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Research article

D.C. programming approach to generalized Nash equilibrium with quadratic objective functions

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Abstract: In this paper, we present a new approach for solving the generalized Nash equilibrium problem (GNEP), where players have nonconvex quadratic objective functions and are subject to jointly convex shared constraints. We showed that this problem can be reformulated as a nonconvex unconstrained optimization problem based on a regularized Nikaido-Isoda function. This reformulation reduces to a difference of two convex functions (D.C.) programming problem, which enables us to apply local and global search methods developed in A. S. Strekalovsky. Numerical results are presented.

Keywords: nonconvex games; Nikaido-Isoda function; gap functions; D.C. optimization; local and global search methods
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1. Introduction

The Generalized Nash Equilibrium Problem (GNEP) is a non-cooperative game involving N players, in which the strategy set available to each player depends on the strategies chosen by all other players. This formulation was first introduced by Debreu [1] in 1952. GNEPs have widespread applications in economics, engineering, mathematics, operational research, telecommunications, and climate change. Game theory plays an important role in Industrial management for analyzing strategic business decisions in competitive markets. We refer to the survey papers [2, 3].

The development of computational algorithms for convex GNEPs has led to several approaches. Notable examples are the Nikaido-Isoda-type function [4], the gap function method [5], penalty methods [6], and the parametrized variational inequality approach [7]. Conjugate duality has also been applied to this problem in [8, 9].

Moreover, in recent years, applications of GNEPs have been increasing in many fields. GNEP's approach to Malfatti's problem [10] was considered in [11]. Some applications of GNEPs for the banking and insurance sectors were been considered in [12] and [13], respectively.

In [14], the author investigated regret-based reinforcement learning frameworks for computing Nash equilibria in multi-agent stochastic environments. Similarly, the researchers in [15] introduced an adaptive Nash equilibrium search method incorporating better-response dynamics to address nonlinearity and convergence issues in strategic interaction models. Furthermore, the researchers in [16] proposed a structured variational formulation for noncooperative games, demonstrating that decomposition-based optimization can significantly reduce computational burden while ensuring feasibility under nonlinear payoff relationships.

In recent years, in addition to the Nash equilibrium, research on the Berge equilibrium has been conducted intensively. We refer to [17] for the regularized function approach and to [18] for the bimatrix game.

It seems that with the convex GNEP, the theory, algorithms, and examples of the nonconvex GNEP are not well developed. To the best of our knowledge, except for a few papers, there are no extensive investigations into nonconvex GNEPs. We refer to [19] for nonconvex games with side constraints and to [20] for weighted potential games.

A local quasi-Nash equilibrium (QNE) for non-convex problems is determined via variational inequality problems in [19]. In [20], local and global search methods were developed by introducing a weighted potential function for the nonconvex quadratic games. These methods differ significantly from our approach.

In this paper, we investigate GNEPs with quadratic nonconvex objective functions and jointly convex constraints. We show that an optimization reformulation of the nonconvex GNEPs is based on the regularized Nikaido-Isoda function. Then, we reduce this optimization problem to a D.C. minimization problem and apply D.C. programming methods [21]. This approach is the main contribution of this paper.

The D.C. programming was widely studied in [22–25]. For instance, in [24], local optimality conditions of D.C. programming and D.C. duality were examined. The survey of D.C. programming is presented in [22, 25, 26].

The paper is organized as follows: In Section 2, we consider some classes of convex GNEPs and reformulations of optimization problems using the Nikaido-Isoda functions. In Section 3, we consider a quadratic nonconvex GNEP with joint constraints and the reformulation of the equivalent optimization problem. Then, we show that the optimization problem reduces to a D.C. programming problem. In Section 4, we recall the local and global methods for the D.C. minimization problem and apply these methods to the problem. Some numerical results are provided in Section 5.

2. Generalized Nash equilibrium problem and equivalent optimization formulations

Let us consider an N-player game. Each player k ($k = 1, ..., N$) controls his strategy vector

$$x_k := (x_k^1, ..., x_k^{n_k})^T \in \mathbb{R}^{n_k}$$

of n_k decision variables. The vector

$$x := (x_1, ..., x_N)^T \in \mathbb{R}^n$$

contains the $n = \sum_{k=1}^{N} n_k$ decision variables of all players. To emphasize the $k-$th player's variables within x, one can write (x_k, x_{-k}) instead of x, where x_{-k} is formed by all the players' decision variables except those of player k.

In the generalized Nash equilibrium problem, player k ($k = 1, ..., N$) controls $x_k \in \mathbb{R}^{n_k}$ and tries to solve the following optimization problem

$$P_k(x_{-k}) : \quad \min_{x_k} \theta_k(x_k, x_{-k})$$

$$s.t. \;\; x_k \in X_k(x_{-k}).$$

Here, a cost function $\theta_k : \mathbb{R}^n \to \mathbb{R}$ for each player k depends on the player's own decision variables x_k and the decision variables x_{-k} of the other players.

Each player's strategy x_k belongs to a feasible set $X_k(x_{-k}) \subset \mathbb{R}^{n_k}$, which depends on the strategies of the rival players. $X_k(x_{-k})$ is called a feasible set of player k. In many applications, these feasible sets are defined by inequality constraints. For each player $k = 1, ..., N$, and a given continuous function $g_k : \mathbb{R}^n \to \mathbb{R}^{m_k}$, the set $X_k(x_{-k})$ is defined as:

$$X_k(x_{-k}) := \{x_k \mid g_k(x_k, x_{-k}) \leq \mathbf{0}\}. \tag{2.1}$$

For any given $x \in \mathbb{R}^n$, let us define the set $X(x)$:

$$X(x) := \prod_{k=1}^{N} X_k(x_{-k}) = \{y \in \mathbb{R}^n \mid y_k \in X_k(x_{-k}),\; k = 1, ..., N\}. \tag{2.2}$$

A feasible point $\bar{x}$ is a solution of the GNEP if, for all players $k = 1, ..., N$, we have

$$\theta_k(\bar{x}_k, \bar{x}_{-k}) \leq \theta_k(x_k, \bar{x}_{-k}), \quad \forall x_k \in X_k(x_{-k}).$$

Point $\bar{x}$ is also known as a generalized Nash equilibrium. Now we consider some important subclasses of GNEP.

Player convex GNEPs are an important subclass of GNEPs. In these games, each player's feasible strategy set $X_k(x_{-k})$ remains closed and convex regardless of rivals' choices x_{-k}, and their objective function $\theta_k(\cdot, x_{-k})$ is also convex.

