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*Theory article*

# Extragradient Methods for Solving Variational Inequalities with Summable Errors

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**Abstract:** We study, in the setting of a Hilbert space, weak convergence of the sequences generated by extragradient methods for solving variational inequalities in the presence of summable computational errors. Our results enhance earlier results which were obtained for exact iterates of such methods.

**Keywords:** Extragradient method; Hilbert space; iteration; variational inequality
**Mathematics Subject Classification:** 47H05, 47H14

## 1. Introduction

Gradient-type methods and variational inequalities have recently been and continue to be important topics in optimization theory and its applications. See, for example, [1–12] and references mentioned therein. In the present paper we study, in the setting of a Hilbert space, weak convergence of the sequences generated by the extragradient method, introduced in [11] for solving variational inequalities, in the presence of summable computational errors. Our results enhance earlier results, which were obtained by Censor, Gibali and Reich [5] for exact iterates of this method. It is shown that the weak convergence established in [5] remains in force even in the presence of small computational errors, which are always present in practice.

Let $H$ be a Hilbert space equipped with an inner product $\langle \cdot, \cdot \rangle$ which induces the norm

$$\|x\| = \langle x, x \rangle^{1/2}, \ x \in H.$$

Let $D$ be a nonempty, closed, and convex subset of $H$. It is well known that for each $x \in H$, there exists a unique nearest point in $D$ which is denoted by $P_D(x)$. That is,

$$\|x - P_D(x)\| \le \|x - y\|, \ y \in D.$$

It is also well known that $P_D : H \to H$ is a nonexpansive operator [13, 14], that is,

$$\|P_D(x) - P_D(y)\| \le \|x - y\|, \ x, y \in H,$$

and that for each $x \in H$, and each $y \in D$, we have

$$P_D(x) \in D,$$

$$\langle x - P_D(x), P_D(x) - y \rangle \geq 0, \tag{1.1}$$

and

$$\|x - y\|^2 \geq \|x - P_D(x)\|^2 + \|y - P_D(x)\|^2. \tag{1.2}$$

Recall that a map $B : H \to 2^H$ is said to be monotone if

$$\langle u - v, x - y \rangle \geq 0$$

for each $x, y \in H$, each $u \in B(x)$ and each $v \in B(y)$.

Note that the following fact, called the Opial property, holds in $H$.

If a sequence $\{x_k\}_{k=1}^{\infty} \subset H$ converges weakly to $x \in H$, then for each $y \in H \setminus \{x\}$, we have

$$\liminf_{k\to\infty} \|x_k - x\| < \liminf_{k\to\infty} \|x_k - y\|.$$

## 2. The first main result

Assume that $C \subset H$ is a nonempty, closed, and convex set, and let $f : H \to H$ be an operator. Assume that

$$\langle f(x) - f(y), x - y \rangle \geq 0 \text{ for each } x, y \in C. \tag{2.1}$$

In other words, the operator $f$ is monotone. Assume also that $L > 0$ and that

$$\|f(x) - f(y)\| \leq L\|x - y\|, \;\; x, y \in C. \tag{2.2}$$

In other words, the operator $f$ is Lipschitz with a Lipschitz constant $L$. Denote by $\mathrm{SOL}(C, f)$ the set of all points $x_* \in X$ such that

$$\langle f(x_*), x - x_* \rangle \geq 0, \;\; x \in C. \tag{2.3}$$

We assume that $\mathrm{SOL}(C, f) \neq \emptyset$. Note that for each $x_* \in C$, we have [15]

$$x_* \in \mathrm{SOL}(C, f) \text{ if and only if } \langle f(y), y - x_* \rangle \geq 0, \;\; y \in C. \tag{2.4}$$

Indeed, if $x_* \in \mathrm{SOL}(C, f)$, then for each $y \in C$, we have

$$\langle f(y), y - x_* \rangle \geq \langle f(x_*), y - x_* \rangle \geq 0.$$

If for each $y \in C$,

$$\langle f(y), y - x_* \rangle \geq 0,$$

then for each $x \in C$ and each $t \in (0, 1)$, we have

$$0 \leq \langle f((1 - t)x_* + tx), (1 - t)x_* + tx - x_* \rangle$$

$$= t\langle f((1 - t)x_* + tx), x - x_* \rangle$$

and

$$0 \leq \langle f((1-t)x_* + tx), x - x_* \rangle \to \langle f(x_*), x - x_* \rangle$$

as $t \to 0^+$.

