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Research article

Production and inventory rationing in an unreliable Make-to-Stock System under preventive maintenance policy

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Abstract: This paper proposes a joint optimization approach for the production decision-making and inventory allocation in a fault-prone machine Make-to-Stock System, considering both production-side uncertainties and demand-side uncertainties, these factors often cause operational inefficiencies, higher costs and lower customer satisfaction, thus highlighting the need to integrate maintenance strategies into production-inventory management. A model based on Markov decision theory is presented to more accurately reflect real-world production processes, utilizing a preventive repair maintenance strategy, which focuses on repairing the machine only when it fails, as opposed to traditional fixed-cycle or fixed-threshold maintenance strategies. To tackle the complex structure of the optimal control strategy, the paper employs a numerical algorithm for updating the optimal strategy. Computational experiments are conducted to explain the properties of the optimal control strategies and emphasize the importance of considering a maintenance factor in the system. The study highlights the significance of effective production inventory system management, with the conclusion that the optimal control strategy is a machine-state-dependent threshold strategy.

Keywords: preventive maintenance; optimal control strategy; fault-prone machine; Markov decision theory; stationary analysis
Mathematics Subject Classification: Primary: 90B30, 91-10; Secondary: 65-11

1. Introduction

In today's dynamic business environment, production and inventory systems are pivotal to ensuring the seamless operation of modern enterprises, thereby enhancing customer satisfaction and reducing operational costs. However, the management and maintenance of such systems remain inherently challenging due to the pervasive uncertainties and complexities associated with demand forecasting, supply chain coordination, and production planning. Uncertainties exert a substantial impact on production-inventory systems, often leading to elevated operational costs, diminished efficiency, and decreased customer satisfaction. In particular, production-related uncertainties—such as unforeseen equipment failures and prolonged maintenance durations—can disrupt the production flow and trigger stock-outs. Similarly, demand uncertainties, characterized by stochastic variations in customer arrival rates and order quantities, may result in either overstocking, which increases holding costs and risks product obsolescence or spoilage, or understocking, which contributes to higher shortage costs and service-level deterioration. Consequently, effective management of production-inventory systems must explicitly account for these uncertainties in order to optimize performance and minimize total system cost.

A substantial body of literature has proposed diverse methodologies to address uncertainty in production-inventory management. For instance, Jin et al. (2021) [1] developed a reliability index and option pricing framework based on uncertain fractional-order differential equations, offering novel insights into the integration of uncertainty within inventory systems. Similarly, Li et al. (2022) [2] introduced a piecewise parameterization approach for multifactor uncertain systems and examined optimization strategies for uncertain inventory-promotion interactions. In a different domain, Shi et al. (2022) [3] investigated dynamic pricing and production control for perishable goods under uncertainty. Additionally, Kim et al. (2023) [4] applied model-based robust optimization techniques to closed-loop supply chains, accounting for uncertain carbon tax rates and fluctuating demand. Taleizadeh, Ata Allah, et al. (2017) [5] focused on stochastic machine breakdown and discrete delivery in imperfect inventory-production systems, providing targeted insights into addressing production-side uncertainties (i.e., random machine failures) while integrating discrete delivery constraints into inventory-production optimization. Collectively, these studies offer valuable perspectives on managing uncertainty in production-inventory systems.

Among the prevailing strategies in this field, controlled inventory allocation has emerged as a widely adopted approach. To accurately represent customer demand and analyze the behavior of production-inventory systems, researchers frequently employ queuing theory and queuing network models. Zhou SX and Yu Y (2008) [6], Zhou SX and Tao Z (2011) [7] provide comprehensive reviews on this topic, modeling the conversion of products into inventory to satisfy customer demand, where unmet demand is immediately lost. Under the assumption of exponential distributions, they characterize the machine state as a Markov decision process [8, 9] and analyze the joint management of maintenance and inventory allocation strategies in production-inventory systems. Neubauer (2019) [10] analyzes the application of semi-Markov processes to constructing a four-state system for preventive replacements with minimal repair. Research addressing production uncertainty spans discrete models [11–14] and fluid models [15–17], collectively supporting the conclusion that the optimal production policy typically follows a threshold structure dependent on the machine's operational state. Porteus et al. (1986) [18] examines production systems with deteriorating machine conditions; Pierskalla and Voelker (1976) [19] develop maintenance models for deteriorating systems; Iravani et al. (2000) [20] investigate the joint optimization of production and maintenance policies in unreliable settings, concluding that the optimal production strategy mirrors a basic inventory replenishment rule; Iravani and Duenyas (2002) [21] extend this line of research by analyzing a multi-state make-to-order system with machine degradation, jointly optimizing production and maintenance decisions.

Preventive and corrective maintenance are widely regarded as primary strategies for managing fault-prone machines in production-inventory systems [22, 23], often implemented through age-based maintenance policies. Navarro (2020) and Ito (2019) [24, 25] investigate minimal repair strategies within coherent systems. Contrasting traditional approaches that treat preventive maintenance and inventory management separately, Cheung and Hausman (1997) [26] propose an integrated framework that jointly optimizes both aspects. Zhao (2021) [27] further suggests that effective replacement policies should adopt a collaborative structure. Charlot et al. (2007) [28] extend the application of preventive maintenance to accident prevention by introducing lockout/tagout and non-lockout/tagout strategies. Regarding the structural properties of optimal control strategies, Dong and You (1999) [13] analyze the convexity, monotonicity, and continuity of the optimal cost function; Boukas et al. (1995) [29] formulate joint production and maintenance planning within a stochastic manufacturing framework as a piecewise deterministic Markov decision process; Das and Sarkar (1999) [30] determine the optimal timing for executing preventive maintenance activities; Hopp et al. (1989) [31] and Salameh and Ghattas (2001) [32] emphasize the role of periodic preventive maintenance in creating a temporary buffer between production and demand, thereby mitigating uncertainty; Kyriakidis and Dimitrakos (2006) [33] develop a Markov decision algorithm tailored for control-limited strategies in systems with deteriorating machinery. Furthermore, Pang et al. (2014) [34] address an inventory rationing problem in a lost sales make-to-stock (MTS) production system with batch ordering and multiple demand classes; Shi et al. (2014) [35] consider a production control problem with a failure-prone machine. Meanwhile, inspired by Cheng et al. (2011) [36] and Gao et al. (2010) [37], this study proposes a novel approach that jointly optimizes production and distribution strategies in a production-inventory system serving multiple customers.

In recent years, research on the maintenance and reliability of multi-state manufacturing systems has continued to deepen, providing crucial support for modeling multi-state failing machines in this paper. Yang.L et al. (2024) [38] proposed a structure-reliability-oriented reliability assessment framework for multi-state systems with dependent components and imprecise parameters. Their approach to handling random parameters aligns with the logic of modeling machine degradation rates and maintenance durations in this paper. Yang.X et al. (2024) [39] introduced an opportunistic maintenance approach centered on task reliability, integrating physical and functional failure assessments to optimize maintenance portfolios, thereby addressing the shortcomings of traditional maintenance research that overlooks production task requirements. Liao et al. (2023) [40] constructed a risk-oriented predictive maintenance model, clarifying the operational risk dimensions driven by task reliability, providing practical reference for this paper's integration of "maintenance-production-inventory" to mitigate risks.

Although the existing literature has made significant contributions to the field, gaps in knowledge and methodology remain. Previous studies have considered various aspects of production and demand uncertainties, but there is still a need for a more integrated and systematic approach to optimize production-inventory systems in modern enterprises. In particular, a framework that deeply integrates inventory allocation with maintenance strategies is still lacking to comprehensively address the challenges arising from the interplay of production and demand uncertainties.

To fill this gap, this paper proposes a novel approach that jointly optimizes inventory allocation using a model based on Markov decision theory, while taking into account the cost of maintenance activities. By considering both production and demand uncertainties, this study aims to provide a more integrated and systematic approach to optimize make-to-stock (MTS) [41] production inventory systems in modern enterprises. This approach is expected to contribute significantly to the field by providing a more comprehensive solution to the challenges of production and demand uncertainty.

This paper presents three key innovations: First, we propose an integrated framework that jointly optimizes dynamic production control, multi-level inventory allocation, and adjustable preventive maintenance strategies within a prone-to-failure make-to-stock production system—a first in this field. Second, we establish a continuous-time Markov decision process model, demonstrating that the optimal strategy is a threshold policy dependent on machine state and analyzing its structural monotonicity properties. Finally, we reveal the intrinsic relationship between maintenance and inventory, demonstrating that increasing preventive maintenance intensity effectively reduces both optimal inventory levels and production costs. This provides enterprises with a management solution that balances resilience and cost-effectiveness.

The structure of this paper is organized as follows. Section 2 presents a comprehensive review of the existing literature on production-inventory systems under uncertainty, identifies critical methodological gaps, and positions the novel contributions of this study. Section 3 outlines the proposed methodology, which is based on a Markov Decision Process (MDP) framework incorporating an expected discounted total cost criterion. This formulation enables the optimization of inventory allocation strategies over an infinite planning horizon while accounting for fixed maintenance operations. To solve the MDP problem, dynamic programming algorithms are employed to derive the optimal policy and corresponding value function. Section 4 reports the results of computational experiments, demonstrating the effectiveness and advantages of the proposed model and control strategy. Finally, Section 5 concludes the paper by summarizing key findings and discussing their managerial implications for the design and operation of production-inventory systems.

2. Model Formulation and Optimal Control Policy

In this section, we present a Markov decision model for a production-inventory system that aims to minimize the expected discounted total cost within an infinite range.

2.1. *Summary of Parameter's Meaning*

In order to better understand our model, Table 1 lists the explanatory meanings of all symbols in the text.

2.2. *Model Description*

Suppose there exists a filtered complete probability space $(\Omega, \mathbf{F}, \{\mathbf{F}_t\}, \mathbf{P})$ that encompasses all relevant stochastic elements of the system. Consider a production-inventory system consisting of a single production unit, which manufactures one type of finished product to satisfy the demands of n customer classes. The system state is denoted by a tuple (x, i), where $x \in \mathbb{N}^+$ represents the current inventory level, and $i \in \{0, 1, \ldots, m\}$ indicates the operational status of the machine (consistent with Table 1: $i = 0$ for complete failure, $i = m$ for full functionality, and $1 \leq i \leq m-1$ for partial functionality). Specifically, $i = 0$ corresponds to a fully failed machine, while $i = m$ denotes a fully functional state. The transition dynamics between machine states are governed by the parameter ξ_i, for $i \in \{1, 2, \ldots, m\}$, $\xi_i = i(1-s)$ quantifies the deterioration rate from state i to $i-1$ (dependent on preventive maintenance s); for $i = 0$, $\xi_0 > 0$ quantifies the repair rate from complete failure (state 0) to full functionality (state m) independent of s.

