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Research article

Some results on directionally differentiable multicriteria interval-valued optimization problems with vanishing constraints

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Abstract: In this article, we examine the theoretical properties of interval-valued multiobjective optimization problems that are directionally differentiable and include equality, inequality, and vanishing constraints. The objective functions in these problems are considered interval-valued. Necessary conditions for optimality in nondifferentiable multiobjective optimization are derived under the Abadie and a modified Abadie constraint qualification. Furthermore, sufficient optimality conditions are proven under appropriate convexity assumptions. A Numerical example is also presented to validate the theoretical results of this work.

Keywords: pareto solution; directionally differentiable multiobjective optimization problems; interval valued functions
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1. Introduction

Mathematical programs with vanishing constraints (MPVCs) have garnered significant attention in recent years due to their relevance across diverse fields and the unique mathematical challenges they pose. These optimization problems, characterized by constraints that may vanish under certain conditions, bear a close resemblance to mathematical programs with equilibrium constraints (MPECs) [1, 2]. The problem that is being referred to as mathematical programs with vanishing constraints was initially researched by Achtziger and Kanzow in [3] and this model can be used as a reference for various problems related to topology and structural optimization ([3–5]). For further information on mathematical programs involving vanishing constraints, we refer to (see [6–9]) and references therein.

Multiobjective optimization problems involve the optimization of multiple objective functions and are also known as multicriteria optimization problems. Multiobjective programming problems emerge across various practical contexts, spanning fields such as economic systems, strategic decision analysis, financial planning and investment portfolios, data communication networks, distribution of assets, mechanical systems, and control engineering applications [10–12]. A classical example in financial planning is mean-variance portfolio optimization, which captures the fundamental trade-off between maximizing returns and minimizing risk. This approach generates a Pareto-optimal frontier, allowing investors to choose portfolios based on their risk-return preferences [13]. Similarly, in the asset distribution networks, such as urban water distribution systems, often require designs that minimize installation and operational costs while maximizing reliability and resilience to failure. Multiobjective evolutionary algorithms have been successfully applied to benchmark networks, including those in Hanoi and New York [14].

In mechanical and structural engineering, Pareto fronts emerge when optimizing weight against stiffness or deflection in steel truss design. Multiobjective genetic algorithms assist engineers in navigating these nonlinear trade-offs [15]. In control and mechatronic systems, multiobjective tuning of PID controllers (or nonlinear MIMO controllers) seeks optimal trade-offs between integrated error and actuator effort or control rate. This enables the selection of "best compromise" parameter sets from the Pareto front [16]. These concepts also extend into data science and machine learning applications, where fairness, accuracy, and energy consumption constitute competing objectives in hyperparameter tuning. This leads to diverse yet Pareto-ideal configurations, illustrating how multiobjective programming provides a coherent framework across economics, engineering, control, and data-driven decision-making [17]. Over the last 50 years, the field of multiobjective optimization has seen significant growth in various areas, particularly in the development of duality theory and optimality conditions. In recent years, significant attention has been directed toward a specific class of nondifferentiable multiobjective optimization problems, namely those that are directionally differentiable. These problems have emerged as a focal point for researchers aiming to address challenges arising from nonsmooth structures in optimization. Huang and Zhu [18] focused on establishing stationary conditions for Borwein proper efficient solutions to nonsmooth multiobjective problems with vanishing constraints. Their work is significant because it extends the concept of Borwein's proper efficiency, a refinement of efficiency that avoids pathological solutions to the nonsmooth context. By utilizing advanced variational analysis tools, they provided necessary conditions that ensure proper efficiency even in the presence of nonsmoothness and vanishing constraints. This contribution enhances the robustness of solution concepts in multiobjective optimization. Wang, Kang, and Zhang [19] explored the optimality conditions and duality in multiobjective fractional optimization problems that involve vanishing constraints. Their work specifically addresses the challenges posed by the fractional structure, commonly found in problems related to ratios, such as efficiency measures or return-to-risk ratios. They established both necessary and sufficient conditions for optimality, along with duality results. This contribution expands the theoretical framework for fractional multiobjective optimization while accounting for vanishing constraints. Numerous authors have established foundational results in optimization theory for this class of problems (see, [20–26]).

The methodology for solving optimization problems under interval-valued constraints has been advanced by several researchers (see, e.g., [27–30]. In a parallel development, Wu established the foundational optimality conditions for problems featuring interval-valued objective functions [31–33]. The study of optimality conditions in interval-valued optimization was advanced by Bhurjee and Panda [34], who presented a methodology for efficient solutions when the objective function is differentiable. Later, Singh et al. [35] explored this by formulating the necessary Karush-Kuhn-Tucker (KKT) conditions for the more complex case involving both interval-valued objective and constraint functions. Here are some recent and highly relevant research works on interval-valued multi-objective optimization, particularly in contexts involving nonsmooth objectives, vanishing constraints, and duality (see, [36–40]). Later, Antczak [41], studied a class of directionally differentiable multiobjective programming problems with equality, inequality, and vanishing constraints. Under the Abadie Constraint Qualification (ABCQ) and its modified form, they established necessary KKT optimality conditions. Their approach leveraged the nonlinear Gordan theorem of the alternative for convex functions and specifically considered problems with standard, non-interval-valued objective functions. To the best of our knowledge, no existing work has derived optimality conditions for convex multi-objective optimization problems with directionally differentiable components and vanishing constraints, particularly when objectives are expressed as interval-valued functions.

The structure of this work is outlined as follows: Section 2, introduces the necessary preliminaries and foundational concepts. In Section 3, we investigate a multiobjective interval-valued optimization problem with vanishing constraints (MPIVC), where the objectives are directionally differentiable. Our primary contribution is the development of a comprehensive optimality theory for (MPIVC). This begins with the derivation of KKT-type necessary conditions for (weak) Pareto solutions. To overcome the potential failure of classical constraint qualifications, we propose a novel VC-Abadie Constraint Qualification (VC-ABCQ), which is less restrictive. Using the VC-ABCQ, we then establish valid KKT necessary conditions. We also present sufficient conditions for optimality under convexity assumptions and validate the entire framework with an illustrative example. The paper concludes with a summary in Section 4.

2. Preliminaries and definitions

This section of the article contains definitions, results, and frequently used equalities and inequalities.

Throughout this article, we adopt standard notations for vectors, distinguishing between column and row representations only when necessary for clarity. Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space, and let $\mathbb{R}^n_+$ represent its non-negative orthant. Additionally, let $\mathbb{I}(\mathbb{R})$ denote the set of all closed and bounded intervals in $\mathbb{R}$. For simplicity, we use $\Lambda_1 \in \mathbb{I}(\mathbb{R})$ to represent a closed interval, written as $\Lambda_1 = [\varkappa_0^L, \varkappa_0^U]$, where $\varkappa_0^L$ and $\varkappa_0^U$ are the lower and upper bounds of the interval, respectively.

For two intervals $\Lambda_1 = [\varkappa_0^L, \varkappa_0^U]$ and $\Lambda_2 = [\beta_0^L, \beta_0^U]$, the order relation $\leq_{LU}$ is as follows:

- $\Lambda_1 \leq_{LU} \Lambda_2$ iff $\varkappa_0^L \leq \beta_0^L$ and $\varkappa_0^U \leq \beta_0^U$.
- $\Lambda_1 <_{LU} \Lambda_2$ iff $\Lambda_1 \leq_{LU} \Lambda_2$ and $\Lambda_1 \neq \Lambda_2$.

It is straightforward to verify that $\Lambda_1 <_{LU} \Lambda_2$ holds if one of the following conditions is satisfied:

$$\text{(i)} \quad \varkappa_0^L \leq \beta_0^L \text{ and } \varkappa_0^U < \beta_0^U,$$
$$\text{(ii)} \quad \varkappa_0^L < \beta_0^L \text{ and } \varkappa_0^U \leq \beta_0^U.$$

Definition 1. *Relative interior of a set $C \subseteq \mathbb{R}^n$, denoted by* relint C *and defined as:*

$$\text{relint } C = \{a \in C : (a, r) \cap \text{aff } C \subseteq C \text{ for some } r > 0\},$$

where:

- (a, r) *represents the closed ball of radius $r > 0$ centered at a in $\mathbb{R}^n$, given by:*

$$(a, r) = \{y \in \mathbb{R}^n : \|y - a\| \leq r\}.$$

- aff C *denotes the affine hull of C, which is the smallest affine subspace containing C. Formally, it is defined as:*

$$\text{aff } C = \left\{ \sum_{i=1}^{k} \varkappa_i a_i : \varkappa_i \in \mathbb{R}, \sum_{i=1}^{k} \varkappa_i = 1, \ a_1, \ldots, b_k \in C \right\}.$$

Definition 2. *[41] Let $F : C \to \mathbb{R}$ be a mapping defined on $C \subseteq \mathbb{R}^n$. The mapping F is said to possess a directional derivative at a point $b \in C$ in the direction $c \in \mathbb{R}^n$ if the following limit exists and is finite:*

$$F^{+}(b; c) = \lim_{\varkappa \to 0^{+}} \frac{F(b + \varkappa c) - F(b)}{\varkappa}.$$

Furthermore, F is said to be directionally differentiable at b if the directional derivative $F^{+}(b; c)$ exists finitely for all directions $c \in \mathbb{R}^n$.

Definition 3. *A mapping $F : C \to \mathbb{I}(\mathbb{R})$, defined on $C \subseteq \mathbb{R}^n$, is called directionally differentiable at a point $b \in C$ in the direction $c \in \mathbb{R}^n$ if the following two limits exist as finite values:*

$$F^{L+}(b; c) = \lim_{\varkappa \to 0^{+}} \frac{F^{L}(b + \varkappa c) - F^{L}(b)}{\varkappa},$$
$$F^{U+}(b; c) = \lim_{\varkappa \to 0^{+}} \frac{F^{U}(b + \varkappa c) - F^{U}(b)}{\varkappa}.$$

Here, F^L and F^U represent the lower and upper bounds of the interval-valued function F, respectively. The mapping F is said to be directionally differentiable at b if both $F^{L+}(b; c)$ and $F^{U+}(b; c)$ exist finitely for all directions $c \in \mathbb{R}^n$.

