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Research article

Optimal portfolio choice with capital gains tax under Heston's stochastic volatility model

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Abstract: In this paper, we studied optimal investment problems with capital gains tax. The price process of the risky asset was assumed to be governed by Heston's stochastic volatility model. Furthermore, the tax evasion behavior of investors was also considered in our model. However, investors will be punished when their tax evasion behaviors are noticed by the audit. By maximizing the expected utility of terminal wealth, the corresponding Hamilton-Jacobi-Bellman (HJB) equation was obtained by using the principle of stochastic dynamic programming. Then, we obtained analytical solutions of the optimal investment strategies by using the first-order optimality conditions and solving the HJB equations. Finally, some numerical results and economic explanations were provided to make further illustrations.

Keywords: capital gains tax; Heston's stochastic volatility model; optimal investment strategy
Mathematics Subject Classification: 91G10, 93E20

1. Introduction

In recent decades, the optimal portfolio optimization problem has been a hot topic in the fields of financial engineering and financial mathematics. Merton [1, 2] first studied the optimal investment-consumption problem in a continuous-time framework and derived the optimal policy that maximizes the total expected discounted utility from both consumption and investment. He provided the foundation for continuous-time financial theory. In the financial market, unexpected events frequently occur that affect the prices of risky assets. Therefore, this leads to discontinuous jumps in the prices of risky assets. Since then, many studies have considered the problems of optimal investment under jump diffusion models. In [3], it was assumed that the price process of risky assets follows a jump diffusion model and a problem regarding the optimal investment portfolio was considered when investors have an aversion to ambiguity. In [4], multidimensional Hawkes jump diffusion was used to describe the price process of risky assets. They studied the optimal investment portfolio and consumption problems with different utility functions. In [5], the authors supposed that the price process of risky assets follows a generalized jump diffusion model, and the model parameters depend on the state variables. Using the HJB equations, they obtained a semi-closed solution for the optimal investment strategy when the investor's utility function is a hyperbolic absolute risk aversion (HARA) utility function. These studies considered the discontinuity of market risk asset prices, but they did not take the volatility risk into account or study the effects of volatility risk on the optimal investment strategy. The aforementioned papers assumed constant volatility in risky asset prices. In contrast to the Black-Scholes model's constant volatility assumption, Heston's stochastic volatility model allows the volatility of risky assets to follow a stochastic process, which makes it more realistic for capturing market behavior [6]. Some empirical studies have validated the relevance of Heston's stochastic volatility model in real-world investment scenarios. For instance, [7] provided empirical evidence that Heston's stochastic volatility model outperforms the Black-Scholes model in capturing the dynamics of option prices. [8] derived closed-form solutions for optimal portfolio weights under Heston's stochastic volatility model. Their results showed that investors should allocate more to risky assets when volatility is low and reduce risk exposure during periods of high volatility. Liu [9] examined optimal portfolio decisions in Heston's stochastic volatility framework, and showed that investors should adjust their portfolios dynamically to hedge against volatility risk. In [10], the optimal investment problem was studied under a power utility function and Heston's stochastic volatility model. It provided a verification theorem, which illustrates a unique solution to the optimal investment problem only when the model parameters satisfy certain restrictive conditions. In addition, robust portfolio decision problems with model uncertainty have also been studied under Heston's stochastic volatility model. For example, in [11], it was supposed that the risky asset price process follows a stochastic volatility jump diffusion process, and the investor is ambiguous. They derived an approximate solution for robust consumption and investment strategies of an ambiguity averse investor with recursive preference.

There is also a stream of literature that considered optimal investment problems from the viewpoint of capital gains tax [12, 13]. Capital gains refer to the profit earned from the sale of an asset that has increased in value during the period of holding, such as stocks, bonds, real estate, or other investments. Capital gains are a key component of investment returns and are subject to taxation in most jurisdictions. Capital gains tax is closely related to investors' wealth, and studying the impact of capital gains tax on investment is also of great significance. Many researchers have considered the problems of optimal investment capital gains tax. They also showed that capital gains tax has a significant impact on the investment's optimal strategy. In [14, 15], the problems of optimal investment with capital gains tax under discrete time models were studied. In [16], the optimal tax-timing and asset allocation problem was studied when tax rebates on capital losses are limited. Different from these studies, in [17, 18], optimal investment problems with capital gains tax under continuous-time models were considered. Furthermore, in [19], it was supposed that the investor's utility is a recursive utility and the problem of optimal investment and consumption with capital gains tax under regime switching models was studied.

Tax evasion refers to the illegal practice of deliberately underreporting income, inflating deductions, hiding assets, or otherwise falsifying financial information to reduce tax liability. It is a willful violation of tax laws, punishable by fines, penalties, or even criminal prosecution. Due to the seminal papers [20, 21], the problem of optimal dynamic tax evasion has attracted more and more attention. In [22], the problem of optimal investment with capital gains was considered. In addition, the tax evasion behavior of investors was taken into account in their paper. They supposed that investors conceal some asset gains in order to evade taxes, and once discovered, they are punished. [23] adopted some assumptions from [22] and considered the problem of optimal investment and consumption with capital gains tax and tax evasion. Different from [22], they supposed that the risky assets follow a jump diffusion process. They illustrated that the jump risk has a significant impact on the optimal investment strategies of investors. Additional studies on optimal dynamic tax evasion can be found in [24–27].

However, the above studies do not consider optimal investment problems with capital gains tax under stochastic volatility models. These research papers have examined these ingredients largely in isolation. For example, [8] analyzed optimal asset allocation choice under Heston's stochastic volatility model without capital gains tax or tax evasion, while [17, 18] investigated capital gains taxation in continuous time without stochastic volatility and tax evasion. Closer to our setting, [22] incorporated tax evasion under constant volatility, and [23] considered tax evasion with jump risk but without stochastic volatility risk. To the best of our knowledge, the joint characterization of optimal portfolio allocation and tax evasion decisions under the triple interaction of stochastic volatility, capital gains taxation, and endogenous evasion with an audit risk remains unexplored. We incorporate capital gains taxation and tax evasion into Heston's stochastic volatility model. In particular, the stochastic volatility is also correlated with asset returns. This extension allows market uncertainty to affect the investors' portfolio choice, leading to state-dependent optimal strategies that cannot be captured by a constant volatility model. Our research results also complement [23], in which the optimal investment and consumption with capital gains tax and jump risk was analyzed. While their focus was on jump risk as a source of time-varying uncertainty, we show that stochastic volatility alone generates rich comparative statics for optimal portfolio decisions. The main purpose of this paper is to obtain the closed-form formulae of optimal investment strategies of an investor with a constant relative risk aversion (CRRA) utility and exponential utility, when the price process of risky assets follows Heston's stochastic volatility model. Furthermore, we adopt some model assumptions from [22] and [23]. The risk-free asset and risky asset are both subject to taxation. Investors may evade taxes on a risky asset return, but cannot evade taxes on risk-free asset returns. In addition, investors will have to pay fines once tax evasion is detected by tax authorities. We illustrate that the investor's optimal investment strategy is highly sensitive to the risk aversion coefficient. Compared to the constant volatility, numerical examples reveal that stochastic volatility risk has great effect on the optimal portfolio policies. From an intuitive perspective, investors' incentives to conceal capital gains are likely to depend on the prevailing volatility environment. Periods of elevated volatility may generate large transient gains that increase the temptation to evade taxes, while our analytical results show that the optimal evasion decisions are independent of stochastic volatility risk, highlighting a sharp separation between market risk and regulatory risk. In particular, we find that punishment severity and audit intensity can significantly affect investors' tax evasion behavior. This separation result highlights that enforcement policies, rather than market conditions, are the primary determinants of concealment behavior. These findings suggest that a regulatory agency can strengthen enforcement along these two dimensions.

