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Theory article

Scalarization, convergence and well-posedness in set optimization

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Abstract: In this paper, we introduce the notions of weak m-minimal approximate solutions and m-minimal approximate solutions for constrained set optimization problems, based on a novel set order relation that involves the Minkowski difference. We derive scalarization results for the sets of weak m-minimal approximate solutions and m-minimal approximate solutions in the context of set optimization. Utilizing these scalarizations, we analyze the Painlevé-Kuratowski convergence properties of both classes of approximate solutions. Furthermore, new notions of well-posedness for set optimization problems are proposed, and relationships between these notions are rigorously established. Finally, we establish equivalences between the well-posedness of set optimization problems and their scalar counterparts through carefully constructed optimization frameworks.

Keywords: approximate solution; scalarization; Painlevé-Kuratowski convergence; set optimization; well-posedness
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1. Introduction

Set-valued optimization constitutes a fundamental subfield of applied mathematics. In recent years, problems in this area have attracted considerable research interest due to their broad utility in various disciplines, including vector variational inequalities, optimal control scenarios, game theory, mathematical economics, and others. For an in-depth understanding, refer to [1–3] and the citations contained therein. When tackling set-valued optimization problems, two prevalent methodologies for solution identification are the vector criterion and the set criterion. Among these, the set criterion, which was introduced by Kuroiwa [4], appears particularly intriguing and practical. This criterion hinges on each value of the set-valued map, thus necessitating direct comparisons of objective function values. Leveraging the Minkowski difference, Karaman et al. [5] introduced novel partial order relations within the realm of nonempty, bounded sets.

Approximate solutions are of great importance in the optimization theory, as algorithms typically produce solutions that are only approximate. A variety of concepts for approximate solutions in set-valued optimization have been explored in the literature, as referenced in [6–11]. Li and Xu [10] introduced the scalarization theorem, Lagrangian multiplier theorem, and saddle point theorems for ϵ-strictly efficient solutions. In [7], Dhingra and Lalitha proposed an approximate solution concept based on set-less order relations and further derived its stability and compactness properties. Han et al [9] derived the Painlevé-Kuratowski convergence for approximate solution sets in set optimization problems, thereby assuming convexity and continuity of the objective mappings. Anh et al. [6] studied internal and external stability along with the Hausdorff convergence of solution sets in set optimization using set-less order relations. Their analysis included internal stability for ideal and weak minimal solutions under assumptions of continuity and compactness. Gupta and Srivastava [8] introduced an approximate solution concept in set optimization and examined properties including closedness and convergence of the solution sets.

The concept of well-posedness in optimization problems, which has now become well-established, has been delved into within the realms of vector and set-valued optimization. This concept finds numerous applications in the stability theory, sensitivity analysis, and other fields, as referenced in [1, 12–16]. Gupta and Srivastava [17] proposed new well-posedness concepts for set optimization based on the upper set-less relation, establishing necessary and sufficient conditions for their characterization. Long et al. [18] investigated three forms of pointwise well-posedness in set optimization and analyzed their interrelationships. Using a generalized nonlinear scalarization approach, they proved the equivalence of these well-posedness types with corresponding ones in scalar optimization problems. Gupta and Srivastava [8] introduced a Levitin-Polyak well-posedness framework for set optimization, thereby providing sufficient conditions and characterizations for this property. Li and Xu [19] introduced fresh types of well-posedness for set optimization problems using the closed unit ball and derived the necessary and sufficient conditions through the application of the m-relation.

Motivated by the preceding research, this paper introduces novel concepts of approximate solutions for constrained set optimization problems and characterizes these solutions through scalarizations. We further examine the Painlevé-Kuratowski convergence of approximate solutions under perturbations in set-valued optimization. To address the concerns raised in Remark 5.8 of [19], we propose several new types of well-posedness.

The paper is organized as follows: Section 2 summarizes key notations and fundamental lemmas. Sections 3 re-examines scalarization functions while proposing two novel scalarization theorems applicable to set optimization problems; Section 4 rigorously formulates the Painlevé-Kuratowski convergence for both weak minimal and minimal approximate solutions under parametric perturbations of set-valued systems; in Section 5, we introduce the concepts of L-well-posedness and DH-well-posedness for set optimization problems and discuss the interrelationships among these well-posedness types; and finally, by employing a generalized nonlinear scalarization approach, we establish an equivalence between the well-posedness of set optimization and corresponding scalar optimization problems.

2. Preliminaries

Throughout this paper, we assume that X and Y are real normed linear spaces, with $K \subseteq Y$ being a convex, closed, pointed cone that has a nonempty interior. Let $P(Y)$ and $B(Y)$ denote the families of nonempty proper subsets and nonempty bounded proper subsets of Y, respectively. Additionally, for any subset $A \subseteq Y$, we use $\mathrm{int}A$ to denote its topological interior and $\mathrm{cl}A$ to denote its topological closure.

The Minkowski difference of sets $A, B \in P(Y)$, introduced in [20], is defined as follows:

$$A \dot{-} B := \{y \in Y : y + B \subseteq A\}.$$

Karaman et al. [5] proposed the following order relations on $P(Y)$:

$$A \leq^m B \Leftrightarrow (B \dot{-} A) \cap K \neq \emptyset,$$

and

$$A <^m B \Leftrightarrow (B \dot{-} A) \cap (\mathrm{int}K) \neq \emptyset.$$

Clearly, $\leq^m$ is a partial order relation on $B(Y)$ [5], meaning that $\leq^m$ is reflexive, transitive, and antisymmetric.

Inspired by [9,21], we introduce two types of m-relations. Let e be a fixed point in $\mathrm{int}K$. For $\epsilon \geq 0$, the $\epsilon - m$ relation, denoted as " $\leq^m_\epsilon$ ", and the weak $\epsilon - m$ relation, denoted as " $<^m_\epsilon$ ", are defined as follows:

$$A \leq^m_\epsilon B \Leftrightarrow A \leq^m (B - \epsilon e) \Leftrightarrow ((B - \epsilon e) \dot{-} A) \cap K \neq \emptyset,$$

and

$$A <^m_\epsilon B \Leftrightarrow A <^m (B - \epsilon e) \Leftrightarrow ((B - \epsilon e) \dot{-} A) \cap (\mathrm{int}K) \neq \emptyset,$$

respectively.

Let $F : X \to 2^Y$ be a set-valued function, and let $G \subseteq X$ be a nonempty set. We consider the following set optimization problem:

$$\begin{array}{ll} (\mathrm{SOP}) & \min \; F(x) \\ & \text{s.t. } \; x \in G. \end{array}$$

Throughout this work, we assume $F(x) \neq \emptyset$ for every $x \in G$ and define $F(G) := \bigcup_{x \in G} F(x)$.

Now, we introduce the concepts of (approximate) minimal and weak minimal solutions for (SOP). To formulate these solution concepts, we assume $F(x) \in B(Y)$ holds for every $x \in G$.

Definition 2.1. [5] An element $\bar{x} \in G$ is said to be as follows:

(i) an m-minimal solution of (SOP) if there does not exist any $x \in G$ such that $F(x) \leq^m F(\bar{x})$ and $F(x) \neq F(\bar{x})$, that is, for any $x \in G$, either $F(x) \nleq^m F(\bar{x})$ or $F(x) = F(\bar{x})$; and

(ii) a weak m-minimal solution of (SOP) if there does not exist any $x \in G$ such that $F(x) <^m F(\bar{x})$.

Let $E_m(G)$ and $W_m(G)$ denote the m-minimal solution set of (SOP) and the weak m-minimal solution set of (SOP), respectively.

Definition 2.2. For $\epsilon \geq 0$, a point $\bar{x} \in G$ is called the following:

(i) an ϵ-m-minimal approximate solution of (SOP) if there is no $x \in G$ such that $F(x) \leq^m_\epsilon F(\bar{x})$ and $F(x) \neq F(\bar{x})$, that is, either $F(x) \nleq^m_\epsilon F(\bar{x})$ or $F(x) = F(\bar{x})$, for any $x \in G$; and

(ii) a weak ϵ-m-minimal approximate solution of (SOP) if there is no $x \in G$ such that $F(x) <^m_\epsilon F(\bar{x})$.

Let $E_m(\epsilon, G)$ and $W_m(\epsilon, G)$ denote the ϵ-m-minimal approximate solution set of (SOP) and the weak ϵ-m-minimal approximate solution set of (SOP), respectively.

