Journal of Industrial and Management Optimization

https://www.aimspress.com/journal/jimo

JIMO, 22(2): 923–943.
DOI:10.3934/jimo.2026034
Received: 18 September 2025
Revised: 24 December 2025
Accepted: 05 January 2026
Published: 19 January 2026

Research article

Ramsey model of optimal growth with Allee effect

Zihan Wang and Vladimir Shikhman*

Chemnitz University of Technology, Reichenhainer Str. 41, 09126 Chemnitz, Germany

* **Correspondence:** Email: vladimir.shikhman@mathematik.tu-chemnitz.de.

Abstract: For the Ramsey model of economic growth, which describes the optimal allocation of consumption and saving over time, we assume the underlying population dynamics due to the Allee effect. The so-called Allee threshold separates two regimes from each other. If starting below the threshold, the population decreases to zero. Above this threshold, it gradually saturates. We show that the corresponding consumption per labor stabilizes at two different levels. As for our main result, the steady state consumption per labor is higher in cases of population's decrease rather than of saturation. This is in line with our previous results on the capital per labor for the Solow–Swan model of economic growth with the Allee effect. However, the comparison of consumption-to-capital ratios at the both steady states crucially depends on the curvature of the production function.

Keywords: economic growth; Ramsey model; Allee effect; Pontryagin's maximum principle; consumption-to-capital ration
Mathematics Subject Classification: Primary: 91B62

1. Introduction

The Ramsey model, since its formulation in 1928 [1] and further popularization, has served as a foundational framework in intertemporal macroeconomic theory. It describes the optimal allocation of consumption and saving over time, providing insights into long-run economic growth and capital accumulation. The classical model assumes a representative household that adjusts consumption in order to maximize its discounted utility, subject to a capital accumulation constraint and a production function. However, because the underlying population dynamics exhibits constant and positive population growth rate, more complex ecological or demographic factors are neglected.

In this paper, we introduce the Allee effect [2] into the Ramsey framework to investigate how varying and negative population growth rates influence optimal consumption paths and long-run capital dynamics. The Allee effect, originating in ecology, describes a phenomenon in which a population's growth rate can be negative when the population size falls below a critical threshold. This effect can lead to population extinction if the size remains too small. We incorporate this feature by modifying the population dynamics within the Ramsey model. The main aim of this study is to explore how the Allee effect alters the optimal behavior of agents and the stability properties of the resulting dynamical system. Specifically, we analyze how the presence of an Allee threshold affects capital accumulation, optimal consumption, and the long-run survival of the economy. For the demographic justification of considering the Allee effect within the economic growth models, we refer to [3] and references therein.

Previous studies have examined extensions of the Ramsey model to include demographic dynamics such as logistic growth law [4] and bounded population growth rates [5], but few have considered critical population thresholds or the implications of ecological feedback. By integrating the Allee effect, we contribute to a growing literature that links ecological constraints with economic decision-making. In particular, in analyzing the long-run behavior of the capital per capita, we need to establish rigorous upper and lower bounds. For this purpose, we make use of the results in [3], where Akhalaya and Shikhman studied the Solow–Swan model with Allee effect and derived bounding techniques that we adapt in the Ramsey setting. Moreover, our work also relies on the findings in [5], where Guerrini investigated the Ramsey model under varying but positive population growth rate. While his framework did not include the case of negative growth, his contribution on the functional family of the consumption-to-capital ratio provides valuable analytical tools. In particular, Guerrini obtained closed-form solutions under certain conditions, whereas our focus lies on studying the stability of the consumption-to-capital ratio and characterizing its asymptotic behavior. Overall, our modeling enlarges the scope of [3] by dealing with consumption, as is standard for Ramsey models, rather than just with capital per labor, as is standard for Solow–Swan models. Our results also go beyond the analysis from [5], in that we incorporate diminishing population regimes into the Ramsey model. The latter allows us to compare how consumption per labor performs at equilibrium if the population declines or saturates. Both aspects are novel and contribute to the literature on economic growth models. Finally, we mention the paper [6], where the authors incorporate the Allee effect for biodiversity into a variant of the Ramsey model in order to study pollution and mass extinction phenomena.

The paper is structured as follows. Section 2 presents the classical Ramsey model and introduces the modified population dynamics incorporating the Allee effect. Then, we analyze the resulting optimal control problem, including the application of Pontryagin's maximum principle [7]. Section 3 discusses bounding techniques and long-term behavior, while Section 4 concludes with numerical simulations.

2. Model description

The classical Ramsey model [1] is a fundamental framework in economic growth theory. It describes the intertemporal allocation of consumption and savings in an economy with a representative agent maximizing utility over an infinite time horizon. The standard Ramsey model assumes the production function

$$Y(t) = F(K(t), L(t)), \tag{2.1}$$

where Y is output, K represents the aggregate capital stock, and L represents the labor force with respect to the time t. We assume throughout the paper that F is homogeneous of degree one, that is,

$$F(aK, aL) = a \cdot F(K, L) \quad \text{for all } a > 0.$$

Moreover, we assume F to be strictly concave with respect to (K, L) and to satisfy the well-known Inada conditions; see, for example, [5]:

$$\lim_{K\to 0}\frac{\partial F}{\partial K}=\lim_{L\to 0}\frac{\partial F}{\partial L}=\infty,\quad \lim_{K\to\infty}\frac{\partial F}{\partial K}=\lim_{L\to\infty}\frac{\partial F}{\partial L}=0.$$

These assumptions are consistent with standard economic reasoning. Concavity guarantees diminishing marginal returns to capital. The Inada conditions ensure that the marginal productivity of capital is sufficiently high when capital is scarce, and vanishes when capital is abundant.

As in [1, 8], the economy's output Y is further composed of savings S and consumption $C \geq 0$. The savings rate will be denoted by $s \geq 0$. This implies that there is a fixed proportion of savings in each unit of output, that is,

$$S = s\cdot Y = Y - C. \tag{2.2}$$

Note that the savings S go completely for investment I, that is,

$$I = S. \tag{2.3}$$

The capital per labor is defined as

$$k(t) = \frac{K(t)}{L(t)},$$

and consumption per labor as

$$c(t) = \frac{C(t)}{L(t)}.$$

The representative agent in the original Ramsey model chooses consumption per labor $c(t)$ in order to maximize the total welfare function

$$W = \int_0^\infty \exp(-\rho t)\cdot u(c(t))dt, \tag{2.4}$$

subject to the capital accumulation dynamics:

$$\dot{k}(t) = f(k(t)) - c(t) - (\delta + r)\cdot k(t), \tag{2.5}$$

where u used in the Ramsey model is the isoelastic utility function:

$$u(c(t)) = \frac{c^{1-\frac{1}{\sigma}}(t) - 1}{1 - \frac{1}{\sigma}}, \tag{2.6}$$

and

$$f(k) = \frac{F(K, L)}{L} = F\left(\frac{K}{L}, 1\right) = F(k, 1)$$

is the output per labor, $\sigma > 0$ is constant rate of intertemporal elasticity of substitution, $\rho > 0$ is the discount rate, $\delta > 0$ is the depreciation rate, and $r > 0$ is the constant growth rate of population from

$$\dot{L}(t) = r\cdot L(t).$$

Incorporating the Allee effect into the Ramsey model leads to a modified population dynamics equation. The Allee effect describes a phenomenon in population dynamics where a population has a critical threshold below which it cannot sustain itself [2]. In our model, we propose the following labor dynamics according to the Allee effect:

$$\dot{L}(t) = r \cdot L(t) \cdot \left(1 - \frac{L(t)}{M}\right) \cdot \left(\frac{L(t)}{N} - 1\right), \tag{2.7}$$

where M is the carrying capacity, and N is the critical population threshold. This leads to the modified version of (2.5),

$$\dot{k}(t) = f(k(t)) - c(t) - (\delta + n(t)) \cdot k(t), \tag{2.8}$$

with the now variable population growth rate

$$n(t) = \frac{\dot{L}(t)}{L(t)} = r \cdot \left(1 - \frac{L(t)}{M}\right) \cdot \left(\frac{L(t)}{N} - 1\right).$$