The remainder of this paper is organized as follows. Section 2 presents the model formulation. In Section 3, we derive the explicit expressions of the optimal investment strategies of an investor with a power utility function and exponential utility function when the risky asset follows Heston's stochastic volatility model.

In Section 4, we provide some numerical examples to illustrate and analyze the impact of some model parameters and capital gains tax on the optimal portfolio weights. Finally, Section 5 concludes the paper.

2. The financial model

2.1. Model assumptions

In this section, we consider a frictionless and arbitrage-free financial market. All transactions in the financial market can be traded continuously during $[0, T]$ without transaction fees. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbf{P})$ be a complete filtered probability space, where $\mathbf{P}$ is a probability measure and $\{\mathcal{F}_t\}_{t\in[0,T]}$ is a filtration satisfying the usual conditions. Let $\{W_i(t), i = 1, 2\}_{t\in[0,T]}$ be a standard two-dimensional Brownian motion and $N(t)$ be a Poisson process with constant arrival intensity λ_1. Furthermore, stochastic processes $N(t)$ and $\{W_i(t), i = 1, 2\}_{t\in[0,T]}$ are defined on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbf{P})$ and are assumed to be independent.

Suppose that there are two assets in the financial market, a risk-free bond and a risky asset. The price dynamics of the risk-free bond $B := B(t)$ are governed by

$$dB(t) = rB(t)dt, \tag{2.1}$$

where $r > 0$ is a constant and denotes the instantaneous market interest rate.

The price dynamics of the risky asset are described by the following Heston's stochastic volatility model:

$$dS(t) = S(t)\left[(r + \lambda m(t))dt + \sqrt{m(t)}dW_1(t)\right], \tag{2.2}$$

and

$$dm(t) = k(\theta - m(t))dt + \sigma\rho\sqrt{m(t)}dW_1(t) + \sigma\sqrt{(1 - \rho^2)m(t)}dW_2(t), \tag{2.3}$$

where $k > 0, \theta > 0, \sigma > 0, \lambda$ are constants, λ is the market price of volatility risk of the risky asset, σ denotes the volatility coefficient of the instantaneous variance process $m(t)$, k is the mean reversion rate, and θ is the long-term mean level. To ensure $m(t) > 0$ for all $t \geq 0$, we assume that the Feller condition $2k\theta > \sigma^2$ is satisfied. In addition, the risky asset price and volatility are assumed to be correlated, and the correlated coefficient is $\rho \in (-1, 1)$.

In this paper, we make some assumptions about capital gains tax that are also presented in [22] and [23]. We assume that capital gains are taxed continuously and the investor pays taxes when the asset price changes. Furthermore, investors will pay taxes when the change in the asset price is positive. However, investors get a refund when the change in the asset price is negative. Here, the tax rate on the returns of both risk-free and risky assets are represented by $\tau \in (0, 1)$. $r_1 = (1 - \tau)r$ represents the after-tax return on risk-free assets. In this paper, to simplify the calculations, the tax rates on the income of risk-free and risky assets are assumed to be the same. Similar approaches can be adopted to make optimal investment decisions under the situations that tax rates are different. Furthermore, we assume that investors may conceal part of the investment in the risky assets to evade taxes. However, investors will be fined if their tax evasion is detected by an audit. We suppose that the audit of risky assets is described by the Poisson process $N(t)$ with the intensity λ_1, and the fine is a percentage of the total amount of concealed risky assets. Note that returns on risk-free assets (e.g., government bonds or bank deposits) are typically subject to automatic third-party reporting, so there is limited scope for tax evasion. Consequently, tax evasion is assumed to not be feasible for risk-free assets in this paper. Let $\alpha > 0$ denote the penalty rate, and assume that the fine is proportional to the value of the concealed risky assets. This assumption is also seen in [20, 22, 23]. From a regulatory viewpoint, the enforcement instruments considered in this paper, namely audit intensity and penalty severity, are consistent with real-world capital gains tax enforcement practices. For example, tax authorities such as the Internal Revenue Service (IRS) in the United States rely on probabilistic audits, where only a small fraction of individual tax returns are examined in every year, but detected violations may result in substantial monetary penalties proportional to the amount of concealed gains.

The financial model (2.2)–(2.3) in this paper is relatively general, with several existing models in the literature being special cases of our model. When taxation and evasion are excluded, the model reduces to Heston's portfolio choice setting as in [8]. When the volatility of risky assets is constant and evasion is present, it aligns with constant-volatility tax-evasion formulations such as those proposed by [22]. When evasion and stochastic volatility are excluded but capital gains taxation remains, it reduces to a continuous-time model in [17, 18]. Furthermore, the continuous-time formulation of tax evasion in this paper should be interpreted as a reduced form description. Although capital gains are reported at discrete points in time, it usually occurs at the end of the tax year. The investor's exposure to detection risk and the incentives to conceal gains accumulate continuously over the investment horizon. Therefore, we adopt a continuous-time model to describe the investor's investment and tax evasion behavior.

Let $\pi_0(t), \pi_1(t)$, and $\omega(t) = 1 - \pi_0(t) - \pi_1(t)$ denote the proportions of wealth invested in the concealed risky asset, non-concealed risky asset, and risk-free bond, respectively, at time $t \in [0, T]$. Denote by $X^\pi(t)$ the investor's wealth at time $t \in [0, T]$. Then the wealth process $X^\pi(t)$ satisfies the following stochastic differential equation:

$$\frac{dX^\pi(t)}{X^\pi(t-)} = (1-\tau)\pi_1(t)\frac{dS(t)}{S(t)} + \pi_0(t)\frac{dS(t)}{S(t)} + \frac{\omega(t)}{B(t)}(1-\tau)dB(t) - \alpha\pi_0(t)dN(t). \tag{2.4}$$

Combining Eqs. (2.1) and (2.2) with (2.4), we have

$$\begin{aligned}\frac{dX^\pi(t)}{X^\pi(t-)} &= \Big\{\pi_1(t)(1-\tau)\lambda m(t) + \pi_0(t)\left[r + \lambda m(t) - r_1\right] + r_1\Big\}dt \\ &\quad + \left[(1-\tau)\pi_1(t) + \pi_0(t)\right]\sqrt{m(t)}dW_1(t) - \alpha\pi_0(t)dN(t),\end{aligned} \tag{2.5}$$

with $X^\pi(0) = x_0 > 0$, $r_1 = (1-\tau)r$.

2.2. The optimization problem

In this subsection, we introduce the definition of admissible strategies and the corresponding optimization problem.

Definition 2.1. *(Admissible strategies) For any $t \in [0, T]$, assume that $\pi(t) := (\pi_0(t), \pi_1(t))$ is an $\mathcal{F}_t$-progressive measure process and satisfies:*

$$\mathrm{E}\Big[\int_0^T \Big(\pi_0^2(t) + \pi_1^2(t)\Big)m(t)dt\Big] < \infty, \qquad \mathrm{E}\Big[\,|U(X^\pi(T))|\,\Big] < \infty,$$

where $\mathrm{E}[\cdot]$ *denotes the expectation under the probability measure* $\mathbf{P}$. *Furthermore, if Eq. (2.5) has a unique strong solution, then the strategy $\pi(t)$ is called an admissible strategy.*

Denote by Π the set of all admissible strategies π. The objective of this investor is to maximize the expected utility of the terminal wealth at time T by investing into risk-free assets and risky assets, so the optimization problem he faces can be denoted as follows:

$$\max_{(\pi_0,\pi_1)\in\Pi} \mathrm{E}\Big[U(X^\pi(T))\Big], \tag{2.6}$$

where $U(x)$ is the utility function.