Proposition 2.3. *The following assertions are true:*

(i) If $\epsilon \geq 0$*, then* $E_m(\epsilon, G) \subseteq W_m(\epsilon, G)$*;*

(ii) If $\epsilon \geq 0$*, then* $E_m(G) \subseteq E_m(\epsilon, G)$ *and* $W_m(G) \subseteq W_m(\epsilon, G)$*; and*

(iii) If $\epsilon > 0$*, then* $W_m(G) \subseteq E_m(\epsilon, G)$*.*

Proof. (i) Let $\bar{x} \in E_m(\epsilon, G)$. Suppose that there exists $y \in G$ such that $F(y) <^m_\epsilon F(\bar{x})$; then, there exists $k \in \text{int}K \subseteq K$ such that

$$k + F(y) \subseteq F(\bar{x}) - \epsilon e.$$

This shows that $F(y) \leq^m_\epsilon F(\bar{x})$ and $F(y) \neq F(\bar{x})$. This contradicts the fact that $\bar{x} \in E_m(\epsilon, G)$. Therefore, $\bar{x} \in W_m(\epsilon, G)$ and so $E_m(\epsilon, G) \subseteq W_m(\epsilon, G)$.

(ii) Let $\bar{x} \in E_m(G)$. Suppose that there exists $y \in G$ such that $F(y) \leq^m_\epsilon F(\bar{x})$ and $F(y) \neq F(\bar{x})$. Then, there exists $k \in K$ such that

$$k + F(y) \subseteq F(\bar{x}) - \epsilon e.$$

This shows that

$$\epsilon e + k + F(y) \subseteq F(\bar{x}).$$

By $\epsilon e + k \in K$, one has $F(y) \leq^m F(\bar{x})$ and $F(y) \neq F(\bar{x})$. This contradicts the fact that $\bar{x} \in E_m(G)$. Therefore, $\bar{x} \in E_m(\epsilon, G)$ and so $E_m(G) \subseteq E_m(\epsilon, G)$. Similarly, we can prove that $W_m(G) \subseteq W_m(\epsilon, G)$.

(iii) Let $\bar{x} \in W_m(G)$. Suppose that there exists $y \in G$ such that $F(y) \leq^m_\epsilon F(\bar{x})$ and $F(y) \neq F(\bar{x})$. Then there exists $k \in K$ such that

$$k + F(y) \subseteq F(\bar{x}) - \epsilon e.$$

This shows that

$$\epsilon e + k + F(y) \subseteq F(\bar{x}).$$

By $\epsilon e + k \in \text{int}K$, one has $F(y) <^m F(\bar{x})$. This contradicts the fact that $\bar{x} \in W_m(G)$. Therefore, $\bar{x} \in E_m(\epsilon, G)$, and so $W_m(G) \subseteq E_m(\epsilon, G)$. □

Remark 2.4. (i) $E_m(\epsilon, G)$ and $W_m(\epsilon, G)$ depend on the choice of $e \in \text{int}K$.

(ii) When $\epsilon = 0$, the concept of a (weak) ϵ-m-minimal approximate solution reduces to the concept of a (weak) m-minimal solution for (SOP).

(iii) When F is single-valued, Definition 2.2 coincides with the concept of an approximate efficient solution in vector optimization, as introduced in [22].

(iv) The reverse subset inclusions stated in Proposition 2.3 may not necessarily be true, as illustrated by Example 2.5.

Example 2.5. *Let* $\epsilon = 0.2$, $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $e = (1, 1)$*, and* $G = [0, 4]$*. Define the set-valued mapping* $F : G \to 2^Y$ *by the following:*

$$F(x)=\begin{cases}\{(0,0)\}, & if\ 0\le x\le 1,\\ (-1,0]\times(-1,0], & if\ 1<x\le 2,\\ \{(0.15,0.15)\}, & if\ 2<x\le 3,\\ \{(0.2,0.2)\}, & if\ 3<x\le 4.\end{cases}$$

Here, $E_m(G)=[0,1]$, $W_m(G)=[0,2]$, $E_m(\epsilon,G)=[0,3]$, and $W_m(\epsilon,G)=[0,4]$.

Definition 2.6. [2] Let $F: X\to 2^Y$ be a set-valued mapping. Then, F is said to be the following:

(i) upper semicontinuous (u.s.c.) at $\bar{x}$ if, for any neighborhood U of $F(\bar{x})$, there is a neighborhood $N(\bar{x})$ of $\bar{x}$ such that for every $x\in N(\bar{x})$, $F(x)\subseteq U$; and

(ii) lower semicontinuous (l.s.c.) at $\bar{x}$ if, for any open subset U of Y with $F(\bar{x})\cap U\neq\emptyset$, there is a neighborhood $N(\bar{x})$ of $\bar{x}$ such that $F(x)\cap U\neq\emptyset$ for all $x\in N(\bar{x})$.

We say that F is u.s.c. and l.s.c. on X if it is u.s.c. and l.s.c. at each point $x\in X$, respectively. We say that F is continuous on X if it is both u.s.c. and l.s.c. on X.

Lemma 2.7. *[2] A set-valued mapping $F: X\to 2^Y$ is l.s.c. at $\bar{x}\in X$ if and only if for any sequence $\{x_n\}\subseteq X$ with $x_n\to\bar{x}$ and for any $\bar{y}\in F(\bar{x})$, there exists $y_n\in F(x_n)$ such that $y_n\to\bar{y}$.*

Lemma 2.8. *[2] Let $F: X\to 2^Y$ be a set-valued mapping. For any given $\bar{x}\in X$, if $F(\bar{x})$ is compact, then F is u.s.c. at $\bar{x}\in X$ if and only if for any sequence $\{x_n\}\subseteq X$ with $x_n\to\bar{x}$ and for any $y_n\in F(x_n)$, there exist $\bar{y}\in F(\bar{x})$ and a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k}\to\bar{y}$.*

3. Scalarization for the approximate solution sets

To address set optimization challenges, Karaman et al. [5] introduced a scalarization function denoted as $I^m_e(.,.): P(Y)\times P(Y)\to\mathbb{R}\bigcup\{\pm\infty\}:=\overline{\mathbb{R}}$. This function is defined as follows:

$$I^m_e(A,B)=\inf\{t\in\mathbb{R}\mid A\le^m te+B\},$$

for all $A,B\in P(Y)$, where $e\in \mathrm{int}K$.

The following propositions enumerate some of the properties associated with I^m_e.

Proposition 3.1. *[5] Let $A,B,C\in P(Y)$. The function $I^m_e(.,.): P(Y)\times P(Y)\to\overline{\mathbb{R}}$ satisfies the following properties:*

(i) If $A\in B(Y)$, then $I^m_e(A,A)=0$;

(ii) For any $r\in\mathbb{R}$, $I^m_e(A,B)<r$ if and only if $A<^m re+B$;

(iii) Suppose $B\dot{-}A$ is compact; then $A\le^m B$ if and only if $I^m_e(A,B)\le 0$;

(iv) $I^m_e(.,B)$ is m-increasing on $P(Y)$, that is, if $A\le^m C$, then $I^m_e(A,B)\le I^m_e(C,B)$; and

(v) If B is compact, then $I^m_e(.,B)$ is strictly m-increasing on $P(Y)$, that is, if $A<^m C$, then $I^m_e(A,B)<I^m_e(C,B)$.

Proposition 3.2. *Let $A,B,C,D\in P(Y)$. The following assertions hold:*

(i) $I^m_e(A,B+me)=I^m_e(A,B)-m$ for every $m\in\mathbb{R}$;

(ii) $I^m_e(A+me,B)=I^m_e(A,B)+m$ for every $m\in\mathbb{R}$;

(iii) Suppose $I^m_e(,)$ is finite; then $I^m_e(A,B)\le I^m_e(A,C)+I^m_e(C,B)$; and

(iv) Suppose $I^m_e(,)$ is finite; then $I^m_e(A+D,B+C)\le I^m_e(A,B)+I^m_e(D,C)$.

Proof. (i) Let $\Upsilon^m_B = \{t \in \mathbb{R} \mid A \leq^m te + B\}$; then,

$$t \in \Upsilon^m_{B+me} \Leftrightarrow A \leq^m te + B + me \Leftrightarrow A \leq^m (t+m)e + B \Leftrightarrow t + m \in \Upsilon^m_B \Leftrightarrow t \in \Upsilon^m_B - m.$$

Hence, $I^m_e(A, B + me) = I^m_e(A, B) - m$ for every $m \in \mathbb{R}$.

(ii) The proof can be obtained similar to (i).

(iii) Let $\alpha := I^m_e(A, C)$ and $\beta := I^m_e(C, B)$. For any $\epsilon > 0$, it is clear that $I^m_e(A, C) < \alpha + \epsilon$ and $I^m_e(C, B) < \beta + \epsilon$. Thanks to Proposition 3.1 (ii), we get $A <^m (\alpha + \epsilon)e + C$ and $C <^m (\beta + \epsilon)e + B$. Then, $A <^m (\alpha + \beta + 2\epsilon)e + B$. By the arbitrariness of $\epsilon > 0$, we obtain $I^m_e(A, B) \leq \alpha + \beta = I^m_e(A, C) + I^m_e(C, B)$. This completes the proof.