Remark 1. *For an interval-valued function $F : C \to \mathbb{I}(\mathbb{R})$, directional differentiability at $b \in C$ in a direction $v \in \mathbb{R}^n$ is defined by the existence of finite limits for its lower and upper bounds, denoted $F^{L+}(b; v)$ and $F^{U+}(b; v)$.*

Definition 4. *A mapping $F : C \to \mathbb{I}(\mathbb{R})$ is convex if F^L and F^U are convex on the nonempty convex set $C \subset \mathbb{R}^n$.*

Proposition 1. *[42] Let* $F : \mathbb{R}^n \to \mathbb{R}$ *be a convex mapping. Then, for any point* $u \in \mathbb{R}^n$ *and any direction* $c \in \mathbb{R}^n$*, the directional derivative* $F^+(u; c)$ *exists.*
Moreover, because the convexity of F *guarantees that the directional derivative along* $b - u$ *exists* $\forall b \in \mathbb{R}^n$*, the subsequent inequality holds:*

$$F(b) - F(u) \geq F^+(u; b - u). \tag{2.1}$$

Lemma 1. *[42] Let* $C \subseteq \mathbb{R}^n$ *be an open set, and let* $u \in C$ *be given. Consider the functions* $F, \varphi : C \to \mathbb{R}$ *and a direction* $c \in \mathbb{R}^n$ *and assuming that directional derivatives of* F *and* φ *at* u *in the direction* c *exist, i.e.,* $F^+(u; c)$ *and* $\varphi^+(u; c)$ *are well-defined. Then, the directional derivative of the product* $F \cdot \varphi$ *also exists, and it satisfies the following equation:*

$$(F \cdot \varphi)^+(u; c) = F(u)\varphi^+(u; c) + F^+(u; c)\varphi(u).$$

Theorem 1. *Let* $C \subseteq \mathbb{R}^n$ *be a nonempty convex set, and consider the interval-valued map* $F : C \to \mathbb{IR}^\tau$*, the convex function* $\varphi : C \to \mathbb{R}^j$*, and the linear function* $h : \mathbb{R}^n \to \mathbb{R}^q$*. Suppose there exists a point* $b_0 \in \operatorname{relint} C$ *such that*

$$\varphi_\eta(b_0) \leq 0, \quad \eta = 1, \ldots, j, \quad \textit{and} \quad h_\rho(b_0) \leq 0, \quad \rho = 1, \ldots, q.$$

Then,

$$\begin{aligned} &F_k^L(b) < 0, \quad k = 1, \ldots, \tau, \\ &F_k^U(b) < 0, \quad k = 1, \ldots, \tau, \\ &\varphi_\eta(b) \leq 0, \quad \eta = 1, \ldots, j, \\ &h_\rho(b) \leq 0, \quad \rho = 1, \ldots, q, \end{aligned}$$

has no solution if and only if $\exists\, (\theta, \vartheta, \beta)^\varsigma \in \mathbb{R}_+^\tau \times \mathbb{R}_+^j \times \mathbb{R}^q$ *with* $\theta \neq 0$ *such that:*

$$\begin{aligned} &\theta^T F^L(b) + \vartheta^T \varphi(b) + \beta^T h(b) \geq 0, \quad \forall b \in C, \\ &\theta^T F^U(b) + \vartheta^T \varphi(b) + \beta^T h(b) \geq 0, \quad \forall b \in C. \end{aligned} \tag{2.2}$$

Proof. The result is a direct consequence of a nonlinear alternative theorem (Theorem 2.8 in [44]). Specifically, applying that result with the convex constraints φ and linear constraints h yields the equivalence between the infeasibility of the system and the existence of multipliers $(\theta, \vartheta, \beta)$ satisfying (2.2). □

Definition 5. *[41] The cone of sequential linear directions to a set* $E \subset \mathbb{R}^n$ *at* $\bar{b} \in E$ *is denoted by* $Z(E; \bar{b})$ *and is defined as:*

$$Z(E; \bar{b}) := \left\{ c \in \mathbb{R}^n \mid \exists(\varkappa_k) \subset \mathbb{R}_+,\ \varkappa_k \downarrow 0 \textit{ such that } \bar{b} + \varkappa_k c \in E,\ \forall k \in \mathbb{N} \right\}.$$

The tangent cone to E *at* $\bar{b} \in \operatorname{cl} E$ *is defined as:*

$$\begin{aligned} T(E; \bar{b}) &:= \left\{ c \in \mathbb{R}^n \mid \exists(b_k) \subseteq E,\ (\varkappa_k) \subseteq \mathbb{R}_+ \textit{ such that } \varkappa_k \downarrow 0,\ b_k \to \bar{b}, \textit{ and } \frac{b_k - \bar{b}}{\varkappa_k} \to c \right\} \\ &= \left\{ c \in \mathbb{R}^n \mid \exists c_k \to c,\ \varkappa_k \downarrow 0 \textit{ such that } \bar{b} + \varkappa_k c_k \in E,\ \forall k \in \mathbb{N} \right\}. \end{aligned}$$

Here, $\operatorname{cl} E$ *represents the closure of the set* E*. It is worth noting that these cones are always nonempty. While cone* $T(E; \bar{b})$ *is closed, but not necessarily be convex. Additionally, the following inclusion is always satisfied:*

$$Z(E; \bar{b}) \subseteq T(E; \bar{b}).$$

3. Interval-valued multiobjective optimization problem with vanishing constraints

In this article, we consider following constrained multiobjective interval-valued optimization problem with vanishing constraints defined as

$$
\begin{aligned}
(MPIVC)\quad \text{V -minimize } & F(b) := (F_1(b), ..., F_\tau(b))\\
& \varphi_\eta(b) \leqq 0, \eta = 1, ..., j,\\
& h_\rho(b) = 0,\ \rho = 1, ..., q,\\
& \zeta_\varsigma(b) \geqq 0,\ \varsigma = 1, ..., r,\\
& \zeta_\varsigma(b) D_\varsigma(b) \leqq 0,\ \varsigma = 1, ..., r,\ b \in C,
\end{aligned}
$$

here:

- Each $F_k : \mathbb{R}^n \to \mathbb{I}(\mathbb{R})$, $k \in I = \{1, \dots, \tau\}$, is an interval-valued function, written as

$$F_k(b) = [F_k^L(b), F_k^U(b)];$$

- $\varphi_\eta : \mathbb{R}^n \to \mathbb{R}, \eta \in M = \{1, \dots, j\}$;
- $h_\rho : \mathbb{R}^n \to \mathbb{R}, \rho \in S = \{1, \dots, r\}$;
- $\zeta_\varsigma, D_\varsigma : \mathbb{R}^n \to \mathbb{R}, \varsigma \in V = \{1, \dots, r\}$;
- $C \neq \emptyset$ is a convex open subset of $\mathbb{R}^n$.

To make our presentation easier to understand, we will be using some consistent notations throughout this article. Let

$$\Omega = \{b \in C : \varphi_\eta(b) \leqq 0, \eta \in M, h_\rho(b) = 0, \rho \in S, \zeta_\varsigma(b) \geqq 0, \zeta_\varsigma(b) D_\varsigma(b) \leqq 0, \varsigma \in V\}$$

be the feasible set for (MPIVC). Further, we define the following sets at the point $\bar{b} \in \Omega$:

- The active set: $M(\bar{b}) := \{\eta \in M : \varphi_\eta(\bar{b}) = 0\}$;
- The inactive set: $M^<(\bar{b}) = \{\eta \in 1, 2, ..., j : \varphi_\eta(\bar{b}) < 0\}$.

These sets form a partition of M, such that $M(\bar{b}) \cup M^<(\bar{b}) = M$.

Definition 6. *We say that a feasible point $\bar{b} \in \Omega$ is a Pareto solution to (MPIVC) iff there does not exist any point $b \in \Omega$ such that*

$$F(b) \leq_{LU} F(\bar{b}).$$

Definition 7. *A feasible point $\bar{b} \in \Omega$ is called a* beak Pareto solution *(or* weakly efficient solution*, or* weak minimum*) for (MPIVC) if there exists no other feasible point $b \in \Omega$ satisfying*

$$F(b) <_{LU} F(\bar{b}).$$

Now, for any feasible solution $\bar{b}$, we give the following index sets

$$
\begin{aligned}
V_+(\bar{b}) &= \{\varsigma \in V : \zeta_\varsigma(\bar{b}) > 0\},\\
V_0(\bar{b}) &= \{\varsigma \in V : \zeta_\varsigma(\bar{b}) = 0\}.
\end{aligned}
$$

Moreover, let us divide the index set $V_+(\bar{b})$ into these two index subsets:

$$V_{+0}(\bar{b}) = \{\varsigma \in V : \zeta_\varsigma(\bar{b}) > 0, D_\varsigma(\bar{b}) = 0\},$$
$$V_{+-}(\bar{b}) = \{\varsigma \in V : \zeta_\varsigma(\bar{b}) > 0, D_\varsigma(\bar{b}) < 0\}.$$

Likewise, we can divide the index set $V_0(\bar{b})$ into three distinct subgroups in the following manner:

$$V_{0+}(\bar{b}) = \{\varsigma \in V : \zeta_\varsigma(\bar{b}) = 0, D_\varsigma(\bar{b}) > 0\},$$
$$V_{00}(\bar{b}) = \{\varsigma \in V : \zeta_\varsigma(\bar{b}) = 0, D_\varsigma(\bar{b}) = 0\},$$
$$V_{0-}(\bar{b}) = \{\varsigma \in V : \zeta_\varsigma(\bar{b}) = 0, D_\varsigma(\bar{b}) < 0\}.$$

Further, we represent using the notation $V_{HG}(\bar{b})$ the set of indexes $\varsigma \in V$ defined by

$$V_{HG}(\bar{b}) = V_{00}(\bar{b}) \cup V_{0+}(\bar{b}) \cup V_{0-}(\bar{b}) \cup V_{+0}(\bar{b}).$$

Next, we will introduce Abadie constraint qualification for directionally differentiable multi-objective interval-valued optimization problem with vanishing constraints.

We now introduce the constraint qualification discussed previously. For a given point $\bar{b} \in \Omega$, we define the sets $E^l(\bar{b})$ (where $l = 1, 2, ...\tau$) and $E(\bar{b})$ as follows:

$$\begin{aligned} E^l(\bar{b}) := \{b \in C :\, & F_k(b) \leqq_{LU} F_k(\bar{b}), \forall k = 1, 2, ..., \tau, k \neq l, \\ & \varphi_\eta(b) \leqq 0, \forall \eta \in M, \\ & h_\rho(b) = 0, \forall \rho \in S, \\ & \zeta_\varsigma(b) \geqq 0, \forall \varsigma \in V, \\ & \zeta_\varsigma(b) D_\varsigma(b) \leqq 0, \forall \varsigma \in V\}, \end{aligned}$$

$$\begin{aligned} E(\bar{b}) := \{b \in C :\, & F_k(b) \leqq_{LU} F_k(\bar{b}), \forall k = 1, 2, ..., \tau, \\ & \varphi_\eta(b) \leqq 0, \forall \eta \in M, \\ & h_\rho(b) = 0, \forall \rho \in S, \\ & \zeta_\varsigma(b) \geqq 0, \forall \varsigma \in V, \\ & \zeta_\varsigma(b) D_\varsigma(b) \leqq 0, \forall \varsigma \in V\}. \end{aligned}$$

We now present *almost linearizing cone* in the context of the MPIVC problem, which incorporates interval-valued functions. Preda and Chitescu [45] first introduced the almost linearizing cone for directionally differentiable multiobjective problems with inequality constraints. The concept presented here is a generalization of their work.

Definition 8. *We define* $\varpi(\Omega; \bar{b})$ *as the* almost linearizing cone *to the set* Ω *at the point* $\bar{b}$ *if it is given by*

$$\begin{aligned} \varpi(\Omega; \bar{b}) = \Big\{ c \in \mathbb{R}^n :\, & F_k^{L+}(\bar{b}; c) \leq 0 \text{ and } F_k^{U+}(\bar{b}; c) \leq 0, \ \forall k \in I, \\ & \varphi_\eta^+(\bar{b}; c) \leq 0, \ \forall \eta \in M(\bar{b}), \\ & h_\rho^+(\bar{b}; c) = 0, \ \forall \rho \in S, \end{aligned}$$

$$\zeta_\varsigma^+(\bar{b};c) \geq 0,\ \forall \varsigma \in V,$$
$$(\zeta_\varsigma D_\varsigma)^+(\bar{b};c) \leq 0,\ \forall \varsigma \in V\Big\}. \tag{3.1}$$

We now establish result that provides formulation of almost linearizing cone *for sets* $E^l(\bar{b})$, $l = 1, 2, \ldots, p$, *incorporating interval-valued functions.*

Proposition 2. *Let $\bar{b} \in \Omega$ be a Pareto solution of directionally-differentiable multiobjective interval-valued programming problem (MPIVC) with vanishing constraints. Then, for each $l = 1, 2, \ldots, p$, the linearizing cone to $E^l(\bar{b})$ satisfies the following equality:*

$$\begin{aligned}\varpi(E^l(\bar{b}); \bar{b}) = \Big\{ & c \in \mathbb{R}^n : F_k^{L+}(\bar{b};c) \leq 0 \text{ and } F_k^{U+}(\bar{b};c) \leq 0\ \forall k \in I,\ k \neq l,\\ & \varphi_\eta^+(\bar{b};c) \leq 0\ \forall \eta \in M(\bar{b}), \quad h_\rho^+(\bar{b};c) = 0\ \forall \rho \in S,\\ & \zeta_\varsigma^+(\bar{b};c) = 0\ \forall \varsigma \in V_{0+}(\bar{b}), \quad \zeta_\varsigma^+(\bar{b};c) \geq 0\ \forall \varsigma \in V_{00}(\bar{b}) \cup V_{0-}(\bar{b}),\\ & D_\varsigma^+(\bar{b};c) \leq 0\ \forall \varsigma \in V_{+0}(\bar{b})\Big\}. \end{aligned} \tag{3.2}$$