Whether and how the transition from state (x, i) occurs depends on three dynamic factors, which can be described as follows:

1. Natural breakdown and repairment. For $0 < i \leq m$, the state (x, i) tends to decline to $(x, i-1)$, with the time between failures following an exponential distribution with parameter ξ_i. Here, ξ_i is controlled by s, and $\xi_i = r(i)(1-s)$ is only valid for $1 \leq i \leq m$ (deterioration process), where $r(i)$ is fixed. For $i = 0$ (complete failure), the state (x, i) has two potential transition modes after maintenance: one is full repair that restores the machine to the fully functional state (x, m) with transition rate ξ_{0m}; the other is partial repair that transitions the machine to an intermediate operational state (x, k) (where $1 \leq k \leq m-1$) with transition rate ξ_{0k}. The sum of all transition rates from state $i = 0$ satisfies $\sum_{k=1}^{m} \xi_{0k} = \xi_0$, and the repair time for each transition mode follows an exponential distribution with the corresponding rate parameter.
2. Production. The controller can choose a production rate $\mu_i \in [0, \bar{\mu}_i]$ to product. In such rate, the interval between the production of two consecutive goods follows an exponential distribution with parameter μ_i. When a commodity is produced, the state tuple turns from (x, i) to $(x+1, i)$ for $x > 0$. Accordingly, it costs operating cost of $q\mu_i(t)$, where q represents production cost. And we suppose $\bar{\mu}_m > \bar{\mu}_{m-1} > \cdots > \bar{\mu}_0 = 0$.
3. A lost sale inventory system facing n types of demands. Each demand process is a Poisson process with parameter λ_i. The controller has the power to satisfy the demands or not, based on the principle of minimizing costs. If a demand is satisfied, the state tuple turns from (x, i) to $(x-1, i)$ for $x > 0$. Or the controller pays an out-of-stock cost of c_i. Commonly, we suppose $c_1 > c_2 > \cdots > c_m$. If the inventory level is not positive, the customer will immediately leave the system.

Accordingly, there are three factors controlling the policy π when the system is in state (x, i), which can be described as follows:

Table 1. Interpretation of symbols.

Symbol	Explanation
ξ_i	Deterioration rate from state i to state $i-1$ (for $i \in \{1, 2, \ldots, m\}$); repair rate from state 0 to state m (for $i = 0$). Constraints: $\xi_i = r(i)(1-s)$ for $1 \le i \le m$ (controlled by preventive maintenance s); $\xi_0 > 0$ (constant, independent of s)
$r(i)$	Natural deterioration rate of the machine at state i, $r(i) > 0$ (constant). Constraint: $r(1) < r(2) < \cdots < r(m)$ (deterioration rate increases with machine state deterioration)
s	Preventive maintenance degree, $s \in [0, 1)$. Constraints: $s = 0$ (no preventive maintenance); $s \to 1^-$ (maximum preventive maintenance intensity, minimizing deterioration rate)
c_k	Out-of-stock cost when the demand of the k-th class customers is not satisfied, $c_1 > c_2 > \cdots > c_n > 0$ (constant)
γ	Discount factor, $0 < \gamma < 1$ (constant)
$h(x)$	Inventory holding costs; $h(x)$ is convex, increasing and $h(0) = 0$, $h(x) \ge 0$ for all $x \in \mathbb{N}_0$
$w(s)$	Maintenance costs; $w(s)$ is convex, increasing, $w(0) = 0$ and $\lim_{s \to 1^-} w(s) = \infty$, $w(s) \ge 0$ for all $s \in [0, 1)$
q	Production cost per unit, $q > 0$ (constant)
μ_i	Maximum production rate at state i. Constraints: $\mu_0 = 0$ (no production in failure state); $\mu_1 < \mu_2 < \cdots < \mu_m$ (production rate increases with machine state improvement), $\mu_i \ge 0$
$N_k^\pi(t)$	The total out-of-stock quantity of the k-th class customers by time t under the policy π, $N_k^\pi(t) \ge 0$ (integer)
λ_k	Arrival rate of the k-th class customers, $\lambda_k > 0$ (constant)
$V(x, i)$	The optimal expected discounted present value in state (x, i), $V(x, i) \ge 0$
$S^*(i)$	Production threshold when the system is in state (x, i), $S^*(i) \in \mathbb{N}_0$ (integer)
$R_k^*(x, i)$	Inventory allocation threshold for the k-th class customers in state (x, i), $R_k^*(x, i) \in \mathbb{N}_0$ (integer)
$\mathcal{T}$	The operator to represent the Markov decision process, defining the inventory allocation rule (see Definition 4.1)
x	Current inventory level, a non-negative integer ($x \in \mathbb{N}_0$, where $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$)
i	Operational status of the machine, where $i \in \{0, 1, \ldots, m\}$. Constraints: $i = 0$ (complete failure, no production); $1 \le i \le m-1$ (partial functionality, sorted by decreasing efficiency); $i = m$ (full functionality, maximum production rate)

1. A preventive maintenance degree $s \in [0, 1)$, leading to the deterioration rate $\xi_i = r(i)(1 - s)$.
2. A production rate $\mu_i \in [0, \bar{\mu}_i]$.
3. Whether to satisfy the arriving demands.

3. Expected Total Discounted Cost and Optimality Analysis

Before delving into the expected discounted cost criterion, it is necessary to clarify the rationale for prioritizing this criterion. The expected discounted cost criterion is widely adopted in long-horizon production-inventory optimization due to its ability to reflect the time value of capital—assigning higher weights to near-term costs and lower weights to future costs, which aligns with the financial management logic of enterprises. Additionally, this criterion exhibits favorable mathematical tractability, facilitating the derivation of structural properties of the optimal strategy (e.g., threshold characteristics) and laying a theoretical foundation for subsequent analysis. Section 4 will extend the research to the average cost criterion, which focuses on long-term steady-state cost performance, and compare the two criteria to validate the robustness of the optimal strategy across different decision contexts.

3.1. The Expected Total Discounted Cost Criterion

The total cost of the system includes state transition costs, production costs, and out-of-stock costs. State transition costs occur when the system state changes, and are comprised of holding costs $h(x(t))$ and maintenance costs $w(s)$. Production cost is represented by q, and $\mu(t)$ denotes the production rate. $N_k^\pi(t)$ represents the total out-of-stock quantity of the k-th class of customers by time t under the policy π. If the inventory is insufficient to meet the demand of the k-th class of customers, an out-of-stock cost c_k is incurred. The indefinite expected discounted total cost associated with the system can be calculated over an infinite time horizon from 0 to positive infinity, with a discount factor represented by γ. This total cost is expressed as:

$$V(x, i) = \mathrm{E}\left[\int_0^{+\infty} e^{-\gamma t}\left[[h(x(t)) + w(s) + q\mu(t)]\mathrm{d}t + \sum_{i=1}^{n} c_i \mathrm{d}N_i^\pi(t)\right]\right] \tag{3.1}$$

For the purpose of finding the optimal strategy that results in the minimum expected total cost, we use the set of all viable policies Π, and choose a specific policy $\pi \in \Pi$. The optimal policy is denoted by π^*, and it results in the minimum expected total cost $V(x, i)$, which is represented by the optimal expected discount value in the current state that satisfies:

$$\begin{aligned} V^{\pi^*}(x, i) &= \min_{\pi\in\Pi}\{V^\pi(x, i)\} \\ &= \min\left\{\mathrm{E}\left[\int_0^{+\infty} e^{-\gamma t}\left[[h(x(t)) + w(s) + q\mu(t)]\mathrm{d}t + \sum_{i=1}^{n} c_i \mathrm{d}N_i^\pi(t)\right]\right]\right\} \end{aligned} \tag{3.2}$$

To obtain the structural properties of the optimal policy, it is necessary to get the Bellman Equation of (3.2). As the arrival time intervals of customers, machine maintenance, and production process all follow exponential distributions with different parameters, the decision process conforms to a continuous-time Markov decision processes.

Before getting the Bellman Equation, we need to clarify a property of Poisson process: labelet $N_{\Delta t}$ is a Poisson process with parameter λ, then

$$\begin{cases} P(N_{\Delta t} = 0) = 1 - \lambda\Delta t + o(\Delta t), \\ P(N_{\Delta t} = 1) = \lambda\Delta t \\ P(N_{\Delta t} > 1) = o(\Delta t) \end{cases}$$

Therefore, we can get that the probability of each event (breakdown and repairment, production, arrival of demands) occurring is infinitely small, with a value of Δt. And the state transition process from t to $t + \Delta t$ is shown in Figure 1.

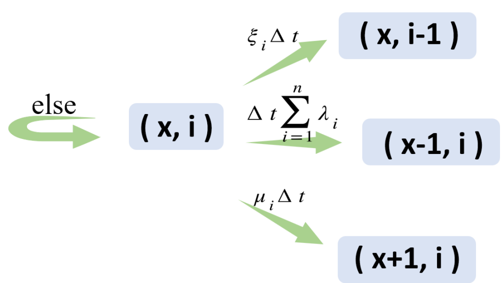

Figure 1. State transition process from t to $t + \Delta$t.

Then we devide the time $[0, \infty)$ into $[0, \Delta t) \cup [\Delta t, \infty)$, that is

$$
\begin{aligned}
V(x,k) &= \min \mathbf{E} \int_0^{\infty} e^{-\gamma t}[(h(x_t) + w(s) + q\mu(t))dt + \sum_{i=1}^{m} c_i dN_i(t)] \\
&= \min \mathbf{E}\{\int_0^{\Delta t} + \int_{\Delta t}^{\infty}\}[e^{-\gamma t}(h(x_t) + w(s) + q\mu(t))dt + \sum_{i=1}^{m} c_i dN_i(t)] \\
&= [h(x_t) + w(s)]\Delta t \\
&+ \min_{0<\mu_k\le\bar{\mu_k}}\{\min\{q\mu_k\Delta t + \mu_k\Delta t e^{-\gamma\Delta t}V(x+1,k), \mu_k\Delta t e^{-\gamma\Delta t}V(x,k)\}\} \\
&+ \sum_{i=1}^{n} \lambda_i\Delta t \min\{V(x,k) + c_i, V(x-1,k)\} + \xi_k\Delta t V(x,k-1) \\
&+ e^{-\gamma\Delta t}V(x,k)[1 - \Delta t(\sum_{i=1}^{n}\lambda_i + \xi_k + \mu_k)] + o(\Delta t) \\
&= [h(x_t) + w(s)]\Delta t + \min_{0<\mu_k\le\bar{\mu_k}}\{\mu_k\Delta t \min\{q + V(x+1,k), V(x,k)\}\} \\
&+ \sum_{i=1}^{n} \lambda_i\Delta t \min\{V(x,k) + c_i, V(x-1,k)\} + \xi_k\Delta t V(x,k-1) \\
&+ V(x,k)[1 - \Delta t(\sum_{i=1}^{n}\lambda_i + \mu_k + \xi_k + \gamma)] + o(\Delta t) \\
&= [h(x_t) + w(s)]\Delta t + \Delta t \min_{0\le\mu_k\le\bar{\mu_k}}\{\mu_k[q + V(x+1,k) - V(x,k)]\} \\
&+ \Delta t \sum_{i=1}^{n} \lambda_i \min\{V(x,k) + c_i, V(x-1,k)\} + \Delta t\xi_k V(x,k-1) \\
&+ V(x,k)[1 - \Delta t(\sum_{i=1}^{n}\lambda_i + \xi_k + \gamma)] + o(\Delta t)
\end{aligned}
$$

