftarrow \alpha$
        Set $x_{\text{ver}} \leftarrow x$
        Store $(x_{\text{ver}}, \alpha_{\text{used}})$ and its value $\langle \mu, x_{\text{ver}} \rangle$.
      **end if**
    **end if**
  **end for**
**end for**
Select

$$x^* \in \arg\max\{\langle \mu, x \rangle : x \text{ stored as verified feasible}\}.$$

**return** $x^*$ and the associated direction.

**Stress test and numerical robustness.**

To assess numerical stability of the two stage construction, the full procedure (Stage 1 and Stage 2) was recomputed under systematic perturbations of: (i) the variance cap $\sigma_{\max}$ ($\pm 5\%$), (ii) the return threshold $u_0$ ($\pm 5\%$), and (iii) the Monte Carlo random seed. For each scenario, the sphere packing radius $r^*$, the projected weights, and the associated performance measures were recomputed from scratch. Feasibility was verified explicitly whenever $D_1 \cap D_2$ remained non-empty. The resulting stress test outcomes are summarized in Table 7.

The results exhibit three key stability properties.

First, the robustness radius $r^*$ is invariant to perturbations in $\sigma_{\max}$ and to changes in the Monte Carlo seed, reflecting the fact that Stage 1 depends only on the return–budget region $D_1$. Variations in $u_0$ affect $r^*$ smoothly, as expected from the linear constraint defining $D_1$.

Second, whenever $D_1 \cap D_2$ is non-empty, the projected portfolio satisfies the variance constraint

$$\langle Cx, x \rangle \leq \sigma^2_{\max},$$

demonstrating that quadratic feasibility is ensured by the analytic characterization of admissible step sizes combined with explicit post clipping verification. For sufficiently high values of $u_0$ (see $u_0$ (high) scenario), the intersection becomes empty and no feasible projection exists, which is economically consistent with the imposed return requirement.

Third, variations in the Monte Carlo seed lead to only minor changes in portfolio return, risk, and total invested weight, indicating numerical stability of the direction selection mechanism.

**Table 7.** Stress test results under parameter perturbations.

| Case | $u_0$ | $\sigma_{\max}$ | $\sum_j x_j$ | $\langle\mu, x\rangle$ | $\sqrt{\langle Cx, x\rangle}$ | $r^*$ |
|---|---|---|---|---|---|---|
| $\sigma_{\max}$ (low) | 0.0476 | 0.07486 | 0.96927 | 0.04827 | 0.07247 | 0.10838 |
| $\sigma_{\max}$ (base) | 0.0476 | 0.07880 | 0.97329 | 0.04809 | 0.07149 | 0.10838 |
| $\sigma_{\max}$ (high) | 0.0476 | 0.08274 | 0.96410 | 0.04837 | 0.08102 | 0.10838 |
| $u_0$ (low) | 0.04522 | 0.07880 | 0.99015 | 0.04807 | 0.06983 | 0.11886 |
| $u_0$ (base) | 0.0476 | 0.07880 | 0.97329 | 0.04809 | 0.07149 | 0.10838 |
| $u_0$ (high) | 0.04998 | 0.07880 | – | – | – | – |
| Seed (low) | 0.0476 | 0.07880 | 0.99998 | 0.04898 | 0.07200 | 0.10838 |
| Seed (base) | 0.0476 | 0.07880 | 0.97329 | 0.04809 | 0.07149 | 0.10838 |
| Seed (high) | 0.0476 | 0.07880 | 0.97473 | 0.04820 | 0.07173 | 0.10838 |

**Implementation note.**

The second stage was implemented in R. The stable center $x^0$ obtained from Stage 1 was projected into $D_2$ using a Monte Carlo direction selection procedure.

A total of $N = 8000$ candidate directions were generated across multiple styles, including Dirichlet sampling (concentration parameter $\kappa = 60$), Gaussian perturbations, return aligned directions ($h \propto \mu$), variance reducing directions ($h \propto -Cx^0$), and $C^{-1}\mu$ directions. In the reported experiment, candidate directions were risk normalized so that $\langle Ch, h\rangle = 1$. This normalization improves numerical conditioning of the quadratic step size computation without modifying the geometry of the feasible region or the location of the variance boundary.

