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Research article

Optimal investment under irreversible consumption and locally risk-seeking preferences

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Abstract: We study an infinite-horizon portfolio choice problem in which an investor cannot tolerate any decline in consumption. The agent also exhibits locally risk-seeking preferences of the Friedman–Savage type, which generate alternating regions of risk aversion and risk loving. The interaction between the non-decreasing consumption constraint and local risk-seeking behavior produces an extreme form of habit formation and yields a dynamic free-boundary structure.
To address the resulting non-concave optimization problem, we transform it into an equivalent concave problem via the concave-hull technique and characterize the optimal consumption and portfolio policies in closed form. The optimal policy features delayed consumption adjustments even when wealth is sufficient to remain within the risk-seeking region, followed by a discrete jump in consumption once a critical threshold is reached. The optimal portfolio displays trend-chasing behavior: risky exposure is higher during booms, while allocation becomes investor conservative after upward consumption adjustments.
Numerical simulations illustrate that the expected duration of the risk-seeking regime increases with the investor patience, the magnitude of the consumption jump, and the market risk premium. Finally, the model admits an actuarial interpretation: the non-decreasing consumption rule parallels ratcheted or guaranteed-increasing payouts in insurance and annuity products, providing a theoretical foundation for analyzing dynamic guarantee and bonus-adjustment mechanisms.

Keywords: irreversible (non-decreasing) consumption; consumption ratcheting; non-concave utility; Friedman–Savage preferences; free-boundary analysis; portfolio optimization; concave-hull method; ratcheted guarantees
Mathematics Subject Classification: 91G10, 93E20, 49L20, 60H30

1. Introduction

In this study, we examine the portfolio choice problem of an infinitely lived agent who cannot tolerate any decline in consumption. [1] demonstrates that consumption is not reversible over time and that this irreversibility generates a ratcheting effect. Motivated by [1]'s the notion of irreversible consumption, [2] applies this idea to portfolio selection, presenting a novel framework for optimal consumption and portfolio allocation that precludes any decrease in consumption over time. Such behavior can be regarded as an extreme form of habit formation due to the strong restriction imposed on feasible consumption paths. In the context of prospect theory, a loss-averse agent exhibits an asymmetric utility structure, assigning greater disutility to losses than the utility gained from equivalent increases in consumption. In this paper, the behavior of an individual who cannot tolerate a decrease in consumption is treated as equivalent to infinite loss aversion [3].

While [2] assumes a concave von Neumann–Morgenstern utility function, we adopt a non-concave utility function that combines two opposing attitudes—risk aversion and risk loving—as proposed by [4]. In our setting, the Friedman–Savage investor exhibits locally risk-seeking preferences, being risk averse except at specific consumption levels.

Existing literature on portfolio selection with partially convex utility functions primarily focuses on maximizing utility from terminal wealth (see [5–8]). These studies explore the risk-taking implications of wealth-based preferences. [9] instead consider utility derived from consumption. We extend this line of work by introducing a non-decreasing consumption constraint, thereby combining non-concave preferences with irreversible consumption decisions.

Our contribution is to provide the optimal solution to this joint problem. To address the non-concavity of the utility function, we replace it with its concave hull (envelope) following [10]. This transformation enables us to employ the dual approach developed in [11] and [12] for continuous-time consumption–investment problems.

We show that the investor optimally defers consumption adjustments even when holding sufficient wealth within the locally risk-seeking region. Once a critical threshold is reached, a discrete jump in consumption occurs along the linear segment of the concave hull. Furthermore, the investor exhibits trend-chasing behavior: becoming more risk-averse in bear markets and more risk-seeking in bull markets. Compared with [13], the agent displays lower overall risk aversion. This trend-following behavior leads to greater risky investment and faster wealth accumulation, while exposure to financial risk decreases sharply following the consumption jump.

Through simulation, we compute the expected time for the Friedman–Savage investor to exit the risk-loving region. The duration of the risk-seeking regime increases with the investor's patience, the size of the consumption jump, and the market risk premium—findings consistent with [9].

There exists a growing literature on portfolio choice under consumption ratcheting. Building on [2], [14] relax the homotheticity assumption and derive a solution in an infinite horizon. [13] analyze trend-chasing behavior in a finite-horizon setting with general concave utility, while [15] study a quadratic utility function incorporating a bliss level of consumption.

For models with Friedman–Savage type preferences, [4] first propose the utility specification to reconcile simultaneous gambling and insurance behavior, and [16] generalize it to allow for multiple concave and convex regions. Finally, [10] investigate a related entrepreneurial decision problem with non-concave objectives under social insurance.

Finally, our framework can also be viewed from an actuarial perspective. The non-decreasing consumption constraint naturally parallels ratcheted or guaranteed-increasing payouts commonly used in insurance and annuity products. In this interpretation, the free boundary governing consumption adjustments corresponds to bonus-declaration or guarantee-adjustment thresholds, thereby providing a theoretical foundation for analyzing such mechanisms within a stochastic control framework.

To connect our mechanism to insurance practice, consider a stylized with-profits annuity (or a guaranteed withdrawal benefit with step-ups) in which the policyholder receives a payout rate that is *ratcheted upward* when the insurer's asset position becomes sufficiently strong. Let c_t denote the policyholder's payout (consumption) rate, which is contractually non-decreasing. The insurer monitors a solvency/asset ratio (or a shadow value of reserves) and declares a permanent bonus step-up only when this indicator crosses a pre-specified threshold.

In our model, the optimal consumption policy exhibits exactly this structure. The free boundary (trigger) $z_I(\xi)$ determines the first time at which it becomes optimal to increase the payout beyond level ξ. Equivalently, the hitting time $\tau_\xi^* = \inf\{t \geq 0 : \; y_t \leq z_I(\xi)\}$ plays the role of the bonus-declaration time: when the shadow value y_t is high indicating (tight resources), the payout is kept unchanged; once y_t falls to the trigger, so that (resources become sufficiently abundant), a permanent step-up is executed. For non-concave (Friedman–Savage) preferences, the concavification implies an indifference band, allowing the payout to optimally remain at the lower boundary until the upper boundary is reached, thereby producing discrete step-ups. This interpretation clarifies how our free-boundary characterization maps naturally to real-world contract features such as bonus thresholds and guaranteed step-up rules.

Roadmap. The remainder of the paper is organized as follows. Section 2 introduces the market setting, formulates the irreversible (ratcheting) consumption constraint, and describes the concavification step via the concave hull. Section 3 develops the dual-martingale formulation, reduces the monotone consumption problem to a continuum of optimal stopping problems indexed by the target consumption level, and verifies optimality through a trigger (free-boundary) characterization. Section 4 specializes to the shifted-CRRA case to obtain closed-form policies and discusses the economic interpretation of the resulting "pause-and-jump" consumption behavior within the risk-seeking region. Section 5 concludes by outlining limitations and directions for future research.

2. Model Setup

We consider a continuous-time frictionless financial market with no such as transaction costs, taxes, or short-selling constraints. The market consists of a risk-free asset and a single risky asset. The risk-free asset earns a constant interest rate $r > 0$. The price process S_t of the risky asset follows a geometric Brownian motion with drift $\mu > r$ and volatility $\sigma > 0$:

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dB_t,$$

where B_t is a standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. The filtration $\mathbb{F} \equiv \{\mathcal{F}_t\}_{t\geq 0}$ is generated by B_t and augmented by all $\mathbb{P}$-null sets.

Wealth dynamics. The agent's wealth process X_t evolves according to

$$dX_t = [rX_t + (\mu - r)\pi_t - c_t]\, dt + \sigma\pi_t\, dB_t, \tag{2.1}$$

where π_t denotes the dollar amount invested in the risky asset at time t, and c_t is the consumption rate. The admissible strategy $(c_t, \pi_t)_{t\geq 0}$ is progressively measurable and satisfies

$$\int_0^t c_s\, ds < \infty, \qquad \int_0^t \pi_s^2\, ds < \infty, \quad \text{for every } t \geq 0, \text{ a.s.} \tag{2.2}$$

Preferences and constraints. The investor has an infinite horizon and derives utility from the discounted stream of consumption over time. The objective is to maximize

$$U = \mathbb{E}\left[\int_0^\infty e^{-\beta t} u(c_t)\, dt\right], \tag{2.3}$$

where $\beta > 0$ is the subjective discount rate and $u(\cdot)$ is the instantaneous utility function. The utility function u is continuously differentiable, strictly increasing, and satisfies $\lim_{c\to\infty} u'(c) = 0$. It is *not globally concave*, reflecting the agent's locally risk-seeking preferences of the Friedman–Savage type.

The consumption process $\{c_t\}_{t\geq 0}$ is required to be non-decreasing, i.e.,

$$c_s \geq c_t, \quad \forall\, s \geq t,$$

and is right-continuous with left limits (RCLL). The initial consumption level c_{0-} is given.

We write c_{0-} for the given pre-decision consumption level immediately before time 0, and c_0 for the post-decision level after the agent chooses the initial adjustment at time 0. Accordingly, a jump at $t = 0$ is allowed and, in general, $c_0 \neq c_{0-}$.

To obtain the budget constraint from (2.1), we define

$$\theta \equiv \frac{\mu - r}{\sigma}, \quad H_t \equiv e^{-rt} Z_t, \quad Z_t \equiv e^{-\frac{1}{2}\theta^2 t - \theta B_t}.$$

Here, θ represents the market price of risk, Z_t is an exponential martingale, and H_t denotes the stochastic discount factor. Using these definitions, the static budget constraint can be expressed as

$$\mathbb{E}\left[\int_0^\infty H_t\, c_t\, dt\right] \leq X_0, \tag{2.4}$$

where the left-hand side represents the present value of lifetime consumption, discounted by the stochastic discount factor H_t. This condition requires that the expected discounted value of total consumption cannot exceed the agent's initial wealth X_0.

To ensure that the agent's consumption plan is admissible, we impose the following assumption.