Define the value function for problem (2.6) as follows:

$$J(t,x,m) = \max_{(\pi_0,\pi_1)\in\Pi} \mathrm{E}\left[U(X^\pi(T)|X^\pi(t) = x, m(t) = m\right], \tag{2.7}$$

where $J(t,x,m)$ satisfies the boundary condition $J(T,x,m) = U(x)$, and $\mathrm{E}\left[\cdot|X^\pi(t) = x, m(t) = m\right]$ represents conditional expectation.

Let us assume $J(t,x,m)$ is once continuously differentiable with respect to t (C^1 in t) and twice continuously differentiable with respect to x and m (C^2 in x and m). Then we use Itô's formula and obtain that for any $0 \le t < u$,

$$J(u, X^\pi(u), m(u)) = J(t,x,m) + \int_t^u \mathcal{L}^\pi J(s, X^\pi(s), m(s))ds + \mathcal{M}^J(u) - \mathcal{M}^J(t), \tag{2.8}$$

where the process $\mathcal{M}^J = (\mathcal{M}^J(t))_{t\ge0}$ is a local martingale, and the operator $\mathcal{L}^\pi$ is defined by

$$\begin{aligned}\mathcal{L}^\pi J(t,x,m) = J_t &+ J_x\left[\pi_1(t)(1-\tau)\lambda m + \pi_0(t)(r + \lambda m - r_1) + r_1\right]x \\ &+ \frac{1}{2}J_{xx}[(1-\tau)\pi_1(t) + \pi_0(t)]^2x^2m + J_m k(\theta - m) + \frac{1}{2}J_{mm}\sigma^2 m \\ &+ \lambda_1\left[J(t, x(1-\alpha\pi_0(t)), m) - J(t,x,m)\right] + J_{mx}\left[(1-\tau)\pi_1(t) + \pi_0(t)\right]\sigma\rho x m,\end{aligned} \tag{2.9}$$

where J_t, J_x, J_m, J_{xx}, J_{mm}, and J_{mx} are partial derivatives of $J(t,x,m)$ with respect to corresponding variables. Using the stochastic dynamic programming principle (see [28] and [29] for details), we derive the following HJB equation:

$$\begin{aligned}J_t + J_m k(\theta - m) + \frac{1}{2}J_{mm}\sigma^2 m + \max_{(\pi_0,\pi_1)\in\Pi}\Big\{&J_x\left[\pi_1(t)(1-\tau)\lambda m + \pi_0(t)(r + \lambda m - r_1) + r_1\right]x \\ &+ \frac{1}{2}J_{xx}[(1-\tau)\pi_1(t) + \pi_0(t)]^2x^2m + \lambda_1\left[J(t, x(1-\alpha\pi_0(t)), m) - J(t,x,m)\right] \\ &+ J_{mx}\left[(1-\tau)\pi_1(t) + \pi_0(t)\right]\sigma\rho x m\Big\} = 0,\end{aligned} \tag{2.10}$$

with the terminal condition $J(t,x,m) = U(x)$.

2.3. Verification theorem

We provide a verification theorem to rigorously justify that a sufficiently smooth solution to the HJB equation indeed coincides with the value function of problem (2.6), and that any maximizer of the HJB operator yields an optimal strategy.

Theorem 2.1 (Verification theorem)**.** *For any* $(t,x,m) \in [0,T] \times (0,\infty) \times (0,\infty)$*, let* Π *be the admissible set in Definition 2.1 and* $J(t,x,m)$ *be the value function defined in (2.7). Suppose that there exists a function* $V(t,x,m) : [0,T] \times (0,\infty) \times (0,\infty) \to \mathbb{R}$ *such that:*

(i) $V(t,x,m) \in C^{1,2,2}([0,T) \times (0,\infty) \times (0,\infty))$*;*

(ii) $V(T,x,m) = U(x)$ *for all* $x > 0$, $m > 0$*;*

(iii) $V(t,x,m)$ *satisfies the HJB Eq. (2.10), i.e.,*

$$V_t + V_m k(\theta - m) + \frac{1}{2}V_{mm}\sigma^2 m + \max_{(\pi_0,\pi_1)\in\Pi}\Big\{V_x[\pi_1(t)(1-\tau)\lambda m + \pi_0(t)(r + \lambda m - r_1) + r_1]x$$
$$+ \frac{1}{2}V_{xx}[(1-\tau)\pi_1(t) + \pi_0(t)]^2 x^2 m + \lambda_1[V(t,x(1-\alpha\pi_0),m) - V(t,x,m)]$$
$$+ V_{mx}[(1-\tau)\pi_1(t) + \pi_0(t)]\sigma\rho x m\Big\} = 0;$$

(iv) For each $\pi \in \Pi$ *and its corresponding state process* $(X^\pi(s), m(s))_{s\in[t,T]}$*, the local martingale term* $\mathcal{M}^V$ *appearing in Itô's formula for* $V(s, X^\pi(s), m(s))$ *is a martingale on* $[t,T]$*, so that*

$$\mathrm{E}\big[\mathcal{M}^V(T) - \mathcal{M}^V(t)\,\big|\,X^\pi(t) = x,\ m(t) = m\big] = 0.$$

Then $V(t,x,m) = J(t,x,m)$ *for all* (t,x,m)*, i.e., V coincides with the value function.*

In addition, if there exists an admissible strategy $\pi^* = (\pi_0^*, \pi_1^*) \in \Pi$ *attaining the maximum in the HJB operator, then* π^* *is optimal for problem (2.6), i.e.,*

$$V(t,x,m) = J(t,x,m) = \mathrm{E}\big[U(X^{\pi^*}(T))\,\big|\,X^{\pi^*}(t) = x,\ m(t) = m\big].$$

For the proof, see the Appendix.

In Lemma A.2 of the Appendix, we show that under the power utility setting in Section 3.1, the local martingale arising in the Itô–Lévy decomposition of the solution obtained from the HJB equation (2.10) is a martingale. Together with the verification theorem, this proves that the solution derived from the HJB equation (2.10) coincides with the solution of the optimization problem (2.6). In addition, we demonstrate that the optimal portfolio strategies presented in Theorem 3.1 are admissible in Proposition 3.1. Moreover, using an analogous proof approach, it can be shown that the corresponding results under exponential utility also hold.

3. Optimal investment strategy

In this section, we discuss the optimal investment strategy associated with the optimization problem (2.10). To solve problem (2.10), we need to choose a suitable utility function according to the investor's risk aversion. In optimal portfolio selection problems, exponential utility and power utility functions are the most widely adopted specifications, owing to their analytical tractability, well-established economic interpretations, and empirical relevance in modeling investor behavior. The absolute risk aversion coefficient of exponential utility is constant and independent of wealth levels. The exponential form proves particularly advantageous in continuous-time stochastic control problems, as it enables the decoupling of wealth effects from optimal policy functions, thereby simplifying the derivation of closed-form solutions. This property makes it especially suitable for short-term investment horizons and Gaussian return distributions. The relative risk aversion coefficient of the power function is constant, but absolute risk aversion decreases as wealth increases. This feature is consistent with observed long-term investment patterns. The power utility specification frequently admits explicit solutions in dynamic portfolio optimization problems, particularly when asset returns follow geometric Brownian motion, making it the preferred choice for intertemporal consumption-investment frameworks. Therefore, we consider two kinds of utility functions, one is the power utility function, and the other is the exponential utility function. Applying the first-order optimality conditions, we can obtain the explicit expressions of the investor's optimal investment strategies. The main results are provided in this section.

3.1. Power utility

In this subsection, the utility function $U(x)$ is assumed to be a power utility function in the following form:

$$U(x) = \frac{x^{1-\gamma}}{1-\gamma},$$

where constant γ represents the risk aversion parameter and $\gamma > 1$.