A particularly important subclass of GNEPs arises when there exists a nonempty set $X \subseteq \mathbb{R}^n$, satisfying the condition

$$X_k(x_{-k}) = \{x_k \in \mathbb{R}^{n_k} \mid (x_k, x_{-k}) \in X\} \tag{2.3}$$

for every player $k = 1, \ldots, N$ and all x_{-k}. Such problems are commonly referred to as GNEPs with *shared* or *common constraints*.

Proposition 1. *[3] For GNEPs satisfying* (2.3), *the following equivalences hold:*

$$x \in X(x) \Leftrightarrow x_k \in X_k(x_{-k}) \; \textit{for all} \; k = 1, ..., N \Leftrightarrow x \in X.$$

An important subclass of convex GNEPs with shared constraints is known as jointly convex GNEPs. In a jointly convex GNEP, for every player k and all rival strategies x_{-k}, the cost function $\theta_k(\cdot, x_{-k})$ is convex. Furthermore, the shared constraint set $X \subseteq \mathbb{R}^n$ from condition (2.3) is nonempty, closed, and convex. This set X is defined as:

$$X = \{x \in \mathbb{R}^n \mid G(x) \leq \mathbf{0}\}, \tag{2.4}$$

where $G : \mathbb{R}^n \to \mathbb{R}^M$ is a componentwise convex function. Thus, (2.4) becomes:

$$X_k(x_{-k}) = \{x_k \in \mathbb{R}^{n_k} \mid G(x_k, x_{-k}) \leq \mathbf{0}\}.$$

This definition implies that all jointly convex GNEPs are also player convex.
We now turn to the optimization reformulations of convex GNEPs derived from Nikaido-Isoda functions.

Let the function $\psi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be defined by

$$\psi(x, y) := \sum_{k=1}^{N} [\theta_k(x_k, x_{-k}) - \theta_k(y_k, x_{-k})]$$

and consider the corresponding optimal value function

$$V(x) = \sup_{y \in X(x)} \psi(x, y), \tag{2.5}$$

defined for all $x \in \mathbb{R}^n$. Note that the supremum in (2.5) may be attained at multiple points in $X(x)$. It is verified that $V(x) \geq 0$ for all $x \in X(x)$ and that a point $\bar{x}$ is a generalized Nash equilibrium (GNE) of the GNEP if and only if $\bar{x} \in X(x)$ and $V(\bar{x}) = 0$. This establishes V as a gap function for the GNEP. Consequently, solving the GNEP is equivalent to solving the following constrained optimization problem:

$$\min_{x \in X(x)} V(x) \tag{2.6}$$

and achieving an optimal value of zero.

To ensure the existence of a unique maximizer in the evaluation of $V(x)$ for the GNEP, we regularize the problem by replacing ψ with the regularized Nikaido-Isoda function:

$$\psi_\alpha(x, y) = \sum_{k=1}^{N} [\theta_k(x_k, x_{-k}) - \theta_k(y_k, x_{-k})] - \frac{\alpha}{2}\|x - y\|^2 \tag{2.7}$$

where $\alpha > 0$ is a given parameter. We consider the optimal value function

$$V_\alpha(x) = \sup_{y \in X(x)} \psi_\alpha(x, y), \tag{2.8}$$

where for each fixed $x \in \mathbb{R}^n$, the function ψ_α is strongly concave with respect to y.

Therefore, solving a GNEP is equivalent to solving a smooth optimization problem with zero optimal value:

$$\min_{x \in X(x)} V_\alpha(x). \tag{2.9}$$

Problems (2.6) and (2.9) are so-called *quasi-optimization* problems [2]. By Theorem 2.3 and Theorem 2.4 [3], the player convex GNEP (2.6) and (2.9) (or the jointly convex GNEP) reduce to the following equivalent optimization problems

$$\min_{x \in X} V(x), \tag{2.10}$$

and

$$\min_{x \in X} V_\alpha(x). \tag{2.11}$$

3. Generalized Nash equilibrium with quadratic objective functions

In this section, we examine a GNEP featuring nonconvex quadratic objective functions and jointly convex constraints. Now we consider the following game:

$$P_k^Q(x_{-k}): \qquad \min_{x_k} \theta_k(x_k, x_{-k}) = x_k^T\Big(\frac{1}{2}B_{kk}x_k + d_k\Big) + \sum_{j\in N}^{j\neq k} x_k^T C_{kj} x_j,$$

$$s.t.\ \ x_k \in X_k(x_{-k}),$$

for specified matrices $C_{kj} \in \mathbb{R}^{n_k\times n_j}$ with symmetric B_{kk} (without loss of generality) and jointly convex constraints $X_k(x_{-k})$ as reformulated in (2.4). We employ the regularized Nikaido-Isoda function (2.7) to formulate the equivalent optimization problem and examine the properties of the gap function.

Lemma 1. *Let $\alpha > 0$ be chosen. Then the regularized Nikaido-Isoda function ψ_α^Q for the problem $P_k^Q(x_{-k})$ is strictly concave in y for any given $x \in X$.*

Proof. Let $x \in \mathbb{R}^n$ be fixed. We consider Nikaido-Isoda function for the problem $P_k^Q(x_{-k})$:

$$\psi_\alpha^Q(x,y) = \sum_{k=1}^{N}\Big[x_k^T\Big(\frac{1}{2}B_{kk}x_k + d_k\Big) + \sum_{j\in N}^{j\neq k} x_k^T C_{kj}x_j - y_k^T\Big(\frac{1}{2}B_{kk}y_k + d_k\Big) -$$

$$- \sum_{j\in N}^{j\neq k} y_k^T C_{kj}x_j\Big] - \frac{\alpha}{2}\|x-y\|^2 \qquad (3.1)$$

The Hessian of $\psi_\alpha^Q(x,\cdot)$ in variable y for any given $x \in X$ is denoted by $J_\alpha(x)$:

$$J_\alpha(x) = \begin{pmatrix} S_{11} & O & \cdots & O \\ 0 & S_{22} & \cdots & O \\ \cdots & \cdots & \cdots & \\ O & O & \cdots & S_{NN} \end{pmatrix} \qquad (3.2)$$

where $S_{kk} = -(B_{kk} + \alpha I_k),\ k = 1,...,N$ are n_k order square block matrices and I_k are unit matrices of n_k order. Since the matrices $B_{kk},\ k = 1,...,N$ are indefinite and symmetric, the matrices $S_{kk},\ k = 1,...,N$ are diagonal dominated negative definite matrices for chosen $\alpha > 0$. Therefore,

$$\langle J_\alpha(x)x, x\rangle < 0, \quad \forall x \in X.$$

This completes the Lemma. □

By analogy (2.8), we formulate the max-value function as follows:

$$V_\alpha^Q(x) = \sup_{y\in X}\psi_\alpha^Q(x,y), \quad x \in X. \qquad (3.3)$$

Further properties of $V_\alpha^Q(x)$ are given in the following assertions:

Proposition 2. *For the problem ($P_k^Q(x_{-k})$), the following statements hold:*
(a) $V_\alpha^Q(x) \geq 0$ for all $x \in X$.
(b) x^* is a generalized Nash equilibrium if and only if $x^* \in X$ and $V_\alpha^Q(x^*) = 0$.
(c) For every $x \in X$, there exists a unique vector $z_\alpha = (z_\alpha^1, ..., z_\alpha^N)$, such that

$$z_\alpha(x) = arg \min_{z \in X} \Big(\sum_{k=1}^{N} \theta_k(z_k, x_{-k}) + \frac{\alpha}{2} \|x - z\|^2 \Big) \tag{3.4}$$

(d) $x^*$is a generalized Nash equilibrium if and only if $x^* = z_\alpha(x^*)$ holds, that is, x^* is a fixed point of the mapping $x \to z_\alpha(x)$.

Proof. a) For any $x \in X$, we have

$$V_\alpha^Q(x) = \max_{y \in X} \psi_\alpha^Q(x, y) \geq \psi_\alpha^Q(x, x) = 0.$$

b) Suppose that x^* is a solution of the GNEP. Then $x^* \in X$ and

$$\theta_k(x_k^*, x_{-k}^*) \leq \theta_k(x_k, x_{-k}^*), \quad \forall x_k \in X_k(x_{-k}^*)$$

for all $k = 1, ..., N$. Hence,

$$\psi_\alpha^Q(x^*, z) = \sum_{k=1}^{N} [\theta_k(x_k^*, x_{-k}^*) - \theta_k(z_k, x_{-k}^*)] - \frac{\alpha}{2} \|x_k^* - z_k\|^2 \leq 0$$

for all $z \in X$. This implies

$$V_\alpha^Q(x^*) = \max_{z \in X} \psi_\alpha^Q(x^*, z) \leq 0.$$

Together with part (a), we have $V_\alpha^Q(x^*) = 0$.
Conversely, assume that $x^* \in X$ and $V_\alpha^Q(x^*) = 0$. Then

$$\psi_\alpha^Q(x^*, z) \leq 0 \tag{3.5}$$

holds for all $z \in X$. Assume there is a vector $\bar{y} \in X$ such that $\psi_\alpha^Q(x^*, \bar{y}) > 0$.
Then $\lambda x^* + (1 - \lambda \bar{y}) \in X$ for all $\lambda \in (0, 1)$, and Lemma 1 implies

$$\psi_\alpha^Q(x^*, \lambda x^* + (1 - \lambda)\bar{y}) > \lambda \psi_\alpha^Q(x^*, x^*) + (1 - \lambda)\psi_\alpha^Q(x^*, \bar{y}) = (1 - \lambda)\psi_\alpha^Q(x^*, \bar{y}) > 0$$

for all $\lambda \in (0, 1)$ sufficiently close to 1. This is a contradiction to (3.3).
c) From Lemma 1, the mapping $z_k \to \theta_k(z_k, x_{-k}) + \frac{\alpha}{2} \|x_k - z_k\|^2$ is strongly convex for any given x, while taking into account that X is a nonempty, closed, and convex set.
d) See proof Proposition 2.3 [27]. □

Therefore, $V_\alpha^Q(x)$ serves as a gap function for problem $P_k^Q(x_{-k})$. By Proposition 2, solving $P_k^Q(x_{-k})$ is equivalent to finding a global minimum of the following smooth constrained optimization problem:

$$\min_{x \in X} V_\alpha^Q(x). \tag{3.6}$$

Using δ_X, the indicator function of X, we reduce (3.6) to the unconstrained problem:

$$\min_{x\in\mathbb{R}^n}[V_\alpha^Q(x) + \delta_X] \tag{3.7}$$

with zero optimal value.
We consider the following D.C. reformulation of the function V_α^Q with a parameter $\alpha > 0$ [5]:

$$V_\alpha^Q(x) + \delta_X = l_\alpha(x) - \phi_\alpha(x) \tag{3.8}$$

with the functions $l_\alpha : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and $\phi_\alpha : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ defined by

$$l_\alpha(x) := \frac{\alpha}{2}\|x\|^2 + \sum_{k=1}^{N} \theta_k(x) + \delta_X(x), \tag{3.9}$$

$$\phi_\alpha(x) := \frac{\alpha}{2}\|x\|^2 + \inf_{z\in X}\Big(\sum_{k=1}^{N} \theta_k(z_k, x_{-k}) + \frac{\alpha}{2}\|x - z\|^2\Big). \tag{3.10}$$

The following lemma has been formulated in [5] for convex GNEPs. This result is shown to extend to the case of nonconvex quadratic problems.

Lemma 2. *Let l_α and ϕ_α be defined by* (3.9) *and* (3.10)*, respectively. The following properties hold:*
(a) The functions l_α and ϕ_α are lower semicontinuous (lsc) and strongly convex on $\mathbb{R}^n$, with dom(l_α) = dom(ϕ_α) = X.
(b) A point $\bar{x}$ solves the GNEP if and only if it solves the unconstrained optimization problem

$$\min_{x\in\mathbb{R}^n}[l_\alpha(x) - \phi_\alpha(x)] \tag{3.11}$$

and the optimal value of this problem is zero.

Proof. a) We prove the three properties sequentially.
Lower semicontinuity: From equations (3.9) and (3.10), l_α and ϕ_α are constructed as sums of continuous quadratic functions. Since continuous functions are lower semicontinuous, and finite sums of lower semicontinuous functions remain lower semicontinuous, it follows that l_α and ϕ_α are lower semicontinuous on $\mathbb{R}^n$.
Strong convexity: The definitions of l_α and ϕ_α contain terms of the form:

$$l_\alpha(x) = [\text{convex terms}] + \frac{\alpha}{2}\|x\|^2, \quad \phi_\alpha(x) = [\text{convex terms}] + \frac{\alpha}{2}\|x\|^2,$$

where $\frac{\alpha}{2}\|x\|^2$ is strongly convex for $\alpha > 0$. Since adding a strongly convex function to a convex function preserves strong convexity, l_α and ϕ_α are strongly convex on $\mathbb{R}^n$.
Domain: The definitions of l_α and ϕ_α include the indicator function $\delta_X(x)$, which equals 0 if $x \in X$ and $+\infty$ otherwise. Therefore, the domain of both functions is X. Thus, l_α and ϕ_α are lower semicontinuous, strongly convex functions with domain X.
b) First assume that x^* is a Nash equilibrium. Then Proposition 2 (d) implies $x^* = z_\alpha(x^*)$. Hence,
$l_\alpha(x^*) - \phi_\alpha(x^*) = \frac{\alpha}{2}\|x^*\|^2 + \sum_{k=1}^{N} \theta_k(x^*) + \delta_X(x^*) - \frac{\alpha}{2}\|x^*\|^2 - \inf_{z\in\Omega(x^*)}\Big(\sum_{k=1}^{N} \theta_k(z_k, x^*_{-k}) - \frac{\alpha}{2}\|x^* - z\|^2\Big) = \frac{\alpha}{2}\|x^*\|^2 +$