Assumption 2.1.

$$X_0 > \frac{c_{0-}}{r}.$$

If $X_0 < \frac{c_{0-}}{r}$, the consumption process cannot be non-decreasing. When $X_0 = \frac{c_{0-}}{r}$, the agent's consumption rate must remain constant at c_{0-}, satisfying

$$\mathbb{E}\left[\int_0^\infty H_t\, c_t\, dt\right] = X_0.$$

In this degenerate case, the agent cannot invest in the risky asset because any drop in wealth would necessitate a reduction in consumption, violating the monotonicity constraint. Assumption 2.1 thus rules out the trivial case and guarantees the feasibility of a non-trivial optimal solution.

3. Optimization

3.1. Concavification via the concave hull

We introduce the utility maximization problem as follows:

Problem 3.1. *Given $c_{0-} \geq 0$ and $X_0 = x > 0$,*

$$V(x, c_{0-}) \equiv \max_{(c,\pi)\in\mathcal{A}(x,c_{0-})} \mathbb{E}\left[\int_0^\infty e^{-\beta t} u(c_t)\, dt \,\middle|\, X_0 = x,\ c_0 = c_{0-}\right], \tag{3.1}$$

where $\mathcal{A}(x, c_{0-})$ is the set of all admissible strategies such that (i) the consumption process c_t is $\mathbb{F}$-adapted, non-decreasing, positive RCLL with prescribed left limit c_{0-} at $t = 0$ (so a jump at $t = 0$ is allowed), (ii) π_t is an $\mathbb{F}$-progressively measurable process satisfying the following integrability condition: for all $T > 0$, $\int_0^T \pi_t^2\, dt < \infty$ a.s., (iii) the wealth process X_t in (2.1) corresponding to (c, π) satisfies $X_t \geq 0$ for all $t \geq 0$ a.s.

To address the issue of non-concavity, we follow the approach of [10], which replaces the non-concave parts of the utility function with its concave hull. Let $u_E(c)$ denote the concave envelope (or concave hull) of $u(c)$, defined as

$$u_E \equiv \min_f \{f : (0, \infty) \to \mathbb{R} \mid f \geq u \text{ and } f \text{ is concave}\}. \tag{3.2}$$

That is, $u_E(c)$ is the smallest concave function that dominates $u(c)$. Since $u(c)$ is strictly increasing and continuous, its concave hull $u_E(c)$ is also strictly increasing and continuous. This implies that the set of intervals where u is strictly dominated by u_E is open:

$$\{\, c > 0 \mid u_E(c) > u(c)\,\} = \bigcup_{m=1}^{\infty} (\underline{c}_m, \bar{c}_m).$$

On each interval $(\underline{c}_m, \bar{c}_m)$, the concave hull u_E is linear and coincides with the chord connecting $(\underline{c}_m, u(\underline{c}_m))$ and $(\bar{c}_m, u(\bar{c}_m))$. Equivalently, there exists a constant slope (depending on m) such that

$$u(\bar{c}_m) - u(\underline{c}_m) = (\bar{c}_m - \underline{c}_m)\, u_E'(c), \qquad c \in (\underline{c}_m, \bar{c}_m), \quad m = 1, 2, \ldots.$$

Moreover,

$$\lim_{c\to\underline{c}_m^-} u_E'(c) > \lim_{c\to\underline{c}_m^+} u_E'(c),$$

since the concave envelope contains linear segments over the non-concave regions. Figure 1 illustrates the shapes of the original utility function u and its concave hull u_E.

And, we assume the following condition to guarantee the existence of solutions.

Assumption 3.1. *Let $n_2 < 0$ denote the negative root of the quadratic equation*

$$\frac{\theta^2}{2} n^2 + \left(\beta - r - \frac{\theta^2}{2}\right) n - \beta = 0.$$

We assume that

$$\int_c^\infty \left(u_E'(s)\right)^{1-n_2} ds < \infty. \tag{3.3}$$

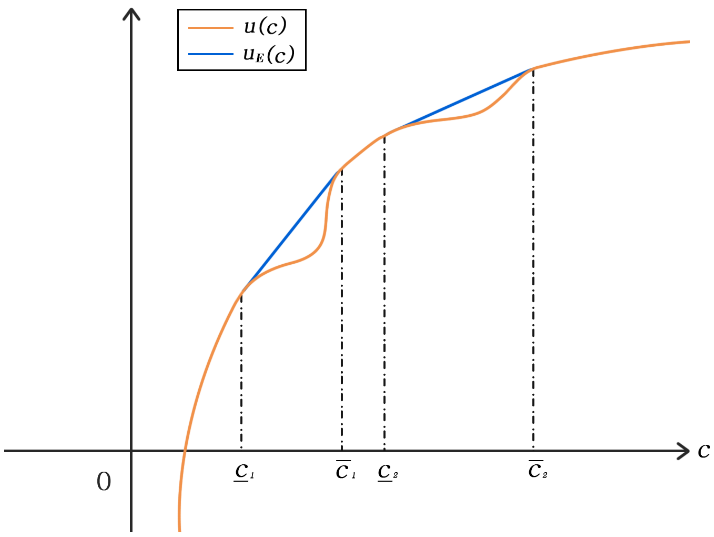


Figure 1. Non-concave function and its concave hull.

This integrability condition guarantees that the key integral terms appearing in the dual representation are finite, ensuring that the dual value function $J(y, c_{0-})$ *and the resulting closed-form expressions for the optimal wealth/portfolio processes are well-defined.*

The concave hull u_E is the *smallest concave majorant* of u. Geometrically, one may think of placing a tight rubber band above the graph of u: on regions where u is already concave, we have $u_E = u$, while on any interval where u fails to be concave, u_E replaces u with the straight chord connecting two *contact points* $c_- < c_+$. More precisely, if u_E differs from u on (c_-, c_+), then

$$u_E(c) = \lambda\, u(c_-) + (1-\lambda)\, u(c_+), \qquad c = \lambda c_- + (1-\lambda)c_+, \qquad c \in [c_-, c_+],$$

so that u_E is linear on $[c_-, c_+]$ and satisfies $u_E \geq u$ with equality at c_- and c_+; see Figure 2.

Example 3.1 (A piecewise-linear illustration)**.** *Consider the following piecewise-linear utility specified by the points*

$$(0,0), \quad (1,1), \quad (2,1.4), \quad (3,2.6),$$

and linear interpolation between consecutive points. Equivalently,

$$u(c) = \begin{cases} c, & 0 \leq c \leq 1, \\ 1 + 0.4(c-1), & 1 \leq c \leq 2, \\ 1.4 + 1.2(c-2), & 2 \leq c \leq 3. \end{cases}$$

This function is not concave because the slope increases from 0.4 *on* $[1,2]$ *to* 1.2 *on* $[2,3]$*. The concave hull* u_E *is obtained by replacing the non-concave "dip" on* $[1,3]$ *with the chord connecting* $(1,1)$ *and* $(3,2.6)$*, i.e.,*

$$u_E(c) = \begin{cases} c, & 0 \leq c \leq 1, \\ 1 + 0.8(c-1), & 1 \leq c \leq 3. \end{cases}$$

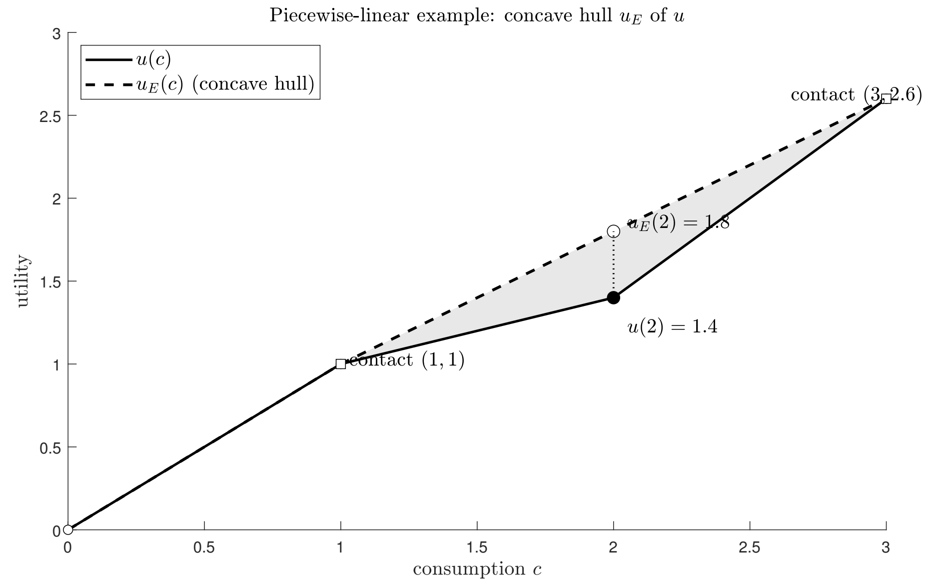


Figure 2. A piecewise-linear utility u (solid) and its concave hull u_E (dashed). The concave hull replaces the non-concave region by the chord between the contact points. At $c = 2$, the hull satisfies $u_E(2) > u(2)$, illustrating how u_E "fills in" the non-concave dip.

In particular, at $c = 2$ *we have* $u(2) = 1.4$ *while* $u_E(2) = 1.8$*, illustrating how* u_E *"fills in" the non-concave region while remaining concave and dominating* u*; see Figure 2.*

The transformation $u \mapsto u_E$ admits a standard economic reading as a *concavification* (or *mixed-strategy* relaxation). When u is non-concave on an interval, the agent prefers to *mix* between two consumption levels $c_- < c_+$ (the contact points), rather than choosing an intermediate level in the non-concave region. Formally, for any $c \in (c_-, c_+)$ there exists $\lambda \in (0, 1)$ such that $c = \lambda c_- + (1 - \lambda)c_+$ and $u_E(c) = \lambda u(c_-) + (1 - \lambda)u(c_+)$, which is the utility attained by a lottery (or time-sharing) that yields c_- with probability λ and c_+ with probability $1 - \lambda$. Thus, u_E represents the *effective* utility once the agent can implement such mixing.