Theorem 3.1. *The optimal investment strategies* $(\pi_0^*(t), \pi_1^*(t), \omega^*(t))$ *are given as follows:*

$$\pi_0^*(t) = \frac{1}{\alpha}\left[1 - \left(\frac{\tau r}{\lambda_1 \alpha}\right)^{-\frac{1}{\gamma}}\right], \tag{3.1}$$

$$\pi_1^*(t) = \frac{1}{\gamma(1-\tau)}\left[\lambda + A(t)\sigma\rho\right] - \frac{1}{\alpha(1-\tau)}\left[1 - \left(\frac{\tau r}{\lambda_1 \alpha}\right)^{-\frac{1}{\gamma}}\right], \tag{3.2}$$

$$\omega^*(t) = 1 - \frac{1}{\gamma(1-\tau)}\left(\lambda + A(t)\sigma\rho\right) + \frac{\tau}{\alpha(1-\tau)}\left[1 - \left(\frac{\tau r}{\lambda_1 \alpha}\right)^{-\frac{1}{\gamma}}\right], \tag{3.3}$$

where the function $A(t)$ *is given as follows:*

$$A(t) = \frac{\nu_1 \nu_2 \mathrm{e}^{-\sqrt{\Delta_1}(T-t)} - \nu_1 \nu_2}{\nu_1 \mathrm{e}^{-\sqrt{\Delta_1}(T-t)} - \nu_2}. \tag{3.4}$$

Moreover, the optimal value function has the following explicit form:

$$J(t, x, m) = \frac{x^{1-\gamma}}{1-\gamma}\mathrm{e}^{A(t)m + B(t)},$$

and the function $B(t)$ *is*

$$B(t) = k\theta \int_t^T A(s)ds + M(T-t), \tag{3.5}$$

where the constant M *is*

$$M = \frac{r\gamma\tau}{\alpha}\left(\frac{r\tau}{\lambda_1 \alpha}\right)^{-\frac{1}{\gamma}} + (1-\gamma)\frac{r\tau}{\alpha} + (1-\gamma)r_1 - \lambda_1.$$

For the proof, see the Appendix.

$\mathrm{E}\left[\exp\left\{u\int_0^T m(s)ds\right\}\right] < \infty$ does not hold for all constants u, while it is crucial for ensuring that $\mathrm{E}\left[|U(X^{\pi}(T))|\right] < \infty$. Theorem 5.1 of [30] addresses the finiteness of the moment-generating function of the stochastic integral $\int_0^T m(s)ds$. As in [30], additional conditions need to be imposed in Proposition 3.1 to ensure $\mathrm{E}\left[|U(X^{\pi}(T))|\right] < \infty$. We first introduce a constant, denoted by ζ. From Eqs. (3.1) and (3.2), it follows that $\pi_0^*(t)$ is a constant and $\pi_1^*(t)$ is bounded on $[0, T]$. For any $t \in [0, T]$, let ζ be defined as follows:

$$\zeta = \max_{t\in[0,T]}\left\{\lambda(1-\gamma)\left[\pi_1^*(t)(1-\tau)+\pi_0^*(t)\right] - \frac{\gamma(1-\gamma)}{2}\left[\pi_1^*(t)(1-\tau)+\pi_0^*(t)\right]^2\right\}. \tag{3.6}$$

The following proposition verifies that optimal investment strategies $(\pi_0^*(t), \pi_1^*(t), \omega^*(t))$ in Theorem 3.1 are admissible strategies.

Proposition 3.1 (Admissible strategy). *Assume that one of the following conditions is satisfied:*

(i) $\zeta \le \frac{k^2}{2\sigma^2}$;

(ii) $\zeta > \frac{k^2}{2\sigma^2}$ *and* $T < \delta_2^{-1}\mathrm{arccot}\left(-\frac{\delta_1}{\delta_2}\right)$, *where* δ_1, δ_2 *are given as follows:*

$$\delta_1 = \frac{k}{2}, \qquad \delta_2 = \frac{\sqrt{-k^2+2\sigma^2\zeta}}{2}.$$

Then the optimal investment strategies $(\pi_0^*(t), \pi_1^*(t), \omega^*(t))$ *given in Theorem 3.1 are admissible strategies.*

For the proof, see the Appendix.

Equation (3.2) yields several important findings. We decompose Eq. (3.2) into three distinct components:

$$\pi_1^*(t) = \underbrace{\frac{\lambda}{\gamma(1-\tau)}}_{I} + \underbrace{\frac{A(t)\sigma\rho}{\gamma(1-\tau)}}_{II} - \underbrace{\frac{1}{\alpha(1-\tau)}\left[1-\left(\frac{\tau r}{\lambda_1\alpha}\right)^{-\frac{1}{\gamma}}\right]}_{III} \tag{3.7}$$

First, part I in Eq. (3.7) implies a volatility risk premium (λ) effect. It can be observed that as λ increases, the value of investment strategy π_1^* also increases, indicating that volatility risk significantly influences portfolio decisions. A higher volatility risk premium improves the risk return of the risky asset, encouraging investors to increase their allocation to it.

Second, part II implies a volatility hedging effect. The investment strategy π_1^* incorporates a hedging demand against stochastic volatility. It shows that investors reduce risk exposure when risky asset returns are negatively correlated with volatility ($\rho < 0$).

Third, part III implies a tax evasion effect, arising from the investor's incentive to evade taxes. The capital gains tax τ reduces the after-tax return of the risky asset, whereas the volatility risk premium improves its appeal. Tax evasion partially mitigates the tax burden. From Eq. (3.3), we find that when the volatility risk premium is sufficiently high, investors become more willing to accept tax and audit risks, shifting their portfolios toward both tax evading and non-evading risky assets while reducing their holdings of the risk-free asset.

In the following, we provide some special cases to illustrate the relationships between $\pi_0^*(t)$, $\pi_1^*(t)$, and some model parameters.

Remark 3.1. *In this paper, we assume $\tau r > \lambda_1\alpha$. This assumption is reasonable because if the tax levied on investing in risk-free securities is greater than the average penalty for investing in concealed risky assets, investors will choose to invest in the concealed risky assets. On the contrary, investors will consider investing in risk-free assets.*

Remark 3.2. *When α is sufficiently large such that investors face high punishment risk for tax evasion, they find that it is not optimal to choose to evade taxes. This is because, it follows from (3.1) that the limit of $\pi_0^*(t)$ is 0 when α approaches $+\infty$. The result is consistent with reality, and this reason is economically intuitive. The severity of punishment will naturally lead investors to not choose to evade taxes.*

Remark 3.3. *When $\alpha \to \infty$ and $\sigma = 0$, we find that the wealth optimally invested in risky assets is $\pi_1^*(t) = \frac{\lambda}{\gamma(1-\tau)}$. Let $\mu = r + \lambda\sqrt{m(t)}$ and $\sigma(t) = \sqrt{m(t)}$ denote the appreciation rate and the volatility of the risky asset, respectively. Then the optimal portfolio weight is given by $\pi_1^*(t) = \frac{\mu - r}{\gamma(1-\tau)\sqrt{m(t)}}$. This optimal allocation is higher than the optimal investment in the risky asset without taxation since $\tau \in (0,1)$. This result is consistent with [22].*

Remark 3.4. *From Eq. (3.1), we find that optimal tax evasion $\pi_0^*(t)$ is a constant. $\pi_0^*(t)$ is affected by neither the return nor the volatility of the risky asset. It is only related to the risk aversion coefficient, penalty coefficient, and audit intensity. This demonstrates that the optimal tax evasion strategy solely reflects taxpayers' trade-off between audit risk and penalty costs, while remaining independent of market parameters. Consequently, it may be more reasonable for regulators to curb evasion behavior by prioritizing adjustments to penalty coefficients rather than market parameters.*

Remark 3.5. *The tax rates on risky assets and risk-free assets are assumed to be identical in this paper. If different tax rates are permitted for risky assets and risk-free assets, although this would lead to corresponding adjustments in the optimal portfolio strategy, the investor's incentive to conceal returns on risky assets would still depend on the trade-off between tax benefits and expected enforcement costs. Therefore, the core logic of optimal tax evasion remains unchanged.*

3.2. Exponential utility

In this subsection, the investor's utility function is assumed to be an exponential utility function given as follows:

$$U(x) = -\frac{1}{\beta}\mathrm{e}^{-\beta x}, \tag{3.8}$$

where $\beta > 0$ is the coefficient of absolute risk aversion. In the following, we derive the optimal investment strategy by using the first-order optimality conditions. The explicit solutions of optimal investment strategies are presented in the following theorem.