$\sum_{k=1}^{N} \theta_k(x^*) + +\delta_X(x^*) - \frac{\alpha}{2}\|x^*\|^2 - \Big(\sum_{k=1}^{N} \theta_k(z_\alpha^k(x^*), x_{-k}^*) - \frac{\alpha}{2}\|x^* - z_\alpha(x^*)\|^2\Big) = 0$, since $\delta_X(x^*) = 0$.
Conversely, let $x^* \in \mathbb{R}^n$ be any solution to problem (3.11)

$$0 = l_\alpha(x^*) - \phi_\alpha(x^*) = \min_{x \in \mathbb{R}^n}[l_\alpha(x) - \phi_\alpha(x)].$$

Therefore, $x^* = z_\alpha(x^*)$ clearly. So x^* is also a solution of $P_k^Q(x_{-k})$. □

Remark 1. *We reduce the generalized Nash equilibrium problem ($P_k^Q(x_{-k})$), which has a non-convex quadratic objective function and shared convex constraints, into a smooth constrained optimization problem using a regularization function, as shown in* (3.6). *Subsequently, by utilizing an indicator function, this problem is further reformulated into an unconstrained optimization problem, which is then equivalently reduced to the unconstrained D.C minimization problem* (3.11). *In the next section, we introduce local and global search algorithms for solving problem* (3.11).

4. A special local and global method for D.C. minimization problem

In recent decades, increasing methodological attention has been directed toward the development of specialized local search strategies within the framework of D.C. optimization [21, 28–41]. In the previous section, we constructed the function $l_\alpha(x)$, $\phi_\alpha(x)$ and proved that $l_\alpha(\cdot)$, $\phi_\alpha(\cdot)$ are strongly convex and function

$$V^Q(x) := V_\alpha^Q(x) + \delta_X = l_\alpha(x) - \phi_\alpha(x) \tag{4.1}$$

is a D.C. function. In formulation (4.1), the objective constitutes a difference-of-convex expression, resulting in an unconstrained D.C. minimization framework. The method incorporates local and global optimality requirements consistent with prior theoretical developments in the field [31, 33–35, 37, 38, 40]. These considerations lead to the construction of a dedicated algorithm for solving (4.1). The corresponding Global Search component follows a two-stage structure [35]: The Global Search framework is organized into two major components [35]:

- A Local Search stage.
- A mechanism designed to escape stationary points based on the global optimality criterion (GOC).

<u>**Local search method.**</u> Rewrite the problem (4.1) as the following D.C. minimization problem:

$$(\mathcal{P}) \qquad V^Q(x) = l_\alpha(x) - \phi_\alpha(x) \to \min, \;\; x \in \mathbb{R}^n \tag{4.2}$$

where $V^Q\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, and functions l_α, ϕ_α are convex by Lemma 3.
Similarly, in [33, 34], reduce problem (4.2) into the partially linearized problem

$$(\mathcal{PL}_s) \qquad V_s^Q(x) = l_\alpha(x) - \langle \nabla\phi_\alpha(x^s), x\rangle \to \min_x, \qquad x \in \mathbb{R}^n \tag{4.3}$$

Here, $\nabla\phi_\alpha(x^s)$ denotes a subgradient of $\phi_\alpha(\cdot)$ evaluated at the point x^s, that is, $\nabla\phi_\alpha(x^s) \in \partial\phi_\alpha(x^s)$. The linearized formulation in (4.3) was examined in the context of D.C. optimization, where a corresponding local search framework was first introduced in [33]. The following observations are relevant for the subsequent analysis:

1. The feasible set of problem (3.6) is nonempty.
2. According to Lemma 2, the optimal value of Problem $(\mathcal{P})$ equals zero, i.e.,

$$\Upsilon(\mathcal{P}) := \inf_{x\in\mathbb{R}^n} V^Q(x) = 0.$$

Since the reformulated subproblem $(\mathcal{PL}_s)$ consists of a convex objective over the full space $\mathbb{R}^n$, it can be solved using standard convex optimization techniques (see, e.g., [42]) with arbitrarily prescribed accuracy. A new iterate x^{s+1} is then generated as an approximate solution of $(\mathcal{PL}_s)$, such that the following condition is satisfied:

$$V_s^Q(x^{s+1}) = l_\alpha(x^{s+1}) - \langle \nabla\phi_\alpha(x^s), x^{s+1}\rangle \leq \Upsilon(\mathcal{PL}_s) + \delta_s, \quad x \in \mathbb{R}^n \tag{4.4}$$

where $\Upsilon(\mathcal{PL}_s)$ denotes the optimal objective value of linearized problem $(\mathcal{PL}_s)$, and the sequence δ_s is assumed to satisfy

$$\sum_{s=0}^{\infty} \delta_s < +\infty. \tag{4.5}$$

It follows directly that the iterate x^{s+1} remains feasible not only for the linearized subproblem $(\mathcal{PL}_s)$, but also for the original formulation $(\mathcal{P})$. This observation motivates the construction of an iterative process in which a sequence $\{x^s\} \subset \mathbb{R}^n$ is generated, beginning from an initial point x^0 and updated by repeatedly solving $(\mathcal{PL}_s)$.

The key structural properties of this sequence are consistent with the results investigated in [33]. In particular, $V_s^Q := V_s^Q(x^s)$ and the corresponding improvement measure $\Delta V_s^Q := V_s^Q(x^s) - V_s^Q(x^{s+1})$ form convergent sequences.

Lemma 3. *The sequence $\{x^s\}$ generated according to update rule (4.4) possesses the following convergence properties:*

$$\begin{aligned} &(i) \quad \lim_{s\to\infty} V_s^Q = V^Q \leq \Upsilon(\mathcal{P}), \\ &(ii) \quad \lim_{s\to\infty} \Delta V_s^Q = 0, \\ &(iii) \quad \lim_{s\to\infty} \left(\Upsilon(\mathcal{PL}_s) - \Delta V_s^Q(x^{s+1})\right) = 0. \end{aligned} \tag{4.6}$$

Proof. The result follows from Theorem 1 in [33] after identifying the functions f, g, h, and Φ_s used therein with $V^Q, l_\alpha, \phi_\alpha$ and V_s^Q, respectively. □

The next auxiliary result will be used in the analysis.