(iv) The proof can be obtained similar to (iii). □

Now, we establish key scalarization principles for weak m-minimal and m-minimal approximate solutions in set optimization. Throughout our analysis, we impose the standing assumption that all set-valued mappings $F(x)$ constitute nonempty bounded sets for every $x \in G$. Therefore, $F(x) \dot{-} F(y)$ is bounded and $F(x) \dot{-} F(y) \neq Y$ for all $x, y \in G$. Given a fixed point $\bar{x} \in G$, we formulate the associated scalar optimization problem that corresponds to (SOP) as follows:

$$(\mathrm{SP}_{\bar{\mathrm{x}}}) \quad \min \ I^m_e(F(x), F(\bar{x})) \\ \text{s.t. } \ x \in G.$$

Building upon this foundation, we propose a novel approximate solution for the scalarized problem $(\mathrm{SP}_{\bar{\mathrm{x}}})$.

Definition 3.3. A point $x_0 \in G$ is said to be an ϵ-minimal solution of $(\mathrm{SP}_{\bar{\mathrm{x}}})$ if

$$I^m_e(F(x_0), F(\bar{x})) \leq I^m_e(F(x), F(\bar{x})) + \epsilon,$$

for all $x \in G$.

By Proposition 3.1 (i), we observe that $x_0 \in G$ is an ϵ-minimal solution of $(\mathrm{SP}_{\mathrm{x}_0})$ if

$$I^m_e(F(x), F(x_0)) \geq -\epsilon,$$

for all $x \in G$.

Definition 3.4. A point $x_0 \in G$ is said to be a weak ϵ-minimal solution of $(\mathrm{SP}_{\bar{\mathrm{x}}})$ if

$$I^m_e(F(x_0), F(\bar{x})) < I^m_e(F(x), F(\bar{x})) + \epsilon,$$

for all $x \in G$.

By Proposition 3.1 (i), we observe that $x_0 \in G$ is a weak ϵ-minimal solution of $(\mathrm{SP}_{\mathrm{x}_0})$ if

$$I^m_e(F(x), F(x_0)) > -\epsilon,$$

for all $x \in G$.

Theorem 3.5. *Let $\epsilon > 0$ and $F : X \to 2^Y$ be compact-valued on $G \subseteq X$. Then, $\bar{x} \in G$ is an ϵ-m-minimal approximate solution of* (SOP) *if and only if $\bar{x}$ is a weak ϵ-minimal solution of* $(\mathrm{SP}_{\bar{\mathrm{x}}})$ *(i.e., $I^m_e(F(x), F(\bar{x})) > -\epsilon$).*

Proof. Assume $\bar{x} \in E_m(\epsilon, G)$. Then, we get $F(x) \nleq^m F(\bar{x}) - \epsilon e$ for all $x \in G \backslash \{\bar{x}\}$ and $F(x) \neq F(\bar{x})$. Applying Proposition 3.1 (i) (iii) with Proposition 3.2 (i), we derive the following:

$$I_e^m(F(x), F(\bar{x})) + \epsilon = I_e^m(F(x), F(\bar{x}) - \epsilon e) > 0 = I_e^m(F(\bar{x}), F(\bar{x})).$$

For $F(x) = F(\bar{x})$, a direct evaluation yields $I_e^m(F(x), F(\bar{x})) = I_e^m(F(\bar{x}), F(\bar{x})) = 0$. Hence,

$$I_e^m(F(x), F(\bar{x})) + \epsilon > 0, \quad \forall x \in G,$$

thus establishing $\bar{x}$ is a weak ϵ-minimal solution of $(\mathrm{SP}_{\bar{x}})$.

Conversely, on the contrary, suppose there exists $x \in G$ such that

$$F(x) \leq^m F(\bar{x}) - \epsilon e, \;\text{ and }\; F(x) \neq F(\bar{x}).$$

Through Propositions 3.1 (i), 3.1 (iv), and 3.2 (ii), we obtain the following:

$$I_e^m(F(x), F(\bar{x})) \leq I_e^m(F(\bar{x}) - \epsilon e, F(\bar{x})) = -\epsilon.$$

This implies $I_e^m(F(x), F(\bar{x})) + \epsilon \leq 0$, thus establishing a contradiction. Consequently, we conclude $\bar{x} \in E_m(\epsilon, G)$. □

Theorem 3.6. *Let $\epsilon \geq 0$ and $F : X \to 2^Y$ be compact-valued on $G \subseteq X$. Then, $\bar{x} \in G$ is a weak ϵ-m-minimal approximate solution of* (SOP) *if and only if $\bar{x}$ is an ϵ-minimal solution of* $(\mathrm{SP}_{\bar{x}})$ *(i.e., $I_e^m(F(x), F(\bar{x})) \geq -\epsilon$).*

Proof. If $\bar{x} \in W_m(\epsilon, G)$ and $\bar{x}$ is not an ϵ-minimal solution of $(\mathrm{SP}_{\bar{x}})$, then there exists $x \in G$ such that $I_e^m(F(\bar{x}), F(\bar{x})) > I_e^m(F(x), F(\bar{x})) + \epsilon$. Applying Proposition 3.1 (ii) with Proposition 3.2 (i), we derive $I_e^m(F(x), F(\bar{x}) - \epsilon e) < 0$. Using Proposition 3.1 (ii), we obtain the following:

$$F(x) <^m F(\bar{x}) - \epsilon e,$$

which contradicts the fact that $\bar{x} \in W_m(\epsilon, G)$.

On the contrary, suppose that there exists $x \in G$ such that

$$F(x) <^m F(\bar{x}) - \epsilon e.$$

By Proposition 3.1 (ii), we obtain the following:

$$I_e^m(F(x), F(\bar{x})) < -\epsilon,$$

which leads to a contradiction. □

By taking $\epsilon = 0$ in Theorem 3.6, we obtain

Corollary 3.7. *Let $F : X \to 2^Y$ be compact-valued on $G \subseteq X$. Then, $\bar{x} \in G$ is a weak m-minimal solution of* (SOP) *if and only if $I_e^m(F(x), F(\bar{x})) \geq 0$, for every $x \in G$.*

Remark 3.8. (i) Similar results concerning l-(weak) minimal and approximate solutions in set optimization were established in Corollary 4.12 of [23], Corollary 5.1 of [7], and Theorem 4.3 of [24].

(ii) A comparison reveals key differences. While the aforementioned results in [23], [7], and [24] are based on the (weak) lower set-less relation, our Theorems 3.5-3.6 and Corollary 3.7 are established under the (weak) m-set relation. According to Remark 2.10 in [19] and Example 2.2 in [25], the set of weak ϵ-m-minimal approximate solutions is broader than that of weak l-minimal approximate solutions, and there is no general inclusion relationship between ϵ-m-minimal and l-minimal approximate solutions. This implies that our Corollary 3.7 and Theorems 3.6 apply to a larger solution space than their counterparts in [23] and [7]. Furthermore, our Theorem 3.5 is derived under a fundamentally different framework and therefore is not directly comparable to Theorem 4.3 in [24].

Let $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ and $\mathbb{R}_+^0 = \{x \in \mathbb{R} : x > 0\}$. Now, we define two set-valued mappings $\Gamma : \mathbb{R}_+ \times G \to 2^G$ and $\Lambda : \mathbb{R}_+^0 \times G \to 2^G$ as follows:

$$\Gamma(\epsilon, x) = \{v \in G : I_e^m(F(y), F(x)) + \epsilon \geq I_e^m(F(v), F(x)), \forall y \in G\} \ \ \forall (\epsilon, x) \in \mathbb{R}_+ \times G,$$

and

$$\Lambda(\epsilon, x) = \{v \in G : I_e^m(F(y), F(x)) + \epsilon > I_e^m(F(v), F(x)), \forall y \in G\} \ \ \forall (\epsilon, x) \in \mathbb{R}_+^0 \times G.$$

We denote $\Gamma(\epsilon, G) := \bigcup_{x \in G} \Gamma(\epsilon, x)$ and $\Lambda(\epsilon, G) := \bigcup_{x \in G} \Lambda(\epsilon, x)$.

Theorem 3.9. *Let $\epsilon \geq 0$. Assume that $F(x)$ is compact for any $x \in G$. Then,*

$$\Gamma(\epsilon, G) = W_m(\epsilon, G).$$

Proof. Let $v_0 \in \Gamma(\epsilon, G)$. Then, there exists $x_0 \in G$ such that $v_0 \in \Gamma(\epsilon, x_0)$; thus,

$$I_e^m(F(y), F(x_0)) + \epsilon \geq I_e^m(F(v_0), F(x_0)), \forall y \in G. \tag{3.1}$$

Suppose that $v_0 \notin W_m(\epsilon, G)$, and there exists $y_0 \in G$ such that $F(y_0) <^m F(v_0) - \epsilon e$. Combining Proposition 3.1(v) with Proposition 3.2 (ii) yields the following:

$$I_e^m(F(y_0), F(x_0)) < I_e^m(F(v_0) - \epsilon e, F(x_0)) = I_e^m(F(v_0), F(x_0)) - \epsilon,$$

which contradicts (3.1). Hence, $v_0 \in W_m(\epsilon, G)$, and so $\Gamma(\epsilon, G) \subseteq W_m(\epsilon, G)$.