We now consider the following algorithm for solving our variational inequality.

Fix a number $\tau \in (0, L^{-1})$.

**Initialization**. Choose any point $x_0 \in H$.

**Iterative step**. Assume that $k \geq 0$ is an integer and that $x_k \in H$ is the current iterate. Set

$$y_k = P_C(x_k - \tau f(x_k)), \ x_{k+1} = P_C(x_k - \tau f(y_k)). \tag{2.5}$$

It was shown in [5] that the sequences $\{x_k\}$ and $\{y_k\}$ both converge weakly to the same limit, which belongs to SOL$(C, f)$. In the present paper, we show that inexact iterates of the above algorithm retain this property provided the sequence of computational errors is summable.

The following lemma is an important ingredient of the proofs in [5] and in the present paper.

**Lemma 2.1.** *Let $\{x_k\}_{k=0}^{\infty}, \ \{y_k\}_{k=0}^{\infty} \subset H$, assume that (2.5) holds for each integer $k \geq 0$, and let $u \in$ SOL$(C, f)$. Then for each integer $k \geq 0$, we have*

$$\|x_{k+1} - u\|^2 \leq \|x_k - u\|^2 - (1 - \tau^2 L^2)\|y_k - x_k\|^2.$$

Let $\{\Delta_k\}_{k=0}^{\infty} \subset (0, \infty)$ and

$$\Delta = \sum_{k=0}^{\infty} \Delta_k. \tag{2.6}$$

We are now ready to state our first main result which establishes the weak convergence of iterates generated by the monotone operator $f$ introduced in this section.

**Theorem 2.2.** *Let $\tau \in (0, L^{-1})$, $\{x_k\}_{k=0}^{\infty}, \ \{y_k\}_{k=0}^{\infty} \subset H$, and for each integer $k \geq 0$, let*

$$\|y_k - P_C(x_k - \tau f(x_k))\| \leq \Delta_k \tag{2.7}$$

*and*

$$\|x_{k+1} - P_C(x_k - \tau f(y_k))\| \leq \Delta_k. \tag{2.8}$$

*Then both the sequences $\{x_k\}_{k=0}^{\infty}$ and $\{y_k\}_{k=0}^{\infty}$ converge weakly to the same limit, which belongs to SOL$(C, f)$.*

We precede the proof of this theorem by an auxiliary result.

## 3. An auxiliary result

The following lemma is an important ingredient in the proof of Theorem 2.2.

**Lemma 3.1.** *Let $\tau \in (0, L^{-1})$ and $\{x_k\}_{k=0}^{\infty}, \ \{y_k\}_{k=0}^{\infty} \subset H$, assume that for each integer $k \geq 0$, both (2.7) and (2.8) hold, and let*

$$u \in SOL(C, f). \tag{3.1}$$

*Then for each integer $k \geq 0$, we have*

$$\|x_k - u\| \leq \|x_0 - u\| + \Delta$$

*and*

$$\|x_{k+1} - u\|^2 \leq \|x_k - u\|^2 - (1 - \tau^2 L^2)\|y_k - x_k\|^2 + \Delta_k(4\|x_0 - u\| + 10\Delta).$$

*Proof.* Let $k \geq 0$ be an integer. Set

$$\tilde{y}_k = P_C(x_k - \tau f(x_k)) \tag{3.2}$$

and

$$\tilde{x}_{k+1} = P_C(x_k - \tau f(\tilde{y}_k)). \tag{3.3}$$

Lemma 2.1, (3.2) and (3.3) imply that

$$\|\tilde{x}_{k+1} - u\|^2 \leq \|x_k - u\|^2 - (1 - \tau^2 L^2)\|\tilde{y}_k - x_k\|^2. \tag{3.4}$$