Lemma 4. *[33] Assume that the sequence $\{x^s\}$ generated by update rule (4.4), together with the corresponding subgradients*

$$y^s := \nabla\phi_\alpha(x^s) \in \partial\phi_\alpha(x^s), \qquad s = 1, 2, \ldots,$$

converges. Then the convergence holds in the following sense:

$$(\mathcal{H}) \quad \begin{cases} \lim_{s\to\infty} x^s = x, \\ \lim_{s\to\infty} y^s = y \in \partial\phi_\alpha(x). \end{cases} \tag{4.7}$$

Then the number sequence $\Upsilon_s = \Upsilon(\mathcal{PL}_s)$ converges so that

$$\lim_{s\to\infty} \Upsilon_s = \Upsilon_* \tag{4.8}$$

as well as the sequence $\left\{V_s^Q(x^{s+1})\right\}$:

$$\lim_{s\to\infty} V_s^Q(x^{s+1}) = V_*^Q \tag{4.9}$$

From (4.6.(ii)) it follows that $\Upsilon_* = V_*^Q$. ($\mathcal{P}$).

Proposition 3. *[33] In addition to assumption ($\mathcal{H}$), suppose that the following supplementary condition is satisfied:*

$$(\mathcal{H}1) \quad \lim_{s\to\infty} y^s = y \in \phi_\alpha(x_*). \tag{4.10}$$

where $y^s := \nabla\phi_\alpha(x^s) \in \partial\phi_\alpha(x^s)$ *for all* $s = 1, 2, \ldots$. *Then the accumulation point* x_* *of the sequence* $\{x^s\}$ *is a solution of the optimization problem*

$$(\mathcal{PL}_*) \qquad V_*^Q(x) = l_\alpha(x) - \langle y, x\rangle \to \min_{x\in\mathbb{R}^n}. \tag{4.11}$$

Proof. From relations (4.2)–(4.3), (4.8), and (4.9), we obtain

$$V^Q(x_*) = l_\alpha(x_*) - \langle y, x_*\rangle = \Upsilon_*, \tag{4.12}$$

which follows from the continuity of both $l_\alpha(\cdot)$ and the inner product operator. Furthermore, using the inequalities

$$\Upsilon_* \geq V_s^Q(x) = l_\alpha(x_*) - \langle y, x_*\rangle, \qquad \forall\, x \in \mathbb{R}^n,$$

and passing to the limit as $s \to \infty$, we arrive at

$$\Upsilon_* \geq V_*^Q(x) = l_\alpha(x_*) - \langle y, x\rangle, \qquad \forall\, x \in \mathbb{R}^n,$$

which completes the proof, since the continuity of $\phi_\alpha(\cdot)$ and the inner product has been used in the limiting process. □

Remark 2. *Following the framework proposed in [33], the stopping rule for the iterative procedure is established using one of the following conditions:*

$$V^Q(x^s) - V^Q(x^{s+1}) \leq \frac{\tau}{2}, \qquad \delta_s \leq \frac{\tau}{2}, \tag{4.13}$$

or equivalently,

$$V_s^Q(x^s) - V_s^Q(x^{s+1}) = l_\alpha(x^s) - l_\alpha(x^{s+1}) - \langle\nabla\phi_\alpha(x^s),\ x^s - x^{s+1}\rangle \leq \frac{\tau}{2},\ \ \delta_s \leq \frac{\tau}{2}.$$

These formulations are sufficient to terminate the procedure defined by (4.8).

A further consequence, analogous to Proposition 1 in [33], is the convergence of successive iterates with respect to the decision variable:

$$\lim_{s\to\infty}\|x^s - x^{s+1}\| = 0. \tag{4.14}$$

This property holds under the assumption that $\phi_\alpha(\cdot)$ is strongly convex, i.e.,

$$\phi_\alpha(x) \geq \phi_\alpha(y) + \langle \nabla\phi_\alpha(y), x - y\rangle + \frac{\mu_0}{2}\|x - y\|^2, \qquad \forall\, x, y \in \mathbb{R}^n, \tag{4.15}$$

consistent with the argument used in Proposition 1 of [33].

Algorithm 1 Local Search Algorithm

Step 0. Choose $x^0 \in \mathbb{R}^n$, a tolerance $\tau > 0$, and a sequence $\{\delta_s\}_{s\geq 0}$ with $\delta_s \geq 0$ and $\sum_{s=0}^{+\infty} \delta_s < +\infty$. Set $s := 0$.
Step 1. Compute a subgradient

$$y^s := \nabla\phi_\alpha(x^s) \in \partial\phi_\alpha(x^s),$$

and consider

$$(\mathcal{PL}_s) \quad \min_{x\in\mathbb{R}^n} V_s^Q(x) := l_\alpha(x) - \langle y^s, x\rangle.$$

Step 2. Compute x^{s+1} such that

$$V_s^Q(x^{s+1}) \leq \Upsilon(\mathcal{PL}_s) + \delta_s.$$

Step 3. If either of the following equivalent conditions holds:

$$V^Q(x^s) - V^Q(x^{s+1}) \leq \frac{\tau}{2} \quad \text{and} \quad \delta_s \leq \frac{\tau}{2},$$

or

$$V_s^Q(x^s) - V_s^Q(x^{s+1}) = l_\alpha(x^s) - l_\alpha(x^{s+1}) - \langle y^s, x^s - x^{s+1}\rangle \leq \frac{\tau}{2}, \quad \delta_s \leq \frac{\tau}{2},$$

then stop and output x^{s+1} as the current stationary point of $(\mathcal{P})$.
Step 4. Set $s := s + 1$ and go to Step 1.