Theorem 3.2. *In case of the exponential utility function, the optimal investment strategies $(\pi_0^*(t), \pi_1^*(t), \omega^*(t))$ are given as follows:*

$$\pi_0^*(t) = \frac{1}{\alpha\beta a(t)x}\ln\frac{r\tau}{\lambda_1\alpha}, \tag{3.9}$$

$$\pi_1^*(t) = \frac{1}{\beta a(t)x(1-\tau)}\left[\lambda + C(t)\sigma\rho - \frac{1}{\alpha}\ln\frac{r\tau}{\lambda_1\alpha}\right], \tag{3.10}$$

$$\omega^*(t) \; = 1 - \tfrac{1}{\beta a(t)x(1-\tau)}(\lambda + C(t)\sigma\rho) + \tfrac{\tau}{\beta a(t)x(1-\tau)\alpha}\ln\tfrac{r\tau}{\lambda_1\alpha}. \tag{3.11}$$

The optimal value function satisfies

$$J(t,x,m) = -\frac{1}{\beta}\mathrm{e}^{-\beta a(t)x + C(t)m + D(t)}.$$

Here, $a(t) = e^{r_1(T-t)}$*, and the function* $C(t)$ *is given as follows:*

$$C(t) = \frac{n_1 n_2 \mathrm{e}^{-\sqrt{\Delta_2}(T-t)} - n_1 n_2}{n_1 \mathrm{e}^{-\sqrt{\Delta_2}(T-t)} - n_2}. \tag{3.12}$$

The function $D(t)$ *is given by:*

$$D(t) = k\theta\int_t^T C(s)ds + \left[\frac{r\tau}{\alpha}\left(1 - \ln\frac{r\tau}{\lambda_1\alpha}\right) - \lambda_1\right](T-t). \tag{3.13}$$

For the proof, see the Appendix.

In the following, we establish several corollaries that highlight the impact of audit intensity, tax evasion punishment, the risk-free interest rate, and volatility risk on the optimal investment strategy. To simplify our analysis, let $\overline{\pi}_0^*(t) = \pi_0^*(t)x$, $\overline{\pi}_1^*(t) = \pi_1^*(t)x$. Here, $\overline{\pi}_0^*(t)$ and $\overline{\pi}_1^*(t)$ denote the amount of wealth invested in the concealed risky assets and risky assets, respectively, at time t.

Corollary 3.1. *The effects of the audit intensity (λ_1) and the severity of punishment for tax evasion (α) on the optimal investment strategies are given as follows:*

$$\frac{\overline{\pi}_0^*(t)}{\partial\lambda_1} < 0, \quad \frac{\overline{\pi}_1^*(t)}{\partial\lambda_1} > 0; \quad \frac{\overline{\pi}_0^*(t)}{\partial\alpha} < 0, \quad \frac{\overline{\pi}_1^*(t)}{\partial\alpha} > 0.$$

Proof. From Theorem 3.2, we have

$$\frac{\partial\overline{\pi}_0^*(t)}{\partial\lambda_1} = \frac{-1}{\alpha\beta a(t)\lambda_1},$$

and

$$\frac{\partial\overline{\pi}_1^*(t)}{\partial\lambda_1} = \frac{1}{\alpha\beta a(t)(1-\tau)\lambda_1}.$$

Note that the parameters α,β,λ_1 are positive constants and $0 < \tau < 1$. Therefore, $\frac{\partial\overline{\pi}_0^*(t)}{\partial\lambda_1} < 0$ and $\frac{\partial\overline{\pi}_1^*(t)}{\partial\lambda_1} > 0$. Furthermore, we find

$$\frac{\partial\overline{\pi}_0^*(t)}{\partial\alpha} = \frac{-1}{\alpha^2\beta a(t)}\left(\ln\frac{r\tau}{\lambda_1\alpha} + 1\right),$$

and

$$\frac{\partial\overline{\pi}_1^*(t)}{\partial\alpha} = \frac{1}{\alpha^2\beta a(t)(1-\tau)}\left(\ln\frac{r\tau}{\lambda_1\alpha} + 1\right).$$

Using the assumption $\tau r > \lambda_1 a$, we obtain $\frac{\partial\overline{\pi}_0^*(t)}{\partial\alpha} < 0$ and $\frac{\partial\overline{\pi}_1^*(t)}{\partial\alpha} > 0$. □

From Corollary 3.1, the optimal investment strategy $\overline{\pi}_0^*(t)$ is a decreasing function of both audit intensity and the severity of punishment for tax evasion. In contrast, $\overline{\pi}_1^*(t)$ exhibits the opposite behavior: it increases with higher audit intensity or more tax evasion penalties. This implies that governments or regulatory authorities can reduce investor tax evasion by increasing audit intensity and penalty severity.

Corollary 3.2. *The effects of risk-free interest rate r on optimal investment strategies are given as follows:*

$$\frac{\partial \overline{\pi}_0^*(t)}{\partial r} > 0, \quad \frac{\partial \overline{\pi}_1^*(t)}{\partial r} < 0, \quad \frac{\partial \overline{\omega}^*(t)}{\partial r} > 0.$$

Furthermore, $\overline{\pi}_0^*(t)$ and $\overline{\pi}_1^*(t)$ decrease (in absolute terms) as risk aversion coefficient β increases.

Proof. From Theorem 3.2, we have

$$\frac{\partial \overline{\pi}_0^*(t)}{\partial r} = \frac{1}{r\alpha\beta a(t)} > 0,$$

and

$$\frac{\partial \overline{\pi}_1^*(t)}{\partial r} = \frac{-1}{r\alpha a(t)\beta(1-\tau)} < 0, \quad \frac{\partial \overline{\omega}^*(t)}{\partial r} = \frac{\tau}{r\alpha\beta a(t)(1-\tau)} > 0. \tag{3.14}$$

Moreover, it follows from Eq. (3.9) that $\overline{\pi}_0^*(t)$ is a decreasing function with respect to β. In addition, Eq. (3.10) implies that $\overline{\pi}_1^*(t)$ is decreasing with respect to β if $\overline{\pi}_1^*(t)$ is positive. On the other hand, $\overline{\pi}_1^*(t)$ is an increasing function with respect to β if $\overline{\pi}_1^*(t)$ is negative. This means that $\overline{\pi}_0^*(t)$ and $\overline{\pi}_1^*(t)$ decrease (in absolute terms) as risk aversion coefficient β increases. □

From Corollary 3.2, we find that $\overline{\pi}_0^*(t)$ and $\overline{\omega}^*(t)$ are increasing functions of r. However, $\overline{\pi}_1^*(t)$ is a decreasing function of r. From Eq. (2.2), we find that a higher risk-free interest rate leads to a higher appreciation rate of the risky asset when the market risk price λ is fixed. To maximize profits, investors allocate more to the concealed risky asset and the risk-free bond. Consequently, they must invest less in the non-concealed risky asset. Furthermore, the higher the investors' risk aversion coefficient, the more risk-averse they become, leading them to naturally reduce their exposure to risky assets.