Remark 2.1 ([3])**.** Depending on $L(t_0) = L_0$, the population growth rate will have different behavior:

(1) If $L_0 \in (0, N)$, then $-r < n(t) < 0$, $n'(t) < 0$, and $\lim_{t\to\infty} n(t) = -r$;

(2a) If $L_0 \in \left(N, \frac{N+M}{2}\right]$, then $0 < n(t) \leq n(\bar{t})$, and $\lim_{t\to\infty} n(t) = 0$, where $\bar{t}$ is the unique solution of $L(t) = \frac{N+M}{2}$;

(2b) If $L_0 \in \left(\frac{N+M}{2}, M\right)$, then $0 < n(t) \leq n(t_0)$, and $\lim_{t\to\infty} n(t) = 0$;

(3) If $L_0 \in (M, \infty)$, then $n(t_0) < n(t) < 0$, $n'(t) > 0$, and $\lim_{t\to\infty} n(t) = 0$;

(4) If $L_0 \in \{N, M\}$, then $n(t) \equiv 0$.

In any case in Remark 2.1, the absolute value of the population growth rate $n(t)$ is bounded with respect to t, i.e. there exists $\eta > 0$ with

$$|n(t)| \leq \eta \text{ for all } t \geq t_0. \tag{2.9}$$

To analyze the optimal control problem of maximizing (2.4) subject to (2.8), we use the Pontryagin's maximum principle [7]. The corresponding Hamiltonian is

$$H = \exp(-\rho t) \cdot u(c) + \lambda \left(f(k) - c - (\delta + n) \cdot k\right), \tag{2.10}$$

where λ is a costate variable, also called shadow price, depending on t. The necessary conditions for an optimal solution include:

$$\frac{\partial \mathcal{H}}{\partial c} = 0, \quad \frac{\partial \mathcal{H}}{\partial k} + \dot{\lambda} = 0, \quad \frac{\partial \mathcal{H}}{\partial \lambda} - \dot{k} = 0. \tag{2.11}$$

The standard transversality condition guarantees the optimality for $c(t)$; see, for example, [5]:

$$\lim_{t\to\infty} \lambda(t) \cdot k(t) = 0. \tag{2.12}$$

From the economic perspective, it just prevents the agent from saving forever without consuming or investing, meaning the benefits of saving must eventually diminish to zero. Further, from the first equation in (2.11), we easily obtain

$$\lambda(t) = \exp(-\rho t) \cdot u'(c(t)) = \exp(-\rho t) \cdot c^{-\frac{1}{\sigma}}(t).$$

Hence, the transversality condition (2.12) becomes

$$\lim_{t\to\infty} \exp(-\rho t) \cdot c^{-\frac{1}{\sigma}}(t) \cdot k(t) = 0. \tag{2.13}$$

In what follows, we assume this transversality condition to hold throughout.

Overall, we derive the following system of ordinary differential equations:

$$\begin{aligned} \dot{k}(t) &= f(k(t)) - \left(\delta + r \cdot \left(1 - \frac{L(t)}{M}\right) \cdot \left(\frac{L(t)}{N} - 1\right)\right) \cdot k(t) - c(t), \\ \dot{c}(t) &= \sigma \cdot c(t) \cdot \left[f'(k(t)) - \rho - \delta - r \cdot \left(1 - \frac{L(t)}{M}\right) \cdot \left(\frac{L(t)}{N} - 1\right)\right], \\ \dot{L}(t) &= r \cdot L(t) \cdot \left(1 - \frac{L(t)}{M}\right) \cdot \left(\frac{L(t)}{N} - 1\right). \end{aligned} \tag{2.14}$$

These equations provide insight into the evolution of capital, consumption, and population in case of optimal saving. We see that this system of equations can be divided into two subsystems: one is the independent subsystem of population growth, and the other is the dynamical system of capital and consumption related to population.

3. Analysis of the model

We analyze the behavior of the solution of (2.14) mainly by adopting the method of comparison based on the following result from the literature.

Theorem 3.1 (Comparison theorem, [9]). *Let $u^1(t), u^2(t)$ be solutions of*

$$\dot{u}(t) = \varphi_1(t, u(t)), \quad \dot{u}(t) = \varphi_2(t, u(t)),$$

respectively, with the same initial condition $u^1(0) = u^2(0)$. If $\varphi_1(t,u) \le \varphi_2(t,u)$ for all (t,u), then $u^1(t) \le u^2(t)$ for all t.

3.1. Bounds for the capital

For what follows, it is convenient to consider solutions $k^1(t), k^2(t)$ of

$$\dot{k}(t) = -(\delta + n(t)) \cdot k(t) \quad \text{and} \quad \dot{k}(t) = f(k(t)) - (\delta + n(t)) \cdot k(t),$$

respectively, with the initial value $k^1(t_0) = k^2(t_0) = k_0 \ge 0$. By virtue of Theorem 3.1, the solution $k(t)$ of (2.14) satisfies

$$k^1(t) \le k(t) \le k^2(t).$$

3.1.1. Upper bound

Because here the output is used solely for the capital reproduction, that is, as if $c(t) = 0$, the dynamics for $k^2(t)$ reduce to the Solow–Swan model with Allee effect studied in [3]:

$$\dot{k}(t) = f(k(t)) - (\delta + n(t)) \cdot k(t). \tag{3.1}$$

From there, we know that stability for (3.1) is guaranteed if and only if $\delta > r$. In this case, if $n(t) \to n_\infty$ for $t \to \infty$, then

$$\lim_{t\to\infty} k^2(t) = k^*_{n_\infty},$$

where $k^*_{n_\infty}$ is the steady state of the classical Solow–Swan model [8, 10],

$$\dot{k}(t) = f(k(t)) - (\delta + n_\infty) \cdot k(t), \tag{3.2}$$

and $k_{n_\infty}(t)$ denotes its solution with $k_{n_\infty}(t_0) = k_0$. The economic meaning of the condition $\delta > r$ is straightforward. For guaranteeing stability in the Solow–Swan model with Allee effect, the decrease of population, expressed by the intrinsic population growth rate r, should be rather moderate in comparison to the capital depreciation given by its rate δ; see [3]. In particular, we obtain the following results under $\delta > r$:

- **Case $L_0 \in (0, N)$.**

The population growth rate approaches $n_\infty = -r$. Hence, for any $k_0 > 0$, it holds that

$$\lim_{t\to\infty} k^2(t) = k^*_{-r} \text{ and } k^2(t) \le k_{-r}(t).$$

- **Case $L_0 \in (N, M)$.**

The population growth rate approaches $n_\infty = 0$. Hence, for any $k_0 > 0$, it holds that

$$\lim_{t\to\infty} k^2(t) = k^*_0 \text{ and } k^2(t) \le k_0(t).$$

- **Case $L_0 \in (M, \infty)$.**

The population growth rate approaches $n_\infty = 0$. Hence, for any $k_0 > 0$, it holds that

$$\lim_{t\to\infty} k^2(t) = k^*_0 \text{ and } k^2(t) \le k_{n(t_0)}(t).$$

3.1.2. Lower bound

Here, the capital per labor $k^1(t)$ decreases over time and eventually reaches zero. This is because the entire output is consumed, that is, as if $c(t) = f(k(t))$. We can explicitly solve

$$\dot{k}(t) = -(\delta + n(t)) \cdot k(t), \tag{3.3}$$

and obtain

$$k^1(t) = k^1(0) \cdot \exp\left(-\int_0^t (\delta + n(\tau))\, d\tau\right). \tag{3.4}$$

In view of (3.4), we get for the lower bound under $\delta > r$:

$$\lim_{t\to\infty} k^1(t) = 0. \tag{3.5}$$

3.2. Bounds for the consumption

Now, we turn our attention to the upper and lower bounds for consumption.

3.2.1. Upper bound

From (2.2) and (2.3) we know that $Y = I + C$. If we divide all variables by the quantity of labor L,

$$\frac{Y(t)}{L(t)} = \frac{I(t)}{L(t)} + \frac{C(t)}{L(t)},$$

we obtain the lowercase version

$$f(k(t)) = i(t) + c(t), \tag{3.6}$$

where the investment per labor is set as

$$i(t) = \frac{I(t)}{L(t)}.$$

From (3.6) together with $I = sY \geq 0$, we conclude that $c(t) \leq f(k(t))$. Due to the monotonicity of f and the upper bounds for the capital from Section 3.1.1, we obtain the following for any initial consumption $c(t_0) = c_0 > 0$ under $\delta > r$:

- **Case $L_0 \in (0, N)$.**
$$c(t) \leq f(k^2_{-r}(t)).$$
- **Case $L_0 \in (N, M)$.**
$$c(t) \leq f(k^2_0(t)).$$
- **Case $L_0 \in (M, \infty)$.**
$$c(t) \leq f(k^2_{n(t_0)}(t)).$$

3.2.2. Lower bound

We derive the lower bound from the corresponding equation in (2.14),

$$\frac{\dot{c}(t)}{\sigma c(t)} = f'(k(t)) - \delta - \rho - n(t).$$

Its solution can be written as

$$c(t) = c_0 \cdot \exp\left(\sigma \cdot \int_0^t f'(k(\tau))\, d\tau - \sigma \cdot (\delta + \rho) \cdot t\right) \cdot L^{-\sigma}(t).$$

Next, we use the fact that by our assumptions f is monotonically increasing and concave. The following cases are possible:

- If $k_0 < k^*_{n_\infty}$, then we use $k(t) \leq k^*_{n_\infty}$ to deduce the lower bound as
$$c(t) \geq c_0 \cdot \exp(\sigma \cdot (B - \delta - \rho) \cdot t) \cdot L^{-\sigma}(t).$$
with $B = f'\left(k^*_{n_\infty}\right)$.
- If $k_0 \geq k^*_{n_\infty}$, then analogously we deduce the lower bound as
$$c(t) \geq c_0 \cdot \exp(\sigma \cdot (B - \delta - \rho) \cdot t) \cdot L^{-\sigma}(t)$$
with $B = f'(k_0)$.

Now, we ensure that the derived lower bound of $c(t)$ is not blowing up.

Lemma 3.2. *If $n(t)$ is bounded by $|n(t)| \leq \eta$ for all $t \geq 0$, and $\delta > \eta$ holds, then*

$$\lim_{t\to+\infty} c_0 \cdot \exp\left(\sigma \cdot (B - \delta - \rho) \cdot t\right) \cdot L^{-\sigma}(t) = 0,$$

where B is set as above.

Proof. First, we rewrite the expression for the lower bound:

$$c(t) \geq c_0 \cdot \exp\left(\sigma \cdot (B - \delta - \rho) \cdot t\right) \cdot L_0^{-\sigma} \cdot \exp\left(-\sigma \cdot \int_0^t n(\tau) d\tau\right).$$

We need to distinguish two cases to prove the assertion:

- Case $0 \leq n(t) \leq \eta$ for all $t \geq 0$.

Then, the last term from above is bounded,

$$L^{-\sigma}(t) = L_0^{-\sigma} \cdot \exp\left(-\sigma \cdot \int_0^t n(\tau) d\tau\right) \leq L_0^{-\sigma}.$$

Further, by the assumption that f is strictly concave, we have

$$0 < f(k(t)) - k(t) \cdot f'(k(t)).$$

From here, we get

$$\frac{f(k(t))}{k(t)} > f'(k(t)). \tag{3.7}$$

If $k_0 < k^*_{n_\infty}$, we take the limit in (3.2). Because $\lim_{t\to+\infty} n(t) = 0$ (see Remark 2.1), and, hence, $n_\infty = 0$, we obtain:

$$0 = f(k^*_{n_\infty}) - (\delta + 0) \cdot k^*_{n_\infty}.$$

By means of (3.7), we then deduce:

$$\delta = \frac{f\left(k^*_{n_\infty}\right)}{k^*_{n_\infty}} > f'\left(k^*_{n_\infty}\right) = B.$$

If $k^*_{n_\infty} \leq k_0$, we have again, due to (3.7):

$$\left(\frac{f(k)}{k}\right)' = \frac{f'(k)k - f(k)}{k^2} < 0, \tag{3.8}$$

which means that $\frac{f(k)}{k}$ is a decreasing function. This implies in particular that

$$\delta = \frac{f\left(k^*_{n_\infty}\right)}{k^*_{n_\infty}} \geq \frac{f(k_0)}{k_0} > f'(k_0) = B.$$

Overall, in both cases, we proved $\delta \geq B$. Then, by $\rho > 0$, we have $B - \delta - \rho < 0$. Hence, we obtain

$$\lim_{t\to+\infty} c_0 \cdot \exp\left(\sigma \cdot (B - \delta - \rho) \cdot t\right) \cdot L^{-\sigma}(t) = 0.$$

- Case $-\eta \le n(t) \le 0$.

We set $\bar{\delta} = \delta - \eta$ and $\bar{n}(t) = n(t) + \eta$. Hence, $\bar{\delta} > 0$ and $0 \le \bar{n}(t) \le \eta$. The lower bound above now equals

$$c_0 \cdot \exp\left(\sigma \cdot (B - \bar{\delta} - \rho) \cdot t\right) \cdot L_0^{-\sigma} \cdot \exp\left(-\sigma \cdot \int_0^t \bar{n}(\tau) d\tau\right).$$

Because of $0 \le \bar{n}(t) \le \eta$, we can apply the first case to obtain the assertion. □

Finally, we are ready to analyze the limiting behavior of the lower bound of $c(t)$ from above.