Minus $V(x,k)$, divide both sides by Δt and let $\Delta t \to 0$, and then we can get

$$
\begin{aligned}
&V(x,k)(\gamma + \sum_{i=1}^{n}\lambda_i + \xi_k) \\
&= [h(x) + w(s)] + \min_{0\le\mu_k\le\bar{\mu_k}}\{\mu_k\,[q + V(x+1,k) - V(x,k)]\} \\
&+ \sum_{i=1}^{n}\lambda_i \min\{V(x,k) + c_i, V(x-1,k)\} + \xi_k V(x,k-1)
\end{aligned}
$$

Add both sides by

$$
V(x,k)(\sum_{i=0,i\neq k}^{m}\xi_i + \sum_{i=1}^{m}\mu_i)
$$

We get

$$
\begin{aligned}
&V(x,k)(\gamma+\sum_{i=1}^{n}\lambda_i+\sum_{i=0}^{m}\xi_i+\sum_{i=1}^{m}\mu_i)\\
=&h(x)+w(s)+\min_{0\le\mu_k\le\bar{\mu}_k}\{\mu_k\left[q+V(x+1,k)-V(x,k)\right]\}\\
&+\sum_{i=1}^{n}\lambda_i\min\{V(x,k)+c_i,V(x-1,k)\}+\xi_k V(x,k-1)\\
&+\left[\sum_{i=0,i\neq k}^{m}\xi_i+\sum_{i=1}^{m}\mu_i)\right]V(x,k)\\
=&h(x)+w(s)+\min_{0\le\mu_k\le\bar{\mu}_k}\{\mu_k\min[q+V(x+1,k),V(x,k)]\}\\
&+\sum_{i=1}^{n}\lambda_i\min\{V(x,k)+c_i,V(x-1,k)\}+\xi_k V(x,k-1)\\
&+\left[\sum_{i=0,i\neq k}^{m}\xi_i+\sum_{l=1,l\neq k}^{m}\mu_l)\right]V(x,k)
\end{aligned}
$$

Denote $E_1=\gamma+\sum_{i=1}^{n}\lambda_i+\sum_{i=0}^{m}\xi_i+\sum_{i=1}^{m}\mu_i$, and let $E_1=1$. We will get

$$
\begin{aligned}
V(x,k)&=h(x)+w(s)+\min_{0\le\mu_k\le\bar{\mu}_k}\{\mu_k\min[q+V(x+1,k),V(x,k)]\}\\
&+\sum_{i=1}^{n}\lambda_i\min\{V(x,k)+c_i,V(x-1,k)\}+\xi_k V(x,k-1)\\
&+\left[\sum_{i=0,i\neq k}^{m}\xi_i+\sum_{l=1,l\neq k}^{m}\mu_l)\right]V(x,k)
\end{aligned}
$$

Definition 3.1. *The optimal inventory allocation strategy of the production inventory system is that when $V(x-1,i)<V(x,i)+c_k$, the system decides to deliver in response to the demand of the k-th class of customers. Otherwise, the system does not deliver. The optimal production decision is:*

$$T_kV(x,i)=\min\{V(x-1,i),V(x,i)+c_k\},V(-1,i)=\infty$$

$$T^iV(x,i)=\min\{V(x,i),V(x+1,i)+q\}$$

Definition 3.1 also implies that producing at the maximum rate is the optimal strategy when the system decides to produce, while no production is the best strategy otherwise.

Therefore, based on Equation (3.2), the Bellman Equation for the system can be obtained from as follows:

$$
\begin{aligned}
V(x,i)\;=\;&TV(x,i)=h(x)+w(s)\\
&+\bar{\mu}_i I_{[q+V(x+1,k)-V(x,k)<0]}T^iV(x,i)+\xi_iV(x,i-1)+\sum_{k=1}^{n}\lambda_kT_kV(x,i)\\
&+\left[\sum_{l=1,l\neq i}^{m}\mu_l+\sum_{l=0,l\neq i}^{m}\xi_l\right]V(x,i)
\end{aligned}
\tag{3.3}
$$

The inventory allocation strategy in the production inventory system is represented by the operator T, which determines whether to fulfill the demand of arriving customers or not. Since the deterioration rate ξ_i of machine state i $(1 \leq i \leq m)$ is controlled by the preventive maintenance degree s, satisfying $\xi_i = r(i)(1-s)$ ($r(i)$ is the fixed natural deterioration rate), substitute it into the above Bellman equation to replace the original ξ_i term: The terms related to ξ_i in the original equation are $\xi_i V(x, i-1)$ and $\sum_{l=1, l\neq i}^{m} \xi_l V(x, i)$, where $\xi_l = r(l)(1-s)$ $(l \neq i)$, thus obtaining the specific form of the above equation after substitution.

[Dimensional Consistency Verification]: All terms in the equation have the dimension of "cost per unit time", which is logically consistent to ensure the rigor of the derivation.

When the system is in state (x, i) with $x, i > 1$, the deterioration rate ξ_i can be substituted to obtain

$$\begin{aligned} V(x,i) &= TV(x,i) = h(x) + w(s) \\ &+ \bar{\mu}_i I_{[q+V(x+1,k)-V(x,k)<0]} T^i V(x,i) + r(i)(1-s)V(x,i-1) + \sum_{k=1}^{n} \lambda_k T_k V(x,i) \\ &+ [\sum_{l=0,l\neq i}^{m} \mu_l + \sum_{l=1,l\neq i}^{m} r(l)(1-s) + \xi_0] V(x,i) \end{aligned} \tag{3.4}$$

where the first and second terms represent the current inventory holding cost and maintenance cost, respectively. The remaining terms represent the total expected cost of the system from the next state transitions to infinity.

The optimal equation indicates that if the demand is satisfied with probability ξ_i, the state will be transferred from i to $(i-1)$, while if the demand is not satisfied, the state transitions to (i). Similarly, for the case when $i = 0$, we have

$$\begin{aligned} V(x,0) &= TV(x,0) = h(x) + w(s) + \xi_0 V(x,m) \\ &+ \sum_{k=1}^{n} \lambda_k T_k V(x,i) + [\sum_{l=1}^{m} \mu_l + \sum_{l=1}^{m} r(l)(1-s)] V(x,0) \end{aligned} \tag{3.5}$$

In summary, Definition 3.1 for the optimal inventory allocation strategy in the production inventory system is proposed in our study, which stipulates that if $V(x-1, i) < V(x, i) + c_k$, the system should deliver in response to the demand of the k-th class of customers, otherwise it should not deliver.

3.2. Optimality analysis

The structural properties of the optimal strategy can then be described by the optimal equation. To facilitate the explanation, we introduce a difference operator on the state variables (x, i).

$$D_1 u(x,k) = u(x+1,k) - u(x,k);$$

$$D_2 u(x,k) = u(x,k+1) - u(x,k);$$

$$D_{11} u(x,k) = D_1 D_1 u(x,k);$$

$$D_{12} u(x,k) = D_1 D_2 u(x,k) = D_{21} u(x,k) = D_2 D_1 u(x,k)$$

Meanwhile, for the purpose of obtaining the structural properties of the optimal strategy of the system, we introduce a set $\mathcal{V}$ defined on the state space of the system. We specify that if $u(x,i) \in \mathcal{V}$, then $u(x,i)$ satisfies

$$\begin{array}{lll} \textbf{P1.}\ \text{Convexity} & D_{11}u(x,i) \geq 0 \\ \textbf{P2.}\ \text{Upper bound} & D_{12}u(x,i) \geq 0 \\ \textbf{P3.}\ \text{Lower bound} & D_{1}u(x,i) \geq -c_1 \end{array} \tag{3.6}$$

Proposition 3.1. $\forall V \in \mathcal{V}, TV \in \mathcal{V}$.

Proposition 3.1 states that if V is a set of functions that satisfy **P1–P3**, then the operator T maps V to itself. To prove this proposition, we propose three sub-propositions that demonstrate the satisfaction of the convexity, upper bound, and lower bound conditions, respectively.

The first sub-proposition, Proposition 3.2, states that

Proposition 3.2 (P1)**.** *For $x, i \geq 0$, $D_{11}TV(x,i) \geq 0$.*

If $TV(x,i)$ retains property **P1**, then $D_1TV(x,i)$ is an increasing function of x.

Proof. See proof in Appendix .1. □

After establishing the validity of $D_{11}T_kV(x,i) \geq 0$ in Proposition 3.2, we proceed to the second sub-proposition, which addresses Property **P2**, the upper bound condition, as stated in Proposition 3.3.

Proposition 3.3 (P2)**.** *For $x, i \geq 0$, $D_{12}T^iV(x,i) \geq 0$.*

Proof. See proof in Appendix .2. □

After proving Proposition 3.3, we now turn our attention to the third sub-proposition which deals with the lower bound condition, Property **P3**.

Proposition 3.4 (P3)**.** *For $x, i \geq 0$, $D_1TV(x,i) \geq -c_1$.*

Proof. See proof in Appendix .3. □

Having established all three sub-propositions, we can naturally prove the main proposition. Based on Propositions 3.2, 3.3, and 3.4, we conclude that the operator T maps the set of functions V that satisfy properties **P1–P3** to itself, that is, $T(V) \subset V$. Therefore, we restate Proposition 3.1 as the following theorem.

Theorem 3.1. $\forall\ V \in \mathcal{V}, TV \in \mathcal{V}$.

Where V is the optimal cost function. The optimal strategy is a dynamic control strategy that depends on the equipment status. The optimal production rate control strategy is a threshold strategy that depends on the equipment status. The optimal inventory allocation strategy is a dynamic control strategy that depends on the inventory level and the equipment status. Specifically, when the system is in state (x,i), there exists a production threshold $S^*(i)$, when $x < S^*(i)$, production is carried out at the maximum rate $\mu(i) = \bar{\mu}_i$, otherwise production is not carried out. There exists an inventory allocation threshold $R_k^*(x,i)$, when $x \geq R_k^*(x,i)$, the demand of the k-th type of customer is met, otherwise it is not met. This result is essential to the overall analysis of the inventory allocation system and provides important insights into the system's behavior as follows.