In our dynamic setting, this mixing interpretation aligns with the optimal policy structure. *In particular, when* u_E *is linear on* (c_-, c_+) *(corresponding to the risk-seeking/non-concave region under the original* u*), the agent is indifferent among consumption levels within this interval because the effective marginal utility* $u_E'(c)$ *is constant there.* Equivalently, any $c \in (c_-, c_+)$ delivers the same effective marginal trade-off, and only the endpoints c_- and c_+ matter. Consequently, it is optimal not to finely tune consumption within (c_-, c_+), but instead to keep consumption at one boundary level until a state-dependent trigger is reached, at which point consumption switches (or jumps) to the other boundary level.

3.2. Dual Formulation

Dual-martingale overview. We rewrite the dynamic budget constraint in static form using the stochastic discount factor. Introducing a Lagrange multiplier $y > 0$ yields a dual objective associated with the dual state y_t. The irreversibility (nondecreasing) constraint on consumption implies that raising consumption to a level ξ is an irreversible action. Consequently, the choice of a monotone consumption path can be reduced to a family of stopping decisions indexed by ξ. This formulation leads to a continuum of optimal stopping problems which can be characterized using a trigger (free-boundary) characterization.

We now state the Lagrangian associated with Problem 3.1:

$$\begin{aligned}\mathfrak{L} &= \mathbb{E}\left[\int_0^\infty e^{-\beta t} u(c_t)\, dt\right] + y\left(x - \mathbb{E}\left[\int_0^\infty H_t\, c_t\, dt\right]\right)\\ &\le \mathbb{E}\left[\int_0^\infty e^{-\beta t} u_E(c_t)\, dt\right] + y\left(x - \mathbb{E}\left[\int_0^\infty H_t\, c_t\, dt\right]\right)\\ &= \mathbb{E}\left[\int_0^\infty e^{-\beta t}(u_E(c_t) - y_t c_t)\, dt\right] + yx,\end{aligned} \tag{3.4}$$

where $y > 0$ is the Lagrange multiplier associated with the budget constraint. The inequality in (3.4) follows directly from the definition of the concave hull (3.2).

We define

$$y_t \equiv y\, e^{\beta t} H_t,$$

as the *Lagrange multiplier at time t*, i.e., the (scaled) shadow value of wealth that represents the effective marginal value of relaxing the budget constraint at time t.

Key objects (notation). For the reader's convenience, we summarize several objects that appear repeatedly: (i) y_t is the (scaled) Lagrange multiplier (shadow value) of wealth; (ii) $z_I(\xi)$ is the trigger boundary for the stopping problem associated with level ξ, i.e., the threshold at which it becomes optimal to adjust consumption once y_t falls to $z_I(\xi)$; (iii) Γ is the exponent parameter in the closed-form expressions in Section 4 that governs the relevant integrability/finite-value condition.

Using this process, we can formulate a dual problem that transforms the choice of a non-decreasing consumption process into a collection of optimal stopping problems.

Problem 3.2 (Dual Problem)**.** *Given $y > 0$ and $c_{0-} > 0$, consider the following maximization problem:*

$$J(y, c_{0-}) \;\equiv\; \sup_{c \in \Pi(c_{0-})} \mathbb{E}\left[\int_0^\infty e^{-\beta t}(u_E(c_t) - y_t c_t)\, dt\right], \tag{3.5}$$

where $\Pi(c_{0-})$ denotes the set of all $\mathcal{F}_t$-adapted, non-decreasing, positive RCLL processes c_t starting at c_{0-} and satisfying the integrability condition

$$\mathbb{E}\left[\int_0^\infty e^{-\beta t} \int_{c_{0-}}^{c_t} \left|u_E'(\xi) - y_t\right| d\xi\, dt\right] < \infty. \tag{3.6}$$

Remark 3.1. *The interchange of integrals is justified by* (3.6)*, which ensures the absolute integrability of the relevant integrand. Indeed, by the monotonicity of c_t and the definition of τ_ξ,*

$$\int_0^\infty e^{-\beta t}\left(\int_{c_{0-}}^{c_t} \left|u_E'(\xi) - y_t\right| d\xi\right) dt = \int_{c_{0-}}^\infty \left(\int_{\tau_\xi}^\infty e^{-\beta t}\left|u_E'(\xi) - y_t\right| dt\right) d\xi.$$

Taking expectations and using (3.6) *yields*

$$\mathbb{E}\left[\int_{c_{0-}}^\infty \int_{\tau_\xi}^\infty e^{-\beta t}\left|u_E'(\xi) - y_t\right| dt\, d\xi\right] < \infty,$$

so Tonelli/Fubini applies, and the change of order of integration used above is valid.

From irreversible consumption to a continuum of stopping problems. Because the consumption process is constrained to be non-decreasing, raising consumption from the current level is an *irreversible* action. Hence, one may view a monotone consumption path as being constructed through a sequence of *incremental* upward adjustments: for each target level $\xi \geq c_{0-}$, the decision reduces to choosing the (first) time at which consumption is increased beyond ξ. This leads naturally to a family of optimal stopping problems indexed by ξ, analogous to the incremental irreversible investment formulation in [17].

Lemma 3.1. *The dual value function* $J(y, c_{0-})$ *is given by,*

$$J(y, c_{0-}) \leq \int_{c_{0-}}^{\infty} \sup_{\tau \in \mathcal{S}} \mathbb{E}\left[\int_{\tau}^{\infty} e^{-\beta t} \left(u'_E(\xi) - y_t\right) dt\right] d\xi + J_1(y, c_{0-}), \tag{3.7}$$

where $\mathcal{S}$ *is the set of all* $\mathcal{F}$*-stopping times taking values in* $[0, \infty)$*, and*

$$J_1(y, c_{0-}) = \frac{u_E(c_{0-})}{\beta} - \frac{y}{r} c_{0-}.$$

We will show that the inequality in (3.7) becomes an equality in the verification of Theorem 3.1.

Proof. The dual value function (3.5) satisfies

$$\begin{aligned}
J(y, c_{0-}) &= \sup_{c\, \in \Pi(c_{0-})} \mathbb{E}\left[\int_0^{\infty} e^{-\beta t} \left(u_E(c_t) - y_t c_t\right) dt\right] \\
&= \sup_{c\, \in \Pi(c_{0-})} \mathbb{E}\left[\int_0^{\infty} e^{-\beta t} \left(\int_{c_{0-}}^{c_t} \left(u'_E(\xi) - y_t\right) d\xi + u_E(c_{0-}) - y_t c_{0-}\right) dt\right] \\
&= \sup_{c\, \in \Pi(c_{0-})} \mathbb{E}\left[\int_0^{\infty} e^{-\beta t} \left(\int_{c_{0-}}^{c_t} \left(u'_E(\xi) - y_t\right) d\xi\right) dt\right] + \mathbb{E}\left[\int_0^{\infty} e^{-\beta t} \left(u_E(c_{0-}) - y_t c_{0-}\right) dt\right].
\end{aligned}$$

There are two parts of the above equation, let us define as

$$\begin{aligned}
J_0(y, c_{0-}) &\equiv \mathbb{E}\left[\int_0^{\infty} e^{-\beta t} \left(\int_{c_{0-}}^{c_t} \left(u'_E(\xi) - y_t\right) d\xi\right) dt\right], \\
J_1(y, c_{0-}) &\equiv \mathbb{E}\left[\int_0^{\infty} e^{-\beta t} \left(u_E(c_{0-}) - y_t c_{0-}\right) dt\right].
\end{aligned}$$

Additionally, we define the stopping time τ_ξ which is the right-continuous inverse of c_t as follows:

$$\tau_\xi \equiv \inf\{t > 0 \mid c_t > \xi\},$$

For $J_0(y, c_{0-})$, we can rewrite as:

$$\begin{aligned}
J_0(y, c_{0-}) &= \mathbb{E}\left[\int_0^{\infty} e^{-\beta t} \left(\int_{c_{0-}}^{c_t} u'_E(\xi) - y_t d\xi\right) dt\right] \\
&= \mathbb{E}\left[\int_{c_{0-}}^{\infty} \int_{\tau_\xi}^{\infty} e^{-\beta t} \left(u'_E(\xi) - y_t\right) dt d\xi\right]
\end{aligned}$$

$$= \int_{c_{0-}}^{\infty} \mathbb{E}\left[\int_{\tau_\xi}^{\infty} e^{-\beta t}\,(u_E'(\xi) - y_t)\,dt\right] d\xi$$

where Fubini's theorem is applied in the second equality, which is guaranteed by the condition (3.6).

Note that τ_ξ is an admissible stopping time, i.e., $\tau_\xi \in \mathcal{S}$. Hence, the left-hand side is simply the objective functional evaluated at the particular choice $\tau = \tau_\xi$. Since the right-hand side takes the supremum over all $\tau \in \mathcal{S}$, it must dominate the value at any feasible τ; therefore,

$$\mathbb{E}\left[\int_{\tau_\xi}^{\infty} e^{-\beta t}(u_E'(\xi) - y_t)\,dt\right] \le \sup_{\tau\in\mathcal{S}} \mathbb{E}\left[\int_{\tau}^{\infty} e^{-\beta t}(u_E'(\xi) - y_t)\,dt\right]. \tag{3.8}$$

Next, $J_1(y, c_{0-})$ can be represented as:

$$\begin{aligned} J_1(y, c_{0-}) &= \mathbb{E}\left[\int_0^{\infty} e^{-\beta t}\,(u_E(c_{0-}) - y_t c_{0-})\,dt\right] \\ &= \frac{u_E(c_{0-})}{\beta} - \frac{y}{r} c_{0-}. \end{aligned}$$

Hence, by integrating the $J_0(y, c_{0-})$ and $J_1(y, c_{0-})$, we have the following equation for any $c_t \in \Pi(c_{0-})$

$$\mathbb{E}\left[\int_0^{\infty} e^{-\beta t}\,(u_E(c_t) - y_t c_t)\,dt\right] \le \int_{c_{0-}}^{\infty} \sup_{\tau\in\mathcal{S}} \mathbb{E}\left[\int_{\tau}^{\infty} e^{-\beta t}\,(u_E'(\xi) - y_t)\,dt\right] d\xi + \frac{u_E(c_{0-})}{\beta} - \frac{y}{r} c_{0-}.$$

By taking the supremum on the both-sides of the above equation over $\Pi(c_{0-})$,

$$J(y, c_{0-}) \le \int_{c_{0-}}^{\infty} \sup_{\tau\in\mathcal{S}} \mathbb{E}\left[\int_{\tau}^{\infty} e^{-\beta t}\,(u_E'(\xi) - y_t)\,dt\right] d\xi + \frac{u_E(c_{0-})}{\beta} - \frac{y}{r} c_{0-}.$$

□

Lemma 3.1 yields an inequality because it is derived for *any* admissible stopping time. The key step is to apply Itô's formula to the candidate value function and exploit the variational-inequality structure: in the continuation region, the candidate solves the HJB equation, while at the intervention region, the obstacle/gradient condition prevents any instantaneous adjustment from improving upon the candidate. Therefore, the associated discounted process is a supermartingale, and optional sampling gives the desired inequality.