Corollary 3.3. *The effects of volatility risk on optimal investment strategies are given as follows:*

$$\frac{\partial \overline{\pi}_0^*(t)}{\partial \sigma} = 0, \quad \frac{\partial \overline{\pi}_1^*(t)}{\partial \sigma} < 0 \quad if \quad \rho > 0, \quad \frac{\partial \overline{\pi}_1^*(t)}{\partial \sigma} > 0 \quad if \quad \rho < 0.$$

Furthermore, $\frac{\partial \overline{\pi}_1^*(t)}{\partial \sigma} = 0$ when $\rho = 0$.

Proof. It is easy obtain $\frac{\partial \overline{\pi}_0^*(t)}{\partial \sigma} = 0$ from Eq. (3.9). In addition, note that $\hat{a} < 0$ and $\hat{c} > 0$, and then $n_1 n_2 = \frac{\hat{c}}{\hat{a}} < 0$, $n_1 < 0$, and $n_2 > 0$. This implies that $C(t) < 0$. Hence, $\rho > 0$ implies $\frac{\partial \overline{\pi}_1^*(t)}{\partial \sigma} = \frac{1}{\beta a(t)(1-\tau)} C(t)\rho < 0$. Additionally, $\rho < 0$ implies $\frac{\partial \overline{\pi}_1^*(t)}{\partial \sigma} > 0$ and $\rho = 0$ implies $\frac{\partial \overline{\pi}_1^*(t)}{\partial \sigma} = 0$. □

From Corollary 3.3, we find that volatility risk does not affect the optimal investment strategy invested in concealed risky assets. However, volatility risk negatively affects the optimal investment strategy invested in non-concealed risky assets when the correlation coefficient $\rho > 0$. Conversely, volatility risk has a positive impact on the optimal investment strategy invested in non-concealed risky assets when $\rho < 0$.

4. Numerical examples

In this section, we present numerical examples to illustrate the theoretical results derived in Section 3. Our focus is on the sensitivity analysis of model parameters and their impact on optimal investment strategies since the qualitative effects of model parameters on optimal investment strategies are identical under both power utility and exponential utility functions. Additional numerical checks under the exponential utility function (not reported for brevity) verify identical qualitative effects. Hence, we focus our numerical analysis in Section 4 exclusively on the power utility case for computational clarity and to avoid redundancy in economic interpretation. The corresponding analytical results for exponential utility are formally established through rigorous mathematical proofs in Section 3.2 (see Corollaries 3.1–3.3), which demonstrate the model parameters' effect on the optimal investment strategy.

In this section, we first establish baseline parameters for our model based on findings from existing empirical studies. As demonstrated in [31] and [32], Heston's stochastic volatility model's empirical characteristics have been investigated through two primary approaches: analysis of S&P 500 index time-series data ([33,34]) and combined analysis of S&P 500 index and options data ([35,36]). However, parameter estimates may vary across studies due to differing sample periods or methodologies. In the following Table 1, we provide the baseline parameter values used in the numerical analysis, along with these parameters' economic interpretations. Furthermore, our selected model parameters are chosen to be consistent with the existing portfolio choice and tax evasion literature (see [23, 31, 32]).

Table 1. The values of model parameters.

Symbol	Description	Value	Economic interpretation
λ_1	Audit intensity	0.1	The intensity of tax audits
τ	Capital gains tax rate	0.27	Proportional tax rate on risky asset returns
γ	Relative risk aversion coefficient	4	Measures the investor's aversion to risk
r	Risk-free interest rate	0.05	Constant risk-free return
$T-t$	Time to maturity	20	Remaining investment horizon
k	Speed of mean reversion	5	Speed at which volatility reverts to its long-term mean
λ	Market price of volatility risk	4	The risk premium per unit of stochastic volatility
σ	Volatility of variance	0.25	Controls the variability of stochastic volatility
ρ	Correlation coefficient	-0.4	Captures the leverage effect
α	Penalty coefficient	0.1	Severity of penalties upon detection of tax evasion

In Figure 1, we vary the risk aversion coefficient γ from 4 to 5. This illustrates the effect of the risk aversion coefficient γ and the tax rate τ on the optimal investment proportion strategy $\pi_0^*(t)$. The results show that as γ increases, investors allocate a smaller proportion of their wealth to the concealed risky assets S. This reflects a more conservative investment strategy under higher risk aversion, which naturally leads to decreased investments in risky assets. Furthermore, Figure 1 also illustrates the relationship between $\pi_0^*(t)$ and τ. It appears that parameter τ is positively correlated with the proportion of the wealth invested in the concealed risky assets. Since τ represents the tax rate on risky assets, investors are willing to take greater risks to conceal their holdings when the tax rate rises.

Figure 2 shows the impact of the audit intensity λ_1 and tax evasion punishment coefficient α on the optimal investment strategy $\pi_0^*(t)$. It is easy to find that these effects are very significant. When λ_1

is varied from 0.1 to 0.12, the optimal investment strategy $\pi_0^*(t)$ decreases as λ_1 increases. Since λ_1 represents the intensity of auditing risky assets, a higher λ_1 implies a greater likelihood of tax evasion being detected. Consequently, as λ_1 increases, investors face higher audit risk for the same α. Hence, investors would reduce their investment proportion in the concealed risky assets. Figure 2 further demonstrates how the optimal investment strategy $\pi_0^*(t)$ varies with different values of α. From Figure 2, it can be observed that parameter α is negatively correlated with the portfolio proportion allocated to concealed risky assets. Here, parameter α reflects the severity of punishment for tax evasion. When the penalties for tax evasion become more severe, investors naturally decrease their holdings of concealed risky assets. Therefore, we find that increasing the tax evasion punishment coefficient or the frequency of tax audits can effectively reduce the tax evasion behavior. However, Figure 2 also reveals that increasing the tax evasion punishment coefficient can more effectively suppress tax evasion compared to increasing the frequency of tax audits, which is consistent with [22] and [23].

As shown in Figure 3, there exists a statistically significant positive correlation between the volatility risk premium λ and non-concealed risky asset allocation $\pi_1^*(t)$. As the volatility risk premium increases, investors are compensated with higher risk adjusted returns, which allows them to actively increase their positions in risky assets. Moreover, audit intensity λ_1 exerts a positive effect on allocation in non-concealed risky assets. Numerical results in Figure 2 show that audit intensity leads to a significant reduction in tax evasion behavior among investors. This subsequently drives capital reallocation toward compliant assets, thus increasing the optimal allocation weight to non-concealed risky assets.

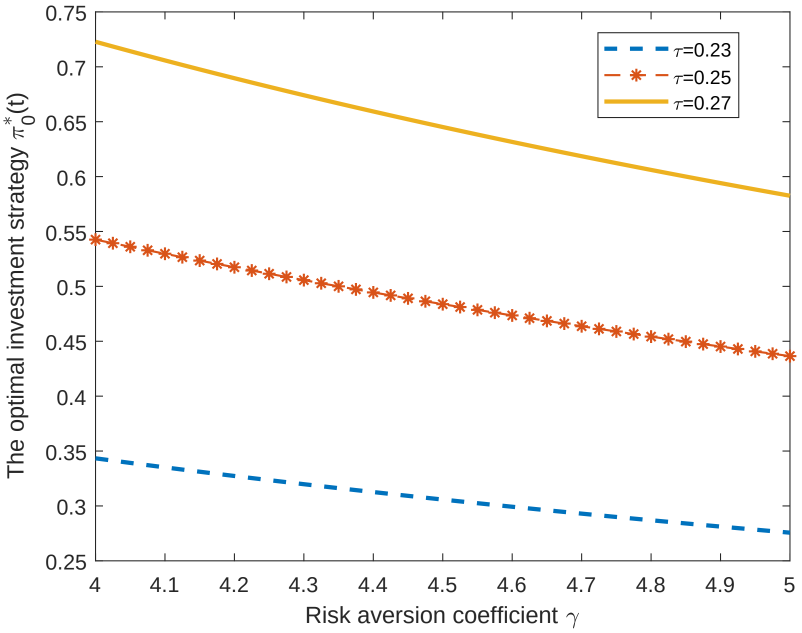

Figure 1. Impacts of τ and γ on the optimal investment strategy $\pi_0^*(t)$.