In view of (2.7) and (3.2), we have

$$\|\tilde{y}_k - y_k\| = \|y_k - P_C(x_k - \tau f(x_k))\| \leq \Delta_k. \tag{3.5}$$

By (2.2), (2.8), (3.3), (3.5), and the inequality $L\tau < 1$, and since the operator $P_C$ is nonexpansive, we have

$$\begin{aligned}\|\tilde{x}_{k+1} - x_{k+1}\| &= \|P_C(x_k - \tau f(\tilde{y}_k)) - P_C(x_k - \tau f(y_k))\| \\ &+\|P_C(x_k - \tau f(y_k)) - x_{k+1}\| \\ &\leq \tau\|f(\tilde{y}_k) - f(y_k)\| + \Delta_k \leq L\tau\|\tilde{y}_k - y_k\| + \Delta_k \leq 2\Delta_k.\end{aligned} \tag{3.6}$$

By (3.5) and (3.6), for each integer $k \geq 0$, we have

$$\|\tilde{y}_k - y_k\| \leq \Delta_k, \ \|\tilde{x}_{k+1} - x_{k+1}\| \leq 2\Delta_k. \tag{3.7}$$

Using (3.4), (3.7), and the inequality $L\tau < 1$, we see that for each integer $k \geq 0$,

$$\|\tilde{x}_{k+1} - u\| \leq \|x_k - u\|$$

and

$$\|x_{k+1} - u\| \leq \|x_{k+1} - \tilde{x}_{k+1}\| + \|\tilde{x}_{k+1} - u\| \leq \|x_k - u\| + 2\Delta_k. \tag{3.8}$$

It follows from (2.6) and (3.8) that for each integer $k \geq 1$,

$$\|x_k - u\| \leq \|x_0 - u\| + 2\sum_{i=0}^{\infty} \Delta_i \leq \|x_0 - u\| + 2\Delta. \tag{3.9}$$

Let $k \geq 0$ be an integer. In view of (3.4) and (3.9),

$$\|\tilde{y}_k - x_k\| \leq (1 - \tau^2 L^2)^{-1/2}\|x_k - u\| \leq (1 - \tau^2 L^2)^{-1/2}(\|x_0 - u\| + 2\Delta). \tag{3.10}$$

By (2.8), (3.3), and (3.9), we have

$$|\|\tilde{x}_{k+1} - u\|^2 - \|x_{k+1} - u\|^2|$$

$$\leq |\|\tilde{x}_{k+1} - u\| - \|x_{k+1} - u\||(\|\tilde{x}_{k+1} - u\| + \|x_{k+1} - u\|)$$
$$\leq \|\tilde{x}_{k+1} - x_{k+1}\|(2\|x_{k+1} - u\| + \|\tilde{x}_{k+1} - x_{k+1}\|) \leq \Delta_k(2\|x_0 - u\| + 5\Delta). \tag{3.11}$$

It follows from (3.5), (3.9), (3.10), and the inequality $\tau L < 1$ that

$$(1 - \tau^2 L^2)|\|y_k - x_k\|^2 - \|\tilde{y}_k - x_k\|^2|$$
$$\leq (1 - \tau^2 L^2)|\|y_k - x_k\| - \|\tilde{y}_k - x_k\||(\|\tilde{y}_k - x_k\| + \|y_k - x_k\|)$$
$$\leq (1 - \tau^2 L^2)\|\tilde{y}_k - y_k\|(2\|\tilde{y}_k - x_k\| + \|\tilde{y}_k - y_k\|)$$
$$\leq (1 - \tau^2 L^2)\Delta_k(2\|\tilde{y}_k - x_k\| + \Delta_k) \leq \Delta_k(2\|x_0 - u\| + 5\Delta). \tag{3.12}$$

Using (3.4), (3.11), and (3.12), we find that

$$\|x_{k+1} - u\|^2 \leq \|\tilde{x}_{k+1} - u\|^2 + \Delta_k(2\|x_0 - u\| + 5\Delta)$$
$$\leq \|x_k - u\|^2 - (1 - \tau^2 L^2)\|\tilde{y}_k - x_k\|^2 + \Delta_k(2\|x_0 - u\| + 5\Delta)$$
$$\leq \|x_k - u\|^2 - (1 - \tau^2 L^2)\|y_k - x_k\|^2 + \Delta_k(4\|x_0 - u\| + 10\Delta).$$