In Theorem 3.1, we take the specific stopping rule τ_c^* in (3.16), i.e., the first hitting time of the free boundary. On $[0, \tau_c^*)$ the HJB equation holds with equality, and at τ_c^* the boundary (value-matching and, when applicable, smooth-fit) conditions eliminate any slack. Hence, the stopped process becomes a martingale and the bound in Lemma 3.1 is attained, which turns the inequality into equality in Theorem 3.1.

Lemma 3.1 also clarifies the economic structure of the dual value function. In (3.7), the second term corresponds to the dual value when the consumption rate remains fixed at c_{0-}, whereas the first term captures the incremental utility from raising consumption. The integrand $u_E'(\xi) - y_t$ represents the marginal net gain from increasing consumption by an infinitesimal amount, from $c - dc$ to c. Accordingly, the maximization in the first term determines the optimal timing at which this marginal gain should be realized.

Equivalently, Problem 3.2 can be viewed as a continuum of optimal stopping problems: for each target level $c \geq c_{0-}$, one chooses the optimal stopping time τ_c at which consumption is adjusted to c. This structure is analogous to incremental irreversible investment problems (see [17, 18]), where a decision maker chooses the optimal time to expand capacity from $K - dK$ to K for each $K \geq K_{0-}$. Here, the agent chooses the optimal time to increase consumption from $c - dc$ to c for each $c \geq c_{0-}$.

3.3. Reduction to a continuum of optimal stopping problems

We now proceed to introduce the problem of maximizing the dual value function $J(y, c_{0-})$.
Problem 3.3. Optimal Stopping Problem *We consider the following optimal stopping problem for given $\xi > 0$:*

$$Q_\xi(y) \equiv \sup_{\tau \in \mathcal{S}} \mathbb{E}\left[\int_\tau^\infty e^{-\beta t}\left(u_E'(\xi) - y_t\right) dt\right]. \tag{3.9}$$

For the stopping problem $Q_\xi(y)$, we can rewrite as:

$$\begin{aligned} Q_\xi(y) &= \sup_{\tau \in \mathcal{S}} \mathbb{E}\left[\int_\tau^\infty e^{-\beta t}\left(u_E'(\xi) - y_t\right) dt\right] \\ &= \sup_{\tau \in \mathcal{S}} \mathbb{E}\left[e^{-\beta\tau}\mathbb{E}_\tau\left[\int_\tau^\infty e^{-\beta(t-\tau)}\left(u_E'(\xi) - y_t\right) dt\right]\right]. \end{aligned}$$

Let us denote $\psi_\xi(y)$ as follows:

$$\psi_\xi(y_\tau) \equiv \mathbb{E}_\tau\left[\int_\tau^\infty e^{-\beta(t-\tau)}\left(u_E'(\xi) - y_t\right) dt\right]. \tag{3.10}$$

The equation (3.10) satisfies

$$\psi_\xi(y) = \mathbb{E}\left[\int_0^\infty e^{-\beta t}\left(u_E'(\xi) - y_t\right) dt\right] = \frac{u_E'(\xi)}{\beta} - \frac{y}{r}.$$

Then, applying the strong Markov property, the optimal stopping problem is represented as

$$Q_\xi(y) = \sup_{\tau \in \mathcal{S}} \mathbb{E}\left[e^{-\beta\tau}\psi_\xi(y_\tau)\right]. \tag{3.11}$$

The problem is similar to finding the optimal stopping time of a perpetual American put option. $y_t = ye^{\beta t}H_t$ satisfies the following stochastic differential equation (SDE):

$$dy_t = (\beta - r)y_t dt - \theta y_t dB_t.$$

Hence, the process $\{y\}_{s=t}^\infty$ is a strong Markov process with an infinitesimal generator $\mathcal{L}$ given by

$$\mathcal{L}(\cdot) \equiv \frac{1}{2}\theta^2 z^2 \frac{d^2}{dz^2}(\cdot) + (\beta - r)z\frac{d}{dz}(\cdot) - \beta(\cdot).$$

Thus, we can derive the variational inequality (VI) from the optimal stopping problem.

Lemma 3.2. *The standard theory of the optimal stopping time problem ([19]), the Q_ξ satisfies the following VI:*

$$\begin{cases} \mathcal{L}(Q) = 0, & \text{if } Q_\xi > \psi_\xi, \\ \mathcal{L}(Q) \leq 0, & \text{if } Q_\xi = \psi_\xi. \end{cases} \tag{3.12}$$

And, it satisfies a transversality condition:

$$\liminf_{t\to\infty} e^{-\beta t}\mathbb{E}\left[Q(y_t, \xi)\right] = 0. \tag{3.13}$$

Since we consider an infinite-horizon problem, the VI is expressed as an ordinary differential equation (ODE) and we can obtain the solution of the VI explicitly. The subsequent lemma provides a solution to Problem 3.3.

Lemma 3.3. *The solution $Q_\xi(y)$ of Problem 3.3 is given by*

$$Q_\xi(y) = \begin{cases} \dfrac{u_E'(\xi)}{\beta(1-n_2)}\left(\dfrac{y}{z_I(\xi)}\right)^{n_2} & \text{for } y > z_I(\xi), \\ \dfrac{u_E'(\xi)}{\beta} - \dfrac{y}{r} & \text{for } y \leq z_I(\xi) \end{cases}$$

where

$$z_I(\xi) = \frac{r}{\beta}\cdot\frac{n_2}{n_2-1}u_E'(\xi).$$

Proof. We define the continuous region (**CR**) and adjustment region (**AR**) as follows:

$$\mathbf{CR} = \{y \in \mathbb{R}^+ : y > z_I(\xi)\}, \quad \mathbf{AR} = \{y \in \mathbb{R}^+ : 0 < y \leq z_I(\xi)\}.$$

In the continuous region, $Q_\xi(y)$ follows the homogeneous equation,

$$\frac{1}{2}\theta^2 z^2\frac{d^2}{dz^2}Q_\xi''(y) + (\beta - r)z\frac{d}{dz}Q_\xi'(y) - \beta Q_\xi(y) = 0. \tag{3.14}$$

Then, a general solution of the above equation (3.14) is given by the following form:

$$Q_\xi(y) = A_1 y^{n_1} + A_2 y^{n_2},$$

where $n_1 > 1$ and $n_2 < 0$ are positive and negative roots of the equation (3.14), respectively. Note that A_1 must be 0, since $Q_\xi(y)$ satisfies the transversality condition (3.13). Hence,

$$Q_\xi(y) = A_2 y^{n_2}.$$

By smoothing condition at $y = z_I(\xi)$,

$$Q_\xi(z_I(\xi)) = \frac{u_E'(\xi)}{\beta} - \frac{z_I(\xi)}{r},$$

$$Q'_\xi(z_I(\xi)) = -\frac{1}{r}.$$

Then, we get the free boundary $z_I(\xi)$ and the coefficient A_2 as follows:

$$z_I(\xi) = \frac{r}{\beta} \cdot \frac{n_2}{n_2 - 1} u'_E(\xi), \quad A_2 = \frac{u'_E(\xi)}{\beta(1-n_2)} (z_I(\xi))^{-n_2}.$$

Therefore, for $y > z_I(\xi)$,

$$Q_\xi(y) = \frac{u'_E(\xi)}{\beta(1-n_2)} \left(\frac{y}{z_I(\xi)}\right)^{n_2}.$$

□

3.4. Verification and trigger boundary

By Lemma 3.3, we obtain the following main theorem.

Theorem 3.1 (Verification)**.** *Let us define c_t^* as the right-continuous inverse of τ_ξ^*, i.e.,*

$$c_t^* = \inf\left\{\xi \geq c_{0-} : \ \tau_\xi^* > t\right\}, \qquad t \geq 0, \tag{3.15}$$

where τ_ξ^* is the optimal stopping time from Lemma 3.3, defined by

$$\tau_\xi^* = \tau_\xi^*(y) = \inf\{t > 0 : \ y_t \leq z_I(\xi)\}. \tag{3.16}$$

We adopt the convention

$$\inf \emptyset := c_{0-}. \tag{3.17}$$

The c_t^* satisfies the following:

(a) The dual value function for the optimal consumption process in Lemma 3.1 satisfies that

$$J(y, c_{0-}) = \int_{c_{0-}}^{\infty} Q_\xi(y) d\xi + \frac{u_E(c_{0-})}{\beta} - \frac{y}{r} c_{0-}. \tag{3.18}$$

(b) When we select the optimal consumption process, the investor's expected utility of the non-concave function is equivalent to the expected utility of the concave hull,

$$\mathbb{E}\left[\int_0^\infty e^{-\beta t} u(c_t^*) dt\right] = \mathbb{E}\left[\int_0^\infty e^{-\beta t} u_E(c_t^*) dt\right]. \tag{3.19}$$

Proof. (a) It is sufficient to prove that the inequality (3.8) becomes equality for the optimal consumption process. We know that the c^* is derived by the optimal stopping time τ^*. That is, τ^* represents the time that maximizes the right-side of the equation (3.8). Hence,

$$\mathbb{E}\left[\int_{\tau_\xi^*}^\infty e^{-\beta t} \left(u'_E(\xi) - y_t\right) dt\right] = \sup_{\tau \in \mathcal{S}} \mathbb{E}\left[\int_\tau^\infty e^{-\beta t} \left(u'_E(\xi) - y_t\right) dt\right].$$