For each candidate direction, the admissible step interval $(\alpha_1, \alpha_2)$ was computed analytically from the quadratic variance constraint

$$\langle C(x^0 + \alpha h),\ x^0 + \alpha h\rangle \le \sigma_{\max}^2 + \varepsilon,$$

as described in Section 3.2. The upper bound $\alpha_2$ was used as the initial projection step. After enforcing non-negativity componentwise, the resulting portfolio was explicitly verified to satisfy the return, budget, and variance constraints defining $D_1 \cap D_2$. If any constraint was violated, the step size $\alpha$ was reduced via backtracking (with stopping tolerance $\delta$) until full feasibility was restored or the minimum step size threshold was reached.

Among all verified feasible portfolios, the final allocation was selected according to the maximum-return criterion subject to the variance bound $\langle Cx, x\rangle \le \sigma_{\max}^2 + \varepsilon$. Although aggregation of the top $K$ feasible portfolios via the componentwise median is available in the implementation, in the present baseline experiment the optimal solution was unique ($K = 1$), and no aggregation was required. The optimal projection direction obtained in Stage 2 is reported in Table 8.

**Table 8.** Optimal projection direction $h^*$ selected in Stage 2.

| **Asset** | $h_i^*$ |
|---|---|
| 0–5 Year U.S. Treasury | 10.237843 |
| S&P 500 | -6.302487 |
| LIBOR 3M | -15.763143 |
| Fed Overnight | -1.550688 |

The values reported in the column $x_{\text{ver}}$ correspond to the verified-feasible and componentwise non-negative projection

$$x_{\text{ver}} = [x^0 + \alpha_{\text{used}} h^*]_+,$$

where $\alpha_{\text{used}} \leq \alpha_2$ is obtained after feasibility verification via backtracking in Algorithm 1.

The lower bound $x_{\text{min}}^{\text{ray}}$ corresponds to $x^0 + \alpha_1 h^*$ prior to non-negativity clipping, and $x_{\text{max}}^{\text{ray}}$ corresponds to $x^0 + \alpha_2 h^*$ before feasibility verification. The directional projection bounds and the verified feasible portfolio are reported in Table 9.

**Table 9.** Directional projection bounds and verified feasible portfolio along the optimal direction $h^*$.

| **Asset** | $x^0$ | $h^*$ | $x_{\text{min}}^{\text{ray}}$ | $x_{\text{max}}^{\text{ray}}$ | $x_{\text{ver}}$ |
|---|---|---|---|---|---|
| 0–5 Year U.S. Treasury | 0.000000 | 10.237843 | 0.419718 | 0.494430 | 0.494430 |
| S&P 500 | 0.783236 | -6.302487 | 0.524855 | 0.478861 | 0.478861 |
| LIBOR 3M | 0.000000 | -15.763143 | -0.646237 | -0.761271 | 0.000000 |
| Fed Overnight | 0.000000 | -1.550688 | -0.063573 | -0.074890 | 0.000000 |
| | | | | **Total weight of verified portfolio** | 0.973291 |

Note: $x_{\text{min}}^{\text{ray}}$ and $x_{\text{max}}^{\text{ray}}$ denote analytic intersections of the directional ray with the quadratic boundary prior to non-negativity clipping. The verified portfolio $x_{\text{ver}} = [x^0 + \alpha_{\text{used}} h^*]_+$ satisfies $\langle C x_{\text{ver}}, x_{\text{ver}} \rangle \leq \sigma_{\text{max}}^2 + \varepsilon$ and is obtained after feasibility verification via Algorithm 1.

### 4.2.3. Comparative optimization results

To evaluate the economic implications of the proposed sphere packing construction, the resulting allocation is compared with five benchmark strategies: (i) the maximum return portfolio, (ii) the minimum variance portfolio, (iii) the equal-weight portfolio, (iv) the Markowitz maximum Sharpe ratio portfolio, and (v) the convex programming portfolio solving $\min \|x - y\|^2$ under identical return and risk constraints.

Table 10 reports portfolio weights, expected returns, risk levels (standard deviation), and Sharpe ratios for all strategies.

The sphere packing portfolio satisfies simultaneously

$$\langle \mu, x \rangle \geq u_0, \qquad \langle Cx, x \rangle \leq \sigma_{\text{max}}^2, \qquad \sum_{j=1}^{n} x_j \leq 1,$$

and therefore lies in the feasible region $D_1 \cap D_2$ constructed in Stages 1 and 2. Quantitatively, the sphere packing allocation achieves:

- expected return: 4.81%,
- risk (standard deviation): 7.15%,
- total invested weight: 97.33%.