Global search procedure. [34] For problem $(\mathcal{P})$, the global optimality condition (GOC) can be stated as follows: If z is a global solution of $(\mathcal{P})$, then for every pair $(y, \beta) \in \mathbb{R}^n \times \mathbb{R}$, the following system holds:

$$\begin{cases} \phi_\alpha(y) = \beta - \xi, & \xi := l_\alpha(z) - \phi_\alpha(z) \equiv l(z), \\ l_\alpha(x) - \beta \geq \langle \nabla\phi_\alpha(y),\ x - y\rangle, & \forall\, x \in \mathbb{R}^n. \end{cases} \tag{4.16}$$

If there exists a pair $(\hat{y}, \hat{\beta})$ satisfying (4.16) and a point $\hat{x} \in \mathbb{R}^n$ such that

$$l_\alpha(\hat{x}) < \hat{\beta} + \langle \nabla\phi_\alpha(\hat{y}),\ \hat{x} - \hat{y}\rangle,$$

then, using the convexity of $\phi_\alpha(\cdot)$, we obtain

$$l(\hat{x}) = l_\alpha(\hat{x}) - \phi_\alpha(\hat{x}) < \phi_\alpha(\hat{y}) + \xi - \phi_\alpha(\hat{y}) = l(z),$$

which implies that $\hat{x}$ provides an improvement over z.

By varying the pair (y,β) in (4.16) and solving the corresponding linearized subproblems of the form

$$\min_{x\in\mathbb{R}^n} \left(l_\alpha(x) - \langle \nabla\phi_\alpha(y),\ x\rangle\right), \tag{4.17}$$

(where y is not necessarily feasible), a set of candidate points $x(y,\beta)$ for the local search stage is obtained. In practice, it is not required to examine all possible pairs (y,β); identifying a single violated inequality in (4.16) is sufficient to generate a new improving starting point.

Extensive computational studies reported in [34] provide evidence of the effectiveness of this global search strategy.

Algorithm 2 Global Search Algorithm

Step 0. Choose an initial point $x^{0,0} \in \mathbb{R}^n$ and a tolerance $\varepsilon > 0$. Set the outer iteration counter $k := 0$.
Step 1. Starting from $x^{k,0}$, apply the Local Search Algorithm 1 to problem $(\mathcal{P})$ and obtain a stationary point z^k. Set

$$\xi^k := l_\alpha(z^k) - \phi_\alpha(z^k) = V^Q(z^k).$$

Step 2. Generate a finite set of test points $Y^k \subset \mathbb{R}^n$ (not necessarily feasible), for example perturbations of z^k.
Step 3.
for each $y \in Y^k$ **do**
Compute

$$\beta(y) := \phi_\alpha(y) + \xi^k,$$

so that the first relation in (4.16) is satisfied: $\phi_\alpha(y) = \beta(y) - \xi^k$.
Solve the linearized subproblem

$$(\mathcal{PL}(y)) \quad \min_{x\in\mathbb{R}^n} \left(l_\alpha(x) - \langle \nabla\phi_\alpha(y), x\rangle\right),$$

and compute an approximate minimizer $\hat{x}(y)$ and the value

$$\Lambda(y) := l_\alpha(\hat{x}(y)) - \langle \nabla\phi_\alpha(y), \hat{x}(y)\rangle.$$

Define

$$\Theta(y) := \beta(y) - \langle \nabla\phi_\alpha(y), y\rangle.$$

if $\Lambda(y) < \Theta(y) - \varepsilon$ **then**
(violation of GOC found)
Set $x^{k+1,0} := \hat{x}(y)$, $k := k + 1$, and go to Step 1.
end if
end for
Step 4. If no $y \in Y^k$ satisfies $\Lambda(y) < \Theta(y) - \varepsilon$, accept z^k as an (approximate) global solution of $(\mathcal{P})$ and stop.

5. Numerical Results

Example 1. *Consider a two-player nonconvex game with quadratic payoffs and scalar strategies [20]:*

$$\begin{cases} f_1(x_1, x_2) = x_1^2 + x_1 x_2 \to \max\limits_{x_1}, \\ x_1 \in X_1 = [-1, 1], \end{cases} \qquad \begin{cases} f_2(x_1, x_2) = -x_2^2 + \dfrac{1}{2} x_1 x_2 \to \max\limits_{x_2}, \\ x_2 \in X_2 = [-1, 1]. \end{cases} \tag{5.1}$$

We reformulate problem (5.1) as the constrained minimization problem

$$\theta_1(x_1, x_2) = -x_1^2 - x_1 x_2 \to \min_{x_1 \in X_1}, \qquad \theta_2(x_1, x_2) = x_2^2 - \frac{1}{2} x_1 x_2 \to \min_{x_2 \in X_2}, \tag{5.2}$$

and denote $X := X_1 \times X_2$.

The Nikaido-Isoda regularization function for problem (5.2) is given by

$$l_\alpha^Q(x) = \sum_{k=1}^{2} [\theta_k(x_k, x_{-k}) - \theta_k(z_k, x_{-k})] - \frac{\alpha}{2} \|x - z\|^2.$$

Now we construct the D.C. formulation of problem (5.2). We obtain

$$V_\alpha^Q(x) = -x_1^2 - \frac{3}{2} x_1 x_2 + x_2^2 - \min_{z \in X} \eta_\alpha(z_1, z_2), \tag{5.3}$$

where

$$\eta_\alpha(z_1, z_2) = -z_1^2 - z_1 x_2 + z_2^2 - \frac{1}{2} x_1 z_2 + \frac{\alpha}{2}(x_1 - z_1)^2 + \frac{\alpha}{2}(x_2 - z_2)^2. \tag{5.4}$$

We solve the following problem for a fixed point $(x_1, x_2) \in X$:

$$\min_{z \in X} \eta_\alpha(z_1, z_2) \tag{5.5}$$

with

$$X = \{(z_1, z_2) \in \mathbb{R}^2 \mid -z_1 - 1 \le 0,\ z_1 - 1 \le 0,\ -z_2 - 1 \le 0,\ z_2 - 1 \le 0\}.$$

The Lagrangian function associated with this problem is

$$\begin{aligned} L(z_1, z_2, \lambda) = & -z_1^2 - z_1 x_2 + z_2^2 - \frac{1}{2} x_1 z_2 + \frac{\alpha}{2}(x_1 - z_1)^2 + \frac{\alpha}{2}(x_2 - z_2)^2 \\ & + \lambda_1(z_1 - 1) + \lambda_2(z_2 - 1) + \lambda_3(-z_1 - 1) + \lambda_4(-z_2 - 1) \to \min. \end{aligned}$$

The KKT conditions are therefore

$$\begin{cases} \dfrac{\partial L}{\partial z_1} = -2z_1 - x_2 + \alpha(z_1 - x_1) + \lambda_1 - \lambda_3 = 0, \\ \dfrac{\partial L}{\partial z_2} = 2z_2 - \dfrac{1}{2} x_1 + \alpha(z_2 - x_2) + \lambda_2 - \lambda_4 = 0, \\ \lambda_1(z_1 - 1) = 0, \quad \lambda_2(z_2 - 1) = 0, \\ \lambda_3(-z_1 - 1) = 0, \quad \lambda_4(-z_2 - 1) = 0, \\ \lambda_i \ge 0, \quad i = 1, 2, 3, 4. \end{cases} \tag{5.6}$$

The solution of system (5.6) is $z^* = (1, 0)$.