Theorem 3.2. *The optimal strategies hold the below structural properties:*

1. $S^*(i) = \min_x\{V(x+1,i) - V(x,i) \geq -q\}$
2. $R_k^*(i) = \min_x\{V(x,i) - V(x-1,i) \geq -c_k\}$
3. $S^*(z) \leq S^*(z-1) \leq \cdots \leq S^*(1)$
4. $R_1^*(i) \leq R_2^*(i) \leq \cdots R_m^*(i)$
5. $R_k^*(z) \leq R_k^*(z-1) \leq \cdots R_k^*(0)$
6. *The demand of the first type of customer must be met.*

Proof. Properties 1 and 2 follow directly from the convexity of $V(x,i)$ (P1), which implies the optimal production/inventory allocation strategies are state-dependent threshold policies.
Property 3 is derived from the upper bound property (P2): $V(S^*(i)+1, i+1) - V(S^*(i), i+1) \geq V(S^*(i)+1, i) - V(S^*(i), i) \geq -q$, so we have $S^*(i+1) \leq S^*(i)$.
Property 4 stems from $V(R_k^*(i), i) - V(R(i)-1, i) \geq -c_i \geq -c_{i-1}$, leading to $R_{k-1}^*(x,i) \leq R_k^*(x,i)$.
Property 5 is obtained via the definition of $R^*(k)$ and P1, ensuring $R^*(i+1) \leq R(i)$.
Property 6 is a direct consequence of Property 3.
Detailed formula derivations and full scenario analyses are provided in Appendix .4 □

Theorem 3.3. *Let maintenance factors s_1 and s_2 respectively corresponding to $V_{s_1}(x,i)$ and $V_{s_2}(x,i)$, such that $D_1V(0,i,s_1) \leq -c_1$ and $\forall x: 0 \leq x \leq x_0, D_1(x,i,s_2) \leq -q, D_1(x+1,i,s_2) > -q$. If $D_1V(x,i,s_2) - D_1V(x,i,s_1) \geq 0$, then $D_1TV(x,i,s_2) - D_1TV(x,i,s_1) \geq 0$.*

Proof. We begin by observing that if $D_1(0,i,s_1) \leq -c_1$, then $\forall x : 0 \leq x \leq x_0, D_1(x,i,s_2) \leq -q, D_1(x+1,i,s_2) > -q$ holds true. Therefore, to prove the theorem, it suffices to demonstrate that if $D_1V(x,i,s_2) - D_1V(x,i,s_1) \geq 0$, then $D_1TV(x,i,s_2) - D_1V(x,i,s_1) \geq 0$. This is equivalent to proving the following inequalities:

$$D_1 \min_x \{V(x+1,i,s_2) + q, V(x,i,s_2)\} - D_1 \min_x \{V(x+1,i,s_1) + q, V(x,i,s_1)\} \geq 0 \tag{3.7}$$

$$D_1 \min_x \{V(x,i,s_2) + c_k, V(x-1,i,s_2)\} - D_1 \min_x \{V(x,i,s_1) + c_k, V(x-1,i,s_1)\} \geq 0 \tag{3.8}$$

Detailed algebraic expansions and scenario-specific verifications are provided in Appendix .5 □

Based on the preceding theorem, we conclude that an increase in the maintenance factor s leads to a reduction in the optimal stock level for a given machine state. This result underscores the critical role of preventive maintenance in lowering production thresholds and enhancing the overall efficiency of the inventory allocation system. By elucidating the relationship between maintenance intensity and optimal control policies, decision-makers can better schedule maintenance activities and improve system performance.

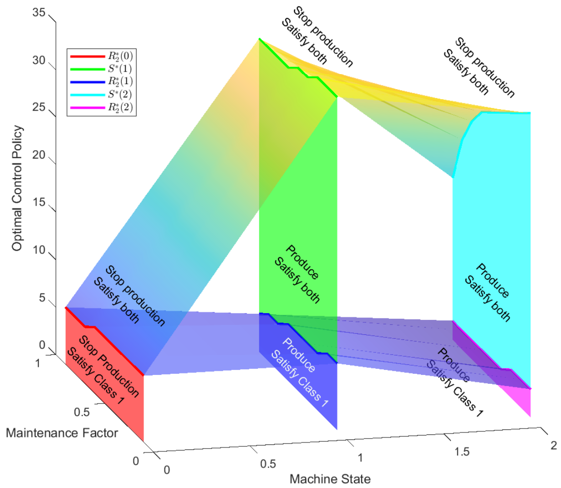

Figure 2. Structure of the Optimal Control Policy.

Figure 2 offers a three-dimensional visualization of the optimal control structure, illustrating both the production and inventory allocation thresholds as functions of machine state i and maintenance factor s. In the figure, $S^*(i)$ denotes the production threshold associated with machine state i, determining whether production should occur. The green and blue lines represent $S^*(1)$ and $S^*(2)$, respectively, indicating that production proceeds at the maximum allowable rate when $x < S^*(i)$. Notably, we observe that $S^*(2) < S^*(1)$, implying that the production threshold declines as the machine condition improves. Moreover, for a fixed machine state, the optimal stock level consistently decreases with increasing maintenance intensity.

The three lines at the bottom of the graph represent the inventory allocation threshold, suggesting whether to allocate or not. When $x > R_k^*(z)$, the demand of the k-th class customers is satisfied. Taking state 1 as an example, we can see that only the demand of the 1-th class cutstomers is satisfied and the demand of the 2-th class customers isn't when optimal control policy is under inventory alloction threshold $R_2^*(1)$. Obviously, $R_2^*(0) > R_2^*(1) > R_2^*(2)$, indicating the inventory alloction threshold decreases with the machine state improves. Furthermore, with the increase of the maintenance factor s, the optimal control policy decreases.

4. The Average Expected Cost Criterion

Referring to [37], we can express the average cost function of the system operating in π mode with initial state (x, i) as follows:

$$g^{\pi}(x,i) = \lim_{\tau\to+\infty} \frac{1}{\tau}\mathrm{E}\left[\int_0^{\tau}\left[[h(x(t)) + w(s) + q\mu(t)]\mathrm{d}t + \sum_{i=1}^{n} c_i \,\mathrm{d}N_i^{\pi}(t)\right]\right]$$

If there is an optimal strategy π^*, the optimal strategy satisfies the following conditions:

$$g^{\pi^*}(x,i) = \min_{\pi} g^{\pi}(x,i)$$

Consider a control strategy denoted by π' where the preventive maintenance degree is set at s. Full production is initiated in all states following a first-come-first-served (FCFS) customer strategy if the inventory level falls below this threshold. Under this condition, the system satisfies ergodicity and has an average cost constant g independent of the initial state. The set of (x, i) satisfying $h(x) + q\mu_i < g - w(s)$ is finite.

Similarly, there exists an optimal average cost constant g^* and a set $f(x, i)$ such that $x \geq 1$.

$$\begin{cases} E^* f(x,2)+g=h(x)+w(s)+\mu_2 T^2 f(x,2)+r(2)(1-s)f(x,1)+\left[\sum_{j=1}^{n}\lambda_j T_j+\sum_{m\neq i}^{k}\mu_m+\sum_{m\neq i,0}^{k}r(m)(1-s)+\xi_0\right]f(x,2) \\ E^* f(x,1)+g=h(x)+w(s)+\mu_1 T^1 f(x,1)+r(2)(1-s)f(x,0)+\left[\sum_{j=1}^{n}\lambda_j T_j+\sum_{m\neq i}^{k}\mu_m+\sum_{m\neq i,0}^{k}r(m)(1-s)+\xi_0\right]f(x,1) \\ E^* f(x,0)+g=h(x)+w(s)+\xi_0 f(x,2)+\sum_{j=1}^{n}\lambda_j T_j f(x,1)+\left[\sum_{m=1}^{k}\mu_m+\sum_{m=1}^{k}r(i)(1-s)\right]f(x,0) \end{cases} \tag{4.1}$$

The above equations can be rewritten as

$$f(x,i) = T' f(x,i) = \frac{1}{E^*}\left[T f(x,i) - g^*\right] \quad x \geq 1$$

where $E^* = E_1 - \gamma$. It is clear that the formula still satisfies properties **P1–P3**.

The case where $x = 0$ will not be further elaborated.

Based on the findings in [37], we can draw a similar conclusion that the optimal control strategy for the average cost criterion exhibits analogous characteristics to that of the expected discounted cost criterion. Specifically, it adopts a threshold strategy that is decided by the state of the system as well.

4.1. Structural properties

In this section, we will present the structural properties of the optimal policy for our system.

As we have proven that $\forall V \in \mathcal{V}, TV \in \mathcal{V}$, it leads to the result below:

Theorem 4.1. *The optimal strategy is a dynamic control strategy that depends on the system state, including the optimal maintenance and inventory allocation strategies.*

Specifically, this theorem highlights that the optimal maintenance strategy is a threshold strategy that depends on the state of the equipment, which decides whether or not to produce. The optimal inventory allocation strategy is a dynamic control strategy that depends on the inventory level, which decides whether or not to deliver.

5. Stationary Analysis

In this section, we analyze the steady-state behavior of the system, which involves two customer classes and three machine states, including failure. Based on the state space defined in Section 2.2, we conduct a stationary analysis of the system dynamics.

The system considers two types of customers and three discrete machine states: $i = 0$ (complete failure), $i = 1$ (partial functionality), and $i = 2$ (full functionality). Let x denote the inventory level and i the machine state. Accordingly, the system state is represented by the tuple $(x, i) \in \{(x, i) \mid 0 \leq x \leq S_1,\ i = 0, 1, 2\}$, where S_1 denotes the upper bound of the inventory level.

The system's control strategy is determined jointly by the current machine state and inventory level. Production decisions depend on whether the inventory has reached a state-dependent threshold, and the inventory is updated accordingly. Similarly, the decision to serve second-class customers is based on whether the current inventory exceeds a corresponding supply threshold.

Five threshold parameters govern the control policy: S_1, S_2, R_0, R_1, and R_2. Here, S_1 and S_2 represent the production thresholds for machine states 1 and 2, respectively. The parameters R_0, R_1, and R_2 denote the inventory thresholds at which supply to second-class customers commences, corresponding to machine states 0, 1, and 2, respectively.

The system strategy and control thresholds together form the control policy that determines the system behavior over time. Furthermore, the balance equations for the system in steady-state will also be derived and discuss the partial constraints involved in the calculation. Let $p_{x,i}$ be the probability that the system is steady in state (x, i), which can be also described as the limit state of the finite-state Markov chain.