(b) We know that $u(\xi) = u_E(\xi)$ whenever u is concave, i.e., for $\xi \notin \bigcup_{m=1}^{\infty}(\underline{c}_m, \bar{c}_m)$. Hence, it suffices to verify the identity on each non-concave interval $(\underline{c}_m, \bar{c}_m)$.
Fix $m \geq 1$. By the definition of the concave hull, u_E coincides with the chord connecting the two contact points $(\underline{c}_m, u(\underline{c}_m))$ and $(\bar{c}_m, u(\bar{c}_m))$. Equivalently, u_E is linear on $[\underline{c}_m, \bar{c}_m]$ and satisfies

$$u_E(\xi) = \lambda\, u(\underline{c}_m) + (1-\lambda)\, u(\bar{c}_m), \qquad \xi = \lambda \underline{c}_m + (1-\lambda)\bar{c}_m, \quad \xi \in [\underline{c}_m, \bar{c}_m]. \tag{3.20}$$

In particular, $u_E \geq u$ on $[\underline{c}_m, \bar{c}_m]$ and $u_E = u$ at the endpoints $\underline{c}_m$ and $\bar{c}_m$.
Let us consider $\tau^*_{\underline{c}_m-}$ and $\tau^*_{\bar{c}_m+}$ as the first hitting times of the optimal consumption process that reach $\underline{c}_m-$ and $\bar{c}_m+$, respectively. More precisely, we may define

$$\tau^*_{\underline{c}_m} := \inf\{t \geq 0 : \ c^*_t \geq \underline{c}_m\}, \qquad \tau^*_{\bar{c}_m} := \inf\{t \geq 0 : \ c^*_t \geq \bar{c}_m\}.$$

During the time from $\tau^*_{\underline{c}_m-}$ to $\tau^*_{\bar{c}_m+}$, the consumption is fixed at $\underline{c}_m-$ due to the definition (3.17). Moreover, the optimal policy does not take intermediate values in $(\underline{c}_m, \bar{c}_m)$: it stays at the lower boundary $\underline{c}_m$ until the upper boundary is triggered, at which point consumption jumps to $\bar{c}_m$. Therefore, along the optimal path, c^*_t takes values in the non-concave region only through the endpoints, at which u and u_E coincide. Hence $u(c^*_t) = u_E(c^*_t)$ for all $t \geq 0$. It implies that the utility of the original function is equal to the concave hull's one for $\xi \in \bigcup_{m=1}^{\infty}(\underline{c}_m, \bar{c}_m)$. Therefore, it satisfies the following:

$$\mathbb{E}\left[\int_0^{\infty} e^{-\beta t} u(c^*_t) dt\right] = \mathbb{E}\left[\int_0^{\infty} e^{-\beta t} u_E(c^*_t) dt\right].$$

□

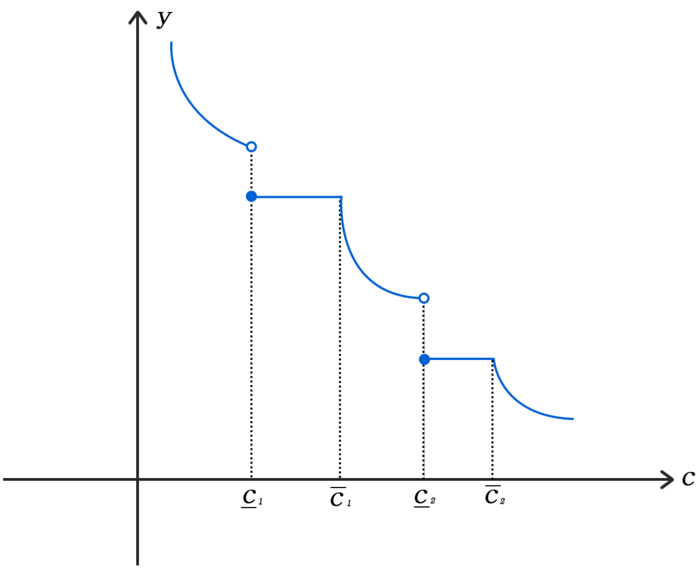


Figure 3. Relationship marginal utility of wealth y and consumption c.

Remark 3.2. *On each interval* $(\underline{c}_m, \bar{c}_m)$ *where the original utility is non-concave, the concave hull* u_E *is linear on* $[\underline{c}_m, \bar{c}_m]$*, so the corresponding marginal utility is constant there, and the classical inverse is not single-valued. In this case, the stopping rule* τ^*_ξ *(equivalently, the trigger* $z_I(\xi)$*) does not depend on the specific* $\xi \in (\underline{c}_m, \bar{c}_m)$*, and the right-continuous inverse in* (3.15) *selects the lower endpoint* $\underline{c}_m$*. Economically, this means that during a non-concave (risk-seeking) region, consumption remains fixed at the lower boundary* $\underline{c}_m$ *until the upper boundary is triggered, at which point consumption jumps to* $\bar{c}_m$*.*

Remark 3.3. *Recall that the dual state* y_t *can be interpreted as a (scaled) shadow value of wealth, i.e., the effective marginal value of relaxing the budget constraint. Accordingly, the boundary* $z_I(\xi)$ *serves as a* trigger *for consumption adjustment: the agent maintains the current consumption regime while* $y_t > z_I(\xi)$ *and adjusts consumption once* y_t *reaches the boundary. Hence,*

$$\tau^*_\xi = \inf\{t \geq 0 : \ y_t \leq z_I(\xi)\}$$

is the first hitting time of the trigger at which the optimal consumption policy switches (or jumps).

The relationship between y and c is determined by marginal utility, and Figure 3 illustrates that the inverse marginal utility mapping is decreasing. Discontinuities at each $\underline{c}$ arise because the left and right derivatives of the (concavified) utility differ at $\underline{c}$, reflecting the presence of linear segments in the concave hull. In terms of the y-process, when y_t reaches the trigger level z_I, consumption is adjusted upward. Once consumption attains $\underline{c}$, it remains fixed at that level by (3.17) until the corresponding threshold associated with $\bar{c}$ is reached, after which y_t resumes its decrease. Beyond this point, the consumption dynamics coincide with those of the baseline ratcheting model.

Theorem 3.2 (Duality)**.** *The value function* (3.1) *and dual value function* (3.5) *have the following duality relationship for all* $t > 0$*:*

$$V(x, c_{t-}) = \min_{y>0} J(y, c_{t-}) + yx \tag{3.21}$$

From the duality, we get the following results:

(a) The optimal wealth process X^*_t *is given by*

$$X^*_t = -\frac{\partial J(y_t, c^*_t)}{\partial y_t} = \frac{1}{\beta} \cdot \frac{n_2}{n_2 - 1} \cdot \frac{1}{\bar{z}} \cdot \left(\frac{y_t}{\bar{z}}\right)^{n_2 - 1} \int_{c^*_t}^{\infty} (u'_E(\xi))^{1-n_2} \, d\xi. \tag{3.22}$$

(b) The optimal portfolio process π^*_t *is expressed as*

$$\pi^*_t = \frac{\theta}{\sigma} y_t \frac{\partial^2 J(y_t, c^*_t)}{\partial^2 y_t} = -\frac{\theta}{\sigma} \cdot \frac{n_2}{\beta} \cdot \left(\frac{y_t}{\bar{z}}\right)^{n_2 - 1} \int_{c^*_t}^{\infty} (u'_E(\xi))^{1-n_2} \, d\xi. \tag{3.23}$$

[13] establish the duality relationship between the value function of Problem 3.1 and the dual value function of Problem 3.2 in Proposition 3.1. In the next section, we examine a specific utility function and analyze the behavior of the optimal strategies.

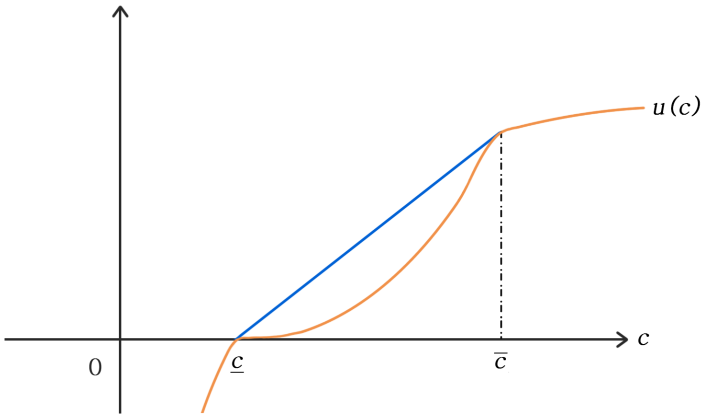


Figure 4. Shifted CRRA function with concave hull.

4. Implications

To illustrate our theoretical framework and its implications for stochastic optimal control, we analyze the optimal strategies under a specific utility specification. We adopt a shifted Constant Relative Risk Aversion (CRRA) utility function, characterized by a single non-concave segment $(\underline{c}, \bar{c})$ (i.e., $m = 1$).

Outside this interval, the agent's utility $u(\xi)$ follows the standard CRRA form. For simplicity and tractability, the function is shifted upward to satisfy $u(\underline{c}) = 0$, which is given by

$$u(\xi) = \frac{\xi^{1-\gamma}}{1-\gamma} - \frac{\underline{c}^{1-\gamma}}{1-\gamma}, \quad \gamma \neq 1.$$

This structural feature is illustrated in Figure 4, where the utility function $u(\xi)$ (orange line) lies strictly below the chord connecting the endpoints $u(\underline{c})$ and $\bar{c}$. As discussed in Section 3, the original objective function $u(\xi)$ is replaced by its concave hull, $u_E(\xi)$. In this setup, $u_E(\xi)$ is a piecewise function: it coincides with $u(\xi)$ in the concave regions and is represented by the linear chord spanning the non-concave region:

$$u_E(\xi) = \begin{cases} u(\xi), & \text{if } \xi \notin (\underline{c}, \bar{c}) \\ u'(\underline{c})(\xi - \underline{c}), & \text{if } \xi \in (\underline{c}, \bar{c}) \end{cases}$$

This linear segment, shown as the blue line in Figure 4, effectively bridges the non-concave gap. The transformation restores the required concavity to the problem, allowing us to apply the dual-martingale approach.