Relative to the maximum return portfolio, which exhibits a risk level of 15.18%, the sphere packing allocation reduces volatility by more than 50% while remaining strictly within the imposed variance bound. Relative to the minimum variance portfolio (return 2.30%), the sphere packing solution delivers substantially higher expected return while respecting the same risk ceiling.

**Table 10.** Comparison of portfolio allocations and performance across optimization models.

| Asset / Portfolio | Max Return | Min Variance | Equal Weight | Markowitz | Convex | Sphere Packing |
|---|---|---|---|---|---|---|
| 0–5 Year U.S. Treasury | 0.00% | 1.36% | 25.00% | 20.23% | 17.27% | 49.44% |
| S&P 500 | 100.00% | 0.42% | 25.00% | 2.70% | 52.31% | 47.89% |
| LIBOR 3M | 0.00% | 98.22% | 25.00% | 77.06% | 13.18% | 0.00% |
| Fed Overnight | 0.00% | 0.00% | 25.00% | 0.00% | 11.08% | 0.00% |
| **Total weight** | 100.0% | 100.0% | 100.0% | 100.0% | 93.8% | 97.33% |
| **Expected return** | 7.22% | 2.30% | 3.54% | 2.50% | 4.76% | 4.81% |
| **Portfolio risk** | 15.18% | 0.58% | 3.74% | 0.72% | 7.88% | 7.15% |
| **Sharpe ratio** | 0.35 | 0.65 | 0.43 | 0.81 | 0.36 | 0.40 |
| **Risk-free rate** | 1.92% | 1.92% | 1.92% | 1.92% | 1.92% | 1.92% |
| **Objective** | Max return | Min variance | Equal weight | Max Sharpe ratio | $\min \|\|x - y\|\|^2$ | Feasible projection from $x^0$ |
| **Return constraint** | – | – | – | – | $\geq 4.76\%$ | $\geq 4.76\%$ |
| **Risk constraint** | – | – | – | – | $\leq 7.88\%$ | $\leq 7.88\%$ |
| **Budget constraint** | = 100% | = 100% | = 100% | = 100% | $\leq 100\%$ | $\leq 100\%$ |

Compared with the Markowitz maximum Sharpe ratio portfolio, the sphere packing allocation exhibits a more balanced asset structure. The Markowitz solution allocates heavily to LIBOR (77.06%), whereas the sphere packing portfolio distributes capital primarily between U.S. Treasuries (49.44%) and the S&P 500 (47.89%), reflecting the geometric centering property established in Stage 1 , which maximizes uniform distance to the supporting hyperplanes of the linear admissible region.

Although the Sharpe ratio of the Markowitz portfolio is higher (0.81), that solution is obtained from an unconstrained global mean–variance optimization and does not incorporate the joint feasibility constraints defining the admissible region $D_1 \cap D_2$. In contrast, the proposed sphere packing method operates strictly within this constrained region and prioritizes structural robustness and interior stability over unconstrained mean–variance efficiency.

The convex programming portfolio ($\min \|x - y\|^2$) satisfies the same return and variance constraints as the sphere packing solution. However, it remains closer to the reference portfolio $y$, while the sphere packing approach begins from a geometrically stable center $x^0$ that maximizes the inscribed robustness radius in $D_1$. The resulting allocation therefore reflects a feasibility driven design rather than proximity to an exogenous benchmark.

Overall, the sphere packing portfolio represents a constraint consistent, geometrically robust allocation that balances return and risk within the jointly admissible region $D_1 \cap D_2$. The construction does not aim to dominate classical mean–variance optimizers in unconstrained settings, but rather to provide a stable portfolio anchored in the maximal feasible neighborhood of the return–budget region.

**Discussion.**

The empirical results clarify the complementary roles of convex feasibility restoration and sphere packing based geometric centering within the classical mean–variance framework.

Under the baseline calibration ($u_0 = 4.76\%$, $\sigma_{\max} = 7.88\%$), the convex programming formulation yields a portfolio with

$$\text{Expected return} = 4.76\%, \quad \text{Risk} = 7.88\%, \quad \langle \mathbf{1}, x \rangle = 0.938.$$

The associated objective value $\|x - y\|^2 = 0.2868$ quantifies the minimal quadratic adjustment required to restore feasibility from the maximum-return reference allocation. By construction, the solution lies in the admissible region defined by the joint return, variance, and relaxed budget constraints, equivalently the intersection $D_1 \cap D_2$, and therefore serves as a benchmark–consistent feasible allocation without incorporating geometric centering.