- **Case** $L_0 \in (0, N)$**.**

Here, we set $\eta = r$ (see Remark 2.1), and by means of Lemma 3.2, we immediately obtain

$$\lim_{t\to+\infty} c_0 \cdot \exp\left(\sigma \cdot (B - \delta - \rho) \cdot t\right) \cdot L^{-\sigma}(t) = 0.$$

- **Case** $L_0 \in (N, M)$**.**

Here, the population growth rate approaches $n_\infty = 0$. Then, we have

$$\lim_{t\to+\infty} L^{-\sigma}(t) \le L_0^{-\sigma}.$$

By using Lemma 3.2, we also have $B - \delta - \rho < 0$. Overall,

$$\lim_{t\to+\infty} c_0 \cdot \exp\left(\sigma \cdot (B - \delta - \rho) \cdot t\right) \cdot L^{-\sigma}(t) = 0.$$

- **Case** $L_0 \in (M, \infty)$**.**

It holds due to Remark 2.1 that

$$n(t) < 0, \quad n'(t) > 0, \quad \lim_{t\to\infty} n(t) = 0.$$

We just set $\eta = n(\hat{t})$ with $\hat{t}$ being sufficiently large to guarantee $\delta + n(\hat{t}) > 0$. By means of Lemma 3.2 again, we immediately obtain

$$\lim_{t\to+\infty} c_0 \cdot \exp\left(\sigma \cdot (B - \delta - \rho) \cdot t\right) \cdot L^{-\sigma}(t) = 0.$$

3.3. Limiting behavior

Next, we study the limiting behavior of the solution in (2.14). First, by assumption on the population growth rate due to the Allee effect (2.1), we recall that

$$\lim_{t\to+\infty} n(t) = n_\infty.$$

However, the convergence of $k(t)$ cannot be guaranteed in general. Here, we shall assume that $k(t)$ is convergent, and prove the following result on the long-run behavior of $k(t)$ and $c(t)$ at the equilibrium.

Theorem 3.3. *Assume that* $\lim_{t\to\infty} k(t) = k_\infty$ *exists and* $\delta > r$*. Then,* $\lim_{t\to\infty} c(t) = c_\infty$ *also exists, and exactly one the following situations is possible:*

(I) $k_\infty = k^*_{n_\infty}, c_\infty = 0,$
(II) $k_\infty = (f')^{-1}(\rho + \delta + n_\infty), c_\infty = f(k_\infty) - (\delta + n_\infty) \cdot k_\infty.$

Moreover, if $c_0 > 0$*, we have case (II).*

Proof. By Section 3.1 on the lower and upper bounds for the capital, we have

$$k^1(t) \le k(t) \le k^2(t).$$

Then, it holds that

$$0 \le k_\infty \le k^*_{n_\infty}.$$

Equality on the left holds only when $c_0 = f(k_0)$, in which case both $k(t)$ and $c(t)$ converge to zero. Because we do not consider this degenerate case (we assume $f(k_0) > c_0 > 0$, with $k_0 > 0$), the equality is not attained. By taking t to infinity in (2.14), $c(t)$ converges to

$$c_\infty = f(k_\infty) - (\delta + n_\infty) \cdot k_\infty. \tag{3.9}$$

Additionally, (2.14) also provides in the limit

$$0 = \sigma \cdot c_\infty \cdot (f'(k_\infty) - \rho - \delta - n_\infty). \tag{3.10}$$

From here, it follows that

$$[f(k_\infty) - (\delta + n_\infty) \cdot k_\infty] \cdot [f'(k_\infty) - \rho - \delta - n_\infty] = 0. \tag{3.11}$$

It turns out that both terms on the left-hand side of (3.11) cannot simultaneously vanish. Otherwise, we would have:

$$\frac{f(k_\infty)}{k_\infty} = \delta + n_\infty, \quad f'(k_\infty) = \rho + \delta + n_\infty.$$

Because f is strictly concave by assumption on F, it holds that

$$\frac{f(k_\infty)}{k_\infty} - f'(k_\infty) > 0.$$

Altogether, we would have $\rho < 0$, which contradicts our assumption on ρ. Now, let us discuss the corresponding cases:

- Case (I) with $[f(k_\infty) - (\delta + n_\infty) \cdot k_\infty] = 0$.

We know that $k^2(t)$ is the solution of Solow model

$$\dot{k}(t) = f(k(t)) - (\delta + n(t)) \cdot k(t).$$

By recalling

$$k^*_{n_\infty} = \lim_{t \to +\infty} k^2(t),$$

we then obtain:

$$0 = f(k^*_{n_\infty}) - (\delta + n_\infty) \cdot k^*_{n_\infty}.$$

The strict concavity of the function f together with the Inada conditions ensure that there exists the unique solution for

$$\frac{f(k)}{k} = \delta + n_\infty.$$

This is because the function $\frac{f(k)}{k}$ monotonically decreases (see (3.8)), and $\delta + n_\infty > 0$ due to the assumption $\delta > r$. Overall, we then get $k_\infty = k^*_{n_\infty}$. In view of (3.9), we have $c_\infty = 0$.

- Case (II) with $[f'(k_\infty) - \rho - \delta - n_\infty] = 0$.

We rewrite the equation into

$$f'(k_\infty) = \rho + \delta + n_\infty.$$

Because $f'' < 0$, the derivative f' is strictly monotone; hence, it admits the inverse. Then, we obtain

$$k_\infty = (f')^{-1}(\rho + \delta + n_\infty).$$

By combining with the equation (3.9), it holds that

$$c_\infty = f(k_\infty) - (\delta + n_\infty) \cdot k_\infty.$$

Now, let us show that if $c_0 > 0$, we have case (II). This will be deduced from the transversality condition, (2.13),

$$\lim_{t\to\infty} \exp(-\rho t) \cdot c^{-\frac{1}{\sigma}}(t) \cdot k(t) = 0.$$

By using the explicit formula for consumption

$$c(t) = c_0 \cdot \exp\left(\sigma \cdot \int_0^t [f'(k(s)) - \delta - n(s) - \rho]\, ds\right),$$

we obtain after simplifications

$$\lim_{t\to\infty} c_0^{-\frac{1}{\sigma}} \cdot k(t) \cdot \exp\left(-\int_0^t [f'(k(s)) - \delta - n(s)]\, ds\right) = 0.$$

Because $k(t)$ converges toward $k_\infty > 0$, as can be easily seen from cases (I) or (II), and $c_0 > 0$, the integral above has to blow up:

$$\int_0^t [f'(k(s)) - \delta - n(s)]\, ds \to +\infty. \tag{3.12}$$

This condition can be further analyzed. For that, assume we are in Case (I), that is, $k_\infty = k^*_{n_\infty}$. Because $k^*_{n_\infty}$ is a steady-state solution of the standard Solow model with n_∞, we have:

$$f(k^*_{n_\infty}) - (\delta + n_\infty) \cdot k^*_{n_\infty} = 0.$$

Due to the strict concavity of f, we obtain:

$$\delta + n_\infty = \frac{f(k^*_{n_\infty})}{k^*_{n_\infty}} > f'\left(k^*_{n_\infty}\right).$$

Now, because $\lim_{t\to\infty} n(t) = n_\infty$, for sufficiently large t, $n(t)$ lies within a neighborhood of n_∞, say $(n_\infty - \varepsilon_1, n_\infty + \varepsilon_1)$ with $\varepsilon_1 > 0$. Similarly, because $\lim_{t\to\infty} k(t) = k^*_{n_\infty}$ and f' is continuous, it holds for sufficiently large t that

$$f'(k(t)) \in \left(f'(k^*_{n_\infty}) - \varepsilon_2,\ f'(k^*_{n_\infty}) + \varepsilon_2\right),$$

where $\varepsilon_2 > 0$ is arbitrary. Because of $\delta + n_\infty > f'(k^*_{n_\infty})$, we can choose $\varepsilon_3 > 0$ such that:

$$\delta + n_\infty - \varepsilon_3 > f'(k^*_{n_\infty}) + \varepsilon_3.$$

Therefore, by setting $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$, it holds for $t > T$ that

$$f'(k(t)) \le f'(k^*_{n_\infty}) + \varepsilon < \delta + n_\infty - \varepsilon \le \delta + n(t),$$

which implies:

$$f'(k(t)) - \delta - n(t) < 0.$$

Hence, it holds for the integral

$$\int_T^\infty [f'(k(s)) - \delta - n(s)]\, ds \le 0.$$

This provides a contradiction to the blow up in (3.12), because

$$\int_0^t [f'(k(s)) - \delta - n(s)]\, ds \le \int_0^T [f'(k(s)) - \delta - n(s)]\, ds,$$

where on the right-hand side we have a constant. Overall, Case (I) is not possible, and, hence, Case (II) occurs.