Applying the general solutions from Theorem 3.1 and Theorem 3.2 to this specific functional form yields explicit expressions for the optimal consumption, wealth, and portfolio processes, which are presented in the following corollary.

Corollary 4.1. *For the shifted CRRA function, the optimal solution is given as follows.*
(Hitting-time notation). *For each target level $\xi > 0$, define the optimal stopping time as the first hitting time of the trigger boundary*

$$\tau_\xi^* := \inf\{t > 0 : \ y_t \leq z_I(\xi)\}.$$

In particular, for the endpoints of the non-concave interval $[\underline{c}, \bar{c}]$*, define*

$$\tau^*_{\underline{c}} := \inf\{t > 0 : \; y_t \le z_I(\underline{c})\}, \qquad \tau^*_{\bar{c}} := \inf\{t > 0 : \; y_t \le z_I(\bar{c})\}.$$

Equivalently, $\tau^*_{\underline{c}}$ *and* $\tau^*_{\bar{c}}$ *are the first times at which the optimal policy reaches* $\underline{c}$ *and* $\bar{c}$*, respectively.*

(a) The optimal consumption process c^*_t *is given by*

$$c^*_t = \begin{cases} \max\left\{c_{0-}, \; \min_{0 \le s \le t} \left(\frac{y_s}{\bar{z}}\right)^{-\frac{1}{\gamma}}\right\}, & 0 \le t < \tau^*_{\underline{c}}, \\ \underline{c}, & \tau^*_{\underline{c}} \le t < \tau^*_{\bar{c}}, \\ \max\left\{\bar{c}, \; \min_{\tau^*_{\bar{c}} \le s \le t} \left(\frac{y_s}{\bar{z}}\right)^{-\frac{1}{\gamma}}\right\}, & t \ge \tau^*_{\bar{c}}. \end{cases} \tag{4.1}$$

(b) The optimal wealth process X^*_t *is represented as*

$$X^*_t = \frac{1}{\beta} \cdot \frac{n_2}{(n_2 - 1)} \cdot \frac{1}{\Gamma \, \bar{z}} \cdot \left(\frac{y_t}{\bar{z}}\right)^{n_2 - 1} \left[\frac{1}{\Gamma}\left(\underline{c}^{\Gamma} - (c^*_t)^{\Gamma} - \bar{c}^{\Gamma}\right) + \underline{c}^{\Gamma - 1}(\bar{c} - \underline{c})\right] + \frac{c^*_t}{r}. \tag{4.2}$$

(c) The optimal portfolio process π^*_t *is expressed by*

$$\pi^*_t = \frac{\theta}{\sigma}(1 - n_2)\left(X^*_t - \frac{c^*_t}{r}\right). \tag{4.3}$$

(Definitions of constants). *The constants* Γ *and* $\bar{z}$ *are defined by*

$$\Gamma := 1 - \gamma(1 - n_2), \qquad \bar{z} := \frac{r}{\beta} \cdot \frac{n_2}{n_2 - 1}.$$

Proof. (a) By the definition of the optimal stopping time (3.16), we can derive the

$$\{c^*_t \ge \xi\} = \left\{\min_{0 \le s \le t} \frac{y_s}{\bar{z}} \le \xi^{-\gamma}\right\} = \left\{\xi \ge \left(\min_{0 \le s \le t} \frac{y_s}{\bar{z}}\right)^{-\frac{1}{\gamma}}\right\}. \tag{4.4}$$

The above equation (4.4) gives us the optimal consumption process,

$$c^*_t = \max\left\{c_{0-}, \; \left(\min_{0 \le s \le t} \frac{y_s}{\bar{z}}\right)^{-\frac{1}{\gamma}}\right\}.$$

Therefore, using the definition of consumption in the non-concave region (3.17), we can obtain the following optimal consumption process.

$$c^*_t = \begin{cases} \max\left\{c_{0-}, \; \left(\min_{0 \le s \le t} \frac{y_s}{\bar{z}}\right)^{-\frac{1}{\gamma}}\right\}, & \text{for } \; 0 \le t < \tau^*_{\underline{c}} \\ \underline{c}, & \text{for } \; \tau^*_{\underline{c}} \le t < \tau^*_{\bar{c}} \\ \max\left\{\bar{c}, \; \left(\min_{\tau^*_{\bar{c}} \le s \le t} \frac{y_s}{\bar{z}}\right)^{-\frac{1}{\gamma}}\right\} & \text{for } \; t \ge \tau^*_{\bar{c}} \end{cases}$$

(b) The dual value function $J(y, c_t^*)$ is given by

$$J(y, c_t^*) = \int_{c_t^*}^{\infty} Q_\xi(y) d\xi + \frac{u_E(c_t^*)}{\beta} - \frac{y}{r} c_t^*. \tag{4.5}$$

For $c_t^* < \underline{c}$, the dual value function is expressed by

$$\begin{aligned} J(y, c_t^*) &= \int_{c_t^*}^{\infty} \frac{1}{\beta} \cdot \frac{1}{1-n_2} \left(\frac{y}{\bar{z}}\right)^{n_2} (u_E'(\xi))^{1-n_2} d\xi + \frac{u_E(c_t^*)}{\beta} - \frac{y_t}{r} c_t^* \\ &= \frac{1}{\beta} \cdot \frac{1}{1-n_2} \left(\frac{y}{\bar{z}}\right)^{n_2} \left[\int_{c_t^*}^{\underline{c}} (u_E'(\xi))^{1-n_2} d\xi + \int_{\underline{c}}^{\bar{c}} \left(u'(\underline{c})\right)^{1-n_2} d\xi + \int_{\bar{c}}^{\infty} (u_E'(\xi))^{1-n_2} d\xi \right] + \frac{u_E(c_t^*)}{\beta} - \frac{y}{r} c_t^* \end{aligned}$$

Then, we get the following form of the dual value function.

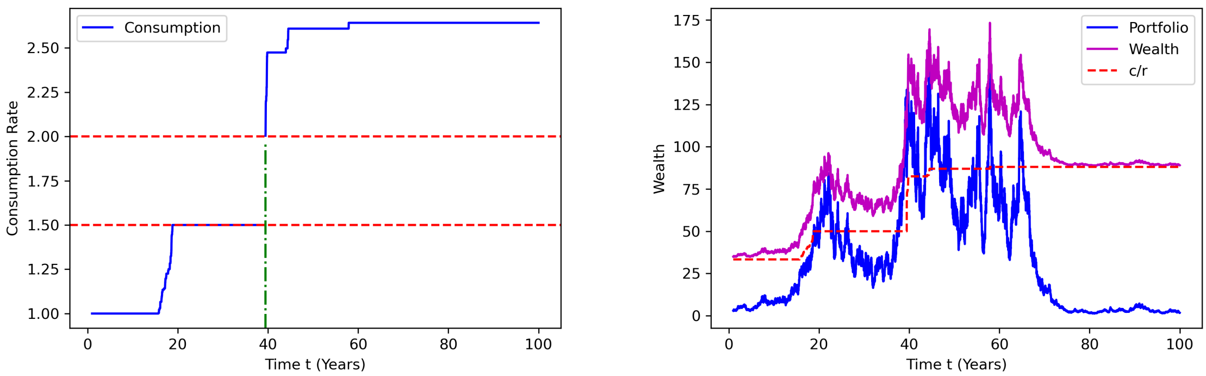
$$J(y, c_t^*) = \frac{1}{\beta} \cdot \frac{1}{1-n_2} \cdot \left(\frac{y_t}{\bar{z}}\right)^{n_2} \left[\frac{1}{\Gamma} \left(\underline{c}^{\Gamma} - (c_t^*)^{\Gamma} - \bar{c}^{\Gamma} \right) + \underline{c}^{\Gamma-1} \left(\bar{c} - \underline{c} \right) \right] + \frac{u_E(c_t^*)}{\beta} - \frac{y_t}{r} c_t^*.$$

Therefore, we obtain the optimal process X_t^* by equation (3.22) in Theorem 3.2,

$$X_t^* = \frac{1}{\beta} \cdot \frac{n_2}{(n_2 - 1)} \cdot \frac{1}{\bar{z}} \cdot \left(\frac{y_t}{\bar{z}}\right)^{n_2 - 1} \left[\frac{1}{\Gamma} \left(\underline{c}^{\Gamma} - (c_t^*)^{\Gamma} - \bar{c}^{\Gamma} \right) + \underline{c}^{\Gamma-1} (\bar{c} - \underline{c}) \right] + \frac{c_t^*}{r}.$$

(c) The optimal portfolio process is obtained by the optimal portfolio (3.23) in Theorem 3.2 and Part (b).

□



(a) Consumption rate

(b) Wealth and portfolio

Figure 5. Optimal policies: (a) consumption path; (b) wealth and portfolio, with the consumption perpetuity c_t^*/r shown as a dashed red line.