Let $\bar{x} \in W_m(\epsilon, G)$; then, we conclude from Theorem 3.6 and Proposition 3.1 (i) that

$$I_e^m(F(y), F(\bar{x})) + \epsilon \geq 0 = I_e^m(F(\bar{x}), F(\bar{x})),$$

that is,

$$I_e^m(F(y), F(\bar{x})) + \epsilon \geq I_e^m(F(\bar{x}), F(\bar{x})), \forall y \in G.$$

This means that

$$\bar{x} \in \Gamma(\epsilon, \bar{x}) \subseteq \Gamma(\epsilon, G);$$

thus, $W_m(\epsilon, G) \subseteq \Gamma(\epsilon, G)$. This completes the proof. □

Similarly to Theorem 3.9, we can get the following theorem.

Theorem 3.10. *Let $\epsilon > 0$. Assume that $F(x)$ is compact for any $x \in G$. Then,*

$$\Lambda(\epsilon, G) = E_m(\epsilon, G).$$

Proof. Let $v_0 \in \Lambda(\epsilon, G)$. Then, there exists $x_0 \in G$ such that $v_0 \in \Lambda(\epsilon, x_0)$; thus,

$$I_e^m(F(y), F(x_0)) + \epsilon > I_e^m(F(v_0), F(x_0)), \forall y \in G. \tag{3.2}$$

Supposing that $v_0 \notin E_m(\epsilon, G)$, there exists $y_0 \in G$ such that $F(y_0) \leq^m F(v_0) - \epsilon e$, and $F(y_0) \neq F(v_0)$. Combining Proposition 3.1 (iv) with Proposition 3.2 (ii) yields the following:

$$I_e^m(F(y_0), F(x_0)) \leq I_e^m(F(v_0) - \epsilon e, F(x_0)) = I_e^m(F(v_0), F(x_0)) - \epsilon,$$

which contradicts (3.2). Hence, $v_0 \in E_m(\epsilon, G)$, and so $\Lambda(\epsilon, G) \subseteq E_m(\epsilon, G)$.

Let $\bar{x} \in E_m(\epsilon, G)$; then, we conclude from Theorem 3.5 and Proposition 3.1 (i) that

$$I_e^m(F(y), F(\bar{x})) + \epsilon > 0 = I_e^m(F(\bar{x}), F(\bar{x})),$$

that is,

$$I_e^m(F(y), F(\bar{x})) + \epsilon > I_e^m(F(\bar{x}), F(\bar{x})), \quad \forall y \in G.$$

This means that

$$\bar{x} \in \Lambda(\epsilon, \bar{x}) \subseteq \Lambda(\epsilon, G);$$

thus, $E_m(\epsilon, G) \subseteq \Lambda(\epsilon, G)$. This completes the proof. □

Remark 3.11. Han [26] presented results that are similar to Theorem 3.9 and Theorem 3.10, specifically concerning l-minimal approximate solutions and l-weak minimal approximate solutions in the context of set optimization problems.

4. Convergence of the approximate solution sets

This section explores the application of nonlinear scalarization to discuss the Painlevé-Kuratowski convergence of weak m-minimal approximate solutions and m-minimal approximate solutions for perturbed set-valued optimization problems.

Let $F : X \to 2^Y$ be a set-valued mapping with nonempty values, and let G_n be a sequence of nonempty subsets of X. The perturbed set-valued optimization problem, denoted by (SOP_n), is given as follows:

$$\begin{aligned} (\mathrm{SOP_n}) \quad & \min \ F(x) \\ & \text{s.t.} \ \ x \in G_n. \end{aligned}$$

Similarly, we denote the set of m-minimal solution set of (SOP_n), the weak m-minimal solution set of (SOP_n), the m-minimal approximate solution set of (SOP_n), and the weak m-minimal approximate solution set of (SOP_n) by $E_m(G_n)$, $W_m(G_n)$, $E_m(\epsilon, G_n)$, and $W_m(\epsilon, G_n)$, respectively.

Let $\{A_n\}$ be a sequence of nonempty subsets of X; the Painlevé-Kuratowski lower and upper limits are, respectively, defined as follows:

$$\mathrm{Li}A_n := \{x \in X : x = \lim_{n\to\infty} x_n, x_n \in A_n, \text{for all large } n, \}$$

$$\mathrm{Ls}A_n := \{x \in X : x = \lim_{k\to\infty} x_{n_k}, x_{n_k} \in A_{n_k}, \{n_k\} \text{ is a subsequence of } \{n\}\}.$$

Definition 4.1. [2] A sequence of nonempty subsets $\{A_n\}$ of X converges to a set A in the sense of Painlevé-Kuratowski, written as $A_n \xrightarrow{K} A$, if and only if $\mathrm{Ls}A_n \subseteq A \subseteq \mathrm{Li}A_n$.

The inclusion $\mathrm{Ls}A_n \subseteq A$ represents the upper Painlevé-Kuratowski convergence, denoted by $A_n \overset{K}{\rightharpoonup} A$, whereas $A \subseteq \mathrm{Li}A_n$ corresponds to the lower Painlev-Kuratowski convergence, denoted by $A_n \overset{K}{\rightharpoondown} A$.

Now, we review the concept of Hausdorff convergence. For $x \in X$ and nonempty subsets $A, B \subseteq X$, we define the following:

$$d(x, A) := \inf_{a \in A} d(x, a), \text{ and } e(A, B) := \sup_{a \in A} d(a, B),$$

$$H(A, B) := \max(e(A, B), e(B, A)).$$

Definition 4.2. [2] A sequence of nonempty subsets $\{A_n\}$ of X converges to a set A in the sense of Hausdorff, written as $A_n \xrightarrow{H} A$, if and only if $H(A_n, A) \to 0$.

The condition $e(A_n, A) \to 0$ is known as the upper part of Hausdorff convergence, written as $A_n \overset{H}{\rightharpoonup} A$, while $e(A, A_n) \to 0$ is the lower part of Hausdorff convergence of sequence of sets $A_n \subseteq X$, written as $A_n \overset{H}{\rightharpoondown} A$.

Lemma 4.3. *[6] Let G be a nonempty compact subset of X and $\{G_n\}$ be a sequence of nonempty subsets of X such that $G_n \overset{H}{\rightharpoonup} G$. Then, for every sequence $\{x_n\}$ with $x_n \in G_n$, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges to some point x in G.*

Lemma 4.4. *[27] If F is continuous with nonempty compact values on X, then $\gamma(x, y) := I_e^m(F(x), F(y))$ is continuous on $X \times X$.*

Now, we define two new set-valued mappings $\Gamma_n : \mathbb{R}_+ \times G_n \to 2^{G_n}$ and $\Lambda_n : \mathbb{R}_+^0 \times G_n \to 2^{G_n}$ as follows:

$$\Gamma_n(\epsilon, x) = \{v \in G_n : I_e^m(F(y), F(x)) + \epsilon \geq I_e^m(F(v), F(x)), \forall y \in G_n\} \ \forall (\epsilon, x) \in \mathbb{R}_+ \times G_n,$$

and

$$\Lambda_n(\epsilon, x) = \{v \in G_n : I_e^m(F(y), F(x)) + \epsilon > I_e^m(F(v), F(x)), \forall y \in G_n\} \ \forall (\epsilon, x) \in \mathbb{R}_+^0 \times G_n.$$

Theorem 4.5. *Let $\{G_n\}$ be a sequence of subsets of X, G be a nonempty compact subset of X and $\{\epsilon_n\} \subseteq \mathbb{R}_+$ with $\epsilon_n \to \epsilon \in \mathbb{R}_+$. Assume the following:*

(i) F is continuous on X with nonempty compact values;
(ii) $G_n \overset{H}{\rightharpoonup} G$; and
(iii) $G_n \overset{K}{\rightharpoondown} G$.
Then, $\Gamma_n(\epsilon_n, G_n) \overset{K}{\rightharpoonup} \Gamma(\epsilon, G)$.