Proof. Assume that $\bar{b} \in \Omega$ is a Pareto solution for the (MPIVC) problem. By applying definitions of almost linearizing cone and the associated index sets, we can characterize the conditions defining $\varpi(E^l(\bar{b}); \bar{b})$. These conditions ensure that the directional derivatives satisfy the constraints imposed by the problem's structure.

$$\begin{aligned}\varpi(E^l(\bar{b}); \bar{b}) := \Big\{ c \in \mathbb{R}^n : & F_k^{L+}(\bar{b};c) \leqq 0,\ F_k^{U+}(\bar{b};c) \leqq 0,\ \forall k \in I, k \neq l,\\ & \varphi_\eta^+(\bar{b};c) \leqq 0,\ \forall \eta \in M(\bar{b}),\\ & h_\rho^+(\bar{b};c) = 0,\ \forall \rho \in S,\\ & \zeta_\varsigma^+(\bar{b};c) = 0,\ \forall \varsigma \in V_{0+}(\bar{b}),\\ & \zeta_\varsigma^+(\bar{b};c) \geqq 0,\ \forall \varsigma \in V_0(\bar{b}),\\ & (\zeta_\varsigma D_\varsigma)^+(\bar{b};c) \leqq 0,\ \forall \varsigma \in V_0(\bar{b}) \cup V_{+0}(\bar{b})\Big\}. \end{aligned} \tag{3.3}$$

Using Lemma 1, we get

$$(\zeta_\varsigma D_\varsigma)^+(\bar{b};c) = D_\varsigma(\bar{b})(\zeta_\varsigma)^+(\bar{b};c) + \zeta_\varsigma(\bar{b})(D_\varsigma)^+(\bar{b};c). \tag{3.4}$$

Using definition of index sets with (3.4), we get

$$\begin{aligned}(\zeta_\varsigma D_\varsigma)^+(\bar{b};c) = \{ & D_\varsigma(\bar{b})\zeta_\varsigma^+(\bar{b};c) \text{ if } \varsigma \in V_{0+}(\bar{b}) \cup V_{0-}(\bar{b})\ 0 \text{ if } \varsigma \in V_{00}(\bar{b})\\ & \zeta_\varsigma(\bar{b})D_\varsigma^+(\bar{b};c) \text{ if } \varsigma \in V \in V_{+0}(\bar{b})\}. \end{aligned} \tag{3.5}$$

Combining (3.3)–(3.5), we have (3.2). The proof is now complete. □

Remark 2. *It is important to see that almost linearizing cone to $E(\bar{b})$ at $\bar{b} \in E(\bar{b})$ is given by*

$$\varpi(E(\bar{b}); \bar{b}) = \cap_{l=1}^{\tau} \varpi(E^l(\bar{b}); \bar{b}). \tag{3.6}$$

Where $\varpi(E(\bar{b}); \bar{b})$ is given as

$$\begin{aligned}\varpi(E(\bar{b}); \bar{b}) :=\{ & c \in \mathbb{R}^n : F_k^{L+}(\bar{b};c) \leqq 0, \text{and } F_k^{U+}(\bar{b};c) \leqq 0\ \forall\ k\ \in I,\\ & \varphi_\eta^+(\bar{b};c) \leqq 0, \forall \eta \in M(\bar{b}), h_\rho^+(\bar{b};c) = 0, \forall \rho \in S, \zeta_\varsigma^+(\bar{b};c) = 0, \forall \varsigma \in V_{0+}(\bar{b}),\\ & \zeta_\varsigma^+(\bar{b};c) \geqq 0, \forall \varsigma \in V_{00}(\bar{b}) \cup V_{0-}(\bar{b}), (D_\varsigma)^+(\bar{b};c) \leqq 0, \forall \varsigma \in V_{+0}(\bar{b})\}. \end{aligned} \tag{3.7}$$

Definition 9. *The Abadie constraint qualification (ABCQ) holds at $\bar{b}$ for (MPIVC) iff*

$$\varpi(E(\bar{b});\bar{b}) \subset \bigcap_{l=1}^{\tau} Z(E^{l}(\bar{b});\bar{b}). \tag{3.8}$$

The following theorem provides a necessary condition for efficiency in (MPIVC).

Theorem 2. *Let $\bar{b} \in \Omega$ be an efficient solution for (MPIVC). Assume the following:*

1. *For every $c \in Z(C,\bar{b})$, the directional derivatives exist for:*
   - *All interval-valued functions $F_k^{+}(\bar{b};c)$, $k = 1,\dots,\tau$*
   - *The constraint functions:*
     - $\varphi_\eta^{+}(\bar{b};c),\ \eta \in M(\bar{b}),\quad h_\rho^{+}(\bar{b};c), \rho \in S,\quad \zeta_\varsigma^{+}(\bar{b};c),\ \varsigma \in V_0(\bar{b})$
     - $\zeta_\varsigma^{+}(\bar{b};c),\ \varsigma \in V_{00}(\bar{b}) \cup V_{0+}(\bar{b}) \cup V_{0-}(\bar{b}),\quad D_\varsigma^{+}(\bar{b};c),\ \varsigma \in V_{+0}(\bar{b})$
2. *The following functions are continuous at $\bar{b}$:*

$$\varphi_\eta,\ \eta \in M(\bar{b}),\quad \zeta_\varsigma,\ \varsigma \in V_{+}(\bar{b}),\quad D_\varsigma,\ \varsigma \in V_{+-}(\bar{b})$$

3. *The (ABCQ) holds at $\bar{b}$ for (MPIVC)*

Then, for every $l = 1,\dots,\tau$, the following system holds:

$$\begin{aligned} &F_k^{L+}(\bar{b};c) \leqq 0,\ F_l^{L+}(\bar{b};c) < 0, k = 1,...,\tau, k \neq l,\\ &F_k^{U+}(\bar{b};c) \leqq 0,\ F_l^{U+}(\bar{b};c) < 0, k = 1,...,\tau, k \neq l, \end{aligned} \tag{3.9}$$

$$\varphi_\eta^{+}(\bar{b};c) \leqq 0, \eta \in M(\bar{b}), \tag{3.10}$$

$$h_\rho^{+}(\bar{b};c) = 0, \rho \in S, \tag{3.11}$$

$$-\zeta_\varsigma^{+}(\bar{b};c) \leqq 0, \varsigma \in V_0(\bar{b}), \tag{3.12}$$

$$(\zeta_\varsigma D_\varsigma)^{+}(\bar{b};c) \leqq 0, \varsigma \in V_{HG}(\bar{b}) \tag{3.13}$$

and has no solution $c \in \mathbb{R}^n$.

Proof. Assume, for the sake of contradiction, that there exist $l_0 \in \{1,...,\tau\}$ such that the system

$$\begin{aligned} &F_k^{L+}(\bar{b};c) \leqq 0, F_{l_0}^{L+}(\bar{b};c) < 0, k = 1,...,\tau, k \neq l_0,\\ &F_k^{U+}(\bar{b};c) \leqq 0, F_{l_0}^{U+}(\bar{b};c) < 0, k = 1,...,\tau, k \neq l_0, \end{aligned} \tag{3.14}$$

$$\varphi_\eta^{+}(\bar{b};c) \leqq 0, \eta \in M(\bar{b}), \tag{3.15}$$

$$h_\rho^+(\bar{b};c) = 0, \rho \in S, \tag{3.16}$$

$$-\zeta_\varsigma^+(\bar{b};c) \leqq 0, \varsigma \in V_0(\bar{b}), \tag{3.17}$$

$$(\zeta_\varsigma D_\varsigma)^+(\bar{b};c) \leqq 0, \varsigma \in V_{HG}(\bar{b}) \tag{3.18}$$

has a solution $c \in \mathbb{R}^n$. Then by (3.5), we have

$$\begin{aligned} &F_k^{L+}(\bar{b};c) \leqq 0,\ F_{l_0}^{L+}(\bar{b};c) < 0, k = 1, ..., \tau, k \neq l_0, \\ &F_k^{U+}(\bar{b};c) \leqq 0,\ F_{l_0}^{U+}(\bar{b};c) < 0, k = 1, ..., \tau, k \neq l_0, \end{aligned} \tag{3.19}$$

$$\varphi_\eta^+(\bar{b};c) \leqq 0, \eta \in M(\bar{b}), \tag{3.20}$$

$$h_\rho^+(\bar{b};c) = 0, \rho \in S, \tag{3.21}$$

$$\zeta_\varsigma^+(\bar{b};c) = 0\ \forall\ \varsigma \in V_{0+}(\bar{b}), \tag{3.22}$$

$$\zeta_\varsigma^+(\bar{b};c) \geqq 0, \varsigma \in V_{00}(\bar{b}) \cup V_{0-}(\bar{b}), \tag{3.23}$$

$$D_\varsigma^+(\bar{b};c) \leqq 0, \varsigma \in V_{+0}(\bar{b}) \tag{3.24}$$

has a solution $c \in \mathbb{R}^n$, it is evident that c belongs to $\varpi(E(\bar{b}), \bar{b})$. Since (ABCQ) is satisfied at $\bar{b}$ for (MPIVC), with Definition 9, we get $c \in \bigcap_{l=1}^{\tau} Z(E^l(\bar{b}); \bar{b})$. Thus, $c \in Z(E^{l_0}; \bar{b})$. Using Definition 5, $\exists\ (\varkappa_k) \subset \mathbb{R}_+, \varkappa_k \downarrow 0$, such that $\bar{b} + \varkappa_k c \in E^{l_0}(\bar{b})$ for all $k \in N$. Hence, $\bar{b} + \varkappa_k c \in C$ and, furthermore,

$$\begin{aligned} &F_k^L(\bar{b} + \varkappa_k c) \leqq F_k^L(\bar{b}), k = 1, ..., \tau, k \neq l_0, \\ &F_k^U(\bar{b} + \varkappa_k c) \leqq F_k^U(\bar{b}), \forall k = 1, ..., \tau, k \neq l_0, \end{aligned} \tag{3.25}$$

$$\varphi_\eta(\bar{b} + \varkappa_k c) \leqq 0, \forall \eta \in M(\bar{b}), \tag{3.26}$$

$$h_\rho(\bar{b} + \varkappa_k c) = 0, \forall \rho = 1, ..., q, \tag{3.27}$$

$$\zeta_\varsigma(\bar{b} + \varkappa_k c) = 0, \forall \varsigma \in V_{0+}(\bar{b}), \tag{3.28}$$

$$\zeta_\varsigma(\bar{b} + \varkappa_k c) \geqq 0, \forall \varsigma \in V_0(\bar{b}), \tag{3.29}$$

$$D_\varsigma(\bar{b} + \varkappa_k c) \leqq 0, \forall \varsigma \in V_{+0}(\bar{b}). \tag{3.30}$$

Using index sets, it can be concluded that

$$\varphi_\eta(\bar{b}) < 0,\ \eta \in M^{<}(\bar{b}),\ \zeta_\varsigma(\bar{b}) > 0,\ \varsigma \in V_+(\bar{b}), D_\varsigma(\bar{b}) < 0,\ \varsigma \in V_{+-}(\bar{b}).$$

Therefore, due to the continuity of

$$\varphi_\eta, \eta \in M^{<}(\bar{b}), D_\varsigma, \varsigma \in V_{+-}(\bar{b}), \zeta_\varsigma, \varsigma \in V_+(\bar{b}),$$

at $\bar{b}$, there exist $k_0 \in N$ such that for all $k > k_0$,

$$\varphi_\eta(\bar{b} + \varkappa_k c) \leqq 0, \forall m \notin M(\bar{b}), \tag{3.31}$$

$$\zeta_\varsigma(\bar{b} + \varkappa_k c) \geqq 0, \forall \varsigma \in V_+(\bar{b}), \tag{3.32}$$

$$D_\varsigma(\bar{b} + \varkappa_k c) \leqq 0, \forall \varsigma \in V_{+-}(\bar{b}). \tag{3.33}$$