For machine state $k = 2$, the behavior of the system can be further classified into seven cases based on the inventory level x. First, when $x = S_1$, the balance equation is given by

$$[\lambda_1 + \lambda_2 + r(2)(1 - s)]p_{S_1,2} = \xi_0 p_{S_1,0}$$

Second, when $x = S_2 + 1, \cdots, S_1 - 1$, the equation is

$$[\lambda_1 + \lambda_2 + r(2)(1 - s)]p_{x,2} = [\lambda_1 + \lambda_2]p_{x+1,2} + \xi_0 p_{x,0}$$

Third, when $x = S_2$, the equation is

$$[\lambda_1 + \lambda_2 + r(2)(1 - s)]p_{S_2,2} = [\lambda_1 + \lambda_2]p_{S_2+1,2} + \xi_0 p_{S_2,0} + \mu_2 p_{S_2-1,2}$$

Fourth, when $x = R_2 + 1, \cdots, S_2 - 1$, the equation is

$$\lambda_1 + \lambda_2 + r(2)(1 - s) + \mu_2]p_{x,2} = [\lambda_1 + \lambda_2]p_{x+1,2} + \xi_0 p_{x,0} + \mu_2 p_{x-1,2}$$

Fifth, when $x = R_2$, the equation is

$$[\lambda_1 + r(2)(1 - s) + \mu_2]p_{R_2,2} = [\lambda_1 + \lambda_2]p_{R_2+1,2} + \xi_0 p_{R_2,0} + \mu_2 p_{R_2-1,2}$$

Sixth, when $x = 1, \cdots, R_2 - 1$, the equation is

$$[\lambda_1 + r(2)(1 - s) + \mu_2]p_{x,2} = \lambda_1 p_{x+1,2} + \xi_0 p_{x,0} + \mu_2 p_{x-1,2}$$

Finally, when $x = 0$, the equation is

$$[r(2)(1 - s) + \mu_2]p_{0,2} = \lambda_1 p_{1,2} + \xi_0 p_{0,0}$$

For machine status $k = 1$, the behavior of the system can be similarly classified into five cases based on the inventory level x. First, when $x = S_1$, the balance equation is given by

$$[\lambda_1 + \lambda_2 + r(1)(1 - s)]p_{S_1,1} = r(2)(1 - s)p_{S_1,2} + \mu_1 p_{S_1-1,1}$$

Second, when $x = R_1 + 1, \cdots, S_1 - 1$, the equation is

$$[\lambda_1 + \lambda_2 + r(1)(1 - s) + \mu_1]p_{x,1} = [\lambda_1 + \lambda_2]p_{x+1,1} + r(2)(1 - s)\pi_{i,2} + \mu_1 p_{x-1,1}$$

Third, when $x = R_1$, the equation is

$$[\lambda_1 + r(1)(1 - s) + \mu_1]p_{R_1,1} = [\lambda_1 + \lambda_2]p_{R_1+1,1} + r(2)(1 - s)p_{R_1,2} + \mu_1 p_{R_1-1,1}$$

Fourth, when $x = 1, \cdots, R_1 - 1$, the equation is

$$[\lambda_1 + r(1)(1 - s) + \mu_1]p_{x,1} = \lambda_1 p_{x+1,1} + r(2)(1 - s)p_{x,2} + \mu_1 p_{x-1,1}$$

Fifth, when $x = 0$, the equation is

$$[r(1)(1 - s) + \mu_1]p_{0,1} = \lambda_1 p_{1,1} + r(2)(1 - s)p_{0,2}$$

For machine status $k = 0$, the behavior of the system can also be further classified into five cases based on the inventory level x. First, when $x = S_1$, the balance equation is given by

$$[\lambda_1 + \lambda_2 + \xi_0]p_{S_1,0} = r(1)(1 - s)p_{S_1,1}$$

Second, when $x = R_0 + 1, \cdots, S_1 - 1$, the equation is

$$[\lambda_1 + \lambda_2 + \xi_0]p_{x,0} = [\lambda_1 + \lambda_2]p_{x+1,0} + r(1)(1 - s)p_{x,1}$$

Third, when $x = R_0$, the equation is

$$[\lambda_1 + \xi_0]p_{R_0,0} = [\lambda_1 + \lambda_2]p_{R_0+1,0} + r(1)(1 - s)p_{R_0,1}$$

Forth, when $x = 1, \cdots, R_0 - 1$, the equation is

$$[\lambda_1 + \xi_0]p_{x,0} = \lambda_1 p_{x+1,0} + r(1)(1 - s)p_{x,1}$$

Fifth, when $x = 0$, the equation is

$$\xi_0 \pi_{0,0} = \lambda_1 p_{1,0} + r(1)(1 - s)p_{0,1}$$

From the above formulas, we can obtain the following results.

For the cases where $x = 0$, we have

$$\begin{bmatrix} p_{1,0} \\ p_{1,1} \\ p_{1,2} \end{bmatrix} = \frac{1}{\lambda_1} \begin{bmatrix} \xi_0 & -r(1)(1-s) & 0 \\ 0 & r(1)(1-s)+\mu_1 & -r(2)(1-s) \\ -\xi_0 & 0 & r(2)(1-s)+\mu_2 \end{bmatrix} \begin{bmatrix} p_{0,0} \\ p_{0,1} \\ p_{0,2} \end{bmatrix}. \tag{5.1}$$

When $x = R_2$,

$$\begin{bmatrix} p_{R_2+1,0} \\ p_{R_2+1,1} \\ p_{R_2+1,2} \end{bmatrix} = \begin{bmatrix} \frac{\lambda_1+\xi_0}{\lambda_1} & -\frac{r(1)(1-s)}{\lambda_1} & 0 \\ 0 & \frac{\lambda_1+r(1)(1-s)+\mu_1}{\lambda_1} & -\frac{r(2)(1-s)}{\lambda_1} \\ -\frac{\xi_0}{\lambda_1+\lambda_2} & 0 & \frac{\lambda_1+r(2)(1-s)+\mu_2}{\lambda_1+\lambda_2} \end{bmatrix} \begin{bmatrix} p_{R_2,0} \\ p_{R_2,1} \\ p_{R_2,2} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{\mu_1}{\lambda_1} & 0 \\ 0 & 0 & -\frac{\mu_2}{\lambda_1+\lambda_2} \end{bmatrix} \begin{bmatrix} p_{R_2-1,0} \\ p_{R_2-1,1} \\ p_{R_2-1,2} \end{bmatrix} \tag{5.2}$$

When $x = R_2+1, \cdots, R_1 - 1$,

$$\begin{bmatrix} p_{x+1,0} \\ p_{x+1,1} \\ p_{x+1,2} \end{bmatrix} = \begin{bmatrix} \frac{\lambda_1+\xi_0}{\lambda_1} & -\frac{r(1)(1-s)}{\lambda_1} & 0 \\ 0 & \frac{\lambda_1+r(1)(1-s)+\mu_1}{\lambda_1} & -\frac{r(2)(1-s)}{\lambda_1} \\ -\frac{\xi_0}{\lambda_1+\lambda_2} & 0 & \frac{\lambda_1+\lambda_2+r(2)(1-s)+\mu_2}{\lambda_1+\lambda_2} \end{bmatrix} \begin{bmatrix} p_{x,0} \\ p_{x,1} \\ p_{x,2} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{\mu_1}{\lambda_1} & 0 \\ 0 & 0 & -\frac{\mu_2}{\lambda_1+\lambda_2} \end{bmatrix} \begin{bmatrix} p_{x-1,0} \\ p_{x-1,1} \\ p_{x-1,2} \end{bmatrix} \tag{5.3}$$

When $x = R_1$,

$$\begin{bmatrix} p_{R_1+1,0} \\ p_{R_1+1,1} \\ p_{R_1+1,2} \end{bmatrix} = \begin{bmatrix} \frac{\lambda_1+\xi_0}{\lambda_1} & -\frac{r(1)(1-s)}{\lambda_1} & 0 \\ 0 & \frac{\lambda_1+r(1)(1-s)+\mu_1}{\lambda_1+\lambda_2} & -\frac{r(2)(1-s)}{\lambda_1+\lambda_2} \\ -\frac{\xi_0}{\lambda_1+\lambda_2} & 0 & \frac{\lambda_1+\lambda_2+r(2)(1-s)+\mu_2}{\lambda_1+\lambda_2} \end{bmatrix} \begin{bmatrix} p_{R_1,0} \\ p_{R_1,1} \\ p_{R_1,2} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{\mu_1}{\lambda_1+\lambda_2} & 0 \\ 0 & 0 & -\frac{\mu_2}{\lambda_1+\lambda_2} \end{bmatrix} \begin{bmatrix} p_{R_1-1,0} \\ p_{R_1-1,1} \\ p_{R_1-1,2} \end{bmatrix} \tag{5.4}$$

When $x = R_1+1, \cdots, R_0 - 1$,

$$\begin{bmatrix} p_{x+1,0} \\ p_{x+1,1} \\ p_{x+1,2} \end{bmatrix} = \begin{bmatrix} \frac{\lambda_1+\xi_0}{\lambda_1} & -\frac{r(1)(1-s)}{\lambda_1} & 0 \\ 0 & \frac{\lambda_1+\lambda_2+r(1)(1-s)+\mu_1}{\lambda_1+\lambda_2} & -\frac{r(2)(1-s)}{\lambda_1+\lambda_2} \\ -\frac{\xi_0}{\lambda_1+\lambda_2} & 0 & \frac{\lambda_1+\lambda_2+r(2)(1-s)+\mu_2}{\lambda_1+\lambda_2} \end{bmatrix} \begin{bmatrix} p_{x,0} \\ p_{x,1} \\ p_{x,2} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{\mu_1}{\lambda_1+\lambda_2} & 0 \\ 0 & 0 & \frac{\mu_2}{\lambda_1+\lambda_2} \end{bmatrix} \begin{bmatrix} p_{x-1,0} \\ p_{x-1,1} \\ p_{x-1,2} \end{bmatrix} \tag{5.5}$$

When $x = R_0$,

$$\begin{bmatrix} p_{R_0+1,0} \\ p_{R_0+1,1} \\ p_{R_0+1,2} \end{bmatrix} = \begin{bmatrix} \frac{\lambda_1+\xi_0}{\lambda_1+\lambda_2} & -\frac{r(1)(1-s)}{\lambda_1+\lambda_2} & 0 \\ 0 & \frac{\lambda_1+\lambda_2+r(1)(1-s)+\mu_1}{\lambda_1+\lambda_2} & -\frac{r(2)(1-s)}{\lambda_1+\lambda_2} \\ -\frac{\xi_0}{\lambda_1+\lambda_2} & 0 & \frac{\lambda_1+\lambda_2+r(2)(1-s)+\mu_2}{\lambda_1+\lambda_2} \end{bmatrix} \begin{bmatrix} p_{R_0,0} \\ p_{R_0,1} \\ p_{R_0,2} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{\mu_1}{\lambda_1+\lambda_2} & 0 \\ 0 & 0 & -\frac{\mu_2}{\lambda_1+\lambda_2} \end{bmatrix} \begin{bmatrix} p_{R_0-1,0} \\ p_{R_0-1,1} \\ p_{R_0-1,2} \end{bmatrix} \tag{5.6}$$

When $x = R_0+1, \cdots, S_2 - 1$,

$$\begin{bmatrix} p_{x+1,0} \\ p_{x+1,1} \\ p_{x+1,2} \end{bmatrix} = \begin{bmatrix} \frac{\lambda_1+\lambda_2+\xi_0}{\lambda_1+\lambda_2} & -\frac{r(1)(1-s)}{\lambda_1+\lambda_2} & 0 \\ 0 & \frac{\lambda_1+\lambda_2+r(1)(1-s)+\mu_1}{\lambda_1+\lambda_2} & -\frac{r(2)(1-s)}{\lambda_1+\lambda_2} \\ -\frac{\xi_0}{\lambda_1+\lambda_2} & 0 & \frac{\lambda_1+\lambda_2+r(2)(1-s)+\mu_2}{\lambda_1+\lambda_2} \end{bmatrix} \begin{bmatrix} p_{x,0} \\ p_{x,1} \\ p_{x,2} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{\mu_1}{\lambda_1+\lambda_2} & 0 \\ 0 & 0 & -\frac{\mu_2}{\lambda_1+\lambda_2} \end{bmatrix} \begin{bmatrix} p_{x-1,0} \\ p_{x-1,1} \\ p_{x-1,2} \end{bmatrix} \tag{5.7}$$