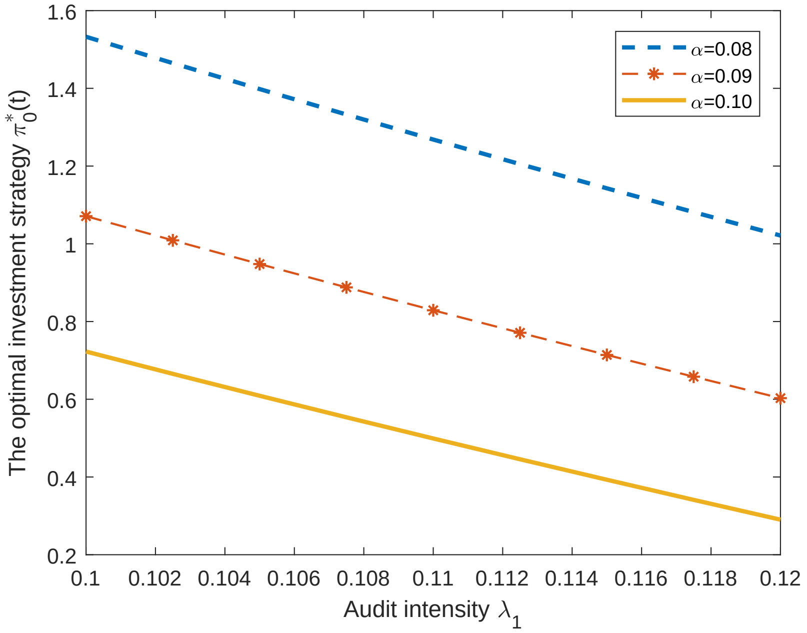

Figure 2. Impacts of α and λ_1 on the optimal investment strategy $\pi_0^*(t)$.

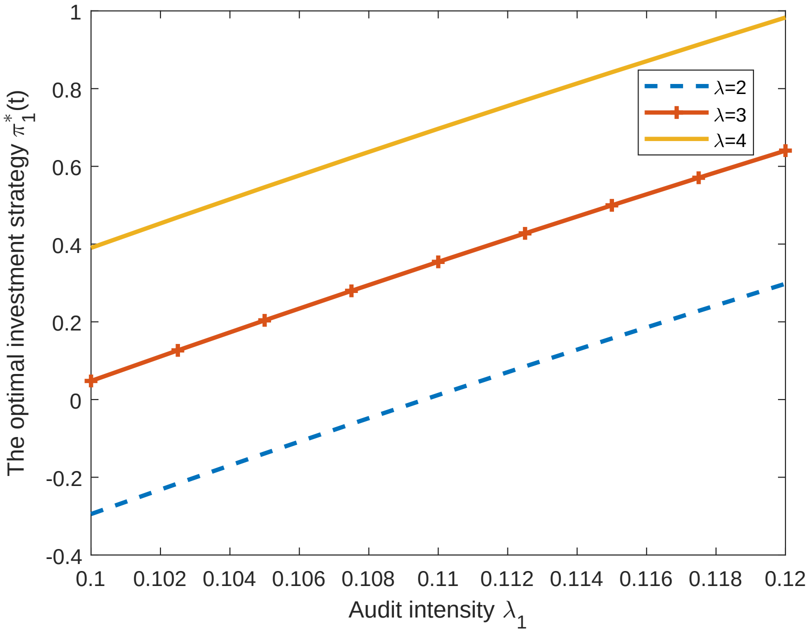

Figure 3. Impacts of λ and λ_1 on the optimal investment strategy $\pi_1^*(t)$.

Figure 4 depicts how the optimal investment strategy $\pi_1^*(t)$ is influenced by different values of α. In Figure 2, we observe that α is negatively correlated with the proportion of the wealth invested in the concealed risky assets. However, Figure 4 shows that, as α increases, investors choose to buy more non-concealed risky assets while reducing investment in the concealed risky assets. This is because a higher α signifies a greater penalty for investing in concealed risky assets, which aligns with the behavior of rational investors. In addition, r represents the risk-free interest rate. A higher interest rate indicates a greater return on the risk-free asset. Hence, investors increase the proportion of wealth used to buy the risk-free bond and reduce their investment in risky assets. This is also consistent with economic intuition.

Furthermore, we analyze the effects of the coefficient of risk aversion γ and the tax rate τ on the optimal investment strategy $\pi_1^*(t)$ in Figure 5. We find that investors are more aggressive when the coefficient of risk aversion is smaller. This is reasonable because a lower risk aversion coefficient indicates a greater willingness to take risks, leading investors to allocate more wealth to risky assets. Additionally, as Figure 6 illustrates, when the tax rate τ increases, investors will reduce the proportion of wealth allocated to non-concealed risky assets. This is because investors face a higher tax rate when τ increases, hence investors are willing to bear more risks of being audited and punished for tax evasion, and choose to invest a greater proportion in concealed risky assets. This leads to a decrease in the proportion of investment in non-concealed risky assets.

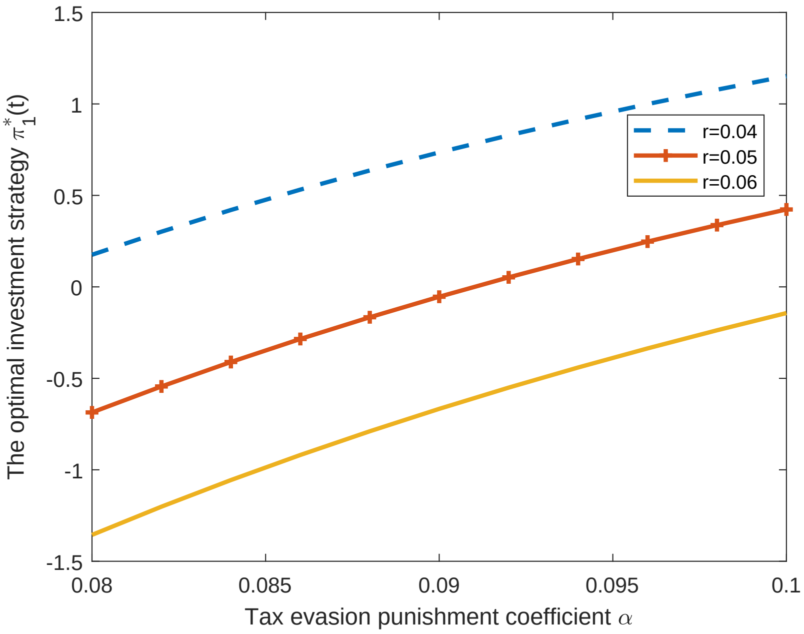

Figure 4. Impacts of r and α on the optimal investment strategy $\pi_1^*(t)$.