Therefore, by (3.26) to (3.33) we conclude that

$$\exists\, \delta > 0\ :\ \bar{b} + \varkappa_k c \in \Omega \cap B(\bar{b}; \delta),$$

which implies that $\bar{b} \in \Omega$ is an efficient solution in (MPIVC). From Definition 6

$$\exists \delta > 0\ :\ w \in \Omega \cap B(\bar{b}; \delta)\ \text{ does not exist which satisfies,}$$

$$F_k(b) \leqq_{LU} F_k(\bar{b}), k = \{1, 2, ..., \tau\}, \tag{3.34}$$

$$F_k(b) <_{LU} F_k(\bar{b}), \text{for some } k \in \{1, 2, ..., \tau\}. \tag{3.35}$$

Since $\bar{b} + \varkappa_k c \in \Omega \cap B(\bar{b}; \delta)$ and (3.25) hold, by (3.34) and (3.35), our conclusion is that the inequality for every $k \in N$

$$F_{l_0}(\bar{b} + \varkappa_k c) >_{LU} F_{l_0}(\bar{b})$$

holds. Therefore, the inequality above indicates that

$$F_{l_0}^{L+}(\bar{b}; c) \geqq 0 \text{ and } F_{l_0}^{U+}(\bar{b}; c) \geqq 0$$

holds. This contradicts (3.14). Thus, the proof is complete. □

Remark 3. *The argument used to prove Theorem* (2) *shows that if the system* (3.9)–(3.13) *has* $c \in \mathbb{R}^n$ *(a solution), then* $\forall\ l = 1, ..., \tau$*, the nonlinear system*

$$F_k^{L+}(\bar{b}; c) \leqq 0, F_k^{L+}(\bar{b}; c) < 0, k = 1, ..., \tau, k \neq l, \tag{3.36}$$

$$F_k^{U+}(\bar{b}; c) \leqq 0, F_k^{U+}(\bar{b}; c) < 0, k = 1, ..., \tau, k \neq l, \tag{3.37}$$

$$\varphi_\eta^+(\bar{b};c) \leqq 0, \forall \eta \in M(\bar{b}), \tag{3.38}$$

$$h_\rho^+(\bar{b};c) = 0 \forall \rho \in S, \tag{3.39}$$

$$\zeta_\varsigma^+(\bar{b};c) = 0 \forall \varsigma \in V_{0+}(\bar{b}), \tag{3.40}$$

$$\zeta_\varsigma^+(\bar{b};c) = 0 \forall \varsigma \in V_{00}(\bar{b}) \cup V_{0-}(\bar{b}), \tag{3.41}$$

$$D_\varsigma^+(\bar{b};c) \leqq 0 \forall \varsigma \in V_{+0}(\bar{b}) \tag{3.42}$$

has no solution $c \in \mathbb{R}^n$.

Now let us define

$F = (F_1, ..., F_k)$ such that $F_k : \mathbb{R}^n \to \mathbb{I}(\mathbb{R}), k = 1, .., \tau,$
$\varphi = (\varphi_1, ..., \varphi_{|M(\bar{b})|+|V_{00}(\bar{b})|+|V_{0-}(\bar{b})|+|V_{+0}(\bar{b})|}) : \mathbb{R}^n \to \mathbb{R}^{|M(\bar{b})|+|V_{00}(\bar{b})|+|V_{0-}(\bar{b})|+|V_{+0}(\bar{b})|}$
and
$h = (h_1, ..., h_{q+|V_{0+}(\bar{b})|}) : \mathbb{R}^n \to \mathbb{R}^{q+|V_{0+}(\bar{b})|}$

as follows

$$\begin{aligned} F_k^L(c) &:= F_k^{L+}(\bar{b};c), \\ F_k^U(c) &:= F_k^{U+}(\bar{b};c), k \in I, \end{aligned} \tag{3.43}$$

$$\begin{aligned} \varphi_\varkappa(c) :=\{ &\varphi_l^+(\bar{b};c) \text{ for } I \in M(\bar{b}), \varkappa = 1, ..., M(\bar{b}), \\ &\zeta_l^+(\bar{b};c) \text{ for } l \in V_0(\bar{b}), \varkappa = |M(\bar{b})| + 1, ..., |M(\bar{b})| + |V_{00}(\bar{b})|, \\ &-\zeta_l^+(\bar{b};c) \text{ for } l \in V_{0-}(\bar{b}), \varkappa = |M(\bar{b})| + |V_{00}(\bar{b})| + 1, ..., |M(\bar{b})| + \\ &|V_{00}(\bar{b})| + |V_{0-}(\bar{b})| + |V_{+0}(\bar{b})|, l \in V_{+0}(\bar{b}), \\ &D_\varsigma^+(\bar{b};c) \text{ for } \varkappa = |M(\bar{b})| + |V_{00}(\bar{b})| + |V_{0-}(\bar{b})| + 1, ..., |M(\bar{b})| + \\ &|V_{00}(\bar{b})| + |V_{00}(\bar{b})| + |V_{+0}(\bar{b})|\}, \end{aligned} \tag{3.44}$$

$$\begin{aligned} h_\beta(c) :=\{ &h_l^+(\bar{b};c) \text{for } l = 1, ..., q, \beta = 1, ..., q, \\ &\zeta_l^+(\bar{b};c) \text{ for } l \in V_{0+}(\bar{b}), \beta = q + 1, ..., q + |V_{0+}(\bar{b})|\}. \end{aligned} \tag{3.45}$$

We can now formulate the Karush-Kuhn-Tucker necessary optimality condition under the Abadie constraints qualification (ABCQ) for a feasible solution $\bar{b}$ to be an efficient solution in (MPIVC).

Theorem 3. *(K.K.T. Necessary Optimality Condition) Let $\bar{b} \in \Omega$ be an efficient solution for the directionally differentiable interval-valued multiobjective optimization problem (MPIVC) with vanishing constraints. Assume the following conditions hold:*

1. *The functions F_k, $k \in I$, are directionally differentiable interval-valued functions, and φ_η ($\eta \in M$), h_ρ ($\rho \in S$), ζ_ς ($\varsigma \in V$), and D_ς ($\varsigma \in V$) are directionally differentiable at $\bar{b}$.*
2. *The directional derivatives $F_k^+(\bar{b};\cdot)$, $k \in I$, are interval-valued convex functions, and the following functions are convex:*

$$\varphi_\eta^+(\bar{b};\cdot),\ \eta \in M, \quad -\zeta_\varsigma^+(\bar{b};\cdot),\ \varsigma \in V_{00}(\bar{b}) \cup V_{0-}(\bar{b}), \quad D_\varsigma^+(\bar{b};\cdot),\ \varsigma \in V_{+0}(\bar{b}).$$

3. *The functions $h_\rho^+(\bar{b};\cdot)$, $\rho \in S$, and $\zeta_\varsigma^+(\bar{b};\cdot)$, $\varsigma \in V_{0+}(\bar{b})$, are linear.*
4. *The functions φ_η, $\eta \in M^<(\bar{b})$, ζ_ς, $\varsigma \in V_+(\bar{b})$, and D_ς, $\varsigma \in V_0(\bar{b}) \cup V_{+-}(\bar{b})$, are continuous at $\bar{b}$.*
5. *The (ABCQ) is satisfied at $\bar{b}$ for (MPIVC).*

Furthermore, suppose there exists $c_0 \in \operatorname{relint} Z(C;\bar{b})$ such that $\varphi(c_0) < 0$ and $h(c) \leq 0$. Then, $\bar{\theta} \in \mathbb{R}^\tau$, $\bar{\mu} \in \mathbb{R}^j$, $\bar{\xi} \in \mathbb{R}^q$, $\bar{\gamma}^\zeta \in \mathbb{R}^r$, and $\bar{\gamma}^D \in \mathbb{R}^r$ exist as multipliers of Lagrange satisfying the necessary optimality conditions.

$$\sum_{k=1}^{\tau} \bar{\theta}_k F_k^{L+}(\bar{b};c) + \sum_{\eta=1}^{j} \bar{\mu}_\eta \varphi_\eta^+(\bar{b};c) + \sum_{\rho=1}^{b} \bar{\xi}_\rho h_\rho^+(\bar{b};c) - \sum_{\varsigma=1}^{r} \bar{\gamma}_\varsigma^\zeta \zeta_\varsigma^+(\bar{b};c) + \sum_{\varsigma=1}^{r} c_\varsigma^D D_\varsigma^+(\bar{b};c) \geqq 0, \forall c \in Z(C;\bar{b}),$$

$$\sum_{k=1}^{\tau} \bar{\theta}_k F_k^{U+}(\bar{b};c) + \sum_{\eta=1}^{j} \bar{\mu}_\eta \varphi_\eta^+(\bar{b};c) + \sum_{\rho=1}^{b} \bar{\xi}_\rho h_\rho^+(\bar{b};c) - \sum_{\varsigma=1}^{r} \bar{\gamma}_\varsigma^\zeta \zeta_\varsigma^+(\bar{b};c) + \sum_{\varsigma=1}^{r} c_\varsigma^D D_\varsigma^+(\bar{b};c) \geqq 0, \forall c \in Z(C;\bar{b}), \tag{3.46}$$

$$\bar{\mu}_\eta \varphi_\eta(\bar{b}) = 0, \eta \in M, \tag{3.47}$$

$$\bar{\gamma}_\varsigma^\zeta \zeta_\varsigma(\bar{b}) = 0, \varsigma \in V, \tag{3.48}$$

$$\bar{\gamma}_\varsigma^D D_\varsigma(\bar{b}) = 0, \varsigma \in V, \tag{3.49}$$

$$\bar{\theta} \geqq 0, \bar{\mu} \geqq 0, \tag{3.50}$$

$$\bar{\gamma}_\varsigma^\zeta = 0, \varsigma \in V_+(\bar{b}), \bar{\gamma}_\varsigma^\zeta \geqq 0, \varsigma \in V_{00}(\bar{b}) \cup V_{0-}(\bar{b}), \bar{\gamma}_\varsigma^\zeta free, \varsigma \in V_{0+}(\bar{b}), \tag{3.51}$$

$$\bar{\gamma}_\varsigma^D = 0, \varsigma \in V_{00}(\bar{b}) \cup V_{0+}(\bar{b}) \cup V_{0-}(\bar{b}) \cup V_{+-}(\bar{b}), \bar{\gamma}_\varsigma^D \geqq 0, \varsigma \in V_{+0}(\bar{b}) \tag{3.52}$$

hold.