Remark 4.1. *Outside the non-concave interval, we use the shifted CRRA utility*

$$u(\xi) = \frac{\xi^{1-\gamma}}{1-\gamma} - \frac{\underline{c}^{1-\gamma}}{1-\gamma}, \qquad \gamma \neq 1,$$

so $u'(\xi) = \xi^{-\gamma}$. On concave parts we have $u_E' = u'$, and therefore (7) is equivalent to

$$\int_{c}^{\infty} s^{-\gamma(1-n_2)} ds < \infty \quad \Longleftrightarrow \quad \gamma(1-n_2) > 1 \quad \Longleftrightarrow \quad \Gamma := 1 - \gamma(1-n_2) < 0.$$

To visualize the optimal processes derived in Corollary 4.1 and analyze the agent's behavior, we conduct numerical simulations. For the subsequent analysis, we adopt a baseline set of market and preference parameters as follows:

$$X_0 = 35,\ \beta = 0.03,\ r = 0.03,\ \mu = 0.07,\ \sigma = 0.2,\ \gamma = 2,\ c_{0-} = 1,\ \underline{c} = 1.5,\ \text{and}\ \ \bar{c} = 2.$$

Part (a) of Corollary 4.1 shows that the optimal consumption policy follows a ratcheting rule driven by the shadow price process y_t. In particular, over the interval $\underline{c}$ on $[\tau^*_{\underline{c}}, \tau^*_{\bar{c}})$ the optimal policy is to hold consumption exactly at $\underline{c}$; only when the free boundary associated with $\bar{c}$ is reached does consumption jump upward to $\bar{c}$. The economic intuition is that, because the marginal utility is flat on $\left(\underline{c}, \bar{c}\right)$ under the concave hull, the investor is locally indifferent within that band, and therefore optimally parks consumption at the boundary until a new upward jump becomes justified. The inverse link between the shadow price y_t and the chosen consumption rate is already visible in Figure 3. Figure 5 reports the simulated paths of the optimal consumption rate, the corresponding wealth process, and the risky allocation.

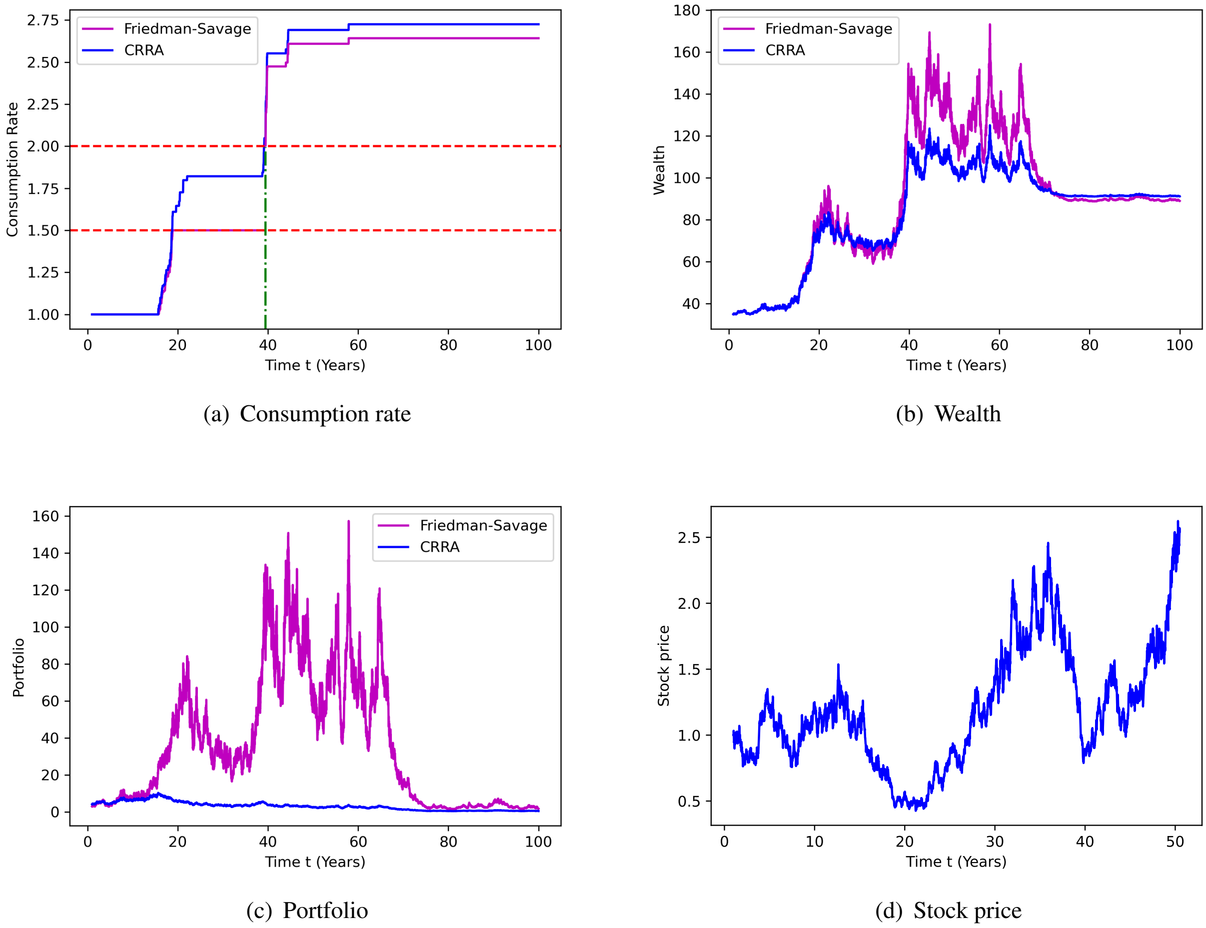
4.1. Economic Interpretation

Pause-and-jump consumption in the risk-seeking region. Panels (a) and Figure 3 highlight the inverse relationship between the shadow price y_t and consumption. When preferences are concavified, the concave hull u_E is linear on $(\underline{c}, \bar{c})$, so the effective marginal utility is flat there. With irreversible consumption, this makes fine-tuning inside the band suboptimal: the agent optimally *parks* consumption at $\underline{c}$ and waits until the trigger associated with $\bar{c}$ is reached, at which point consumption jumps to $\bar{c}$.

Buffer saving and the option value of waiting. Part (b) implies that wealth stays strictly above the perpetuity value c^*_t/r. The gap $X^*_t - c^*_t/r$ can be interpreted as discretionary (buffer) wealth. Because raising consumption is irreversible; the agent delays upgrading the living standard until the buffer is sufficiently large, reflecting an option value of waiting and precautionary behavior.

Risk taking scales with discretionary wealth. Part (c) shows that the risky position is proportional to discretionary wealth. During the pause at $\underline{c}$, the consumption floor remains low, so discretionary wealth tends to be larger, which mechanically generates higher risk exposure relative to the smooth-adjustment CRRA benchmark.

Part (b) states that the optimal wealth process X^*_t is always strictly above the perpetuity value of the current consumption, c^*_t/r. Algebraically, X^*_t is decomposed into c^*_t/r plus an explicit term obtained from the dual representation, which remains finite under $\Gamma < 0$, and strictly positive. Economically, this gap between X^*_t and c^*_t/r, which is the investor's discretionary wealth, reflects precaution: even if wealth is already sufficient to afford a higher consumption rate $c > \underline{c}$ at that instant, the investor does not immediately raise consumption. The decision to lift consumption, for example from $\underline{c}$ to $\bar{c}$, is irreversible because consumption does not decrease later. This irreversibility creates an option value of waiting. The investor delays the upward adjustment until wealth is sufficiently above the threshold needed to fund the higher standard of living on a continuing basis. Thus, a Friedman–Savage type investor will choose to accumulate a financial buffer before exercising the *option* to step up consumption, precisely to avoid being forced to cut it in the future.



(a) Consumption rate (b) Wealth

(c) Portfolio (d) Stock price

Figure 6. Comparison of optimal policies under non-concave (Friedman–Savage) versus concave (CRRA) preferences: (a) consumption, (b) wealth, (c) portfolio; (d) simulated risky-asset price path.

Part (c) gives the optimal portfolio. The risky holding π_t^* is proportional to the investor's discretionary wealth $X_t^* - c_t^*/r$, that is the portion of wealth in excess of the perpetuity value of current consumption. The proportional constant is $\frac{\theta}{\sigma}(1 - n_2)$, which coincides with the constant-share result in [14]. First, portfolio choice is pinned down by investment opportunities and discretionary wealth, but not by the specific curvature of utility beyond the fact that we work with its concave hull. Second, risk-taking exhibits effective state-dependent and trend-following: when discretionary wealth is high, the investor takes a larger position in the risky asset; when discretionary wealth is low, the risky allocation is scaled back. After a consumption jump, discretionary wealth falls because the consumption floor has moved up, and its perpetuity value c_t^*/r is larger. As a result, even if the market price of risk θ/σ is unchanged, the investor becomes more conservative going forward. This mechanism is visible in Panel (b) of Figure 5, which plots the wealth and portfolio weight implied by Corollary 4.1.

Our model shares with [13] the key scaling property that optimal risky demand is proportional to *discretionary wealth* (the component of wealth above the perpetuity value of current consumption), implying procyclical risk-taking when wealth drifts upward. The difference is that, under globally concave utility, the marginal-utility inversion is single-valued, and consumption adjusts more smoothly, so discretionary wealth (and hence risky exposure) evolves without pronounced regime changes. In contrast, with Friedman–Savage-type preferences, the concave hull u_E contains linear segments over the non-concave region, making the effective marginal utility locally flat. This produces an *indifference band* in which the agent optimally *pauses* consumption at the lower boundary until the upper boundary is triggered, followed by a discrete upward jump in consumption. Mechanically, the pause keeps the consumption floor lower for longer, increasing discretionary wealth and thereby amplifying trend-chasing in the risky position, whereas the jump raises the floor sharply and leads to an immediate de-risking episode. Hence, locally risk-seeking preferences generate an endogenous cycle of stronger run-ups in risk exposure and sharper post-jump reductions, beyond what arises under concave utility.

As noted in Theorem 3.1, once consumption reaches $\underline{c}$, the Friedman-Savage investor optimally pauses at that level until the upper boundary $\bar{c}$ is reached, whereas the benchmark CRRA investor adjusts consumption more smoothly as depicted in Figure 6(a). This pause keeps the consumption floor lower for longer. Consequently, discretionary wealth $X_t^* - c_t^*/r$ is larger over this interval, and the risky position, which equals $\frac{\theta}{\sigma}(1 - n_2)$ times discretionary wealth, is mechanically higher than under CRRA (Figure 6(c)). In other words, the elevated risk taking is not merely a manifestation of local risk-seeking on $(\underline{c}, \bar{c})$; it also follows directly from the policy rule that scales risky investment with the slack above the consumption perpetuity.