Proof. Let $v_0 \in \mathrm{Ls}\Gamma_n(\epsilon_n, G_n)$. Then, there exists a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ with $v_n \in \Gamma_n(\epsilon_n, x_n)$ such that $v_{n_k} \to v_0$ and $x_n \in G_n$. Since $G_n \overset{H}{\rightharpoonup} G$ and G is nonempty and compact, by Lemma 4.3, we have $v_0 \in G$, and there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x_0 \in G$. Now, we assert that $v_0 \in \Gamma(\epsilon, x_0)$. For any $y \in G$, since $G_n \overset{K}{\rightharpoondown} G$, there exists a sequence $y_n \in G_n$ such that $y_n \to y$. As $v_n \in \Gamma_n(\epsilon_n, x_n)$, we have the following:

$$\gamma(y_{n_k}, x_{n_k}) + \epsilon_{n_k} \geq \gamma(v_{n_k}, x_{n_k}). \tag{4.1}$$

Taking the limit in (4.1) and using Lemma 4.4, we get the following:

$$\gamma(y, x_0) + \epsilon \geq \gamma(v_0, x_0).$$

Hence, $v_0 \in \Gamma(\epsilon, x_0)$. □

By using Theorem 3.9 and Theorem 4.5, we get the Painlevé-Kuratowski convergence for weak m-minimal solutions and weak m-minimal approximate solutions.

Corollary 4.6. *If the assumptions of Theorem 4.5 are satisfied, then* $W_m(\epsilon_n, G_n) \overset{K}{\rightharpoonup} W_m(\epsilon, G)$.

Corollary 4.7. *Let* G *be a nonempty compact subset of* X *and* $\{\epsilon_n\} \subseteq \mathbb{R}_+$ *with* $\epsilon_n \to \epsilon \in \mathbb{R}_+$. *Assume that* F *is continuous on* X *with nonempty compact values. Then,* $W_m(\epsilon_n, G) \overset{K}{\rightharpoonup} W_m(\epsilon, G)$.

Theorem 4.8. *Let* $\{G_n\}$ *be a sequence of subsets of* X*,* G *be a nonempty compact subset of* X*, and* $\{\epsilon_n\} \subseteq \mathbb{R}_+$ *with* $\epsilon_n \to \epsilon \in \mathbb{R}_+$. *Assume the following:*
(i) F *is continuous on* X *with nonempty compact values;*
(ii) $G_n \overset{H}{\rightharpoonup} G$*; and*
(iii) $G_n \overset{K}{\rightharpoondown} G$.
Then $\Lambda_n(\epsilon_n, G_n) \overset{K}{\rightharpoondown} \Lambda(\epsilon, G)$.

Proof. Let $v_0 \in \Lambda(\epsilon, G)$. Then, there exists $x_0 \in G$ such that $v_0 \in \Lambda(\epsilon, x_0)$, that is,

$$\gamma(y, x_0) + \epsilon > \gamma(v_0, x_0), \forall y \in G. \tag{4.2}$$

It follows from $G_n \overset{K}{\rightharpoondown} G$ that there exist subsequences $v_n \in G_n$ and $x_n \in G_n$ such that $v_n \to v_0$ and $x_n \to x_0$, respectively. We claim that for a sufficiently large n,

$$\gamma(y, x_n) + \epsilon_n > \gamma(v_n, x_n), \forall y \in G_n. \tag{4.3}$$

In fact, if not, then there exist subsequences $\{n_k\}$ and $y_{n_k} \in G_{n_k}$ such that

$$\gamma(y_{n_k}, x_{n_k}) + \epsilon_{n_k} \leq \gamma(v_{n_k}, x_{n_k}).$$

Without a loss of generality, we assume that

$$\gamma(y_n, x_n) + \epsilon_n \leq \gamma(v_n, x_n). \tag{4.4}$$

Since $G_n \overset{H}{\rightharpoonup} G$ and G is nonempty compact, by Lemma 4.3, without a loss of generality, we assume $y_n \to y_0 \in G$. By taking the limit in (4.4) and applying Lemma 4.4, we obtain $\gamma(y_0, x_0) + \epsilon \leq \gamma(v_0, x_0)$, which contradicts (4.2). □

By using Theorem 3.10 and Theorem 4.8, we get the Painlevé-Kuratowski convergence for m-minimal solutions and m-minimal approximate solutions.

Corollary 4.9. *If the assumptions of Theorem 4.8 are satisfied, then* $E_m(\epsilon_n, G_n) \overset{K}{\rightharpoondown} E_m(\epsilon, G)$.

Corollary 4.10. *Let G be a nonempty compact subset of X and $\{\epsilon_n\} \subseteq \mathbb{R}_+$ with $\epsilon_n \to \epsilon \in \mathbb{R}_+$. Assume that F is continuous on X with nonempty compact values. Then, $E_m(\epsilon_n, G) \xrightarrow{K} E_m(\epsilon, G)$.*

Remark 4.11. (i) Several existing studies, including Dhingra et al. [7], Karuna and Lalitha [28], Ansari et al. [29], and Khushboo et al. [30], have obtained related results for set optimization problems under different set order relations.

(ii) Consistent with the explanation in Remark 3.8, our Theorem 4.5 and its corollaries pertain to a broader solution space than their counterparts in [28] and [7]. Moreover, Theorem 4.8 and its corollaries are derived from a fundamentally different theoretical framework, which makes a direct comparison with the results in [28] and [7] inappropriate.

(iii) Ansari et al. [29] established convergence results similar to our Theorems 4.5 and 4.8, thereby using the m-relation and considering perturbations in both the objective function and the feasible set. However, their proofs do not employ scalarization techniques. In contrast, our methodology investigates approximate solutions generated by perturbing the feasible set through scalarization techniques, thus representing a distinct approach.

The aforementioned Theorem and Corollary are exemplified through the following case study.

Example 4.12. *Consider the scenario where $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, and $X = \mathbb{R}$. Define $\epsilon_0 = 1$, $\epsilon_n = 1 + \frac{1}{2n}$, $e = (1, 1)$, $G = [-\frac{\pi}{2}, \frac{\pi}{2}]$, and $G_n = (-\frac{\pi}{2} + \frac{1}{2n}, \frac{\pi}{2} - \frac{1}{2n})$. Now, introduce the multifunction $F : X \to 2^Y$ as follows:*

$$F(x) = (-x^2, cos x) + \mathbb{B}_Y,$$

where $\mathbb{B}_Y$ represents the closed unit balls centered at 0_Y in Y. Upon verification, it becomes evident that all prerequisites of Theorems 4.5 and 4.8 are fulfilled. Through straightforward calculations, we find that $E_m(\epsilon, G) = [-\frac{\pi}{2}, \frac{\pi}{2})$, $W_m(\epsilon, G) = [-\frac{\pi}{2}, \frac{\pi}{2}]$, $E_m(\epsilon_n, G_n) = (-\frac{\pi}{2} + \frac{1}{2n}, \frac{\pi}{2} - \frac{1}{2n})$, and $W_m(\epsilon_n, G_n) = (-\frac{\pi}{2} + \frac{1}{2n}, \frac{\pi}{2} - \frac{1}{2n})$. Consequently, it follows that $E_m(\epsilon_n, G_n) \xrightarrow{K} E_m(\epsilon, G)$ and $W_m(\epsilon_n, G_n) \overset{K}{\rightharpoonup} W_m(\epsilon, G)$.

Remark 4.13. The following example demonstrates that the continuity of F is essential to the aforementioned theorem and corollary.

Example 4.14. *Let $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, and $X = \mathbb{R}$. Define $\epsilon_0 = 1$, $\epsilon_n = 1 + \frac{1}{2n}$, $e = (1, 1)$, $G = [-\frac{\pi}{2}, \frac{\pi}{2}]$, and $G_n = (-\frac{\pi}{2} + \frac{1}{2n}, \frac{\pi}{2} - \frac{1}{2n})$. Consider the multifunction $F : X \to 2^Y$ given by the following:*

$$F(x) = \begin{cases} (-x^2, cos x) + \mathbb{B}_Y, & if\ x \neq \frac{\pi}{2}, \\ (-x^2, cos x) + \frac{1}{2}\mathbb{B}_Y, & if\ x = \frac{\pi}{2}, \end{cases}$$

where $\mathbb{B}_Y$ represents the closed unit balls centered at 0_Y in Y. Note that F is discontinuous at $x = \frac{\pi}{2}$. A direct computation yields $W_m(\epsilon, G) = [-\frac{\pi}{2}, \frac{\pi}{2})$ and $W_m(\epsilon_n, G_n) = (-\frac{\pi}{2} + \frac{1}{2n}, \frac{\pi}{2} - \frac{1}{2n})$. Since $\frac{\pi}{2} \in LsW_m(\epsilon_n, G_n)$, though $\frac{\pi}{2} \notin W_m(\epsilon, G)$. We conclude that $W_m(\epsilon_n, G_n) \overset{K}{\not\rightharpoonup} W_m(\epsilon, G)$.

5. Well-posedness for (SOP)

In this section, we will define new well-posedness for (SOP). Additionally, we will establish relationships between these new well-posedness. Furthermore, we will explore the equivalences between the well-posedness for (SOP) and the well-posedness of a scalar optimization problem.