Proof. Let $\bar{b} \in \Omega$ be an efficient solution for (MPIVC). Since ABCQ is satisfied at $\bar{b}$ for (MPIVC), using Remark 3, we get that the system (3.36) to (3.42) has no solution $\gamma \in \mathbb{R}^n$. By (3.43) to (3.45), one has

$$\begin{aligned}\{&F_k^L(\gamma) < 0, k = 1,2,...,\tau,\\ &F_k^U(\gamma) < 0, k = 1,2,...,\tau,\\ &\varphi_\eta(\gamma) \leqq 0, \eta = 1,2,...,j,\\ &h_\rho(\gamma) \leqq 0, \rho = 1,2,...,q\}\end{aligned}$$

admit no solutions. Hence by Theorem 1, $\exists\ (\theta,\vartheta,\beta)^\varsigma \in \mathbb{R}_+^\tau \times \mathbb{R}_+^j \times \mathbb{R}^q, \theta \neq 0$, satisfying

$$\begin{aligned}&\theta^T F^L(\gamma) + \vartheta^T \varphi(\gamma) + \beta^T h(\gamma) \geqq 0, \forall \gamma \in \mathbb{R}^n,\\ &\theta^T F^U(\gamma) + \vartheta^T \varphi(\gamma) + \beta^T h(\gamma) \geqq 0, \forall \gamma \in \mathbb{R}^n.\end{aligned}$$

Using (3.43)–(3.45), we get

$$\begin{aligned}&\sum_{k=1}^{\tau} \bar{\theta}_k F_k^{L+}(\bar{b};\gamma) + \sum_{\eta \in M(\bar{b})} \vartheta_\eta \varphi_\eta^+(\bar{b};\gamma) + \sum_{\rho=1}^{b} \beta_\rho h_\rho^+(\bar{b};\gamma) - \sum_{\varsigma \in V_{00}(\bar{b}) \cup V_{0-}(\bar{b})} \vartheta_\varsigma \zeta_\varsigma^+(\bar{b};\gamma) -\\ &\sum_{\varsigma \in V_{0+}(\bar{b})} \beta_\varsigma \zeta_\varsigma^+(\bar{b};\gamma) + \sum_{\varsigma \in V_{+0}(\bar{b})} \vartheta_\varsigma D_\varsigma^+(\bar{b};\gamma) \geqq 0, \forall \gamma \in \mathbb{R}^n,\\ &\sum_{k=1}^{\tau} \bar{\theta}_k F_k^{U+}(\bar{b};\gamma) + \sum_{\eta \in M(\bar{b})} \vartheta_\eta \varphi_\eta^+(\bar{b};\gamma) + \sum_{\rho=1}^{b} \beta_\rho h_\rho^+(\bar{b};\gamma) - \sum_{\varsigma \in V_{00}(\bar{b}) \cup V_{0-}(\bar{b})} \vartheta_\varsigma \zeta_\varsigma^+(\bar{b};\gamma) -\\ &\sum_{\varsigma \in V_{0+}(\bar{b})} \beta_\varsigma \zeta_\varsigma^+(\bar{b};\gamma) + \sum_{\varsigma \in V_{+0}(\bar{b})} \vartheta_\varsigma D_\varsigma^+(\bar{b};\gamma) \geqq 0, \forall \gamma \in \mathbb{R}^n. \qquad (3.53)\end{aligned}$$

Let us set

$$\begin{aligned}\bar{\mu}_\eta = \{&\vartheta_\eta \text{ if } \eta \in M(\bar{b}),\\ &0 \text{ if } m \notin M(\bar{b})\}, \qquad (3.54)\end{aligned}$$

$$\bar{\xi}_\rho = \beta_\rho, s = 1,2...,q, \qquad (3.55)$$

$$\begin{aligned}\bar{\gamma}_\varsigma^\zeta = \{&\vartheta_\varkappa \text{ if } \varsigma \in V_{00}(\bar{b} \cup V_{0-}(\bar{b}), \varkappa = |M(\bar{b})| + 1, ..., |M(\bar{b})| + |V_{00}(\bar{b})| + |V_{0-}(\bar{b})|,\\ &\beta_\varkappa \text{ if } \varsigma \in V_{0+}(\bar{b}), \varkappa = q + 1, ..., q + |V_{0+}(\bar{b})|,\\ &0 \text{ if } \varsigma \in V_+(\bar{b})\} \qquad (3.56)\end{aligned}$$

$$\begin{aligned}\bar{\gamma}_\varsigma^D = \{&\vartheta_\varkappa \text{ if } \varsigma \in V_{+0}(\bar{b}), \varkappa = |M(\bar{b})| + |V_{00}(\bar{b})| + |V_{0-}(\bar{b})| + 1, ..., |M(\bar{b})| + |V_{00}(\bar{b})| + |V_{+0}(\bar{b})|,\\ &0 \text{ if } \varsigma \in V_{00}(\bar{b}) \cup V_{0+}(\bar{b}) \cup V_{0-}(\bar{b}) \cup V_{+-}(\bar{b})\}. \qquad (3.57)\end{aligned}$$

If we substitute (3.54)–(3.57) into (3.53), we obtain the KKT optimality conditions (3.46). Additionally, (3.54)–(3.57) imply the KKT conditions (3.47) to (3.52). Hence proved. □

We substitute the constraint $\zeta_\varsigma D_\varsigma(b) \leqq 0$ for t in V with the constraints

$$\zeta_\varsigma(b) = 0, D_\varsigma(b) \geqq 0, \varsigma \in V_{0+}(\bar{b})$$
$$\zeta_\varsigma(b) \geqq 0, D_\varsigma(b) \leqq 0, \varsigma \in V_{00}(\bar{b}) \cup V_{+0}(\bar{b}) \cup V_{0-}(\bar{b}) \cup V_{+-}(\bar{b}),$$

where the index sets depend on $\bar{b}$.
We now introduce the multiobjective minimization model, which is built from (MPIVC), where several constraints are dependent on the optimal point $\bar{b}$:

$$
\begin{aligned}
(MPI)(\bar{b}) \quad \text{V - minimize } & F(b) := (F_1(b), F_2(b), ..., F_\tau(b)) \\
& \varphi_\eta(b) \leqq 0, \eta = 1, ..., j, \\
& h_\rho(b) = 0, \rho = 1, ..., q, \\
& \zeta_\varsigma(b) \geqq 0, \varsigma = 1, ..., r, \\
& \zeta_\varsigma(b) = 0, D_\varsigma(b) \geqq 0, \varsigma \in V_{0+}(\bar{b}), \\
& \zeta_\varsigma(b) \geqq 0, D_\varsigma(b) \leqq 0, \varsigma \in V_{00}(\bar{b}) \cup V_{+0}(\bar{b}) \cup V_{0-}(\bar{b}) \cup V_{+-}(\bar{b}), b \in C.
\end{aligned}
$$

To present the modified Abadie Constraint Qualification (ABCQ), we define the set $\bar{E}^l(\bar{b})$, $l = 1, \ldots, \tau$, and $\bar{E}(\bar{b})$ as follows, for $\bar{b} \in \Omega$:

$$
\begin{aligned}
\bar{E}^l(\bar{b}) :=\{ & b \in C : F_k(b) \leqq_{LU} F_k(\bar{b}), \forall k = 1, ..., \tau, k \neq l, \\
& \varphi_\eta(b) \leqq 0, \forall \eta = 1, ..., j, \\
& h_\rho = 0, \forall \rho = 1, ..., q, \\
& \zeta_\varsigma(b) = 0, D_\varsigma(b) \geqq 0, \forall \varsigma \in V_{0+}(\bar{b}), \\
& \zeta_\varsigma(b) \geqq 0, D_\varsigma(b) \leqq 0, \forall \varsigma \in V_{00}(\bar{b}) \cup V_{+0}(\bar{b}) \cup V_{0-}(\bar{b}) \cup V_{+-}(\bar{b})\},
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{E}(\bar{b}) :=\{ & b \in C; F_k(b) \leqq_{LU} F_k(\bar{b}), \\
& \varphi_\eta(b) \leqq 0, \forall \eta = 1, ..., j, \\
& h_\rho(b) = 0, \forall \rho = 1, ..., q, \\
& \zeta_\varsigma(b) = 0, D_\varsigma(b) \geqq 0, \forall \varsigma \in V_{0+}(\bar{b}), \\
& \zeta_\varsigma(b) \geqq 0, D_\varsigma(b) \leqq 0, \forall \varsigma \in V_{00}(\bar{b}) \cup V_{+0} \cup V_{0-}(\bar{b}) \cup V_{+-}(\bar{b})\}.
\end{aligned}
$$

Then, almost linearizing cone for set $\bar{E}^l(\bar{b})$ is defined as

$$
\begin{aligned}
\varpi(\bar{E}^l(\bar{b}); \bar{b}) :=\{ & c \in \mathbb{R}^n : F_k^+(\bar{b}; c) \leqq_{LU} 0, \forall k \in I, k \neq l, \\
& \varphi_\eta^+(\bar{b}; c) \leqq 0, \forall \eta \in M(\bar{b}), h_\rho^+(\bar{b}; c) = 0, \forall \rho \in S, \zeta_\varsigma^+(\bar{b}; c) = 0, \forall \varsigma \in V_{0+}(\bar{b}) \\
& \zeta_\varsigma^+(\bar{b}; c) \geqq 0, \forall \varsigma \in V_{00}(\bar{b}) \cup V_{0-}(\bar{b}), D_\varsigma^+(\bar{b}; c) \leqq 0, \forall \varsigma \in V_{00}(\bar{b}) \cup V_{+0}(\bar{b})\}.
\end{aligned}
\tag{3.58}
$$

Thus, almost linearizing cone for set $\bar{E}(\bar{b})$ is defined as

$$\varpi(\bar{E}(\bar{b}); \bar{b}) = \bigcap_{l=1}^{\tau} \varpi(\bar{E}^l(\bar{b}); \bar{b}). \tag{3.59}$$

Remark 4. *The sole distinction between the cones $\varpi(E(\bar{b});\bar{b})$ and $\varpi(\bar{E}(\bar{b});\bar{b})$ lies in the inclusion of the inequality $D^{+}_{\varsigma}(\bar{b};c) \leqq 0,\ \forall\varsigma \in V_{00}(\bar{b})$ in $\varpi(\bar{E}(\bar{b});\bar{b})$. Specifically, we have the containment relation:*

$$\varpi(\bar{E}(\bar{b});\bar{b}) \subset \varpi(E(\bar{b});\bar{b}). \tag{3.60}$$

Proposition 3. *Let $\bar{b}$ be a feasible solution in (MPIVC). Then*

$$\bigcap_{l=1}^{\tau} Z(\bar{E}^{l}(\bar{b});\bar{b}) \subset \varpi(E(\bar{b});\bar{b}). \tag{3.61}$$

Proof. By Proposition 2, we get

$$\bigcap_{l=1}^{\tau} Z(E^{l}(\bar{b});\bar{b}) \subset \varpi(E(\bar{b});\bar{b}). \tag{3.62}$$

Moreover, as inferred from the proof of Proposition 2, we have

$$Z(E^{l}(\bar{b});\bar{b}) \subset \varpi(E^{l}(\bar{b});\bar{b}), \forall l = 1,..,\tau. \tag{3.63}$$

Thus, (3.63) and (3.59) yield

$$\bigcap_{l=1}^{\tau} Z(\bar{E}^{l}(\bar{b});\bar{b}) \subset \bigcap_{l=1}^{\tau} \varpi(\bar{E}^{l}(\bar{b});\bar{b}) = \varpi(E(\bar{b});\bar{b}). \tag{3.64}$$

Since $\bar{E}^{l}(\bar{b}) \subset E^{l}(\bar{b}), l = 1,...,\tau$, therefore, one has

$$Z(\bar{E}^{l}(\bar{b});\bar{b}) \subseteq Z(E^{l}(\bar{b});\bar{b}), \forall l = 1,...,\tau, \tag{3.65}$$

$$\varpi(\bar{E}(\bar{b});\bar{b}) \subseteq \varpi(E(\bar{b});\bar{b}). \tag{3.66}$$

Combining (3.62) to (3.66), we get (3.61). □

We are introducing the VC-Abadie constraints qualification, which is a modified version of the Abadie constraints qualification.

Definition 10. *We say that the VC-Abadie constraint qualification (VC-ABCQ) holds at an efficient solution $\bar{b} \in \Omega$ of (MPIVC) iff*

$$\varpi(\bar{E}(\bar{b});\bar{b}) \subseteq \bigcap_{l=1}^{\tau} Z(E^{l}(\bar{b});\bar{b}). \tag{3.67}$$

We will now define the Abadie constraints qualification for MPI($\bar{b}$). We will also demonstrate that (VC-ABCQ) is valid at $\bar{b}$ for (MPIVC), even in cases where the (ABCQ) is not met.