When $x = S_2$,

$$\begin{bmatrix} p_{S_2+1,0} \\ p_{S_2+1,1} \\ p_{S_2+1,2} \end{bmatrix} = \begin{bmatrix} \frac{\lambda_1+\lambda_2+\xi_0}{\lambda_1+\lambda_2} & -\frac{r(1)(1-s)}{\lambda_1+\lambda_2} & 0 \\ 0 & \frac{\lambda_1+\lambda_2+r(1)(1-s)+\mu_1}{\lambda_1+\lambda_2} & -\frac{r(2)(1-s)}{\lambda_1+\lambda_2} \\ -\frac{\xi_0}{\lambda_1+\lambda_2} & 0 & \frac{\lambda_1+\lambda_2+r(2)(1-s)}{\lambda_1+\lambda_2} \end{bmatrix} \begin{bmatrix} p_{S_2,0} \\ p_{S_2,1} \\ p_{S_2,2} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{\mu_1}{\lambda_1+\lambda_2} & 0 \\ 0 & 0 & -\frac{\mu_2}{\lambda_1+\lambda_2} \end{bmatrix} \begin{bmatrix} p_{S_2-1,0} \\ p_{S_2-1,1} \\ p_{S_2-1,2} \end{bmatrix} \tag{5.8}$$

When $x = S_2+1, \cdots, S_1 - 1$,

$$\begin{bmatrix} p_{x+1,0} \\ p_{x+1,1} \\ p_{x+1,2} \end{bmatrix} = \begin{bmatrix} \frac{\lambda_1+\lambda_2+\xi_0}{\lambda_1+\lambda_2} & -\frac{r(1)(1-s)}{\lambda_1+\lambda_2} & 0 \\ 0 & \frac{\lambda_1+\lambda_2+r(1)(1-s)+\mu_1}{\lambda_1+\lambda_2} & -\frac{r(2)(1-s)}{\lambda_1+\lambda_2} \\ -\frac{\xi_0}{\lambda_1+\lambda_2} & 0 & \frac{\lambda_1+\lambda_2+r(2)(1-s)}{\lambda_1+\lambda_2} \end{bmatrix} \begin{bmatrix} p_{x,0} \\ p_{x,1} \\ p_{x,2} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{\mu_1}{\lambda_1+\lambda_2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_{x-1,0} \\ p_{x-1,1} \\ p_{x-1,2} \end{bmatrix} \tag{5.9}$$

In addition, according to the first balance expression of each state, we have the following constraints:

$$\begin{bmatrix} 1 & -\frac{r(1)(1-s)}{\lambda_1+\lambda_2+\xi_0} & 0 \\ 0 & 1 & -\frac{\lambda_1+\lambda_2+r(2)(1-s)+\mu_1}{\lambda_1+\lambda_2+r(1)(1-s)} \\ -\frac{\xi_0}{\lambda_1+\lambda_2+r(2)(1-s)} & 0 & 1 \end{bmatrix} \begin{bmatrix} p_{S_1,0} \\ p_{S_1,1} \\ p_{S_1,2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{\mu_1}{\lambda_1+\lambda_2+r(1)(1-s)} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_{S_1-1,0} \\ p_{S_1-1,1} \\ p_{S_1-1,2} \end{bmatrix} \tag{5.10}$$

To obtain the final probability distribution, one can add up the probabilities across all possible states and ensure that their sum equals 1.

$$\sum_{x=0}^{S_1} \sum_{i=0}^{2} p_{x,i} = 1 \tag{5.11}$$

Therefore, the instant satisfaction rates f_1, f_2 of the two types of customers can be obtained as

$$f_1 = 1 - \sum_{i=0}^{2} p_{0,i}$$

$$f_2 = 1 - \sum_{x=0}^{R_0} p_{x,0} - \sum_{x=0}^{R_1} p_{x,1} - \sum_{x=0}^{R_2} p_{x,2}$$

By the above formulas, we can solve for the optimal control strategy that minimize the total expected cost, that is, a set of thresholds based on inventory levels:

$$\{S^*(1), S^*(2), R_2^*(0), R_2^*(1), R_2^*(2)\}$$

6. Numerical Simulation

This section presents computational experiments to evaluate the effectiveness of the proposed optimal control policy. A simulated annealing (SA) algorithm is employed to iteratively update the control strategy and identify the optimal policy for the system. Determining the optimal strategy in a production-inventory environment constitutes an NP-hard problem, even under simplified conditions. The objective is to minimize the total cost by treating the control policy as a decision variable within the optimization framework.

Simulated annealing, a Monte Carlo-based stochastic optimization technique, is selected because of its ability to escape local optima—an advantage over traditional gradient-based or hill-climbing algorithms. The optimization problem is formulated with respect to the structural parameters of the system, and the transition probabilities between candidate solutions are explicitly defined to guide the search process [42,43].

To ensure the reproducibility of computational experiments, the core parameter settings of the SA algorithm are supplemented as follows: initial temperature $T_0 = 1000$, cooling rate $\alpha = 0.95$, maximum iterations per temperature $N = 500$, termination condition as temperature drops to 10^{-6} or the relative change of the optimal objective function (total system cost) is less than 1×10^{-6} for 100 consecutive iterations, neighborhood search step for threshold parameters $S^*(i)$ and $R_k^*(x, i)$ set to 1 (integer adjustment) to match the discrete nature of inventory levels and machine states, and acceptance probability function $P(\Delta E) = \exp(-\Delta E/T)$ with ΔE denoting the difference between the new solution cost and the current optimal cost and T being the current temperature.

6.1. Cases Study

To be specific, we discuss a production inventory system that has a single device provided by a single machine with three states ($i = 0, 1, 2$) to meet demands from two classes of customers ($x = 1, 2$), corresponding to the machine failure factor. Two numerical cases are presented to demonstrate the influence of different parameters on the optimal control strategy. Case 6.1 involves implementing threshold strategies for different preventive maintenance policies under a fixed corrective maintenance approach. This scenario is applicable to companies where some equipment, although difficult to repair once damaged, can be adjusted and preemptively maintained during earlier stages. Case 6.2, conversely, involves implementing threshold strategies for various corrective maintenance policies under a fixed preventive maintenance approach, suitable for companies that prefer not to conduct early-stage maintenance but can promptly repair and adjust equipment once failure occurs. Analysis of both cases indicates that under the same parameters, the optimal threshold for adjusting preventive maintenance strategies is lower than that for adjusting corrective maintenance strategies. This suggests that systems like those in Case 6.1 exhibit greater resilience, lower costs, and higher reliability.

Case 6.1. *The production rates of a machine in its three states are* $\mu_0 = 0$, $\mu_1 = 5$, *and* $\mu_2 = 20$. *The two customer classes have arrival rates of* $\lambda_1 = 1$ *and* $\lambda_2 = 10$. *The machine's repair time during a complete breakdown follows an exponential distribution with parameter* $\xi_0 = 0.5$. *The unit production cost is* $q = 0.5$, *and the unit maintenance cost is* $w(s) = 1$ *The shortage costs for the two customer classes are* $c_1 = 100$ *and* $c_2 = 10$. *The machine's natural damage rates in its two operating states are* $r(1) = 1$ *and* $r(2) = 2$. *The maximum inventory level is* $x_{\max} = 50$. *The maintenance degree* s *ranging from 0.1 to 0.9 in increments of 0.1.*

To obtain optimal production and inventory allocation strategies for this case, we formulated the problem as an MDP model and derived the corresponding value function and constraints, as described in the previous sections. However, due to the high complexity of the model, traditional optimization methods were insufficient to obtain an optimal solution. We thus applied intelligent optimization algorithm to search for the optimal policy. After 500 iterations for each temperature, the algorithm converged and provided approximations of the optimal decisions for each system state. The resulting optimal policy is characterized by a set of thresholds that define switching points between different production and inventory allocation strategies, as illustrated in Figure 3, to show how it varies with changes in the maintenance parameter s.

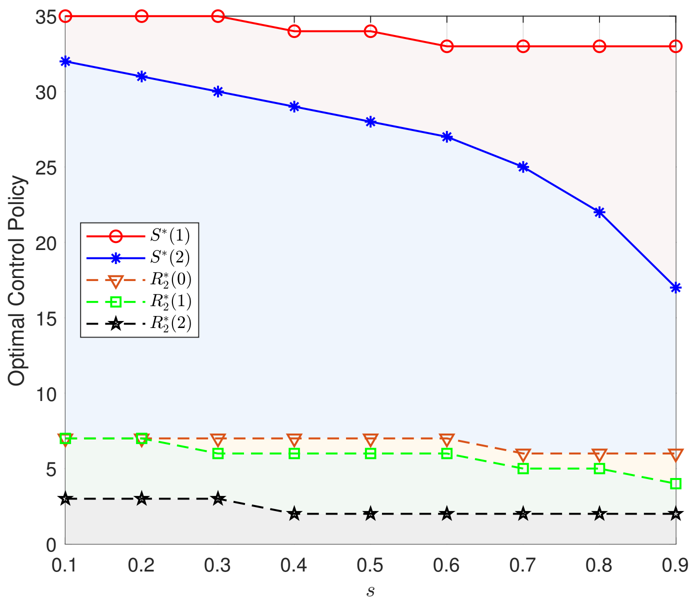

Figure 3. Optimal production policy structure for varying maintenance parameter s.

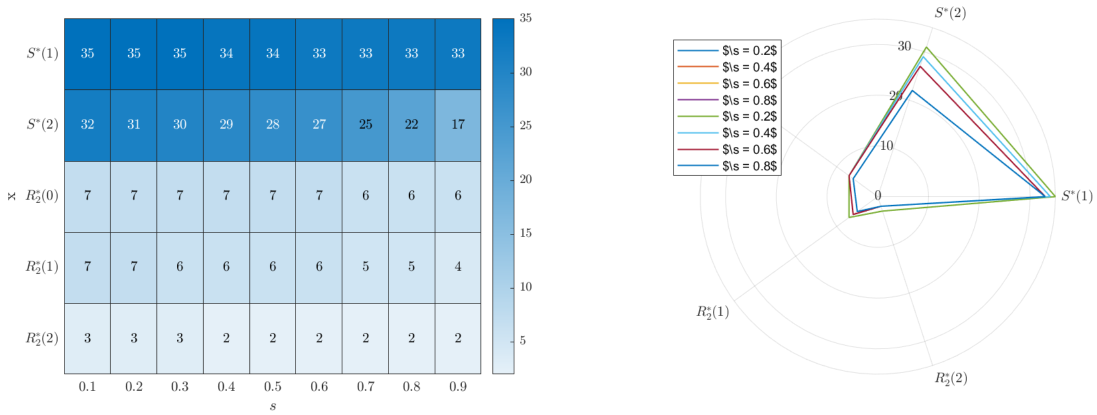

Figure 4. Heatmaps and radar charts for different s values.