□

Let us clarify how Theorem 3.3 is related to the similar result in [5], where the Ramsey model with a positive and bounded population growth rate has been studied. In addition to the distinguishing of Cases (I) and (II) there, we prove that, under the transversality condition, the capital stock and the corresponding consumption $(k(t), c(t))$ cannot converge to the boundary steady state $(k^*_{n_\infty}, 0)$ if $c_0 > 0$. More precisely, the only limit compatible with both optimality and feasibility is the interior steady state (k_∞, c_∞). This is an improvement compared to the results in [5].

Theorem 3.3 allows to compare the equilibrium capital and consumption per labor corresponding to the Allee effect. As above, we distinguish Cases (I) and (II), by additionally assuming $\delta > r$.

Remark 3.4 (Case (I))**.** Here, we start from the zero consumption level with $c_0 = 0$. If $L_0 < N$, we have for the equilibrium capital per capita $k_{\infty,-r} = k^*_{-r}$ because $n_\infty = -r$. If $L_0 \ge N$, it holds that $k_{\infty,0} = k^*_0$ because $n_\infty = 0$ here; see also Remark 2.1. Note that, due to Theorem 3.1, $k^*_{-r} > k^*_0$, meaning that the capital per labor stabilizes at a higher level in case of diminishing rather than of growing population, that is, $k_{\infty,-r} > k_{\infty,0}$. This is in accordance with the results from [3]. In both cases, the consumption per labor approaches zero, that is, $c_\infty = 0$. □

Whereas Case (I) turns out to be more or less known from the previous literature, Case (II) sheds light on the economic interpretations of the Ramsey model with Allee effect, especially, with respect to its consumption part.

Remark 3.5 (Case (II))**.** Now, we start from the nonzero consumption level with $c_0 > 0$. If $L_0 < N$, we have $n_\infty = -r$ and, thus, for the equilibria of (2.14),

$$k_{\infty,-r} = (f')^{-1}(\rho + \delta - r), \quad c_{\infty,-r} = f(k_{\infty,-r}) - (\delta - r) \cdot k_{\infty,-r}.$$

If $L_0 \geq N$, we have $n_\infty = 0$ and, thus, for the equilibria of (2.14),

$$k_{\infty,0} = (f')^{-1}(\rho + \delta), \quad c_{\infty,-r} = f(k_{\infty,0}) - \delta \cdot k_{\infty,0}.$$

First, we compare the equilibrium capital per labor in these two cases. Because f' is monotonically decreasing, so is its inverse. Therefore, we conclude that

$$(f')^{-1}(\rho + \delta - r) > (f')^{-1}(\rho + \delta).$$

Again, the capital per labor stabilizes at a higher level in cases of diminishing rather than of growing populations, that is, $k_{\infty,-r} > k_{\infty,0}$. How are the corresponding equilibrium consumptions per labor related? For this, we define the difference

$$D_c = c_{\infty,-r} - c_{\infty,0} = f(k_{\infty,-r}) - f(k_{\infty,0}) - (\delta - r) \cdot k_{\infty,-r} + \delta \cdot k_{\infty,0}.$$

By using the mean value theorem, there exists $\xi \in [k_{\infty,0}, k_{\infty,-r}]$ such that

$$D_c = f'(\xi) \cdot (k_{\infty,-r} - k_{\infty,0}) + \delta \cdot k_{\infty,0} - (\delta - r) \cdot k_{\infty,-r}.$$

Because f' is monotonically decreasing, we obtain

$$D_c \geq f'(k_{\infty,-r}) \cdot (k_{\infty,-r} - k_{\infty,0}) + \delta \cdot k_{\infty,0} - (\delta - r) \cdot k_{\infty,-r}.$$

Recall that $k_{\infty,-r} = (f')^{-1}(\rho + \delta - r)$ and substitute this into the previous formula, yielding

$$D_c \geq \rho \cdot (k_{\infty,-r} - k_{\infty,0}) + r \cdot k_{\infty,0}.$$

Due to $k_{\infty,-r} > k_{\infty,0}$, the latter provides $D_c > 0$. Overall, we conclude that the equilibrium consumption per labor stabilizes at a higher level in cases of diminishing rather than growing populations, that is, $c_{\infty,-r} > c_{\infty,0}$. □

Finally, we investigate the limiting behavior of the consumption-to-capital ratio for the Ramsey model with Allee effect, that is,

$$x(t) = \frac{c(t)}{k(t)}.$$

This is to compare in Case (II) how consumption and capital relate to each other, depending on whether the population grows or diminishes. For this, we set the share of capital to production as auxiliary variable

$$z(t) = \frac{k(t)}{f(k(t))}.$$

As elaborated in [5], we obtain from (2.14):

$$\begin{aligned} \frac{\dot{z}(t)}{z(t)} &= (1 - f'(k(t)) \cdot z(t)) \cdot \left[z(t)^{-1} - (\delta + n(t)) - x(t)\right], \\ \frac{\dot{x}(t)}{x(t)} &= \left[\sigma \cdot f'(k(t)) - z(t)^{-1}\right] + (1 - \sigma) \cdot (\delta + n(t)) - \sigma \cdot \rho + x(t). \end{aligned} \tag{3.13}$$

For this transformed version of the Ramsey system, we similarly investigate its stability by identifying the steady states of (3.13). According to the previous results and Theorem 3.3, assuming the existence of k_∞, we also know that c_∞ exists. Consequently, under the assumption $k_0 > 0$ and $c_0 > 0$, that is, in Case (II), we have that $x_\infty = \frac{c_\infty}{k_\infty}$ also exists, and $z_\infty = \frac{k_\infty}{f(k_\infty)}$ exists as well. The next result shows that in the long-run, the consumption-to-capital ratio cannot fall below the individual discounting.