Definition 11. *We say that the modified Abadie constraint qualification (MABCQ) holds at a (weakly) efficient solution $\bar{b} \in \Omega$ for (MPI($\bar{b}$)) iff*

$$\varpi(E(\bar{b});\bar{b}) \subseteq \bigcap_{l=1}^{\tau} Z(\bar{E}^{l}(\bar{b});\bar{b}). \tag{3.68}$$

We will now state the sufficient condition for satisfying the VC-Abadie constraints qualification at an efficient solution in (MPIVC).

Lemma 2. *Let $\bar{b} \in \Omega$ be an efficient solution in MPIVC. If MABCQ holds at $\bar{b}$ for MPI($\bar{b}$), then VC-ABCQ holds at $\bar{b}$ for MPIVC.*

Proof. Assuming $\bar{b} \in \Omega$ is an efficient solution in MPIVC and MABCQ holds at $\bar{b}$ for MPI($\bar{b}$), then, using Definition 11, we get

$$\varpi(E(\bar{b}); \bar{b}) \subseteq \bigcap_{l=1}^{\tau} Z(\bar{E}^l(\bar{b}); \bar{b}). \tag{3.69}$$

Since $\bar{E}^l(\bar{b}) \subseteq E^l(\bar{b}), l = 1, ..., \tau$, we have that

$$Z(\bar{E}^l(\bar{b}); \bar{b}) \subseteq Z(E^l(\bar{b}); \bar{b}), l = 1, ..., \tau, \tag{3.70}$$

$$\varpi(\bar{E}^l(\bar{b}); \bar{b}) \subseteq \varpi(E^l(\bar{b}); \bar{b}), l = 1, ..., \tau. \tag{3.71}$$

Hence, by (3.71), we have

$$\varpi(\bar{E}(\bar{b}); \bar{b}) = \bigcap_{l=1}^{\tau} \varpi(\bar{E}^l(\bar{b}); \bar{b}) \subseteq \bigcap_{l=1}^{\tau} \varpi(E^l(\bar{b}); \bar{b}) = \varpi(E(\bar{b}); \bar{b}). \tag{3.72}$$

Thus, (3.70) gives

$$\bigcap_{l=1}^{\tau} Z(\bar{E}^l(\bar{b}); \bar{b}) \subseteq \bigcap_{l=1}^{\tau} Z(E^l(\bar{b}); \bar{b}). \tag{3.73}$$

Thus, by (3.69), (3.72) and (3.73), we get

$$\varpi(\bar{E}(\bar{b}); \bar{b}) \subseteq \bigcap_{l=1}^{\tau} Z(E^l(\bar{b}); \bar{b}),$$

as was to be shown. □

Theorem 4. *(KKT-Type Necessary Conditions of Optimality) Let $\bar{b} \in \Omega$ be an efficient solution for the (MPIVC) problem. Assume the following:*

- *The interval-valued functions F_k ($k \in I$) are directionally differentiable at $\bar{b}$*
- *The functions φ_η ($\eta \in M$), h_ρ ($\rho \in S$), ζ_ς ($\varsigma \in V$), and D_ς ($\varsigma \in V$) are directionally differentiable at $\bar{b}$*
- *The following are convex functions:*
  - $F_k^+(\bar{b}; \cdot)$ *for* $k \in I$
  - φ_η *for* $\eta \in M(\bar{b})$
  - $-\zeta_\varsigma^+(\bar{b}; \cdot)$ *for* $\varsigma \in V_{00}(\bar{b}) \cup V_{0-}(\bar{b})$
  - $D_\varsigma^+(\bar{b}; \cdot)$ *for* $\varsigma \in V_{00}(\bar{b}) \cup V_{+0}(\bar{b})$

- *The following are linear functions:*
  - $h^+_\rho(\bar{b};\cdot)$ *for* $\rho \in S$
  - $\zeta^+_\varsigma(\bar{b};\cdot)$ *for* $\varsigma \in V_{0+}(\bar{b})$
- *The following are continuous at* $\bar{b}$*:*
  - φ_η *for* $\eta \in M^<(\bar{b})$
  - ζ_ς *for* $\varsigma \in V_+(\bar{b})$
  - D_ς *for* $\varsigma \in V_0(\bar{b}) \cup V_{+-}(\bar{b})$
- *The (VC-ABCQ) constraint qualification is satisfied at* $\bar{b}$ *for (MPIVC).*

If $\exists\, c_0 \in relint\, Z(C;\bar{b}):\ \varphi(c_0) < 0$ *and* $h(c_0) \leq 0$*, then* $\exists\, \bar{\theta} \in \mathbb{R}^\tau,\ \bar{\mu} \in \mathbb{R}^j,\ \bar{\xi} \in \mathbb{R}^q,\ \bar{\gamma^\zeta} \in \mathbb{R}^r$*, and* $\bar{\gamma^D} \in \mathbb{R}^r$ *(multipliers of Lagrange) satisfying the following conditions:*

$$\sum_{k=1}^{\tau} \bar{\theta}_k F^{L+}_k(\bar{b};c) + \sum_{\eta=1}^{j} \bar{\mu}_\eta \varphi^+_\eta(\bar{b};c) + \sum_{\rho=1}^{b} \bar{\xi}_\rho h^+_\rho(\bar{b};c) - \sum_{\varsigma=1}^{r} \gamma^\zeta_\varsigma \zeta^+_\varsigma(\bar{b};c) + \sum_{\varsigma=1}^{r} \gamma^D_\varsigma D^+_\varsigma(\bar{b};c) \geqq 0\ \forall c \in Z(C;\bar{b}),$$

$$\sum_{k=1}^{\tau} \bar{\theta}_k F^{U+}_k(\bar{b};c) + \sum_{\eta=1}^{j} \bar{\mu}_\eta \varphi^+_\eta(\bar{b};c) + \sum_{\rho=1}^{b} \bar{\xi}_\rho h^+_\rho(\bar{b};c) - \sum_{\varsigma=1}^{r} \gamma^\zeta_\varsigma \zeta^+_\varsigma(\bar{b};c) + \sum_{\varsigma=1}^{r} \gamma^D_\varsigma D^+_\varsigma(\bar{b};c) \geqq 0\ \forall c \in Z(C;\bar{b}), \tag{3.74}$$

$$\bar{\mu}_\eta \varphi_\eta(\bar{b}) = 0, \eta \in M, \tag{3.75}$$

$$\bar{\gamma}^\zeta_\varsigma \zeta_\varsigma(\bar{b}) = 0, \varsigma \in V, \tag{3.76}$$

$$\gamma^D_\varsigma D_\varsigma(\bar{b}) = 0, \varsigma \in V, \tag{3.77}$$

$$\bar{\theta} \geq 0,\ \bar{\mu} \geq 0, \tag{3.78}$$

$$\bar{\gamma}^\zeta_\varsigma = 0, \varsigma \in V_+(\bar{b}),\ \bar{\gamma}^\zeta_\eta \geqq 0,\ \varsigma \in V_{00}(\bar{b}) \cup V_{0-}(\bar{b}),\ \bar{\gamma}^\zeta_\varsigma \text{ free } \varsigma \in V_{0+}(\bar{b}), \tag{3.79}$$

$$\bar{\gamma}^\zeta_\varsigma = 0,\ \varsigma \in V_{00}(\bar{b}) \cup V_{0+}(\bar{b}) \cup V_{+-}(\bar{b}),\ \bar{\gamma}^D_\varsigma \geqq 0,\ \varsigma \in V_{+0}(\bar{b}) \cup V_{00}(\bar{b}) \tag{3.80}$$

After establishing the necessary convexity assumptions, we demonstrate that the KKT conditions of optimality are sufficient for solving the directional differential multiobjective optimization model with vanishing constraints and interval values. Presently, we confirm the sufficiency of these conditions for the directional differential multiobjective optimization model under suitable convexity assumptions.

Theorem 5. *Let $\bar{b}$ be a feasible solution of (MPIVC) satisfying the KKT-type necessary optimality conditions* (3.46)–(3.52) *at $\bar{b}$ with Lagrange multipliers:*

$$\bar{\theta} \in \mathbb{R}^{\tau}_{+}, \ \ \bar{\xi} \in \mathbb{R}^{q}, \ \ \bar{\gamma}^{\zeta} \in \mathbb{R}^{r}, \ \ \bar{\gamma}^{D} \in \mathbb{R}^{r}.$$

Furthermore, assume the following convexity properties on Ω:

- F_k *are interval-valued convex functions for all* $k \in I$
- φ_η *are convex for all* $\eta \in M(\bar{b})$
- h_ρ *are convex for all* $\rho \in S^{+}(\bar{b}) := \{\rho \in S : \bar{\xi}_\rho > 0\}$
- $-h_\rho$ *are convex for all* $\rho \in S^{-}(\bar{b}) := \{\rho \in S : \bar{\xi}_\rho < 0\}$
- $-\zeta_\varsigma$ *are convex for all* $\varsigma \in V_{00}(\bar{b}) \cup V_{0+}(\bar{b})$
- D_ς *are convex for all* $\varsigma \in V_{+0}(\bar{b})$

Then $\bar{b}$ is a weak Pareto solution for (MPIVC).

Proof. We proceed by considering a proof by contradiction. To do this, we will assume the opposite: $\bar{b}$ is not a weak Pareto solution in (MPIVC), and explore the implications of this assumption. Definition 6 provides $\tilde{b} \in \Omega$ such that

$$F(\tilde{b}) <_{LU} F(\bar{b}). \tag{3.81}$$

If we suppose that F is an interval-valued convex mapping at $\bar{b}$ on Ω, the Proposition 1 implies that inequality (3.81) yields

$$\begin{aligned} &F_k^{L+}(\bar{b}; \tilde{b} - \bar{b}) < 0, k = 1, ..., \tau, \\ &F_k^{U+}(\bar{b}; \tilde{b} - \bar{b}) < 0, k = 1, ..., \tau. \end{aligned} \tag{3.82}$$

since $\bar{\theta} \geq 0$, the inequality (3.82) gives

$$\begin{aligned} &\sum_{k=1}^{\tau} \bar{\theta}_k F_k^{L+}(\bar{b}; \tilde{b} - \bar{b}) < 0, \\ &\sum_{k=1}^{\tau} \bar{\theta}_k F_k^{U+}(\bar{b}; \tilde{b} - \bar{b}) < 0. \end{aligned} \tag{3.83}$$

From $\bar{b}, \ \tilde{b} \in \Omega$ and the definition of $M(\bar{b})$, it follows that

$$\varphi_\eta(\tilde{b}) \leqq \varphi_\eta(\bar{b}) = 0, \eta \in M(\bar{b}), \tag{3.84}$$

$$h_\rho(\tilde{b}) = h_\rho(\bar{b}) = 0, \rho \in S, \tag{3.85}$$

$$-\zeta_\varsigma(\tilde{b}) \leqq -\zeta_\varsigma(\bar{b}) = 0, \varsigma \in V_{00}(\bar{b}) \cup V_{0-}(\bar{b}) \cup V_{0+}(\bar{b}), \tag{3.86}$$

$$D_\varsigma(\tilde{b}) \leqq D_\varsigma(\bar{b}) = 0, \varsigma \in V_{+0}(\bar{b}). \tag{3.87}$$

By assumption, φ_η, $\eta \in M(\bar{b})$, $h_\rho, \rho \in S^+(\bar{b}) = \{\rho \in S : \bar{\xi}_\rho > 0\}$, $-h_\rho$, $\rho \in S^-(\bar{b}) = \{\rho \in S : \bar{\xi}_\rho < 0\}$, $-\zeta_\varsigma, \varsigma \in V_{00}(\bar{b}) \cup V_{00}(\bar{b}) \cup V_{0+}(\bar{b})$, $D_\varsigma, \varsigma \in V_{+0}(\bar{b})$, are convex on Ω. Then by Definition 1, inequalities (3.84) to (3.87) imply, respectively,

$$\varphi_\eta^+(\bar{b}; \tilde{b} - \bar{b}) \leqq 0, \eta \in M(\bar{b}), \tag{3.88}$$

$$h_\rho^+(\bar{b}; \tilde{b} - \bar{b}) \leqq 0, \rho \in S^+(\bar{b}), \tag{3.89}$$

$$-h_\rho^-(\bar{b}; \tilde{b} - \bar{b}) \leqq 0, \rho \in S^-(\bar{b}), \tag{3.90}$$

$$-\zeta_\varsigma^+(\bar{b}; \tilde{b} - \bar{b}) \leqq 0, \varsigma \in V_{00}(\bar{b}) \cup V_{0-}(\bar{b}) \cup V_{0+}(\bar{b}), \tag{3.91}$$

$$D_\varsigma(\bar{b}; \tilde{b} - \bar{b}) < 0, \varsigma \in V_{+0}(\bar{b}). \tag{3.92}$$