As depicted in Figure 3, the optimal control policy in Case 1 is designed based on thresholds for the inventory level, which depend on the maintenance parameter s. In line with Definition 3.1, the policy is formulated based on $S^*(1) > S^*(2) > R_2^*(0) > R_2^*(1) > R_2^*(2)$. When the inventory level x surpasses $S^*(1)$ and $S^*(2)$, the machines in states 1 and 2 serve both types of customers without producing. If x exceeds $R_2^*(0)$, the machine is in a state of shutdown (state 0) but still delivers to meet customer demand. When x exceeds $R_2^*(1)$ or $R_2^*(2)$, the machine is in state 1 or 2 and produces to satisfy all demand.

Figure 4 presents the heatmap and radar chart of the system's optimal control strategy under different preventive maintenance intensities s. The heatmap visually illustrates the gradual decline in production and inventory allocation thresholds across machine states as s increases, corroborating the conclusion in Theorem 3.3 regarding the negative correlation between maintenance intensity and inventory levels. The radar chart further compares the overall structure of system strategies under different s values from multiple dimensions, highlighting the regulatory role of maintenance strategies on the system's overall performance.

It can be observed that the threshold levels decrease as the maintenance parameter s increases. From a practical perspective, maintaining reasonable inventory levels in conjunction with an effective maintenance strategy can help mitigate uncertainty in production and supply processes. Higher maintenance intensity enables manufacturers to respond more swiftly to machine failures, thereby reducing the risk of operational disruptions and minimizing the need for excessive inventory buffering. However, it is essential to balance inventory holding costs and maintenance expenses against the operational benefits to ensure optimal system performance in real-world applications.

Additionally, the optimal stock level corresponding to machine state 1 is higher than that of state 2, as the production capacity in state 1 exceeds that in state 2. Consequently, the expected demand during state 1 is greater, justifying a higher inventory threshold.

These results demonstrate that the optimal control policy adopts a dynamic threshold strategy that adapts to the current machine state. Specifically, the manufacturer increases production to the maximum allowable rate whenever the inventory level falls below the threshold associated with the machine's current condition. As the maintenance level improves, the adverse impact of machine failure on system operations diminishes, leading to progressively lower threshold levels. Therefore, machine reliability emerges as a critical determinant in the design of optimal production-inventory control policies.

Case 6.2. *The production rates of a machine in its three states are $\mu_0 = 0$, $\mu_1 = 5$, and $\mu_2 = 20$. The two customer classes have arrival rates of $\lambda_1 = 1$ and $\lambda_2 = 10$. The maintenance degree $s = 0.3$.The unit production cost is $q = 0.5$, and the unit maintenance cost is $w(s) = 1$ The shortage costs for the two customer classes are $c_1 = 100$ and $c_2 = 10$. The machine's natural damage rates in its two operating states are $r(1) = 1$ and $r(2) = 2$. The maximum inventory level is $x_{\max} = 50$. The machine's repair time during a complete breakdown follows an exponential distribution with parameter ξ_0 ranging from 0.1 to 1 in increments of 0.1.*

Figure 5 depicts the impact of various xi_0 on the optimal control strategy in Case 6.2. The relationship between stock levels and thresholds remains the same as described in Theorem 1. when the machine is in state 1 and the inventory level is greater than or equal to $S^*(1)$ and $S^*(2)$, the machine does not produce but delivers to satisfy the demands of both customer classes. Similarly, in state 0, when the inventory level is greater than or equal to $R_2^*(0)$ due to machine failure, the machine does not produce but delivers to satisfy the demands of both customer classes. Finally, the machine produces to meet the demands of both customer classes when it is in state 1 and the inventory level is greater than or equal to $R_2^*(1)$ or when it is in state 2 and the inventory level is greater than or equal to $R_2^*(2)$.

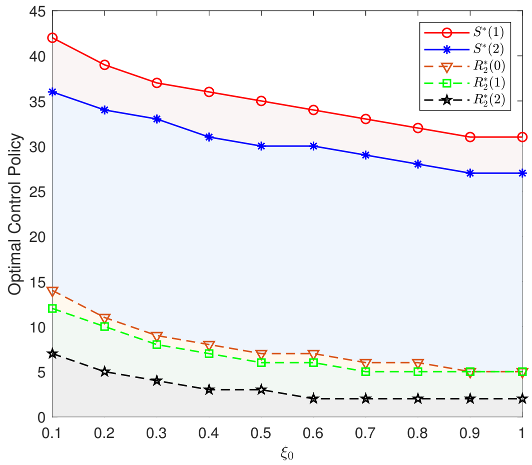

Figure 5. Optimal production policy structure for varying maintenance parameter ξ_0.

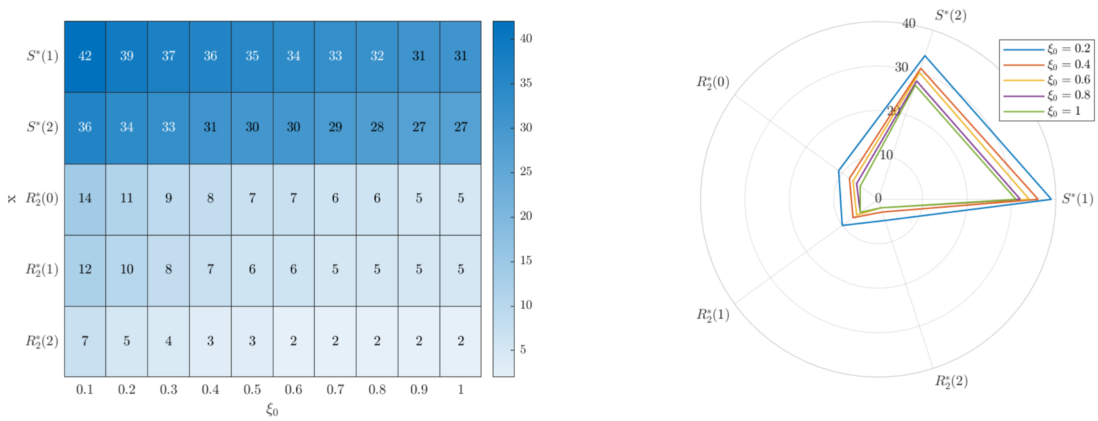

Figure 6. Heatmaps and radar charts for different ξ_0 values

Figure 6 illustrates the influence of the repair rate $\xi_0$on the optimal control strategy. As ξ_0 increases (i.e., the average repair time decreases), both production and inventory allocation thresholds for each machine state exhibit a downward trend. This indicates that the system can maintain service levels with lower inventory levels under a more efficient repair mechanism. The radar chart further reveals coordinated changes in the system strategy across multiple dimensions, demonstrating how repair capability enhances the overall scheduling flexibility of the system.

Furthermore, the numerical results indicate that all threshold levels decrease as the parameter ξ_0 increases. The optimal manufacturertrategy involves suspending production for maintenance while still delivering products to meet customer demand. A shorter maintenance time implies less impact of machine failure on the overall operation of the system.

6.2. *Benefit from The Optimal Control Policy*

In conclusion, the results of the computational experiments demonstrate the validity and precision of the model in this document and the solution approach to the complex problem of production and inventory allocation under machine failure. The numerical results show that the optimal production and inventory allocation strategies are dependent on the machine state and threshold values, and can be obtained through the application of intelligent optimization algorithm combined with the MDP model. Moreover, the numerical experiments indicate that the proposed approach can effectively reduce the impact of machine failures on the system by improving maintenance levels, and thus decrease total costs. In general, the proposed model and solution approach provides a useful tool for decision makers in the field of production and inventory management under machine failure.

6.3. Convergence Stability Analysis

To rigorously validate the stability and robustness of the obtained optimal thresholds, we conducted additional experiments with two distinct sets of initial threshold values:

- Initial Set 1 (High): $S^*(1) = 30$, $S^*(2) = 25$
- Initial Set 2 (Low): $S^*(1) = 20$, $S^*(2) = 15$

For both initial settings, the inventory allocation thresholds $R_2^*(0), R_2^*(1), R_2^*(2)$ were initialized proportionally. The SA algorithm was executed under each initial configuration using the convergence criteria specified in Section 6.

The results, summarized in Table 2, demonstrate that both initializations converge to identical optimal thresholds: $S^*(1) = 28$, $S^*(2) = 22$, $R_2^*(0) = 23$, $R_2^*(1) = 20$, $R_2^*(2) = 16$. The total expected cost difference between the two converged solutions is less than 0.5%. This consistency across different initial settings ultimately verifies that the optimal control strategy is independent of the initial threshold, demonstrating the algorithm's robustness.

Table 2. Convergence Results under Different Initial Thresholds (Case 6.1, $s = 0.5$).

Initial Setting	$S^*(1)$	$S^*(2)$	$R_2^*(0)$	$R_2^*(1)$	$R_2^*(2)$	Total Cost (CNY)
Set 1 (High)	28	22	23	20	16	1089.6
Set 2 (Low)	28	22	23	20	16	1085.2

7. Conclusion and Future Research

This study presents a comprehensive joint optimization framework that integrates production decision-making and inventory allocation in a fault-prone supply chain system. By explicitly modeling production and supply uncertainties alongside the impact of machine failures, the framework emphasizes the strategic value of preventive repair and maintenance. This unified approach provides a holistic solution for managing the complexities of machine-dependent production environments. Through the proactive mitigation of machine failure risks, firms can reduce downtime, stabilize production flows, and enhance overall system resilience.

Leveraging Markov decision theory under an infinite-horizon expected discounted cost criterion, we analytically derive the structural properties of the optimal inventory allocation and maintenance policies. The resulting threshold-based strategies offer critical insights into dynamic system behavior. Stationary analysis further enriches our understanding of the system's long-term performance. Visualizations in three dimensions highlights the interplay between machine states, inventory levels, and maintenance efforts, providing an intuitive representation of the threshold dynamics. Computational experiments using a simulated annealing algorithm confirm the robustness and practical value of the proposed approach. The results demonstrate significant improvements in system stability and reliability while efficiently managing production and demand uncertainties and minimizing long-term maintenance costs.

The optimized inventory strategy not only achieves cost reduction, but also exhibits strong resilience under uncertainty, reaffirming the essential role of preventive maintenance in maintaining operational robustness. By preventing unexpected disruptions, preventive strategies enhance reliability and reduce the cumulative cost of corrective actions. These findings highlight the critical importance of embedding preventive repair in production-inventory systems to maintain long-term operational stability.

Nonetheless, the current model operates under certain assumptions, such as Poisson-distributed demand and fixed batch sizes. While these assumptions provide analytical tractability, they may limit the model's generality. Future research could explore extensions to more realistic settings by relaxing these assumptions, incorporating variable batch sizes, and modeling non-Poisson demand processes. In particular, further investigation is warranted into backlog systems under flexible production conditions and varying demand rates, which may offer additional managerial insights.

Looking ahead, promising research directions include extending the model to multi-product and multi-machine settings, as well as exploring alternative or hybrid maintenance strategies. The integration of predictive and condition-based maintenance with preventive policies may further improve system reliability and cost efficiency.

In essence, this research advances the state of the art by integrating preventive maintenance and inventory optimization into a unified control framework. It demonstrates how such synergy can significantly enhance system resilience in fault-prone environments. The study's main contribution lies in its holistic perspective on managing operational uncertainty, offering both theoretical insights and practical guidance for the design of robust production-inventory systems in dynamic settings.