The concept of well-posedness has been extensively explored in vector and set-valued optimization. Recently, research on the well-posedness of set optimization problems has considerably expanded, particularly for formulations based on the upper and lower set-less relations. Building on the insight from Remark 3.8, we introduce new well-posedness notions grounded in the m set-less relation.

Building on Definition 3.2 in [20] for vector optimization, we introduce the following concept of a minimizing sequence for (SOP).

Definition 5.1. Given $\bar{x} \in W_m(G)$, a sequence $\{x_n\} \subseteq G$ is termed a $\bar{x}$-minimizing sequence for problem (SOP) if there exists a sequence $\{k_n\} \subseteq K$ with $k_n \to 0$, satisfying the following condition:

$$F(x_n) \leq^m F(\bar{x}) + k_n.$$

Definition 5.2. A sequence $\{x_n\} \subseteq G$ is termed a generalized minimizing sequence for problem (SOP) if there exists a sequence $\{k_n\} \subseteq K$ with $k_n \to 0$ and a sequence $\{y_n\} \subseteq G$ such that $y_n \in W_m(G)$ and

$$F(x_n) \leq^m F(y_n) + k_n.$$

Next, we introduce the following three notions of well-posedness: L-well-posedness, DH-well-posedness, and generalized L-well-posedness.

Definition 5.3. Problem (SOP) is said to be L-well-posed at $\bar{x} \in W_m(G)$ if, for any $\bar{x}$-minimizing sequence, there exists a subsequence that converges to a point of $W_m(G)$.

Definition 5.4. Problem (SOP) is said to be DH-well-posed at $\bar{x} \in W_m(G)$ if, for each $k \in K$,

$$\inf_{\alpha > 0} \operatorname{diam} L(\bar{x}, k, \alpha) = 0.$$

Here, $L(\bar{x}, k, \alpha) := \{x \in G : F(x) \leq^m F(\bar{x}) + \alpha k\}$, and diam denotes the diameter of a set.

Definition 5.5. Problem (SOP) is said to be generalized L-well-posed if, for any generalized minimizing sequence, there exists a subsequence that converges to a point of $W_m(G)$.

Subsequently, we present the necessary and sufficient conditions for DH-well-posedness.

Proposition 5.6. *Let $e \in \operatorname{int} K$ and $\bar{x} \in W_m(G)$. Problem (SOP) is DH-well-posed at $\bar{x}$ if and only if* $\lim_{\alpha \to 0}$ *diam*$L(\bar{x}, e, \alpha) = 0$.

Proof. The necessity is self-evident. Now, let's proceed to establish the sufficiency. Given that $e - K$ constitutes a neighborhood of 0 in Y, we can find a positive number $\epsilon > 0$ that satisfies $\epsilon B(0, 1) \subseteq e - K$. Here, $B(x, r)$ denotes the closed ball centered x with radius r. For $k \in K$, we have the following:

$$k \in \|k\| B(0, 1) \subseteq \|k\| \epsilon^{-1}(e - K) = \|k\| \epsilon^{-1} e - K.$$

Taking $\lambda = \|k\| \epsilon^{-1}$, then $\lambda e - k \in K$. Therefore, there exists $k_1 \in K$ such that $\lambda e - k = k_1$. Then, for any $\alpha > 0$, we have the following:

$$F(x) \leq^m F(\bar{x}) + \alpha k \Rightarrow \exists k_0 \in K, \;\; k_0 + F(x) \subseteq F(\bar{x}) + \alpha k \Rightarrow k_0 + \alpha k_1 + F(x) \subseteq F(\bar{x}) + \lambda \alpha e,$$

that is, $F(x) \leq^m F(\bar{x}) + \lambda \alpha e$. Consequently, we have $L(\bar{x}, k, \alpha) \subseteq L(\bar{x}, e, \lambda \alpha)$. Hence,
$\lim_{\lambda \alpha \downarrow 0} \operatorname{diam} L(\bar{x}, e, \lambda \alpha) = 0$ implies that $\inf_{\alpha > 0} \operatorname{diam} L(\bar{x}, k, \alpha) = 0$.

This concludes the proof. □

Now, we establish the relationship between DH-well-posedness and L-well-posedness.

Proposition 5.7. *(i) Let* $\bar{x} \in W_m(G)$. *If problem (SOP) is DH-well-posed at* $\bar{x}$, *then it is L-well-posed at* $\bar{x}$.

(ii) Let $\bar{x} \in W_m(G)$. *If problem (SOP) is L-well-posed at* $\bar{x}$, *and* $W_m(G)$ *is a singleton, then it is DH-well-posed at* $\bar{x}$.

Proof. (i) Assuming that problem (SOP) exhibits DH-well-posedness at $\bar{x}$, consider a sequence $x_n \in G$ that minimizes $\bar{x}$. Then, there exists a sequence $\{k_n\} \subseteq K$ converging to 0 that satisfies the following inequality:

$$F(x_n) \leq^m F(\bar{x}) + k_n. \tag{5.1}$$

Given that $e \in \text{int}K$ and $e - K$ constitutes a neighborhood of 0 in Y, we can find a positive number $\epsilon > 0$ that satisfies $\epsilon B(0,1) \subseteq e - K$. Here, $B(x,r)$ denotes the closed ball centered x with radius r. For $k_n \in K$, we have the following:

$$k_n \in \|k_n\| B(0,1) \subseteq \|k_n\|\epsilon^{-1}(e - K) = \|k_n\|\epsilon^{-1}e - K.$$

Taking $\alpha_n = \|k_n\|\epsilon^{-1}$, then $k_n \in \alpha_n e - K$, and $k_n \to 0$. Therefore, there exists $z_n \in K$ such that $k_n = \alpha_n e - z_n$. This fact together with (5.1) means that there exists $y_n \in K$ such that

$$y_n + F(x_n) \subseteq F(\bar{x}) + k_n = F(\bar{x}) + \alpha_n e - z_n \Rightarrow y_n + z_n + F(x_n) \subseteq F(\bar{x}) + \alpha_n e,$$

that is, $F(x_n) \leq^m F(\bar{x}) + \alpha_n e$. This implies that $x_n \in L(\bar{x}, e, \alpha_n)$. It follows that $\|x_n - \bar{x}\| \leq \text{diam}L(\bar{x}, e, \alpha_n)$. Letting $n \to +\infty$, we get $x_n \to \bar{x}$. Then, problem (SOP) is L-well-posed at $\bar{x}$.

(ii) Alternatively, consider the scenario where problem (SOP) is L-well-posed at $\bar{x}$, $W_m(G)$ consists of a single point, but problem (SOP) is not DH-well-posed at $\bar{x}$. In this case, there must exist some $k \in K$ and a positive ϵ such that $\text{diam}L(\bar{x}, k, \alpha) > \epsilon$ for all $\alpha > 0$. Let's define a sequence $\{\alpha_n\}$ where $\alpha_n = \frac{1}{n+1}$. For each n, we can find $x_n \in L(\bar{x}, k, \alpha_n)$ such that $\|x_n - \bar{x}\| > \frac{\epsilon}{2}$. Since $x_n \in L(\bar{x}, k, \alpha_n)$, it is also an $\bar{x}$-minimizing point within G. Consequently, there must be a subsequence of $\{x_n\}$ that converges to some point $x \in W_m(G) = \{\bar{x}\}$, which contradicts our assumption. Thus, the proof is complete. □

The above Definition and Proposition are illustrated with the following example.

Example 5.8. *Let* $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $e = (1,1)$, $G = [0,1]$, *and define a set-valued mapping* $F : X \to 2^Y$ *by the following:*

$$F(x) = \begin{cases} \{x\} \times [0, 1-x], & if\ x \neq 0, \\ \{(1,0)\}, & if\ x = 0. \end{cases}$$

Here, $W_m(G) = [0,1]$, *we obtain* $L(0,e,\alpha) = \{0,1\}$, *and thus* $diamL(0,e,\alpha) = 1 \quad \forall \alpha > 0$. *Therefore, Problem (SOP) is not DH-well-posed at* $\bar{x} = 0$.

Example 5.9. *Let* $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $e = (1,1)$, $G = [0,1]$, *and define a set-valued mapping* $F : X \to 2^Y$ *by the following:*

$$F(x) = \{x\} \times [0, 1-x], \quad \forall x \in G.$$

Here, $W_m(G) = [0,1]$, *we obtain* $L(0,e,\alpha) = [0,\alpha)$, *and thus* $\lim_{\alpha \downarrow 0} diamL(0,e,\alpha) = 0$. *Therefore, Problem (SOP) is DH-well-posed at* $\bar{x} = 0$*; then, it is L-well-posed at* $\bar{x} = 0$.

We established several equivalence results between the well-posedness of set optimization problems and their associated scalar optimization problems.