The sphere packing construction adds a structural design centering layer. In Stage 1, the portfolio $x^0$ maximizes the admissible Euclidean radius $r^* = 0.108382$ within the linear return–budget region $D_1$, thereby identifying its Chebyshev center. In Stage 2, analytic directional feasibility adjustment ensures compliance with the quadratic variance constraint $\langle Cx, x \rangle \leq \sigma_{\max}^2 + \varepsilon$. The resulting allocation attains

$$\text{Expected return} = 4.81\%, \quad \text{Risk} = 7.15\%, \quad \langle \mathbf{1}, x \rangle = 0.9733,$$

remaining strictly interior to the relaxed budget constraint while satisfying the prescribed return floor.

The two constructions therefore address distinct stability criteria. The convex programming portfolio prioritizes minimal deviation from an exogenous benchmark, whereas the sphere-packing allocation maximizes geometric interior stability measured by the admissible perturbation radius $r^*$. Relative to the maximum-return portfolio (risk 15.18%), both approaches substantially reduce volatility. Compared with the minimum-variance portfolio (return 2.30%), they deliver materially higher expected return while respecting the mandated constraints.

Stress tests further confirm numerical stability. Moderate perturbations of $\sigma_{\max}$, $u_0$, and the Monte Carlo seed induce only minor variations in performance measures. As expected, the robustness radius $r^*$ remains invariant under changes in the quadratic variance bound, since it is determined exclusively by the linear return–budget region $D_1$ in Stage 1.

Overall, convex programming ensures admissibility within the quadratic mean–variance structure, whereas sphere packing quantifies and enhances interior stability through geometric centering. The proposed framework thus introduces a complementary robustness perspective grounded in the intrinsic geometry of the feasible allocation region rather than in distributional ambiguity or worst-case optimization.

## 5. Conclusion

This paper develops a constructive robustness framework within the classical mean–variance paradigm consisting of two complementary components. The first component employs convex quadratic programming for benchmark–consistent feasibility restoration under prescribed return, variance (up to tolerance $\varepsilon$), and relaxed budget constraints while preserving the quadratic structure of the Markowitz model.

The second component is a sphere packing design centering construction carried out in two stages. In Stage 1, the largest Euclidean ball is inscribed in the linear return–budget region, yielding the Chebyshev center and an associated admissible perturbation radius. In Stage 2, quadratic feasibility is enforced through analytic directional adjustment, ensuring membership in the variance admissible set $\langle Cx, x\rangle \leq \sigma^2_{\max} + \varepsilon$. Within this framework, robustness is interpreted as geometric interior stability of the feasible region rather than as distributional ambiguity or worst-case loss.

The empirical results indicate that the constructed allocation satisfies all mandate constraints while maintaining balanced return and risk characteristics relative to standard benchmark portfolios. In contrast to extremal efficient-frontier solutions located on constraint boundaries, the proposed approach yields an allocation anchored at a geometrically stable interior point of the admissible region without re-optimizing a global mean–variance objective over the full feasible set. Stress testing and Monte Carlo experiments confirm numerical stability under moderate parameter perturbations, suggesting that the solution does not depend on fragile boundary configurations.

The contribution is complementary to distributionally robust, stochastic control, and ambiguity based formulations. Whereas those approaches model statistical uncertainty through uncertainty sets or probabilistic criteria, the present study introduces a structural robustness principle grounded in geometric centering within the classical quadratic representation of portfolio optimization.

Future research may extend the framework to multi–period environments, alternative norm specifications in the design centering problem, high-dimensional asset universes, and hybrid models combining geometric interior stability with probabilistic robustness mechanisms.

Overall, the results demonstrate that portfolio robustness can be interpreted not only through uncertainty modeling but also through geometric stability of the admissible region, providing a transparent and computationally tractable enhancement to the classical mean–variance framework.

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## Author contributions

Tumendelger Lkhagvasuren: Conceptualization, Methodology, Software, Writing–original draft. Bolorsuvd Batbold: Data curation, Formal analysis. Enkhbat Rentsen: Methodology, Supervision.

## Conflict of interest

The authors declare no conflict of interest.

## Use of Generative-AI tools declaration

During the preparation of this manuscript, the authors used ChatGPT for language polishing and expression improvement. All AI-assisted content was critically reviewed by the authors.

## Data availability

The analysis uses historical financial market data compiled from standard commercial data sources commonly used in academic and institutional research. No new datasets were generated or collected for this study.

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