Use of Generative-AI tools declaration

The author(s) declare(s) that no Generative Artificial Intelligence (AI) tools were utilized in the creation of this article.

Author contributions

- Ting Jin: Conceptualization, Methodology, Writing - Original Draft
- Yuting Yan: Formal analysis, Validation
- Houcai Shen: Supervision, Writing - Review & Editing

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Data Availability

Data Availability Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Conflict of interest

The authors declare that there are no competing interests, financial or otherwise, that could be perceived as influencing the results or interpretation of this work.

Ethical Statement

All data utilized in this study were obtained through legitimate sources. Their application strictly adhered to institutional ethics protocols and national regulations governing data management, fulfilling all requirements for research integrity.

References

1. T. Jin, X. Yang, H. Xia, H. Ding, Reliability index and option pricing formulas of the first-hitting time model based on the uncertain fractional-order differential equation with caputo type. *Fractals*, **29** (2021), 2150012. https://doi.org/10.1142/S0218348X21500122
2. B. Li, J. Xu, T. Jin, Y. Shu, Piecewise parameterization for multifactor uncertain system and uncertain inventory-promotion optimization, *Knowl.-Based Syst.*, **255** (2022), 109683. https://doi.org/10.1016/j.knosys.2022.109683
3. R. Shi, C. You, Dynamic pricing and production control for perishable products under uncertain environment, *Fuzzy Optim. Decis. Making*, (2022), 1–28.
4. Y. G. Kim, G. H. Yang, B. D. Chung, Estimated model-based robust optimization of closed-loop supply chain under uncertain carbon tax rates and demand, *Comput. Ind. Eng.*, (2023), 109368.
5. A. A. Taleizadeh, H. Samimi, B. Sarkar, B. Mohammadi, STOCHASTIC MACHINE BREAKDOWN AND DISCRETE DELIVERY IN AN IMPERFECT INVENTORY-PRODUCTION SYSTEM, *J. Ind. Manage. Optim.*, **13** (2017), 1511–1535. https://doi.org/10.3934/jimo.2017005
6. S. X. Zhou, Y. Yu, Optimal product acquisition, pricing, and inventory management for systems with remanufacturing, *Oper. Res.*, **59** (2008), 514–521.
7. S. X. Zhou, Z. Tao, X. Chao, Optimal control of inventory systems with multiple types of remanufacturable products, *Manuf. Serv. Oper. Manage.*, **13** (2011), 20–34. https://doi.org/10.1287/msom.1100.0298

8. J. Hu, P. Vakili, G. Yu, Optimality of hedging point policies in the production control of failure prone manufacturing systems, *IEEE Trans. Autom. Control*, **39** (1994), 1875–1880. https://doi.org/10.1109/9.317116

9. J. Hu, D. Xiang, Monotonicity of optimal flow control for failure-prone production systems, *J. Optimiz. Theory Appl.*, **86** (1995), 57–71. https://doi.org/10.1007/BF02193461

10. L. Knopik, K. Migawa, Semi-markov system model for minimal repair maintenance, *Eksploat. Niezawodn.*, **21** (2019), 256–260.

11. M. M. Srinivasan, H. Lee, Random review production/inventory systems with compound poisson demands and arbitrary processing times, *Manage. Sci.*, **37** (1991), 813-833. https://doi.org/10.1287/mnsc.37.7.813

12. D. Song, Optimal production and backordering policy in failure-prone manufacturing systems, *IEEE Trans. Autom. Control*, **51** (2006), 906–911. https://doi.org/10.1109/TAC.2006.872833

13. D. Song, Y. Sun, Optimal control structure of an unreliable manufacturing system with random demands, *IEEE Trans. Autom. Control*, **44** (1999), 619–622. https://doi.org/10.1109/9.751363

14. Y. Feng, B. Xiao, Optimal threshold control in discrete failure-prone manufacturing systems, *IEEE Trans. Autom. Control* , **47** (2002), 1167–1174.

15. E. Boukas, A. Haurie, Manufacturing flow control and preventing maintenance: a stochastic control approach, *IEEE Trans. Autom. Control*, **35** (1990), 1024–1031. https://doi.org/10.1109/9.58530

16. S. W. Chiu, Production lot size problem with failure in repair and backlogging derived without derivatives, *Eur. J. Oper. Res.*, **188** (2008), 610–615. https://doi.org/10.1016/j.ejor.2007.04.049

17. A. K. Sethi, S. P. Sethi, Flexibility in manufacturing: a survey, *Int. J. Flex. Manuf. Syst.*, **2** (1990), 289–328. https://doi.org/10.1007/BF00186471

18. E. L. Porteus, Optimal lot sizing, process quality improvement and setup cost reduction, *Oper. Res.*, **34** (1986), 137–144. https://doi.org/10.1287/opre.34.1.137

19. W. P. Pierskalla, J. A. Voelker, A survey of maintenance models: the control and surveillance of deteriorating systems, *Naval Res. Logistics Q.*, **23** (1976), 353–388. https://doi.org/10.1002/nav.3800230302

20. S. M. Iravani, I. Duenyas, T. L. Olsen, A production/inventory system subject to failure with limited repair capacity, *Oper. Res.*, **48** (2000), 951–964. https://doi.org/10.1287/opre.48.6.951.12389

21. S. Iravani, I. Duenyas, Integrated maintenance and production control of a deteriorating production system, *Iie Trans.*, **34** (2002), 423–435. https://doi.org/10.1023/A:1013596731865

22. S. Sheu, Y. Lin, G. Liao, Optimal policies with decreasing probability of imperfect maintenance, *IEEE Trans. Reliab.*, **54** (2005), 347–357.

23. G. A. Widyadana, H. M. Wee, An economic production quantity model for deteriorating items with preventive maintenance policy and random machine breakdown, *Int. J. Syst. Sci.*, **43** (2012), 1870–1882.

24. K. Ito, S. Mizutani, T. Nakagawa, Optimal inspection models with minimal repair, *Reliab. Eng. Syst. Saf.*, **201** (2020), 106946. https://doi.org/10.1016/j.ress.2020.106946

25. J. Navarro, A. Arriaza, A. Suárez-Llorens, Minimal repair of failed components in coherent systems, *Eur. J. Oper. Res.*, **279** (2019), 951–964. https://doi.org/10.1016/j.ejor.2019.06.013

26. K. L. Cheung, W. H. Hausman, Joint determination of preventive maintenance and safety stocks in an unreliable production environment, *Naval Res. Logistics*, **44** (1997), 257–272. https://doi.org/10.1002/(SICI)1520-6750(199704)44:3¡257::AID-NAV2¿3.0.CO;2-7

27. X. Zhao, J. Cai, S. Mizutani, T. Nakagawa, Preventive replacement policies with time of operations, mission durations, minimal repairs and maintenance triggering approaches, *J. Manuf. Syst.*, **61** (2021), 819–829. https://doi.org/10.1016/j.jmsy.2020.04.003

28. E. Charlot, J. P. Kenné, S. Nadeau, Optimal production, maintenance and lockout/tagout control policies in manufacturing systems, *Int. J. Prod. Econ.*, **107** (2007), 435–450. https://doi.org/10.1016/j.ijpe.2006.09.017

29. E. Boukas, Q. Zhang, G. Yin, Robust production and maintenance planning in stochastic manufacturing systems, *IEEE Trans. Autom. Control*, **40** (1995), 1098–1102. https://doi.org/10.1109/9.388692

30. T. K. Das, S. Sarkar, Optimal preventive maintenance in a production inventory system, *IIE Trans.*, **31** (1999), 537–551. https://doi.org/10.1023/A:1007602423336

31. W. J. Hopp, N. Pati, P. C. Jones, Optimal inventory control in a production flow system with failures, *Int. J. Prod. Res.*, **27** (1989), 1367–1384. https://doi.org/10.1080/00207548908942628

32. M. K. Salameh, R. E. Ghattas, Optimal just-in-time buffer inventory for regular preventive maintenance, *Int. J. Prod. Econ.*, **74** (2001), 157–161. https://doi.org/10.1016/S0925-5273(01)00122-0

33. E. G. Kyriakidis, T. D. Dimitrakos, Optimal preventive maintenance of a production system with an intermediate buffer, *Eur. J. Oper. Res.*, **168** (2006), 86–99. https://doi.org/10.1016/j.ejor.2004.01.052

34. Z. Pang, H. Shen, T. C. E. Cheng, Inventory rationing in a maketo-stock system with batch production and lost sales, *Prod. Oper. Manage.*, **23** (2014), 1243–1257. https://doi.org/10.1111/poms.12190

35. X. Shi, H. Shen, T. Wu, T. C. E. Cheng, Production planning and pricing policy in a make-to-stock system with uncertain demand subject to machine breakdowns, *Eur. J. Oper. Res.*, **238** (2014), 122–129.

36. T. C. E. Cheng, C. Gao, H. Shen, Production planning and inventory allocation of a single-product assemble-to-order system with failure-prone machines, *Int. J. Prod. Econ.*, **131** (2011), 604–617.

37. C. Gao, H. Shen, T. C. E. Cheng, Order-fulfillment performance analysis of an assemble-to-order system with unreliable machines, *Int. J. Prod. Econ.*, **126** (2010), 341–349. https://doi.org/10.1016/j.ijpe.2010.04.014

38. L. Yang, X. Zhang, Z. Lu, Y. Fu, D. Moens, M. Beer, Reliability evaluation of a multi-state system with dependent components and imprecise parameters: A structural reliability treatment, *Reliab. Eng. Syst. Safety*, **250** (2024), 110240. https://doi.org/10.1016/j.ress.2024.110240

39. X. Yang, Y. He, R. Liao, Y. Cai, W. Dai, Mission reliability-centered opportunistic maintenance approach for multistate manufacturing systems, *Reliab. Eng. Syst. Safety*, **241** (2024), 109693. https://doi.org/10.1016/j.ress.2023.109693

40. R. Liao, Y. He, T. Feng, X. Yang, W. Dai, W. Zhang, Mission reliability-driven risk-based predictive maintenance approach of multistate manufacturing system, *Reliab. Eng. Syst. Safety*, **236** (2023), 109273. https://doi.org/10.1016/j.ress.2023.109273

41. S. Rajagopalan, Make to order or make to stock: model and application, *Manage. Sci.*, **48** (2002), 241–256. https://doi.org/10.1287/mnsc.48.2.241.255

42. X. Yang, Z. Cai, T. Jin, Z. Tang, S. Gao, A three-phase search approach with dynamic population size for solving the maximally diverse grouping problem, *Eur. J. Oper. Res.*, **302** (2022), 925–953. https://doi.org/10.1016/j.ejor.2022.02.003

43. A. Corana, M. Marchesi, C. Martini, S. Ridella, Minimizing multimodal functions of continuous variables with the “simulated annealing” algorithm corrigenda for this article is available here, *Acm Trans. Math. Software*, **13** (1987), 262–280. https://doi.org/10.1145/29380.29864

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