Consider a real-valued function $f : X \to \mathbb{R}$. We examine the following scalar optimization problem:

$$\begin{array}{ll}(\mathrm{SP}_{\bar{x}}) & \min\ f(x) = I_e^m(F(x), F(\bar{x})) \\ & \text{s.t.}\ \ x \in G.\end{array}$$

Here, $\inf_G f$ represents the infimum of f over the set G, and $\operatorname{argmin}(G, f)$ denotes the solution set for problem $(SP_{\bar{x}})$.

The subsequent definitions and proposition are recalled from references [1] and [13].

Definition 5.10. [1] The problem $(SP_{\bar{x}})$ is called Tykhonov well-posed if and only if f has a unique global minimal point, and if $\{x_n\}$ is any sequence from G such that $f(x_n) \to \inf_G f$, then $\{x_n\} \to \operatorname{argmin}(G, f)$.

Definition 5.11. [1] The problem $(SP_{\bar{x}})$ is called generalized well-posed if and only if $\operatorname{argmin}(G, f) \neq \emptyset$, and every sequence $\{x_n\} \subseteq G$ with $f(x_n) \to \inf_G f$ has a subsequence $\{x_{n_k}\}$ such that $\{x_{n_k}\} \to u$ for some $u \in \operatorname{argmin}(G, f)$.

To investigate the well-posedness of problem $(SP_{\bar{x}})$, Furi and Vignoli [13] proposed the following criterion:

$$\lim_{\alpha \to 0} \operatorname{diam}(\alpha - \arg\min(G, f)) = 0, \tag{5.2}$$

where $\alpha - \arg\min(G, f) = \{x \in G : f(x) \leq \inf_G f + \alpha\}$.

Proposition 5.12. *[1] (i) If problem $(SP_{\bar{x}})$ is Tykhonov well-posed, then arg min(G, f) is a singleton and problem $(SP_{\bar{x}})$ satisfies the Furi-Vignoli condition* (5.2)*.*

(ii) If argmin$(G, f) \neq \emptyset$, then the Furi-Vignoli condition (5.2) *implies the Tykhonov well-posedness of problem $(SP_{\bar{x}})$.*

Now, we present the first main result of this section.

Theorem 5.13. *Let $e \in \operatorname{int}K$ and $\bar{x} \in W_m(G)$. Assume that $F(x)$ is compact for any $x \in G$. Problem (SOP) is DH-well-posed at $\bar{x}$ if and only if problem $(SP_{\bar{x}})$ is Tykhonov well-posed.*

Proof. First, we prove that for every $\alpha > 0$,

$$\alpha - \arg\min(G, f) = L(\bar{x}, e, \alpha). \tag{5.3}$$

Indeed, if $x \in \alpha - \arg\min(G, f)$, then by Corollary 3.7 and Proposition 3.1 (i),

$$0 \leq f(x) \leq \inf_G f + \alpha = \alpha.$$

By Proposition 3.1 (iii) and Proposition 3.2 (i),

$$f(x) \leq \alpha \Leftrightarrow I_e^m(F(x), F(\bar{x})) \leq \alpha \Leftrightarrow I_e^m(F(x), F(\bar{x}) + \alpha e) \leq 0 \Leftrightarrow F(x) \leq^m F(\bar{x}) + \alpha e.$$

It follows that $x \in L(\bar{x}, e, \alpha)$, and thus $\alpha - \arg\min(G, f) \subseteq L(\bar{x}, e, \alpha)$.

Conversely, let $x \in L(\bar{x}, e, \alpha)$. Then, $F(x) \leq^m F(\bar{x}) + \alpha e$. By Corollary 3.7 together with Propositions 3.1 (i), 3.1 (iv), and 3.2 (ii), we obtain the following:

$$f(x) = I_e^m(F(x), F(\bar{x})) \leq I_e^m(F(\bar{x}) + \alpha e, F(\bar{x})) = \alpha \leq \inf_G f + \alpha,$$

which implies that $x \in \alpha - \arg\min(G, f)$. Therefore, Equation (5.3) holds.

Supposing that problem (SOP) is DH-well-posed at $(\bar{x})$, then, by Proposition 5.6, $\lim_{\alpha \to 0} \mathrm{diam} L(\bar{x}, e, \alpha) = 0$. This fact together with (5.3) yields the following:

$$\lim_{\alpha \to 0} \mathrm{diam}(\alpha - \arg\min(G, f)) = 0.$$

Since $\bar{x} \in W_m(G)$, $\arg\min(G, f)) \neq \emptyset$. By Proposition 5.12 (ii), problem $(SP_{\bar{x}})$ is Tykhonov well-posed.

Conversely, suppose that the scalar problem $(SP_{\bar{x}})$ is Tykhonov well-posed. Then, by Proposition 5.12 (i), $\lim_{\alpha \to 0} \mathrm{diam}(\alpha - \arg\min(G, f)) = 0$. By (5.3) and Proposition 5.6, we get that problem (SOP) is DH-well-posed at $\bar{x}$. □

Remark 5.14. Li and Xu [19] introduced various well-posedness concepts for set optimization problems, thereby utilizing the closed unit ball as a tool. It is noteworthy that Theorem 5.4 in their work [19], which also incorporates the m-relation, yields results akin to Theorem 5.13 presented here. However, as pointed out in Remark 5.8 of [19], the converse of Theorem 5.4 does not hold in general. In contrast, Theorem 5.13 in this paper offers a necessary and sufficient condition for the well-posedness in question.

Remark 5.15. While Zhou et al. [31] also employed the m-relation in formulating well-posedness concepts for set optimization, and their definition of m-minimal solutions is distinct from the one adopted in this work. Consequently, our methodological approach diverges from that developed in their study.

The illustration of Theorem 5.13 is provided through the following example.

Example 5.16. *Consider a scenario where* $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, *and* $X = \mathbb{R}$. *Define* $e = (1, 1)$ *and* $G = [0, 1]$. *Now, introduce the multifunction* $F : X \to 2^Y$, *which is defined as follows:*

$$F(x) = [x, 1 + x] \times [1 - x, 2 - x], \quad \forall x \in G.$$

Here, $W_m(G) = [0, 1]$. *Let* $\bar{x} = 1$, *we obtain* $L(1, e, \alpha) = (1 - \alpha, 1]$, *and thus* $\lim_{\alpha \downarrow 0} diam L(1, e, \alpha) = 0$. *Therefore, problem (SOP) is DH-well-posed at* $\bar{x} = 1$. *In addition,* $\alpha - arg\ min(G, f) = [1 - \alpha, 1]$*; hence,* $\lim_{\alpha \to 0} diam(\alpha - arg\ min(G, f)) = 0$. *As a result, problem* $(P_{\bar{x}})$ *is Tykhonov well-posed.*

We conclude this section by discussing, via the following two theorems, the relationship between the L-well-posedness and generalized L-well-posedness of (SOP) and the generalized well-posedness of the corresponding scalar problems.

Theorem 5.17. *Given* $e \in \mathrm{int} K$ *and* $\bar{x} \in W_m(G)$, *and assuming that* $F(x)$ *is compact for every* $x \in G$, *if problem* $(SP_{\bar{x}})$ *is generalized well-posed, then problem (SOP) is L-well-posed at* $\bar{x}$.

Proof. Assume that problem $(SP_{\bar{x}})$ is generalized well-posed. Let $x_n \in G$ be a $\bar{x}$-minimizing sequence. Consequently, there exists a sequence $\{k_n\} \subseteq K$ converging to 0 that satisfies the following inequality:

$$F(x_n) \leq^m F(\bar{x}) + k_n.$$

Analogous to the proof of Proposition 5.7 (i), we can deduce the existence of a sequence $\{\alpha_n\}$ with the property that $\alpha_n \downarrow 0$, and for each n, $k_n \in \alpha_n e - K$. Consequently,

$$F(x_n) \leq^m F(\bar{x}) + \alpha_n e.$$

By Corollary 3.7 and Proposition 3.1 (i), 3.1 (iv), and 3.2 (ii),

$$0 \leq \inf_{x \in G} I_e^m(F(x), F(\bar{x})) \leq I_e^m(F(x_n), F(\bar{x})) \leq I_e^m(F(\bar{x}) + \alpha_n e, F(\bar{x})) = \alpha_n,$$

which implies that $f(x_n) = I_e^m(F(x_n), F(\bar{x})) \to \inf_{x \in G} f(x) = \inf_{x \in G} I_e^m(F(x), F(\bar{x})) = 0$. Hence, $\{x_n\}$ has a subsequence that converges to a point of $\arg\min(G, f)$. By Theorem 3.9 (let $\epsilon = 0$), we have $\arg\min(G, f) \subseteq W_m(G)$. Therefore, $\{x_n\}$ has a subsequence that converges to a point of $W_m(G)$. This shows that problem SOP is L-well-posed at $\bar{x}$. The proof is complete. □

To study the generalized L-well-posedness of problem (SOP), we now consider the following scalar problem:

$$\text{(P)} \quad \begin{array}{l} \min \;\; g(x) = \inf_{v \in W_m(G)} I_e^m(F(x), F(v)) \\ \text{s.t.} \;\; x \in G. \end{array}$$