The Hessian matrix of problem (5.5) is

$$H(z^*) = \begin{pmatrix} \alpha + 2 & 0 \\ 0 & \alpha - 2 \end{pmatrix}.$$

In Lemma 2, we assume that parameter α is positive. Then $H(z^*) > 0$, and, hence, $z^* = (1, 0)$ is a (local) solution of problem (5.5). Therefore,

$$\eta_\alpha(z^*) = \frac{\alpha}{2}(x_1^2 + x_2^2) - \alpha x_1 + x_2 - \frac{\alpha}{2} - 1,$$

and

$$V_\alpha^Q(x) = -\left(\frac{\alpha}{2} + 1\right)x_1^2 - \left(\frac{\alpha}{2} - 1\right)x_2^2 - \frac{3}{2}x_1x_2 + \alpha x_1 - x_2 - \frac{\alpha}{2} + 1.$$

Using indicator functions, we decompose the above function into two convex functions as follows:

$$l_\alpha(x) = \alpha x_1 - x_2 - \frac{\alpha}{2} + 1,$$

$$\phi_\alpha(x) = \left(\frac{\alpha}{2} + 1\right)x_1^2 + \left(\frac{\alpha}{2} - 1\right)x_2^2 + \frac{3}{2}x_1x_2 + \delta_X(x).$$

Then (5.3) is equivalent to the following D.C. minimization problem:

$$V^Q(x) = (l_\alpha(x) - \phi_\alpha(x)) \to \min_{x \in \mathbb{R}^2}, \tag{5.7}$$

where $l_\alpha(\cdot)$ and $\phi_\alpha(\cdot)$ are convex functions for positive and sufficiently large α.

The proposed algorithm described in the previous section is applied to problem (5.6). Fixing the regularization parameter at $\alpha = 2.5$, the corresponding D.C. decomposition takes the form

$$l(x) = \frac{5}{2}x_1 - x_2 - \frac{5}{4} + 1, \qquad \phi(x) = \frac{7}{4}x_1^2 + \frac{1}{4}x_2^2 + \frac{3}{2}x_1x_2 + \delta_X(x).$$

The following parameter values are used in the local and global search phases of the algorithm:

$$\sigma = 50, \qquad \rho_s = 10^{-7}, \qquad \tau_k = 10^{-5}, \qquad \delta_k = 10^{-5}.$$

Using these settings, the algorithm converges to the solution

$$x^* = (1,\ \tfrac{1}{4})^\top, \qquad V_*^Q(x^*) = 0.$$

It can be readily verified that the corresponding equilibrium objective values for problem (5.1) are

$$\theta_1(x^*) = 1.250, \qquad \theta_2(x^*) = 0.625.$$

(x^0, y^0)	$V^Q(x^0, y^0)$	$V(x^0, y^0)$	Loc. S.A.	Glob. S.A.	Time	$V_*^Q(x^*, y^*)$
(0, 1)	−1.50	0.075	48	3	6.21	0.00
(1, 0)	−4.25	−0.241	73	3	6.41	0.00
(1, 1)	−3.25	−2.759	97	2	7.34	0.00

Example 2. *Consider a two-player nonconvex game with matrix form:*

$$A = \begin{pmatrix} 3 & 8 & 14 & 2 & 7 & 19 \\ 5 & 4 & 16 & 4 & 11 & 3 \\ 20 & 0 & 15 & 13 & 10 & 6 \end{pmatrix}, \qquad B = \begin{pmatrix} 14 & 1 & 7 & 20 & 12 & 5 \\ 8 & 26 & 2 & 13 & 11 & 9 \\ 3 & 0 & 6 & 10 & 4 & 11 \end{pmatrix}.$$

Given the payoff functions of the two players corresponding to these matrices, we have

$$\begin{aligned} f_1(x,y) = {} & 3x_1y_1 + 8x_1y_2 + 14x_1y_3 + 2x_1y_4 + 7x_1y_5 + 19x_1y_6 \\ & + 5x_2y_1 + 4x_2y_2 + 16x_2y_3 + 4x_2y_4 + 11x_2y_5 + 3x_2y_6 \\ & + 20x_3y_1 + 15x_3y_3 + 13x_3y_4 + 10x_3y_5 + 6x_3y_6 \ \to \max_{(x,y)\in X\times Y}, \end{aligned}$$

and

$$\begin{aligned} f_2(x,y) = {} & 14x_1y_1 + x_1y_2 + 7x_1y_3 + 20x_1y_4 + 12x_1y_5 + 5x_1y_6 \\ & + 8x_2y_1 + 26x_2y_2 + 2x_2y_3 + 13x_2y_4 + 11x_2y_5 + 9x_2y_6 \\ & + 3x_3y_1 + 6x_3y_3 + 10x_3y_4 + 4x_3y_5 + 11x_3y_6 \ \to \max_{(x,y)\in X\times Y} \end{aligned}$$

with constraints

$$X \times Y = \{(x,y) \in \mathbb{R}^9 \mid \sum_{i=1}^{3} x_i = 1,\ \sum_{j=1}^{6} y_j = 1,\ x_i \geq 0,\ y_j \geq 0,\ i = 1,\dots,3,\ j = 1,\dots,6\}.$$

We now derive an explicit D.C. decomposition for the resulting quadratic objective. As such, we introduce the stacked variable

$$z := (x_1, x_2, x_3, y_1, \dots, y_6)^\top \in Z, \quad Z = \{z \in \mathbb{R}^9 \mid \sum_{i=1}^{3} z_i = 1,\ \sum_{i=4}^{9} z_j = 1,\ z_i \geq 0,\ i = 1,\dots,9\}$$

so that the bilinear payoff $f_1(x,y)$ can be written in quadratic form as

$$f_1(z) = x^\top A y = \frac{1}{2} z^\top Q_1 z,$$

where

$$Q_1 := \begin{pmatrix} O_{3\times3} & A \\ A^\top & O_{6\times6} \end{pmatrix} \in \mathbb{R}^{9\times9}.$$

In order to apply the D.C. programming framework, we consider the minimization of $-f_1$, that is,

$$F(z) := -f_1(z) = -\frac{1}{2} z^\top Q_1 z = z^\top \widetilde{Q} z, \qquad \widetilde{Q} := -\frac{1}{2} Q_1.$$