Considering that $\bar{\mu}_\eta = 0$, $\eta \in M^<(\bar{b})$, $\bar{\xi}_\rho = 0$, $s \notin S^+(\bar{b}) \cup S^-(\bar{b})$, $\bar{\gamma}_\varsigma^\zeta = 0$, $\varsigma \in V_+(\bar{b})$, $\bar{\gamma}^D = 0$, $\varsigma \in V_{00}(\bar{b}) \cup V_{0+}(\bar{b}) \cup V_{0-}(\bar{b}) \cup V_{+-}(\bar{b})$, we get,

$$\sum_{\eta=1}^{j} \bar{\mu}_\eta^+ \varphi_\eta^+(\bar{b}; \tilde{b} - \bar{b}) \leqq 0, \tag{3.93}$$

$$\sum_{\rho=1}^{b} \bar{\xi}_\rho h_\rho^+(\bar{b}; \tilde{b} - \bar{b}) \leqq 0, \tag{3.94}$$

$$\sum_{\varsigma=1}^{r} \bar{\gamma}_\varsigma^\zeta \zeta_\varsigma^+(\bar{b}; \tilde{b} - \bar{b}) \leqq 0, \tag{3.95}$$

$$\sum_{\varsigma=1}^{r} \bar{\gamma}_\varsigma^D D_\varsigma^+(\bar{b}; \tilde{b} - \bar{b}) \leqq 0. \tag{3.96}$$

Combining (3.83) and (3.93)–(3.96), we get the inequalities

$$\begin{aligned}
&\sum_{k=1}^{\tau} \bar{\theta}_k F_k^{L+}(\bar{b}; \tilde{b} - \bar{b}) + \sum_{\eta=1}^{j} \bar{\mu}_\eta \varphi_\eta^+(\bar{b}; \tilde{b} - \bar{b}) + \sum_{\rho=1}^{b} \bar{\xi}_\rho h_\rho^+(\bar{b}; \tilde{b} - \bar{b}) \\
&\qquad - \sum_{\varsigma=1}^{r} \gamma_\varsigma^\zeta \zeta_\varsigma^+(\bar{b}; \tilde{b} - \bar{b}) + \sum_{\varsigma=1}^{r} \gamma_\varsigma^D D_\varsigma^+(\bar{b}; \tilde{b} - \bar{b}) < 0, \\
&\sum_{k=1}^{\tau} \bar{\theta}_k F_k^{U+}(\bar{b}; \tilde{b} - \bar{b}) + \sum_{\eta=1}^{j} \bar{\mu}_\eta \varphi_\eta^+(\bar{b}; \tilde{b} - \bar{b}) + \sum_{\rho=1}^{b} \bar{\xi}_\rho h_\rho^+(\bar{b}; \tilde{b} - \bar{b}) \\
&\qquad - \sum_{\varsigma=1}^{r} \gamma_\varsigma^\zeta \zeta_\varsigma^+(\bar{b}; \tilde{b} - \bar{b}) + \sum_{\varsigma=1}^{r} \gamma_\varsigma^D D_\varsigma^+(\bar{b}; \tilde{b} - \bar{b}) < 0
\end{aligned} \tag{3.97}$$

holds. Which contradicts KKT conditions (3.46). Thus, $\bar{b}$ is weakly Pareto-optimal for (MPIVC). □

To establish the KKT type necessary conditions of optimality required for a feasible solution $\bar{b}$ to qualify as a Pareto solution in (MPIVC), we need to enforce strict convexity on the objective functions.

Theorem 6. *Let $\bar{b}$ bet a feasible solution in (MPIVC) and KKT type necessary optimality condition* (3.46) *to* (3.52) *be satisfied at $\bar{b}$ for (MPIVC) with Lagrange multipliers $\bar{\theta} \in \mathbb{R}^{\tau}_{+}$, $\bar{\mu} \in \mathbb{R}^{j}_{+}$, $\bar{\xi} \in \mathbb{R}^{q}$, $\bar{\gamma}^{\zeta}$ and $\bar{\gamma}^{D} \in \mathbb{R}^{r}$. Furthermore, we suppose that F_k, $k \in I$ are strictly convex interval valued functions on Ω, φ_η, $\eta \in M(\bar{b})$, h_ρ, $\rho \in S^{+}(\bar{b}) = \{\rho \in S : \bar{\xi}_\rho > 0\}$, $-h_\rho$, $\rho \in S^{-}(\bar{b}) := \{\rho \in S : \bar{\xi}_\rho < 0\}$, $-\zeta_\varsigma$, $\varsigma \in V_{00}(\bar{b}) \cup V_{0-}(\bar{b}) \cup V_{0+}(\bar{b})$, D_ς, $\varsigma \in V_{+0}(\bar{b})$, are convex on Ω . Then $\bar{b}$ is Pareto solution in (MPIVC).*

Now, we demonstrate the findings presented in the paper with an example.

Example 1. *Consider the (MPIVC1) problem*
V- minimize

$$
\begin{aligned}
F(b) =&(F_1(b), F_2(b))\\
=&([F_1^L, F_1^U], [F_2^L, F_2^U]) = ([|b_1| - 2|b_2|, |b_1| + |b_2|], [-|b_1| + 2|b_2|, |b_1| + |b_2|]),\\
&\zeta_1(b) = b_1 + |b_2| \geqq 0,\\
&\zeta_1(b)G_1(b) = (b_1 + |b_2|)(-b_1 - |b_2|) \leqq 0.
\end{aligned}
$$

Note that

$$\Omega = \{(b_1, b_2) \in \mathbb{R}^2 : b_1 + |b_2| \geqq 0, (b_1 + |b_2|)(-b_1 - |b_2|) \leqq 0\}, \bar{b} = (0, 0)$$

is a feasible solution in (MPIVC1) and $V_{00}(\bar{b}) = 1$. Now, we define the sets $E^l(\bar{b}), E^2(\bar{b}), E(\bar{b})$, $\bar{E}(\bar{b})$. We get

$$
\begin{aligned}
E^1(\bar{b}) &= \{(b_1, b_2) \in \mathbb{R}^2 : [-|b_1| + 2|b_2|, |b_1| + 2|b_2|] \leqq_{LU} 0,\ \zeta_1(b) = b_1 + |b_2| \geqq 0,\\
&\qquad \zeta_1(b)G_1(b) = (b_1 + |b_2|)(-b_1 - |b_2|) \leqq 0\},\\
E^2(\bar{b}) &= \{(b_1, b_2) \in \mathbb{R}^2 : [|b_1| - 2|b_2|, |b_1| + |b_2|] \leqq_{LU} 0,\ \zeta_1(b) = b_1 + |b_2| \geqq 0,\\
&\qquad \zeta_1(b)G_1(b) = (b_1 + |b_2|)(-b_1 - |b_2|) \leqq 0\},\\
E(\bar{b}) &= \{(b_1, b_2) \in \mathbb{R}^2 : [-|b_1| + 2|b_2|, |b_1| + 2|b_2|] \leqq_{LU} 0,\ [|b_1| - 2|b_2|, |b_1| + |b_2|] \leqq_{LU} 0,\\
&\qquad \zeta_1(b) = b_1 + |b_2| \geqq 0, \zeta_1(b)G_1(b) = (b_1 + |b_2|)(-b_1 - |b_2|) \leqq 0\},\\
\bar{E}(\bar{b}) &= \{(b_1, b_2) \in \mathbb{R}^2 : [-|b_1| + 2|b_2|, b_1 + 2|b_2|] \leqq_{LU} 0,\ [|b_1| - 2|b_2|, |b_1| + |b_2|] \leqq_{LU} 0,\\
&\qquad \zeta_1(b) = b_1 + |b_2| \geqq 0, G_1(b) = (-b_1 - |b_2|) \leqq 0\}.
\end{aligned}
$$

Additionally, using Definition 5 alongside the almost linearizing cone definitions (3.2) and (3.7), we can deduce the following

$$
\begin{aligned}
&Z(E^1(\bar{b}); \bar{b}) = \{(c_1; c_2) \in \mathbb{R}^2 : [-|c_1| + 2|c_2|; c_1 + 2|c_2|] \leqq_{LU} 0; c_1 + |c_2| \geqq 0,\\
&(c_1 + |c_2|)(-c_1 - |c_2|) \leqq 0\},\\
&Z(E^2(\bar{b}); \bar{b}) = \{(c_1; c_2) \in \mathbb{R}^2 : [|c_1| - 2|c_2|, |c_1| + |c_2|] \leqq_{LU} 0; c_1 + |c_2| \geqq 0,\\
&(c_1 + |c_2|)(-c_1 - |c_2|) \leqq 0\},\\
&\varpi(E(\bar{b}); \bar{b}) = \{(c_1; c_2) \in \mathbb{R}^2 : [-|c_1| + 2|c_2|; |c_1| + 2|c_2|] \leqq_{LU} 0, [|c_1| - 2|c_2|, |c_1| + |c_2] \leqq_{LU} 0,\\
&c_1 + |c_2| \geqq 0\},
\end{aligned}
$$

$\varpi(\bar{E}(\bar{b}); \bar{b}) = \{(c_1; c_2) \in \mathbb{R}^2 : [-|c_1| + 2|c_2|; |c_1| + 2|c_2|] \leqq_{LU} 0, [|c_1| - 2|c_2|, |c_1| + |c_2|] \leqq_{LU} 0,$
$c_1 + |c_2| \geqq 0, -c_1 - |c_2| \leqq 0\}.$

The example illustrates that VC-ABCQ is weaker than the classical ABCQ. At the point $(0, 0)$ in problem MPVC1, ABCQ fails because the linearized cone is not contained within the tangent cone. However, VC-ABCQ still holds true because the modified linearized cone for vanishing constraints meets the necessary inclusion. This demonstrates that VC-ABCQ can be applied in situations where ABCQ does not, making it more suitable for handling problems with vanishing constraints.

The KKT type necessary optimality conditions (3.74)–(3.80) are fulfilled at $\bar{b}$ with the Lagrange multipliers $\bar{\theta}_1 = 1, \bar{\theta}_2 = 1, \bar{\gamma}_1^{\zeta} = 1, \bar{\gamma}_1^{D} = -1$. Furthermore, we emphasize that all functions in MPIVC1 are convex on Ω, and the objective F_1 is both strictly convex and interval-valued on Ω. A key limitation of existing literature (see, e.g., [46]) is that its optimality conditions do not hold for problem MPIVC1 with vanishing constraints. The absence of suitable constraint qualifications renders these earlier results invalid in our context.

4. Conclusions

This study explores a new class of nonsmooth multiobjective mathematical programming problems in which functions are directionally differentiable and the objective function is interval-valued. We establish necessary and sufficient optimality conditions for the considered optimization problem. The Abadie constraint qualification (ABCQ) is not guaranteed to hold in multicriteria optimization frameworks of this type. To overcome this limitation, a refined version of the ABCQ is proposed. Under suitable convexity assumptions and directional differentiability, KKT necessary optimality criteria are demonstrated to act as sufficient conditions for the studied class of optmization problem.

This research represents a significant advancement in multiobjective optimization by developing optimality conditions for a new category of problems: directionally differentiable, interval-valued problems with vanishing constraints. The key innovation of this work is its integration of three challenging aspects-uncertainty (through interval values), nonsmoothness (through directional differentiability), and complex constraint structures (via vanishing constraints)-into a unified framework. By refining ABCQ and establishing KKT conditions in these contexts, the study offers a robust theoretical foundation for future research.