Theorem 5.18. *Let $e \in \text{int}K$, and assume that $F(x)$ is compact for any $x \in G$. If problem (SOP) is generalized L-well-posed, then problem (P) is generalized well-posed.*

Proof. First, we show that $W_m(G) \subseteq \arg\min(G, g)$. In fact, if $v \in W_m(G)$, then by Corollary 3.7, we have $I_e^m(F(x), F(v)) \geq 0$ for all $x \in G$. $g(v) = \inf_{y \in W_m(G)} I_e^m(F(v), F(y)) = 0$. Hence, $v \in \arg\min(G, g)$. Let $x_n \in G$ be a sequence such that $g(x_n) \to \inf_{x \in G} g(x) = 0$. Let $\epsilon_n = g(x_n)$ and $\alpha_n' = \epsilon_n + \frac{1}{n}$. Then, $\alpha_n' \to 0$ and there exists $v_{x_n} \in W_m(G)$ such that $I_e^m(F(x_n), F(v_{x_n})) < \alpha_n'$. Write $I_e^m(F(x_n), F(v_{x_n})) = \alpha_n$. Then, $\alpha_n \geq 0, \alpha_n \to 0$. Utilizing Propositions 3.1 (iii) and 3.2 (i), we derive the inequality $F(x_n) \leq^m F(v_{x_n}) + \alpha_n e$. This result indicates that $\{x_n\}$ is a generalized minimizing sequence. Hence, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges to a point $\bar{x} \in W_m(G) \subseteq \arg\min(G, g)$. Therefore, (P) is generalized well-posed. The proof is complete. □

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Author contributions

Li was primarily responsible for drafting the manuscript. All authors participated in reviewing and finalizing the text.

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References

1. A. L. Dontchev, T. Zolezzi, Well-posed optimization problems, *Lecture Notes In Mathematics*, Springer, Berlin, 1993. https://doi.org/10.1007/BFb0084195
2. A. A. Khan, C. Tammer, C. Zalinescu, Set-valued optimization, *Vector Optimization,* Springer, America, 2015. https://doi.org/10.1007/978-3-642-54265-7
3. D. Pallaschke, R. Urbanski, Pairs of compact convex sets, *Mathematics and Its Applications,* Kluwer Academic Publishers, Dordrecht, 2002. https://doi.org/10.1007/978-94-015-9920-7
4. D. Kuroiwa, Some duality theorems of set-valued optimization with natural criteria, *Proc. Int. Conf. Nonlinear Anal. Convex Anal.,* World Scientific River Edge, 1999, 221–228.
5. E. Karaman, M. Güvenç, İ. Soyertem, D. Tozkan, M. Küçük, Y. Küçük, Partial order relations on family of sets and scalarizations for set optimization, *Positivity*, **22** (2018), 783–802. https://doi.org/10.1007/s11117-017-0544-3
6. L. Q. Anh, T. Q. Duy, D. V. Hien, D. Kuroiwa, N. Petrot, Convergence of solutions to set optimization problems with the set less order relation, *J. Optim. Theory Appl.*, **185** (2020), 416–432. https://doi.org/10.1007/s10957-020-01657-2
7. M. Dhingra, C. S. Lalitha, Approximate solutions and scalarization in set-valued optimization, *Optimization*, **66** (2017), 1793–1805. https://doi.org/10.1080/02331934.2016.1271419
8. M. Gupta, M. Srivastava, Approximate solutions and Levitin-Polyak well-posedness for set optimization using weak efficiency, *J. Optim. Theory Appl.*, **186** (2020), 191–208. https://doi.org/10.1007/s10957-020-01683-0
9. Y. Han, K. Zhang, N. J. Huang, The stability and extended well-posedness of the solution sets for set optimization problems via the Painlevé-Kuratowski convergence, *Math. Meth. Oper. Res.,* **91** (2020), 175–196. https://doi.org/10.1007/s00186-019-00695-5
10. T. Y. Li, Y. H. Xu, ϵ-strict efficient solutions of vector optimization problems with set-valued maps, *Asia-Pac. J. Oper. Res.*, **24** (2007), 841–854. https://doi.org/10.1142/S0217595907001577
11. T. Y. Li, Y. H. Xu, The strictly efficient subgradient and the optimality conditions of set-valued optimization, *Bull. Austral. Math. Soc.*, **75** (2007), 361–371. https://doi.org/10.1017/S0004972700039290

12. G. P. Crespi, M. Dhingra, C. S. Lalitha, Pointwise and global well-posedness in set optimization: A direct approach, *Ann. Oper. Res.*, **269** (2018), 49–66. https://doi.org/10.1007/s10479-017-2709-7
13. M. Furi, A. Vignoli, About well-posed optimization problems for functionals in metric spaces, *J. Optim. Theory Appl.*, **5** (1970), 225–229. https://doi.org/10.1007/BF00927717
14. M. Durea, Scalarization for pointwise well-posedness vectorial problems, *Math Meth. Oper. Res.*, **66** (2007), 409–418. https://doi.org/10.1007/s00186-007-0162-0
15. W. Y. Zhang, S. J. Li, K. L. Teo, Well-posedness for set optimization problems, *Nonlinear Anal.*, **71** (2009), 3769–3778. https://doi.org/10.1016/j.na.2009.02.036
16. X. J. Long, J. W. Peng, Generalized B-well-posedness for set optimization problems, *J. Optim. Theory Appl.*, **157** (2013), 612–623. https://doi.org/10.1007/s10957-012-0205-4
17. M. Gupta, M. Srivastava, Well-posedness and scalarization in set optimization involving ordering cones with possibly empty interior, *J. Glob. Optim.*, **73** (2019), 447–463. https://doi.org/10.1007/s10898-018-0695-1
18. X. J. Long, J. W. Peng, Z. Y. Peng, Scalarization and pointwise well-posedness for set optimization problems, *J. Glob. Optim.*, **62** (2015), 763–773. https://doi.org/10.1007/s10898-014-0265-0
19. T. Y. Li, G. H. Xu, Scalarization and well-posedness for set optimization problems using generalized oriented distance function, *J. Ind. Manage. Optim.*, **21** (2025), 1584–1599. https://doi.org/10.3934/jimo.2024139
20. P. Loridan, Well-posedness in vector optimization, *Mathematics and Its Applications,* Springer, Dordrecht, 1995.
21. Y. Han, Connectedness of the approximate solution sets for set optimization problems, *Optimization*, **71** (2022), 4819–4834. https://doi.org/10.1080/02331934.2021.1969393
22. L. Zhu, F. Xia, Scalarization method for Levitin-Polyak well-posedness of vectorial optimization problems, *Math Meth. Oper. Res.*, **76** (2012), 361–375. https://doi.org/10.1007/s00186-012-0410-9
23. E. Hernández, L. Rodríguez-Marín, Nonconvex scalarization in set optimization with set-valued maps, *J. Math. Anal. Appl.,* **325** (2007), 1–18. https://doi.org/10.2298/AADM0702325M
24. M. Gupta, M. Srivastava, Hadamard well-posedness and stability in set optimization, *Positivity*, **7** (2024), 28. https://doi.org/10.1007/s11117-023-01026-z
25. Khushboo, C. S. Lalitha, Scalarizations for a set optimization problem using generalized oriented distance function, *Positivity*, **23** (2019), 1195–1213. https://doi.org/10.1007/s11117-019-00659-3
26. Y. Zeng, Z. Y. Peng, T. Tammer, J. C. Yao, K. Deng, Scalarization and well-posedness for set optimization problems involving general set less relations, *J. Optim. Theory Appl.*, **207** (2025), 37. https://doi.org/10.1007/s10957-025-02784-4
27. T. Y. Li, G. H. Xu, On Hadamard well-posedness and convergence in set optimization, *Optimization*, 2026, 1–21. https://doi.org/10.1080/02331934.2026.2618641
28. Karuna, C. S. Lalitha, External and internal stability in set optimization, *Optimization*, **68** (2019), 833–852. https://doi.org/10.1080/02331934.2018.1556663
29. Q. H. Ansari, N. Hussain, P. K. Sharma, Convergence of the solution sets for set optimization problems, *J. Nonlinear Var. Anal.*, **6** (2022), 165–183. https://doi.org/10.23952/jnva.6.2022.3.01

30. Khushboo, C. S. Lalitha, Scalarization and convergence in unified set optimization, *RAIRO-Oper. Res.*, **55** (2021), 3603–3616. https://doi.org/10.1051/ro/2021169
31. Z. Zhou, K. Feng, Q. H. Ansari, Well-posedness of set optimization problems with set order defined by Minkowski difference, *J. Optim. Theory Appl.*, **204** (2025), 31. https://doi.org/10.1007/s10957-025-02608-5

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