Corollary 3.6 (Consumption-to-capital ratio)**.** *Under assumptions of Theorem 3.3 with $c_0 > 0$, the ratio of consumption and capital at the steady state exceeds the discount rate:*

$$x_\infty > \rho.$$

Proof. Because we already know from the Theorem 3.3 that $z_\infty \neq 0$ and $x_\infty \neq 0$, we get from (3.13) that

$$\begin{aligned} 0 &= (1 - f'(k_\infty) \cdot z_\infty) \cdot \left[z_\infty^{-1} - (\delta + n_\infty) - x_\infty\right], \\ 0 &= \left[\sigma \cdot f'(k_\infty) - z_\infty^{-1}\right] + (1 - \sigma) \cdot (\delta + n_\infty) - \sigma \cdot \rho + x_\infty. \end{aligned}$$

In the first equation, we analyze the product of two terms. Due to the concavity of f, we have as in (3.7),

$$\frac{f(k_\infty)}{k_\infty} > f'(k_\infty),$$

which implies:

$$z_\infty = \frac{k_\infty}{f(k_\infty)} < \frac{1}{f'(k_\infty)} \quad \Rightarrow \quad f'(k_\infty) z_\infty < 1.$$

Therefore, the only way the first equation can hold is if the second term vanishes:

$$z_\infty^{-1} - (\delta + n_\infty) - x_\infty = 0.$$

Substituting this into the second equation yields:

$$\sigma \cdot f'(k_\infty) - (\delta + n_\infty + x_\infty) + (1 - \sigma) \cdot (\delta + n_\infty) - \sigma \cdot \rho + x_\infty = 0.$$

The latter simplifies to

$$f'(k_\infty) = \delta + n_\infty + \rho,$$

which is also known from Theorem 3.3. Overall, the formula for the steady state for the x-part is as follows:

$$x_\infty = \frac{1}{z_\infty} - \delta - n_\infty,$$

or, equivalently,

$$x_\infty = \frac{f(k_\infty)}{k_\infty} - \delta - n_\infty.$$

Using the concavity of f once again, we deduce that

$$x_\infty > f'(k_\infty) - \delta - n_\infty = \rho.$$

□

Now, we start from a strictly positive consumption level $c_0 > 0$; see Case (II). Our goal is to compare the steady-state consumption-to-capital ratio $x_\infty = \frac{c_\infty}{k_\infty}$ under different initial labor values L_0, by distinguishing two separate cases:

- In case $L_0 < N$, population growth rate becomes negative in the long run, that is, $n_\infty = -r < 0$. From the equilibrium condition of the original system (2.14), the corresponding steady-state values satisfy:
$$k_{\infty,-r} = (f')^{-1}(\rho + \delta - r), \quad c_{\infty,-r} = f(k_{\infty,-r}) - (\delta - r)\cdot k_{\infty,-r}.$$
- In case $L_0 \geq N$, the population growth stabilizes at zero, that is, $n_\infty = 0$. The equilibrium condition then yields:
$$k_{\infty,0} = (f')^{-1}(\rho + \delta), \quad c_{\infty,0} = f(k_{\infty,0}) - \delta \cdot k_{\infty,0}.$$

In order to compare the steady-state consumption-to-capital ratios in both cases, we define the difference:
$$D_x := x_{\infty,-r} - x_{\infty,0} = \frac{c_{\infty,-r}}{k_{\infty,-r}} - \frac{c_{\infty,0}}{k_{\infty,0}}.$$
By using formulas above, we obtain
$$D_x = \left[\frac{f(k_{\infty,-r})}{k_{\infty,-r}} - f'(k_{\infty,-r})\right] - \left[\frac{f(k_{\infty,0})}{k_{\infty,0}} - f'(k_{\infty,0})\right].$$
Define
$$h(k) := \frac{f(k)}{k} - f'(k), \qquad k > 0.$$
A direct differentiation yields
$$h'(k) = \frac{k\cdot f'(k) - f(k)}{k^2} - f''(k).$$
The terms on the right-hand side have opposite signs in general. In fact, the strict concavity of f gives $kf'(k) < f(k)$, so the first term is negative, while $f''(k) < 0$ makes the second term positive. Hence, the sign of $h'(k)$ depends on the particular choice of f. In order to illustrate this, we consider the following

- Cobb–Douglas production function $f(k) = k^\alpha$, where $0 < \alpha < 1$. Then,
$$h(k) = \frac{k^\alpha}{k} - \alpha \cdot k^{\alpha-1} = (1-\alpha)\cdot k^{\alpha-1}.$$
Because $\alpha - 1 < 0$, we have $h'(k) < 0$. In this case, h is strictly decreasing, and, thus, $D_x < 0$.
- Logarithmic production function $f(k) = \ln(1+k)$. We have
$$h(k) = \frac{\ln(1+k)}{k} - \frac{1}{1+k}, \qquad h'(k) = \frac{\frac{k}{1+k} - \ln(1+k)}{k^2} + \frac{1}{(1+k)^2}.$$
The derivative $h'(k)$ has the unique zero $k^*_{\log} \approx 2.16258$. Moreover, h is increasing on $(0, k^*_{\log})$ and decreasing on $(k^*_{\log}, \infty)$. Thus, the sign of D_x becomes dependent on whether $k_{\infty,-r}$ and $k_{\infty,0}$ lie in one interval or another. For example, setting the parameters $\delta = 0.075$, $\rho = 0.02$, $r = 0.025$, we have
$$k_{\infty,-r} = 0.042857, \quad k_{\infty,0} = 0.005263, \quad D_x \approx -0.00123.$$
For parameters $\delta = 0.75$, $\rho = 0.2$, $r = 0.25$, it holds that
$$k_{\infty,-r} = 0.428571, \quad k_{\infty,0} = 0.052632, \quad D_x \approx 0.107669.$$

- Constant absolute risk-aversion production function $f(k) = 1 - e^{-k}$. Here,

$$h(k) = \frac{1 - e^{-k}}{k} - e^{-k}, \qquad h'(k) = \frac{k \cdot e^{-k} - 1 + e^{-k}}{k^2} + e^{-k}.$$

  Numerical computations yield that $k^*_{\text{CARA}} \approx 1.79328$ is the unique zero of $h'(k)$. Moreover, h is increasing on $(0, k^*_{\text{CARA}})$ and decreasing on $(k^*_{\text{CARA}}, \infty)$. Again, the sign of D_x depends on $k_{\infty,-r}$ and $k_{\infty,0}$. For example, for the parameters $\delta = 0.075$, $\rho = 0.02$, $r = 0.025$, we have

$$k_{\infty,-r} = 0.035667, \quad k_{\infty,0} = 0.005129, \quad D_x \approx -0.00145.$$

  If we change the parameters to $\delta = 0.75$, $\rho = 0.2$, $r = 0.25$, the situation becomes different:

$$k_{\infty,-r} = 0.356675, \quad k_{\infty,0} = 0.051293, \quad D_x \approx 0.116316.$$

Finally, we study the local stability properties of the three-dimensional optimal growth system (2.14) by means of linearization. Its steady states $(k_\infty, c_\infty, L_\infty)$ satisfy

$$f'(k_\infty) = \rho + \delta + n_\infty, \qquad c_\infty = f(k_\infty) - (\delta + n_\infty) \cdot k_\infty, \tag{3.14}$$

where $L_\infty \in \{0, N, M\}$. The Jacobian matrix at $(k_\infty, c_\infty, L_\infty)$ is

$$J = \begin{pmatrix} \rho & -1 & -k_\infty \cdot n'(L_\infty) \\ \sigma \cdot c_\infty \cdot f''(k_\infty) & 0 & -\sigma \cdot c_\infty \cdot n'(L_\infty) \\ 0 & 0 & \lambda_L(L_\infty) \end{pmatrix}, \tag{3.15}$$

where

$$\lambda_L(L_\infty) = r\left(-1 + 2 \cdot \frac{L_\infty}{N} + 2 \cdot \frac{L_\infty}{M} - 3 \cdot \frac{L_\infty^2}{M \cdot N}\right) \tag{3.16}$$

is the eigenvalue in the population direction. Obviously,

$$\lambda_L < 0 \quad \text{for } L_\infty = 0 \text{ and } L_\infty = M, \qquad \lambda_L > 0 \quad \text{for } L_\infty = N.$$