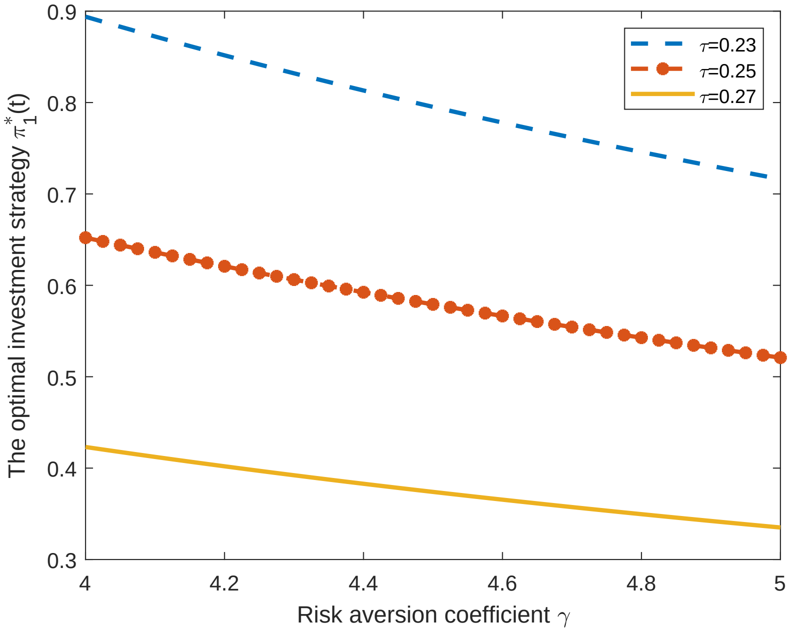

Figure 5. Impacts of τ and γ on the optimal investment strategy $\pi_1^*(t)$.

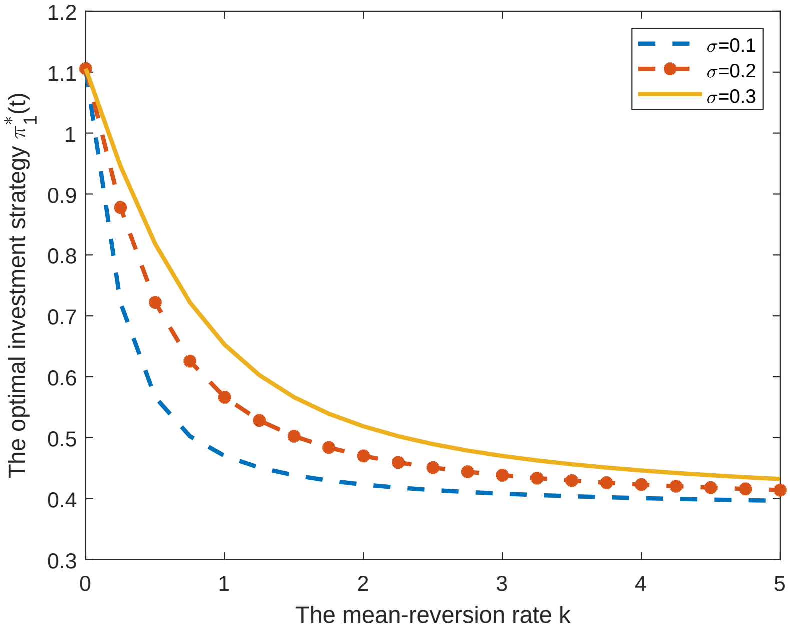

Figure 6. Impacts of k on the optimal investment strategy $\pi_1^*(t)$.

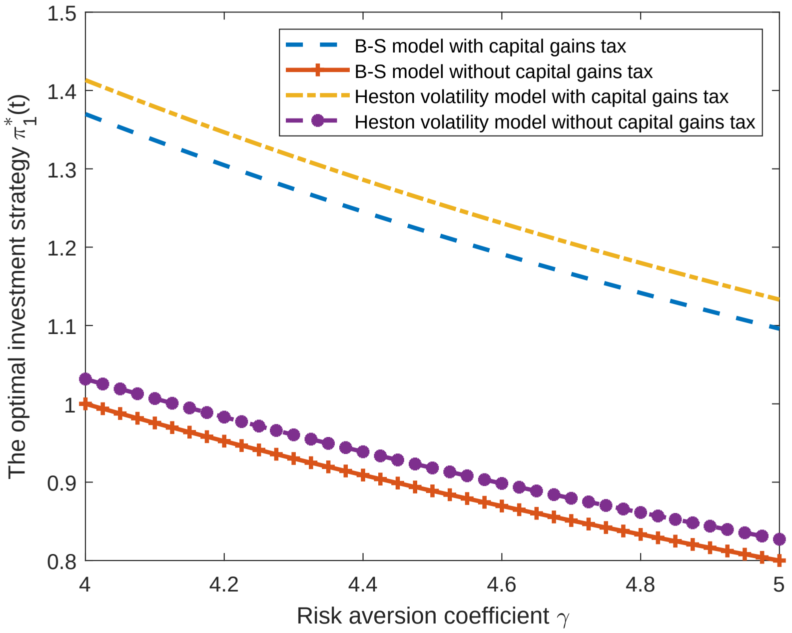

Figure 7. The optimal investment strategy $\pi_1^*(t)$ under different models.

Figure 6 demonstrates how the mean reversion rate k and volatility coefficient σ influence the optimal investment strategy $\pi_1^*(t)$. From this figure, we find that a lower k indicates slower mean reversion in volatility. This persistence increases the additional volatility risk. Higher σ brings more extreme volatility fluctuations of risky asset S. These results align with financial theory: when volatility becomes more persistent (low k) or more variable (high σ), investors require greater compensation for volatility risk. This risk compensation manifests in larger absolute values of the optimal investment strategy $\pi_1^*(t)$. The relationship between the mean reversion rate k, the volatility coefficient σ, and the optimal investment strategy $\pi_1^*(t)$ is consistent with [31] and [32].

Finally, Figure 7 presents a comparative analysis of optimal investment strategies between Heston's stochastic volatility model and the classical Black-Scholes framework. Our results demonstrate that when markets exhibit significant volatility risk premium ($\lambda > 0$), Heston's model recommends substantially higher allocations to risky assets than does the Black-Scholes model. This finding is also supported by [31], which attributes this difference to the volatility risk premium component embedded in Heston's model. Specifically, this premium compensates investors for bearing additional volatility risk. Furthermore, our analysis also demonstrates that capital gains taxation significantly impacts optimal investment strategies. Under the assumption of equal tax rates for both risky and risk-free assets, we find that the optimal allocation to risky assets is higher in the capital-gains-tax scenario compared to the no-capital-gains-tax case.

5. Conclusions

We consider an optimal investment problem incorporating capital gains tax and volatility risk. The volatility of the risky asset follows Heston's stochastic volatility model. The investor dynamically allocates his or her wealth between a risk-free bond and a risky asset. In particular, investors' tax evasion behavior and the audit penalties imposed by the government are also considered in this paper. The goal of the investor is to maximize the utility of the terminal wealth. Optimal investment strategies and the value functions are explicitly derived under the power utility and exponential utility, respectively. The numerical results show that the capital gains tax and volatility risk have a significant impact on optimal policies. The model in this paper extends the benchmark Heston portfolio rule and differs from tax-only or constant-volatility evasion models. Furthermore, the optimal tax evasion strategy is independent of volatility risk, but is related to audit intensity and penalty severity. Hence, the government can reduce investor tax evasion by increasing the intensity of audits and penalties. This is consistent with economic intuition. Although our research results indicate that higher audit intensity and stronger penalties are rather effective in deterring tax evasion, it is important to realize that these enforcement instruments are also associated with supervision and administrative costs. In fact, increasing audit intensity requires additional regulatory resources, and excessively severe penalties may bring diminishing marginal deterrence. Hence, optimal policy design should balance the marginal benefits of tax evasion deterrence against the corresponding enforcement costs. In a more volatile financial market, a higher audit intensity may be justified, but in a more stable financial market, moderate enforcement may lead to a similar compliance result at a lower cost. Hence, it indicates that adopting differentiated enforcement intensities and strategies across different market conditions may be more effective than using a uniform standard.

Author contributions

All authors contributed equally to the paper.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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