This completes the proof of Lemma 3.1. □

## 4. Proof of Theorem 2.2

Assume that

$$u \in \mathrm{SOL}(C, f).$$

Let $Q$ be a natural number. Lemma 3.1 implies that

$$\|x_0 - u\|^2 \geq \|x_0 - u\|^2 - \|x_Q - u\|^2$$
$$= \sum_{k=0}^{Q-1}(\|x_k - u\|^2 - \|x_{k+1} - u\|^2)$$
$$\geq \sum_{k=0}^{Q-1}((1 - \tau^2 L^2)\|x_k - y_k\|^2 - \Delta_k(4\|x_0 - u\| + 10\Delta)),$$
$$\sum_{k=0}^{Q-1}(1 - \tau^2 L^2)\|x_k - y_k\|^2 \leq \|x_0 - u\|^2 + \Delta(4\|x_0 - u\| + 10\Delta). \tag{4.1}$$

In view of (4.1),

$$\sum_{k=0}^{\infty} \|x_k - y_k\|^2 < \infty, \ \lim_{k\to\infty} \|y_k - x_k\| = 0. \tag{4.2}$$

Lemma 3.1 implies that the sequence $\{x_k\}_{k=1}^{\infty}$ is bounded, and so it has a weak accumulation point. Let $\bar{x}$ be the weak limit of a subsequence $\{x_{k_j}\}_{j=1}^{\infty}$ of $\{x_k\}$:

$$\lim_{j\to\infty} x_{k_j} = \bar{x}. \tag{4.3}$$

Clearly, $\bar{x} \in C$. In view of (4.2) and (4.3),

$$\lim_{j\to\infty} y_{k_j} = \bar{x} \tag{4.4}$$

in the weak topology.

Let $v \in C$. By (1.1), for each integer $k \geq 0$,

$$\langle x_k - \tau f(x_k) - P_C(x_k - \tau f(x_k)), P_C(x_k - \tau f(x_k)) - v\rangle \geq 0$$

and

$$\langle \tau^{-1}(P_C(x_k - \tau f(x_k)) - x_k) + f(x_k), v - P_C(x_k - \tau f(x_k))\rangle \geq 0. \tag{4.5}$$

Since the operator $f$ is Lipschitz, it follows from Lemma 3.1 that the sequences $\{P_C(x_k - \tau f(x_k))\}_{k=0}^{\infty}$ and $\{f(x_k)\}_{k=0}^{\infty}$ are bounded. When combined with (2.7), (2.8), and (4.5), this implies that

$$\liminf_{k\to\infty} \langle \tau^{-1}(y_k - x_k) + f(x_k), v - y_k\rangle \geq 0. \tag{4.6}$$

By (4.2) and (4.6),

$$\liminf_{k\to\infty} \langle f(x_k), v - y_k\rangle \geq 0. \tag{4.7}$$

Lemma 3.1, (2.1), (2.2), (4.2), (4.4), and (4.6) imply that

$$\langle f(v), v - \bar{x}\rangle = \lim_{j\to\infty} \langle f(v), v - y_{k_j}\rangle$$

$$\geq \limsup_{j\to\infty} [\langle f(v), v - y_{k_j}\rangle - \langle \tau^{-1}(y_{k_j} - x_{k_j}) + f(x_{k_j}), v - y_{k_j}\rangle]$$

$$= \limsup_{j\to\infty} [\langle f(v) - f(y_{k_j}), v - y_{k_j}\rangle$$

$$+ \langle f(y_{k_j}) - f(x_{k_j}), v - y_{k_j}\rangle$$

$$- \langle \tau^{-1}(y_{k_j} - x_{k_j}), v - y_{k_j}\rangle] \geq 0.$$

So for each $v \in C$, we have

$$\langle f(v), v - \bar{x}\rangle \geq 0. \tag{4.8}$$

By (2.4),

$$\bar{x} \in \mathrm{SOL}(C, f).$$

Thus, we have shown that the limit of any weakly convergent subsequence of $\{x_k\}_{k=0}^{\infty}$ belongs to $\mathrm{SOL}(C, f)$.