However, many important questions remain unanswered. For instance, how can we design efficient algorithms to tackle these problems? How stable are the solutions when there are slight changes in the input data? Additionally, can this approach be extended to more general scenarios, such as set-valued, fuzzy, or stochastic objective functions? Future research should also explore second-order conditions, investigate duality gaps, and apply these theories to real-world problems to enhance their practicality and utility.

Author contributions

Conceptualization, B.C.J. and A.H.; Investigation, M.K.R., B.C.J., S.J.S. and A.H.; Writing–original draft, M.K.R., B.C.J., S.J.S. and A.H.; Writing–review and editing, M.K.R., B.C.J., S.J.S. and A.H.; Supervision, A.H. All authors contributed equally.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

References

1. A. Khare, T. Nath, Enhanced Fritz John stationarity, new constraint qualifications and local error bound for mathematical programs with vanishing constraints, *J. Math. Anal. Appl.*, **472** (2019), 1042–1077. https://doi.org/10.1016/j.jmaa.2018.11.063
2. J. Outrata, M. Kocvara, J. Zowe, Nonsmooth approach to optimization problems with equilibrium constraints: theory, applications and numerical results, *Springer Science and Business Media*, **28** (1998).
3. W. Achtziger, C. Kanzow, Mathematical programs with vanishing constraints: optimality conditions and constraint qualifications, *Math. Program.*, **114** (2008), 69–99. https://doi.org/10.1007/s10107-006-0083-3
4. Y. An, G Ye, D. Zhao, W. Liu, Hermite-Hadamard type inequalities for interval (b_1, b_2)-convex functions. *Mathematics*, **7** (2019), 436. https://doi.org/10.3390/math7050436
5. M. S. Bazaraa, H. D. Sherali, C. M. Shetty, *Nonlinear programming: theory and algorithms*, John wiley and sons, 2006.
6. M. P. Bendsoe, O. Sigmund, Topology optimization: theory, methods, and applications. *Springer Science and Business Media*, 2013.
7. T. Antczak, Optimality conditions and Mond–Weir duality for a class of differentiable semi-infinite multiobjective programming problems with vanishing constraints, *OR-Q J Oper. Res.*, **20** (2022), 417–442. https://doi.org/10.1007/s10288-021-00482-1
8. B. C. Joshi, M. K. Roy, A. Hamdi, On semi-infinite optimization problems with vanishing constraints involving interval-valued functions, *Mathematics*, **12** (2024), 1008. https://doi.org/10.3390/math12071008

9. S. K. Mishra, V. Singh, V. Laha, On duality for mathematical programs with vanishing constraints, *Ann. Oper. Res.,* **243** (2016), 249–272. https://doi.org/10.1007/s10479-015-1814-8

10. K. Miettinen, Nonlinear multiobjective optimization. *Springer Science and Business Media*, 1999. https://doi.org/10.1007/978-1-4615-5563-6

11. T. Stewart, O. Bandte, H. Braun, N. Chakraborti, M. Ehrgott, M. Göbelt, et al., Real-world applications of multiobjective optimization, In: J. Branke, K. Deb, K. Miettinen, R. Słowiński, (eds), *Multiobjective optimization: interactive and evolutionary approaches*, (2008), 285–327. https://doi.org/10.1007/978-3-540-88908311

12. S. Jameii, K. Faez, M. Dehghan, Multiobjective optimization for topology and coverage control in wireless sensor networks. *Int. J. Distrib. Sens. Netw.,* **11** (2015), 363815.

13. A. Ponsich, A. L. Jaimes, C. A. C. Coello, A survey on multiobjective evolutionary algorithms for the solution of the portfolio optimization problem and other finance and economics applications, *IEEE Trans. Evol. Comput.*, **17** (2012), 321–344. https://doi.org/10.1109/TEVC.2012.2196800

14. A. A. Neghabi, N. J. Navimipour, M. Hosseinzadeh, A. Rezaee, Energy-aware dynamic-link load balancing method for a software-defined network using a multi-objective artificial bee colony algorithm and genetic operators, *IET Commun.*, **14** (2020), 3284–3293. https://doi.org/10.1049/iet-com.2019.1300

15. T. S. Cao, T. T. T. Nguyen, Van-Son Nguyen, V. H. Truong, H. H. Nguyen, Performance of six metaheuristic algorithms for multi-objective optimization of nonlinear inelastic steel trusses, *Buildings*, **13** (2023), 868. https://doi.org/10.3390/buildings13040868

16. X. Zhou, X. Zhang, Multi-objective-optimization-based control parameters auto-tuning for aerial manipulators, *Int. J. Adv. Robot. Syst.*, **16** (2019). https://doi.org/10.1177/1729881419828071

17. A. Candelieri, A. Ponti, F. Archetti, Fair and green hyperparameter optimization via multi-objective and multiple information source Bayesian optimization, *Mach. Learn.,* **113** (2024), 2701–2731. https://doi.org/10.1007/s10994-024-06515-0

18. H. Huang, H. Zhu, Stationary condition for Borwein proper efficient solutions of nonsmooth multiobjective problems with vanishing constraints, *Mathematics*, **10** (2022), 4569. https://doi.org/10.3390/math10234569

19. H. Wang, G. Kang, R. Zhang, On optimality conditions and duality for multiobjective fractional optimization problem with vanishing constraints, *Electron. Res. Arch.*, **32** (2024), 5109–5126. https://doi.org/10.3934/era.2024235

20. B.C. Joshi, Mathematical programs with vanishing constraints involving strongly invex functions, *Numer. Algorithms*, **91** (2022), 505–530. https://doi.org/10.1007/s11075-022-01271-5

21. P. Kharbanda, D. Agarwal, D. Sinha, Multiobjective programming under $(\varphi, d) - V-$type I univexity, *Opsearch*, **52** (2015), 168–185. https://doi.org/10.1007/s12597-013-0164-z

22. B. B. Upadhyay, A. Ghosh, S. Treanţă, Optimality conditions and duality for nonsmooth multiobjective semi-infinite programming problems with vanishing constraints on Hadamard manifolds, *J. Math. Anal. Appl.*, **531** (2024), 127785.

23. V. Laha, H. N. Singh, On quasidifferentiable mathematical programs with equilibrium constraints, *Comput. Manag. Sci.*, **20** (2023). https://doi.org/10.1007/s10287-023-00461-3

24. T. Antczak, Optimality results for nondifferentiable vector optimization problems with vanishing constraints, *J. Appl. Anal. Comput.*, **13** (2023), 2613–2629. https://doi.org/10.11948/20220465

25. M. K. Roy, B. C. Joshi, S. Treanţă, C. F. Marghescu, Efficiency criteria and dual models for multi-objective semi-infinite constrained minimization problems with vanishing restrictions, *Comput. Appl. Math.,* **44** (2025), 158.

26. R. R. Sahay, G. Bhatia, Higher order strict global minimizers in non-differentiable multiobjective optimization involving higher order invexity and variational inequality, *Opsearch*, **61** (2024), 226–244. https://doi.org/10.1007/s12597-023-00670-z

27. S. Treanţă, O. M. Alsalami, Results on Solution Set in Certain Interval-Valued Controlled Models, *Mathematics*, **13** (2025), 202. https://doi.org/10.3390/math13020202

28. S. Treanţă, O. M. Alsalami, Characterization Results of Extremization Models with Interval Values, *Axioms*, **14** (2025), 151. https://doi.org/10.3390/axioms14030151

29. S. Treanţă, C. F. Pırje, J. C. Yao, B. B. Upadhyay, Efficiency conditions in new interval-valued control models via modified T-objective functional approach and saddle-point criteria, *Math. Model. Control.*, **5** (2024), 180–192.

30. K. Kummari, R. R. Jaichander, S. Treanţă, C. F. Pirje, Robust parametric E_R Karush–Kuhn–Tucker optimality criteria for fractional interval-valued optimization problems, *Rend. Circ. Mat. Palermo, II. Ser 74*, **121** (2025). https://doi.org/10.1007/s12215-025-01239-z

31. H. C. Wu, On interval-valued nonlinear programming problems, *J. Math. Anal. Appl.*, **338** (2008), 299–316. https://doi.org/10.1016/j.jmaa.2007.05.023

32. X. Tan, Z. Peng, S. Reich, Y. Shehu, Generalized convex interval-valued functions and interval-valued optimization under total order relations, *Fixed Point Methods Optim.*, **2** (2025), 94–109, https://doi.org/10.69829/fpmo-025-0201-ta06

33. H.C. Wu, The Karush–Kuhn–Tucker optimality conditions in multiobjective programming problems with interval-valued objective functions, *Eur. J. Oper. Res.*, **196** (2009), 49–60. https://doi.org/10.1016/j.ejor.2008.03.012

34. A.K. Bhurjee, G. Panda, Efficient solution of interval optimization problem, *Math. Method Oper. Res.*, **76** (2012), 273–288. https://doi.org/10.1007/s00186-012-0399-0

35. D. Singh, B. Dar, and A. Goyal, KKT optimality conditions for interval valued optimization problems, *J. Nonlinear Anal. Optim.*, **5** (2014), 91–103.

36. Z. Y. Peng, C. Y. Deng, Y. O. N. G. Zhao, J. Y. Peng, Optimality conditions and duality for E-differentiable fractional multiobjective interval valued optimization problems with E-invexity, *Appl. Set Valued Anal. Optim.*, **6** (2024), 295–307.

37. T. Antczak, N. Pokharna, A nonparametric approach to nonsmooth vector fractional interval-valued optimization problems, *Chaos Soliton Fract.*, **199** (2025), 116638. https://doi.org/10.1016/j.chaos.2025.116638

38. Z. Y. Peng, J. Y. Peng, D. Ghosh, Y. Zhao, D. Li, Optimality conditions and duality results for generalized-Hukuhara subdifferentiable preinvex interval-valued vector optimization problems, *Fuzzy Sets Syst.*, **515** (2025), 109416. https://doi.org/10.1016/j.fss.2025.109416

39. J. Y. Peng, Z. Y. Peng, C. Y. Deng, M. Wen, $E-\alpha-$ Preinvex Interval-Valued Functions and Optimality Conditions, *J. Chongqing Norm. Univ. Nat. Sci. Ed.*, **41** (2024), 7–15.
40. Z. Y. Peng, J. Y. Peng, E-semi-preinvex interval-valued functions and interval-valued programming. *J. Chongqing Norm. Univ. Nat. Sci. Ed.*, **42** (2025), 117–126.
41. T. Antczak, On directionally differentiable multiobjective programming problems with vanishing constraints, *Ann. Oper. Res.*, **328** (2023), 1181–1212. https://doi.org/10.1007/s10479-023-05368-5
42. J. Jahn, Vector Approximation. *Vector Optimization: Theory, Applications, and Extensions*, (2004), 211–242. Springer. https://doi.org/10.1007/978-3-540-24828
43. J. Zhang, Q. Zheng, X. Ma, L. Li, Relationships between interval-valued vector optimization problems and vector variational inequalities, *Fuzzy Optim. Decis. Mak.*, **15** (2016), 33–55. https://doi.org/10.1007/s10700-015-9212-x
44. G. Giorgi, Osservazioni sui teoremi dell'alternativa non lineari implicanti relazioni di uguaglianza e vincolo insiemistico, *Optim. Econ. Finance Ind.*, Datanova Editrice Srl, (2002), 171–183.
45. V. Preda, I. Chiţescu, On constraint qualification in multiobjective optimization problems: semidifferentiable case, *J. Optim. Theory Appl.*, **100** (1999), 417–433. https://doi.org/10.1023/A:1021794505701
46. I. Ahmad, D. Singh, B. A. Dar, Optimality conditions in multiobjective programming problems with interval valued objective functions, *Control Cybern.*, **44** (2015), 19–45.

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