Moreover, the subsystem (k, c) behaves as in a standard Ramsey model (see [11]), that is, it has one stable and one unstable direction. Hence, the whole system (2.14) is saddle-point stable if $L_\infty = 0$ or $L_\infty = M$, and unstable if $L_\infty = N$. In order to quantify the (local) convergence speed, we consider the exponential estimate. At an equilibrium $(k_\infty, c_\infty, L_\infty)$, the eigenvalues of the Jacobian's 2×2 upper-left block are

$$\lambda_s = \frac{\rho - \sqrt{\rho^2 - 4 \cdot \sigma \cdot c_\infty \cdot f''(k_\infty)}}{2}, \quad \lambda_u = \frac{\rho + \sqrt{\rho^2 - 4 \cdot \sigma \cdot c_\infty \cdot f''(k_\infty)}}{2}.$$

Because f is concave, $\lambda_s < 0$ and $\lambda_u > 0$. For the linearized system, the asymptotic exponential decay along the stable manifold is governed by the magnitudes of the negative eigenvalues, that is, $\min\{|\lambda_s|, |\lambda_L(L_\infty)|\}$. We point out that in [5], the asymptotic stability for the Ramsey model with a positive and bounded population growth rate has been also studied. There, the author explicitly assumes that the capital stock $k(t)$ converges. Under these assumptions, it has been shown in [5] that the optimal consumption path $c(t)$ converges, as well. Thus, the term "asymptotically stable" used there refers to the asymptotic stability along the stable manifold as in our analysis. Additionally, we could quantify the speed of convergence by exploiting the underlying Allee effect.

4. Numerical simulations

We consider the well-known CES production function:

$$F(K,L) = [\alpha \cdot K^{\tau} + (1-\alpha) \cdot L^{\tau}]^{\frac{1}{\tau}}, \tag{4.1}$$

where α is the capital distribution parameter with $0 < \alpha < 1$, and τ is the substitution elasticity parameter with $-\infty < \tau < 1$. Then, the output per labor is

$$f(k) = \frac{F(K,L)}{L} = (\alpha \cdot k^{\tau} + (1-\alpha))^{\frac{1}{\tau}}. \tag{4.2}$$

Then, we can obtain its derivative,

$$f'(k) = \alpha \cdot (\alpha \cdot k^{\tau} + (1-\alpha))^{\frac{1}{\tau}-1} \cdot k^{\tau-1}. \tag{4.3}$$

The capital-related parameters are, for example,

$$\alpha = 0.3, \quad \delta = 0.075, \quad \tau = 0.01, \quad \rho = 0.02, \quad \sigma = 1.01.$$

The population-related parameters are, for example,

$$r = 0.025, \quad N = 1, \quad M = 2.$$

For other parameter choices, the qualitative behavior of the model remains the same. The only requirement is that $\delta > r$. For the simulations to follow, we monitor capital, consumption, welfare, and output per labor.

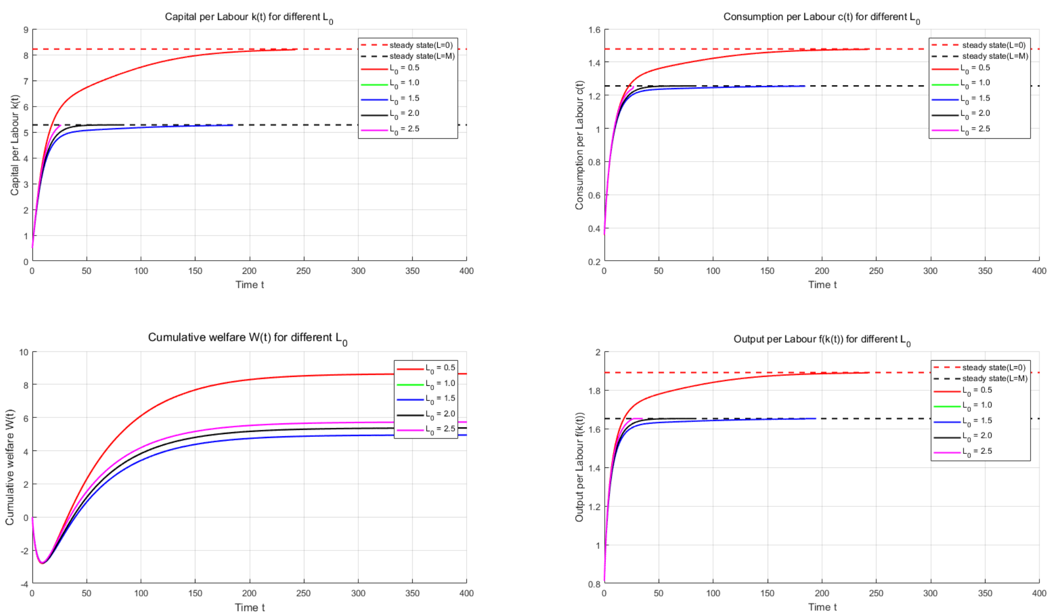

Figure 1. Case (II)

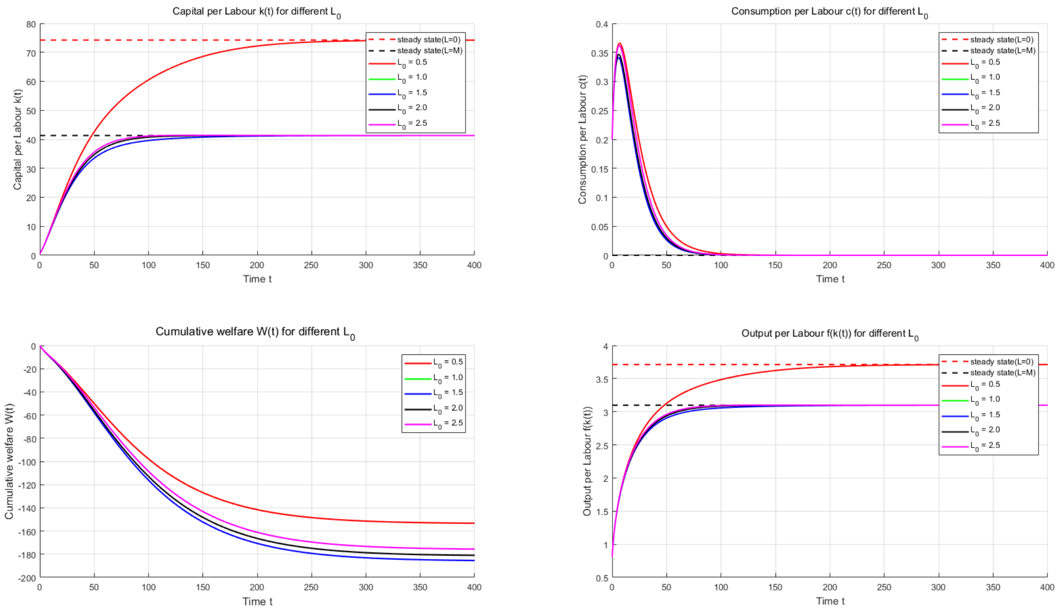

Figure 2. Boundary case

The behavior of capital per labor under the Allee effect aligns as expected in Case (II); see Figure 1. The capital per labor stabilizes at a higher level in cases of $L_0 < N$ compared to $L_0 > N$. Also, the consumption per labor stabilizes at a higher level; see also Figure 1. This is if suitable initial conditions are chosen. For that, the initial capital stock is set to $k_0 = 0.5$ for all simulations here. For each given initial population level,

$$L_0 \in \{0.5,\ 1,\ 1.5,\ 2,\ 2.5\},$$

the corresponding initial consumption level c_0 is determined numerically using a bisection algorithm. The algorithm searches over the admissible interval $(0, f(k_0))$ and iteratively adjusts c_0 until the resulting trajectory converges to the appropriate steady state. This procedure yields the corresponding saddle-path initial values:

$$c_0 = \{0.356995737,\ 0.3657260938,\ 0.3683138086,\ 0.3657260938,\ 0.3596662435\}.$$