Now we show that the sequence $\{x_k\}_{k=0}^{\infty}$ weakly converges to $\bar{x}$. Suppose to the contrary that this does not hold. Then there exists another subsequence $\{x_{\bar{k}_j}\}_{j=1}^{\infty}$ which converges weakly to $\bar{x}' \neq \bar{x}$, where

$$\bar{x}' \in \mathrm{SOL}(C, f).$$

By Lemma 3.1 applied with $u = \bar{x}$, $u = \bar{x}'$ and the Opial property, we have

$$\lim_{k\to\infty} \|x_k - \bar{x}\| = \liminf_{k\to\infty} \|x_k - \bar{x}\|$$

$$= \liminf_{j\to\infty} \|x_{k_j} - \bar{x}\| < \liminf_{j\to\infty} \|x_{k_j} - \bar{x}'\|$$

$$= \lim_{j\to\infty} \|x_{k_j} - \bar{x}'\| = \liminf_{j\to\infty} \|x_{\bar{k}_j} - \bar{x}'\|$$

$$< \liminf_{j\to\infty} \|x_{\bar{k}_j} - \bar{x}\| = \lim_{k\to\infty} \|x_k - \bar{x}\|,$$

a contradiction.

The contradiction we have reached shows that the sequence $\{x_k\}_{k=0}^{\infty}$ converges weakly to $\bar{x}$. In view of (4.2), the sequence $\{y_k\}_{k=0}^{\infty}$ also converges weakly to $\bar{x}$, as asserted. This completes the proof of Theorem 2.2.

## 5. The second main result

In this section we consider a modification of the algorithm studied in the previous sections. This modification was introduced in [5], where the weak convergence of the exact iterates it generates was proved. Here we establish the weak convergence of inexact iterates with summable errors.

Fix a number $\tau \in (0, L^{-1})$.

We consider the following algorithm for solving our variational inequality.

**Initialization**. Choose any $x_0 \in H$.

**Iterative step**. Assume that $k \geq 0$ is an integer and that $x_k \in H$ is the current iterate. Set

$$y_k = P_C(x_k - \tau f(x_k)), \tag{5.1}$$

construct the half-space

$$T_k = \{w \in H : \langle x^k - \tau f(x_k) - y_k, w - y_k\rangle \leq 0\}, \tag{5.2}$$

and set

$$x_{k+1} = P_{T_k}(x_k - \tau f(y_k)). \tag{5.3}$$

In view of (1.1) and (5.1),

$$C \subset T_k,\ k = 1, 2, \ldots.$$

The classical algorithm consists in two calculations of the nearest point projection on the set $C$. In its modification introduced and studied in [5], the first step is the projection on the set $C$ while the second one is the projection on the half-space $T_k$. The reason for this modification is discussed in [5]. Here, we only mention that the set $C$ can be complicated while the projection on the half-space is easily calculated.

It was shown in [5] that the sequences $\{x_k\}$ and $\{y_k\}$ both converge weakly to the same limit, which belongs to SOL$(C, f)$. In the present paper, we show that inexact iterates of the above algorithm retain this property provided the sequence of computational errors is summable.

The following lemma is an important ingredient of the proof in [5].

**Lemma 5.1.** *Let $\{x_k\}_{k=0}^{\infty}$, $\{y_k\}_{k=0}^{\infty} \subset H$, for each integer $k \geq 0$, assume that (5.1)–(5.3) hold, and let $u \in SOL(C, f)$. Then for each integer $k \geq 0$, we have*

$$\|x_{k+1} - u\|^2 \leq \|x_k - u\|^2 - (1 - \tau^2 L^2)\|y_k - x_k\|^2.$$

We are now ready to state our second main result.