Choosing a constant $\lambda > 0$ sufficiently large so that $\lambda I_9 \pm \widetilde{Q}$ are positive semidefinite, we obtain the following D.C. decomposition:

$$l(z) := \frac{1}{2} z^\top (\lambda I_9 + \widetilde{Q}) z, \qquad \phi(z) := \frac{1}{2} z^\top (\lambda I_9 - \widetilde{Q}) z.$$

By construction, $l(\cdot)$ and $\phi(\cdot)$ are convex on $\mathbb{R}^9$ and

$$V^Q(z) = l(z) - \phi(z),$$

which provides an explicit D.C. representation of the objective function in terms of variable z. A fully analogous construction can be performed for the second payoff $f_2(x, y)$ (or for a combined objective based on both players' payoffs) by replacing Q_1 with the corresponding quadratic matrix.
Then, for $\alpha = 2.5$, the corresponding functions $l(\cdot)$ and $\phi(\cdot)$ in (3.9) and (3.10) are given by:

$$\begin{aligned} l(x) = {} & 204x_1^2 + 1 + \frac{489}{4}x_2^2 + \frac{479}{4}x_3^2 + \frac{435}{4}x_4^2 + \frac{51}{2}x_5^2 + \frac{339}{2}x_6^2 + \frac{95}{2}x_7^2 + \frac{271}{4}x_8^2 \\ & + \frac{435}{4}x_9^2 + \frac{571}{2}x_1x_2 + \frac{857}{4}x_1x_3 + \frac{17}{2}x_1x_4 + \frac{9}{2}x_1x_5 + \frac{21}{2}x_1x_6 + 11x_1x_7 + \frac{19}{2}x_1x_8 \\ & + 12x_1x_9 + 221x_2x_3 + 9x_2x_4 + \frac{3}{2}x_2x_5 + 9x_2x_6 + \frac{17}{2}x_2x_7 + 11x_2x_8 + 6x_2x_9 \\ & + \frac{23}{2}x_3x_4 + \frac{13}{2}x_3x_5 + \frac{21}{2}x_3x_6 + \frac{23}{2}x_3x_7 + 7x_3x_8 + \frac{19}{2}x_3x_9 + 77x_4x_5 + 216x_4x_6 \\ & + 143x_4x_7 + 138x_4x_8 + \frac{223}{2}x_4x_9 + 109x_5x_6 + 49x_5x_7 + \frac{127}{2}x_5x_8 + \frac{203}{2}x_5x_9 \\ & + \frac{287}{2}x_6x_7 + 212x_6x_8 + 214x_6x_9 + 94x_7x_8 + 72x_7x_9 + 128x_8x_9, \end{aligned}$$

$$\begin{aligned} \phi(x) = {} & 204x_1^2 + \frac{489}{4}x_2^2 + 161x_3^2 + \frac{435}{4}x_4^2 + \frac{51}{2}x_5^2 + \frac{339}{2}x_6^2 + \frac{95}{2}x_7^2 + \frac{271}{4}x_8^2 \\ & + \frac{435}{4}x_9^2 + 285x_1x_2 + 203x_1x_3 - \frac{17}{2}x_1x_4 - \frac{9}{2}x_1x_5 - 8x_1x_6 - 11x_1x_7 - \frac{19}{2}x_1x_8 \\ & - 12x_1x_9 + \frac{325}{2}x_2x_3 - \frac{13}{2}x_2x_4 - 4x_2x_5 - 9x_2x_6 - \frac{13}{2}x_2x_7 - 11x_2x_8 - 6x_2x_9 \\ & - \frac{23}{2}x_3x_4 - \frac{25}{2}x_3x_5 + \frac{23}{2}x_3x_6 - \frac{23}{2}x_3x_7 - 7x_3x_8 - \frac{19}{2}x_3x_9 + \frac{149}{2}x_4x_5 + 211x_4x_6 \\ & + 143x_4x_7 + 138x_4x_8 + 116x_4x_9 + 109x_5x_6 + 49x_5x_7 + \frac{127}{2}x_5x_8 + \frac{203}{2}x_5x_9 \\ & + \frac{287}{2}x_6x_7 + 212x_6x_8 + 217x_6x_9 + 94x_7x_8 + 77x_7x_9 + 123x_8x_9. \end{aligned}$$

A mixed-strategy equilibrium for this problem is

$$x^* = (0.714286,\ 0.285714,\ 0)^\top, \qquad y^* = (0,\ 0,\ 0.4444444,\ 0,\ 0,\ 0.555556)^\top.$$

The corresponding equilibrium payoffs are

$$f_1(x^*, y^*) = 14.555556, \qquad f_2(x^*, y^*) = 14.888889, \qquad V_*^Q = 2.31 \times 10^{-8}.$$

Example 3. *Consider another two-player nonconvex game in matrix form:*

$$A = \begin{pmatrix} 3 & 5 & 20 \\ 8 & 1 & 6 \\ 14 & 16 & 15 \\ 2 & 4 & 13 \\ 7 & 11 & 10 \\ 19 & 3 & 8 \end{pmatrix}, \qquad B = \begin{pmatrix} 14 & 8 & 3 \\ 1 & 7 & 19 \\ 7 & 2 & 6 \\ 20 & 13 & 10 \\ 12 & 11 & 4 \\ 5 & 9 & 11 \end{pmatrix}.$$

For this example, we again fix the regularization parameter at $\alpha = 2.5$, as in the previous case. Applying the proposed D.C. global search algorithm with the same parameter settings yields the mixed-strategy equilibrium

$$x^* = (0.444447,\ 0,\ 0.555553,\ 0,\ 0,\ 0)^\top, \qquad y^* = (0,\ 0.312501,\ 0.687499)^\top.$$

The corresponding equilibrium payoffs are

$$f_1(x^*, y^*) = 4.666659, \qquad f_2(x^*, y^*) = 15.312499, \qquad V_*^Q = 2.87 \cdot 10^{-8}.$$

6. Conclusion

We examine the GNEP with nonconvex quadratic objective functions and jointly convex constraints. Based on the regularized Nikaido-Isoda function and gap function, we reduce the problem to an unconstrained optimization problem with a D.C. objective function. Then, we apply Local and Global search algorithms developed in [21]. For the first time, an attempt has been made to solve a nonconvex GNEP problem with jointly convex constraints by the D.C. optimization approach.
We can apply the theory and algorithms of the variational inequality problems and penalty methods to problem (3.6), which will be done in the next papers.

Author contributions

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Use of Generative-AI tools declaration

The authors declare that generative AI tools (ChatGPT) were used for translating some sentences during the preparation of this manuscript. All translated content was carefully reviewed and edited by the authors, who take full responsibility for the accuracy, originality, and integrity of the work.

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Conflict of interest

The authors declare that they have no conflict of interest.

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