This mechanism can already be visible before consumption reaches $(\underline{c})$. Even where preferences coincide with CRRA outside the band, the same discretionary-wealth scaling applies, so exposure co-moves with available slack and can exceed the CRRA benchmark depending on the path (Figure 6(c)). As favorable states accumulate, the trend-following behavior of the Friedman–Savage investor accelerates wealth growth relative to CRRA (Figure 6(b)). Note that higher exposure during the pause does not imply wealth dominance. When the risky asset underperforms (Figure 6(d)), the Friedman–Savage investor's larger position can produce temporary reversals in relative wealth, even though the strategy grows wealth faster in favorable states. After an upward ratchet, c_t^*/r rises, the slack shrinks, and the risky share naturally decreases, preserving the same link between risk-taking and discretionary wealth.

Table 1. Average time (years) spent at the baseline consumption $\underline{c}$ before the next consumption jump, under variations in β, $\bar{c}$, and μ. The 'Benchmark' column uses $X_0 = 35$, $\beta = 0.03$, $r = 0.03$, $\mu = 0.07$, $\sigma = 0.2$, $\gamma = 2$, $c_{0-} = 1$, $\underline{c} = 1.5$, and $\bar{c} = 2$. The '(Std.)' row reports standard deviations.

Parameter	Benchmark	β		$\bar{c}$		μ	
		0.01	0.05	2.5	3	0.05	0.1
Mean	14.18	10.65	14.98	21.89	27.35	21.83	7.414
(Std.)	(14.90)	(10.27)	(15.80)	(16.40)	(16.94)	(14.34)	(10.98)

Table 1 reports the average time (in years) the investor remains at the baseline consumption before the next ratchet, under variations in β, $\bar{c}$, and μ. Under the benchmark parameters, the mean duration is 14.18 years with substantial dispersion, reflecting the intrinsic variability of hitting times.

Varying the discount rate shows a clear boundary effect. When $\beta = 0.01$ (more patient), the mean time shortens to 10.65 years, whereas for $\beta = 0.05$ (more impatient), it lengthens to 14.98 years. Since the free boundary satisfies $\bar{z} = \frac{r}{\beta}\frac{n_2}{n_2-1}$, a smaller β raises $\bar{z}$ and makes the ratchet condition easier to meet along a given y_t-path, thereby reducing the expected waiting time. Economically, a lower discount rate increases the present value of the utility gain from a permanent consumption step-up, so the agent is willing to exercise the irreversible "upgrade option" with a smaller buffer (i.e., under less favorable states). This manifests as an earlier boundary hit in the dual trigger formulation. Put differently, although a more patient agent values future utility more, the relevant comparison here is between exercising now and waiting longer under an irreversible rule; a smaller β raises the continuation value of locking in a higher consumption level, lowering the effective hurdle for exercising the step-up.

Changing the target level $\bar{c}$ has the expected effect. Raising $\bar{c}$ from 2.5 to 3 increases the mean duration from 21.89 to 27.35 years. A larger contemplated jump requires a higher wealth (or more favorable states) before adjustment, so the investor remains at $\underline{c}$ longer on average.

Market conditions, captured by μ, move the duration in the opposite direction. A higher drift $\mu = 0.1$ speeds up wealth accumulation and boundary hits, shortening the mean to 7.41 years, whereas a lower drift $\mu = 0.05$ stretches it to 21.83 years. The standard deviations indicate substantial dispersion across all cases, consistent with the stochastic nature of first-passage times.

5. Concluding Remarks

We study a continuous-time consumption–investment problem with an *irreversible* (non-decreasing) consumption constraint and a Friedman-Savage-type utility function that is locally non-concave. To handle the non-concavity, we concavify preferences by replacing u with its concave hull u_E. This transformation preserves the economic content of the problem while restoring concavity, thereby allowing the dual-martingale method and the standard verification machinery for ratcheting problems to apply. A key implication is a *pause-and-jump* structure in optimal consumption: within the locally risk-seeking band, the agent optimally keeps consumption at the lower boundary and upgrades discretely only when the trigger (free boundary) is reached.

Our results further reveal that the agent's exposure to risky assets is generally higher than that of a standard investor with a globally concave utility function. Following a consumption jump, however, risky investment is sharply reduced, reflecting a switch from locally risk-seeking to risk-averse behavior. The resulting portfolio rule is proportional to the agent's *discretionary wealth*, defined as wealth in excess of the perpetuity value of current consumption. During the pause at the lower boundary, discretionary wealth tends to be larger, which can generate higher exposure to the risky asset relative to a benchmark CRRA investor with smoother consumption adjustments. After an upward ratchet, the consumption floor increases and discretionary wealth mechanically shrinks, leading to a sharp reduction in risk-taking. This endogenous pattern links discrete consumption upgrades to alternating phases of elevated risk exposure and subsequent de-risking.

From an actuarial perspective, the non-decreasing consumption rule can be interpreted as a *ratcheted or guaranteed-increasing payout*, as commonly observed in with-profits annuities or guaranteed withdrawal benefit contracts. In this context, the free boundary that triggers consumption adjustments corresponds to the declaration threshold for bonuses or guarantee step-ups. Specifically, the boundary governing optimal consumption adjustments plays the role of a declaration or step-up threshold:

bonuses (or guarantee step-ups) are optimally locked in only when the underlying surplus or buffer becomes sufficiently large. Our framework therefore provides a tractable theoretical foundation for analyzing dynamic guarantee mechanisms under behavioral (Friedman–Savage-type) preferences within a continuous-time stochastic control setting.

Limitations and future research. Our baseline model assumes constant investment opportunities and a single risky asset. Natural extensions include multiple risky assets, stochastic interest rates or time-varying risk premia, and market frictions such as transaction costs or borrowing constraints. Finally, it would be valuable to connect the model to data by testing for step-like consumption upgrades and their association with wealth thresholds, as well as by exploring the calibration implications for ratcheted payout products.

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Author contributions

Seungwon Jeong: Writing – original draft; Formal analysis; Visualization. Junkee Jeon: Conceptualization; Methodology; Formal analysis. Hyeng Keun Koo: Supervision; Writing – review & editing.

Use of Generative-AI tools declaration

The authors declare that they did not utilize any artificial intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no competing interests.

References

1. J. S. Duesenberry, *Income, Saving, and the Theory of Consumer Behavior*. Harvard Economic Studies, No. 87. Harvard University Press, 1949, Cambridge, MA.
2. P. H. Dybvig, Dusenberry's ratcheting of consumption: optimal dynamic consumption and investment given intolerance for any decline in standard of living, *Rev. Econ. Stud.*, **62** (1995), 287–313. https://doi.org/10.2307/2297806

3. K. J. Choi, J. Jeon, H. K. Koo, Intertemporal preference with loss aversion: Consumption and risk-attitude. *J. Econ. Theory*, **200** (2022), 105380. https://doi.org/10.1016/j.jet.2021.105380
4. M. Friedman, L. J. Savage, The utility analysis of choices involving risk, *J. Polit. Econ.*, **56** (1948), 279–304. https://doi.org/10.1086/256692
5. A. B. Berkelaar, R. Kouwenberg, T. Post, Optimal portfolio choice under loss aversion, *Rev. Econ. Stat.*, **86** (2004), 973–987. https://doi.org/10.1162/0034653043125167
6. F. J. Gomes, Portfolio choice and trading volume with loss-averse investors, *J. Bus.*, **78** (2005), 675–706. https://doi.org/10.1086/427643
7. X. D. He, X. Y. Zhou, Portfolio choice under cumulative prospect theory: An analytical treatment, *Manage. Sci.*, **57** (2011), 315–331. https://doi.org/10.1287/mnsc.1100.1269
8. H. Jin, X. Y. Zhou, Greed, leverage, and potential losses: A prospect theory perspective, *Math. Finance*, **23** (2013), 122–142. https://doi.org/10.1111/j.1467-9965.2011.00490.x
9. B. Koo, J. Koo, H. K. Koo, A friedman-savage consumer almost gambles: A continuous time model of consumption and investment with non-concave utility, *Available at SSRN 2039283*, 2012. http://dx.doi.org/10.2139/ssrn.2039283
10. A. Bensoussan, A. Cadenillas, H. K. Koo, Entrepreneurial decisions on effort and project with a nonconcave objective function, *Math. Oper. Res.*, **40** (2015), 902–914. https://doi.org/10.1287/moor.2014.0702
11. I. Karatzas, J. P. Lehoczky, S. E. Shreve, Optimal portfolio and consumption decisions for a "small investor" on a finite horizon, *SIAM J. Control Optim.*, **25** (1987), 1557–1586. https://doi.org/10.1137/0325086
12. J. C. Cox, C. F. Huang, Optimal consumption and portfolio policies when asset prices follow a diffusion process, *J. Econ. Theory*, **49** (1989), 33–83. https://doi.org/10.1016/0022-0531(89)90067-7
13. J. Jeon, H. K. Koo, Y. H. Shin, Portfolio selection with consumption ratcheting, *J. Econ. Dyn. Control*, **92** (2018), 153–182. https://doi.org/10.1016/j.jedc.2018.05.003
14. B. L. Koo, H. K. Koo, J. L. Koo, C. Hyun, A generalization of dybvig's result on portfolio selection with intolerance for decline in consumption, *Econ. Lett.*, **117** (2012), 646–649. https://doi.org/10.1016/j.econlet.2012.08.027
15. J. Jeon, H. K. Koo, Y. H. Shin, Ratcheting with a bliss level of consumption, *Optim. Lett.*, **13** (2019), 1535–1556. https://doi.org/10.1007/s11590-018-1313-3
16. H. Markowitz, The utility of wealth, *J. Polit. Econ.*, **60** (1952), 151–158. https://doi.org/10.1086/257177
17. R. S. Pindyck, Irreversible investment, capacity choice, and the value of the firm. *Am. Econ. Rev.*, **78** (1988), 969–985. https://www.jstor.org/stable/1807160
18. R. K. Dixit, A. K. Dixit, R. S. Pindyck, *Investment under uncertainty*, 1994, Princeton university press. https://doi.org/10.1515/9781400830176
19. G. Peskir, A. Shiryaev, *Optimal stopping and free-boundary problems*, 2006, Springer.

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