With these specially selected initial conditions, the simulated trajectories remain on the stable manifold and converge to their respective steady states, as shown in Figure 1. The steady state is the same for all initial conditions. If $L_0 \geq N$, we have

$$(k_\infty, c_\infty, L_\infty) = (5.276492, 1.256055, 2),$$

and if $L_0 < N$,

$$(k_\infty, c_\infty, L_\infty) = (8.215943, 1.478957, 0).$$

Figure 2 corresponds to the boundary case, where consumption is initially fixed at a low level and the economy does not lie on the stable saddle path. Specifically, the initial capital stock is again set to $k_0 = 0.5$ for all simulations, and the initial consumption level is uniformly chosen as $c_0 = 0.2$. For the same initial population levels as before, the resulting trajectories approach a boundary equilibrium characterized by vanishing consumption. If $L_0 \geq N$, the population converges to $L_\infty = M$, and the boundary steady state is given by

$$(k_\infty, c_\infty, L_\infty) = (41.307566,\ 0,\ 2).$$

If $L_0 < N$, the population drives to extinction, $L_\infty = 0$, and the corresponding boundary steady state becomes

$$(k_\infty, c_\infty, L_\infty) = (74.241787,\ 0,\ 0).$$

These trajectories illustrate that, although the capital stock may converge to a finite positive level and the initial consumption is positive, the associated equilibrium is a boundary solution with zero consumption. Such paths do not satisfy the transversality condition and are therefore excluded from the set of optimal solutions; see Case (I) in Theorem 3.3. Nevertheless, they naturally arise when initial conditions are chosen outside the stable manifold, which is consistent with the theoretical classification of equilibria in the Ramsey framework. Note that here we observe a non-monotonic consumption behavior.

Aiming to illustrate the blow-up effect, which occurs if the stability condition $\delta > r$ is violated, let us take $r = 0.085$ in the previous case. If starting below the Allee threshold N, the capital per labor explodes, and the consumption per labor vanishes in the limit; see Figure 3. In the presence of a rapidly declining population, labor becomes increasingly scarce, and production relies almost exclusively on capital. Additionally, the depreciation of capital is relatively slow, implying that investments pay off more than consumption. As a consequence, resources are continuously shifted towards investments in order to substitute for labor, and current consumption progressively diminishes. This case need further investigation, especially in view of the current political debates on migration policies.

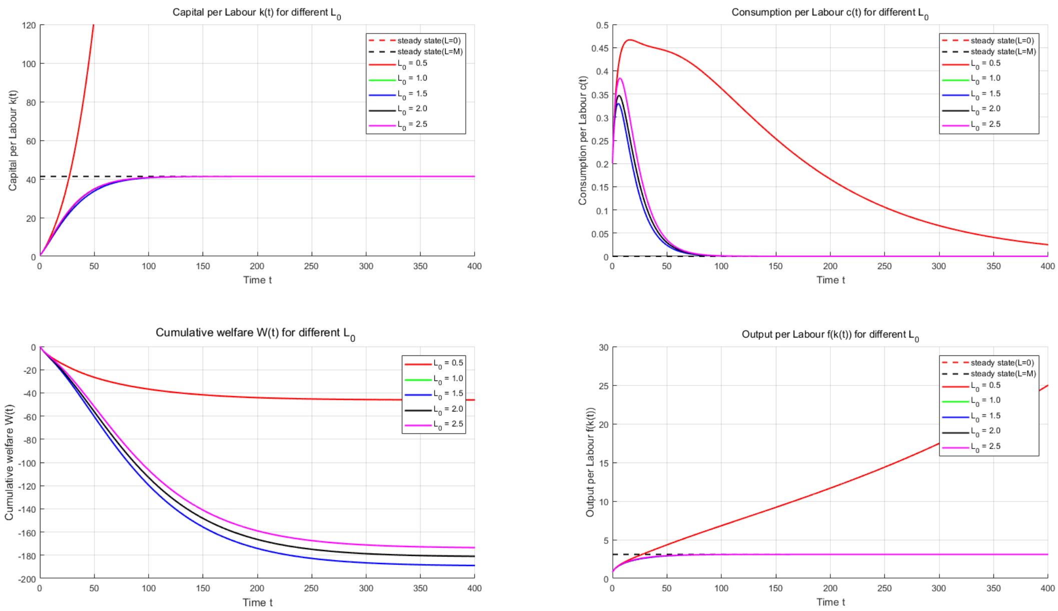

Figure 3. Blow-up effects

5. Conclusions

In this study, we extend the classical Ramsey model of optimal growth by relaxing the assumption on the population growth rate. In the traditional Ramsey framework, the population growth rate is assumed to be a positive constant. Further in [5], this model has been generalized by allowing the growth rate to be a positive and bounded function of time. In contrast, our research investigates the special case where the population growth rate can take negative values, representing a declining population scenario. We thereby extend the model's applicability by incorporating the Allee effect for the population dynamics; see [3]. To so so, we first provide a detailed discussion on the structure of the optimal solution and prove that, under the transversality condition, the solution remains interior and does not approach the boundary. Second, we provide a rigorous characterization of the convergence speed of the Ramsey equilibrium. Third, we demonstrate that under population decline, both capital per labor and consumption per labor converge to higher steady-state levels than in the case of saturation. The main rationale behind this somewhat counterintuitive conclusion is shown to hold under a stability assumption, that is, the decrease of population, expressed by the intrinsic population growth rate, should be rather moderate in comparison to the rate of capital depreciation. Otherwise, although the capital per labor blows up, we encounter by means of numerical simulations that the corresponding consumption may vanish in the limit. The decline of population is then so fast that the economy cannot equilibrate. The growing capital per labor needs to be reinvested completely; thus, less and less is left for consumption. In addition, we prove that along the stable path, the long-run consumption-capital ratio exceeds the discount rate, and we characterize how this ratio varies with different initial population levels. This ensures the economic validity of the model and enables its use in a broader range of population growth dynamics. Overall, our work generalizes the Ramsey model to encompass both positive and negative bounded population growth rates, which expands its theoretical and practical relevance.

Author contributions

Both authors contributed equally to this work.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgment

The authors would like to thank the anonymous referees for their valuable suggestions, which improved the quality of the paper.

Conflict of interest

Vladimir Shikhman is the guest editor for Journal of Industrial and Management Optimization and was not involved in the editorial review or the decision to publish this article. All authors declare that there are no competing interests.

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