**Theorem 5.2.** *Let*

$$\tau \in (0, L^{-1}), \tag{5.4}$$

*and let* $\{\Delta_k\}_{k=0}^{\infty} \subset (0, \infty)$ *with*

$$\Delta = \sum_{k=0}^{\infty} \Delta_k. \tag{5.5}$$

*Assume that* $\{x_k\}_{k=0}^{\infty}, \ \{y_k\}_{k=0}^{\infty} \subset H$*, and that, for each integer* $k \geq 0$*,*

$$y_k = P_C(x_k - \tau f(x_k)),$$

*(5.2) holds, and*

$$\|x_{k+1} - P_{T_k}(x_k - \tau f(y_k))\| \leq \Delta_k. \tag{5.6}$$

*Then, the sequences* $\{x_k\}_{k=0}^{\infty}$ *and* $\{y_k\}_{k=0}^{\infty}$ *both converge weakly to the same limit, which belongs to* $SOL(C, f)$*.*

## 6. An auxiliary result

**Lemma 6.1.** *Let* $\tau \in (0, L^{-1})$ *and* $\{\Delta_k\}_{k=0}^{\infty} \subset (0, \infty)$ *with*

$$\Delta = \sum_{k=0}^{\infty} \Delta_k. \tag{5.7}$$

*Assume that* $\{x_k\}_{k=0}^{\infty}, \ \{y_k\}_{k=0}^{\infty} \subset H$*, that for each integer* $k \geq 0$*, (5.1), (5.2), and (5.6) hold, and that*

$$u \in SOL(C, f).$$

*Then for each integer* $k \geq 0$*, we have*

$$\|x_k - u\| \leq \|x_0 - u\| + \Delta$$

*and*

$$\|x_{k+1} - u\|^2 \leq \|x_k - u\|^2 - (1 - \tau^2 L^2)\|y_k - x_k\|^2 + \Delta_k(2\|x_0 - u\| + 3\Delta).$$

*Proof.* Let $k \geq 0$ be an integer. Set

$$\tilde{x}_{k+1} = P_{T_k}(x_k - \tau f(y_k)). \tag{5.8}$$

Lemma 5.1 and (5.8) imply that

$$\|\tilde{x}_{k+1} - u\|^2 \leq \|x_k - u\|^2 - (1 - \tau^2 L^2)\|y_k - x_k\|^2. \tag{5.9}$$

In view of (5.6) and (5.8),

$$\|\tilde{x}_{k+1} - x_{k+1}\| \leq \Delta_k. \tag{5.10}$$

By (5.7) and (5.10), we have

$$\|x_{k+1} - u\| \le \|x_{k+1} - \tilde{x}_{k+1}\| + \|\tilde{x}_{k+1} - u\| \le \|x_k - u\| + \Delta_k \tag{5.11}$$

and

$$\|x_k - u\| \le \|x_0 - u\| + \sum_{i=0}^{\infty} \Delta_i \le \|x_0 - u\| + \Delta. \tag{5.12}$$

In view of (5.10) and (5.12), we also have

$$\begin{aligned} &|\|\tilde{x}_{k+1} - u\|^2 - \|x_{k+1} - u\|^2| \\ &\le |\|\tilde{x}_{k+1} - u\| - \|x_{k+1} - u\||(\|\tilde{x}_{k+1} - u\| + \|x_{k+1} - u\|) \\ &\le \|\tilde{x}_{k+1} - x_{k+1}\|(2\|x_{k+1} - u\| + \|\tilde{x}_{k+1} - x_{k+1}\|) \\ &\le \Delta_k(2\|x_0 - u\| + 2\Delta + \Delta_k) \le \Delta_k(2\|x_0 - u\| + 3\Delta). \end{aligned}$$

It follows from (5.9) and the above relation that

$$\begin{aligned} &\|x_{k+1} - u\|^2 \le \|\tilde{x}_{k+1} - u\|^2 + \Delta_k(2\|x_0 - u\| + 3\Delta) \\ &\le \|x_k - u\|^2 - (1 - \tau^2 L^2)\|y_k - x_k\|^2 + \Delta_k(2\|x_0 - u\| + 3\Delta). \end{aligned}$$

This completes the proof of Lemma 6.1. □

## 7. Proof of Theorem 5.2

Let

$$u \in \mathrm{SOL}(C, f),$$

and let $Q$ be a natural number. Lemma 6.1 implies that

$$\begin{aligned} &\|x_0 - u\|^2 \ge \|x_0 - u\|^2 - \|x_Q - u\|^2 \\ &= \sum_{k=0}^{Q-1}(\|x_k - u\|^2 - \|x_{k+1} - u\|^2) \\ &\ge \sum_{k=0}^{Q-1}((1 - \tau^2 L^2)\|x_k - y_k\|^2 - \Delta_k(2\|x_0 - u\| + 3\Delta)) \end{aligned}$$

and

$$\sum_{k=0}^{Q-1}(1 - \tau^2 L^2)\|x_k - y_k\|^2 \le \|x_0 - u\|^2 + \Delta(2\|x_0 - u\| + 3\Delta). \tag{7.1}$$

In view of (7.1),

$$\sum_{k=0}^{\infty} \|x_k - y_k\|^2 < \infty, \ \lim_{k\to\infty} \|y_k - x_k\| = 0. \tag{7.2}$$

Lemma 6.1 implies that the sequence $\{x_k\}_{k=1}^{\infty}$ is bounded, and so it has a weak accumulation point. Let $\bar{x}$ be the weak limit of a subsequence $\{x_{k_j}\}_{j=1}^{\infty}$ of $\{x_k\}$:

$$\lim_{j\to\infty} x_{k_j} = \bar{x}. \tag{7.3}$$

It is clear that $\bar{x} \in C$. In view of (7.2) and (7.3),

$$\lim_{j\to\infty} y_{k_j} = \bar{x}$$

in the weak topology. Let $v \in C$. By (1.1) and (5.1), for each integer $k \geq 0$, we have

$$\langle x_k - \tau f(x_k) - y_k, y_k - v\rangle \geq 0. \tag{7.4}$$

In view of (7.4),

$$\langle \tau^{-1}(y_k - x_k) + f(x_k), v - y_k\rangle \geq 0. \tag{7.5}$$

It follows from (2.1) and (7.5) that

$$\begin{aligned}
&\langle f(v), v - y_k\rangle \\
&= \langle f(v) - f(y_k), v - y_k\rangle + \langle f(y_k) - f(x_k), v - y_k\rangle \\
&+\langle \tau^{-1}(y_k - x_k) + f(x_k), v - y_k\rangle - \tau^{-1}\langle y_k - x_k, v - y_k\rangle \\
&\geq \langle f(y_k) - f(x_k), v - y_k\rangle - \tau^{-1}\langle y_k - x_k, v - y_k\rangle.
\end{aligned} \tag{7.6}$$

Since the operator $f$ is Lipschitz, it follows from (7.2) and (7.6) that

$$\lim_{k\to\infty}\langle f(y_k) - f(x_k), v - y_k\rangle = 0 \tag{7.7}$$

and

$$\lim_{k\to\infty}\langle y_k - x_k, v - y_k\rangle = 0. \tag{7.8}$$

By (7.3), (7.6), and (7.8), we have

$$\langle f(v), v - \bar{x}\rangle \geq \liminf_{k\to\infty}\langle f(v), v - y_k\rangle \geq 0$$

and

$$\langle f(v), v - \bar{x}\rangle \geq 0, \ \ v \in C.$$

In view of (2.4),

$$\bar{x} \in \mathrm{SOL}(C, f).$$

Thus we have shown that the limit of any weakly convergent subsequence of $\{x_k\}_{k=0}^{\infty}$ belongs to SOL$(C, f)$. Arguing as in the proof of Theorem 2.2, we can show that the sequence $\{x_k\}_{k=0}^{\infty}$ itself converges weakly to $\bar{x}$. In view of (7.2), the sequence $\{y_k\}_{k=0}^{\infty}$ also converges weakly to $\bar{x}$. This completes the proof of Theorem 5.2. □

## Author contributions

Both authors contributed equally.

## Use of Generative-AI tools declaration

The authors declare that they did not utilize any artificial intelligence (AI) tools in the creation of this article.

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## Conflict of interest

Simeon Reich and Alexander Zaslavski are the guest editors for Journal of Industrial and Management Optimization and were not involved in the editorial review or the decision to publish this article. All authors declare that there are no competing interests.

## Declarations

Ethical approval - Not applicable.
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