Journal of Industrial and Management Optimization

https://www.aimspress.com/journal/jimo

JIMO, 22(12): 6150–6165.
DOI:10.3934/jimo.2026207
Received: 30 June 2026
Revised: 24 October 2026
Accepted: 27 October 2026
Published: 04 November 2026

Research article

A variational inequality approach to Berge equilibrium problems

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Abstract: In this work, we study the Berge equilibrium problem (BEP) using a variational inequality (VI) framework. We establish an exact equivalence between the classical BEP and a corresponding VI problem under a pairwise symmetry condition. Furthermore, we extend this framework to generalized Berge equilibrium problems with shared (jointly convex) constraints, proving that the proposed VI formulation provides a rigorous sufficient condition for generalized Berge equilibria under appropriate assumptions. Illustrative examples are presented to demonstrate the theoretical findings.

Keywords: Berge equilibrium; variational inequalities; shared constraints; pairwise symmetry condition; generalized Berge equilibrium
Mathematics Subject Classification: 91A10, 91A06, 90C33, 90C46

1. Introduction

Game theory has become one of the fundamental mathematical tools for analysing strategic interactions arising in economics, engineering, operations research, social sciences, and many other disciplines. Since the pioneering work of Nash on non-cooperative games [1, 2], equilibrium modelling has received extensive attention, resulting in a wide range of theoretical developments and computational techniques. While Nash equilibrium has remained the dominant solution concept, alternative equilibrium notions have attracted increasing interest in recent years, particularly in situations where cooperation or mutual support among players is emphasized rather than purely individual optimization.

Among these concepts, the Berge equilibrium (BE), originally introduced by Claude Berge [3], provides an alternative view of strategic behavior by assuming that each player is supported by the

remaining participants. Zhukovskii later reformulated this concept within the framework of differential games ([4, 5]), where subsequent studies were developed primarily in the Russian literature. The extension of BE to finite strategic-form games appeared much later. Abalo and Kostreva ([6, 7]) established the first existence results for pure-strategy Berge equilibria, while Nessah et al. [8] proposed further existence theorems under more general assumptions.

Subsequent research has focused on both theoretical and computational aspects of BE. Enkhbat and Batbileg ([9–11]) investigated Berge, anti-Nash, and anti-Berge equilibria for bimatrix games. More recently, Battur et al. [12] proposed a regularized-function approach for computing BE, whereas Zapata et al. [13] introduced several relaxed equilibrium concepts based on altruistic games. Comprehensive treatments of both static and dynamic formulations of BE can be found in the monographs [14, 15].

In recent years, the mathematical formulation and resolution of multi-agent equilibrium problems have gained significant attention through the lens of variational inequalities (VIs) and stochastic control theories. For instance, the application of VIs to address complex decision-making processes in multi-agent environments was comprehensively studied by Pickl [16], who demonstrated the robust connection between optimal control and variational formulations. Further theoretical advancements in variational techniques on structured frameworks have also been investigated by Kropat et al. [17].

In parallel, the modeling of cooperative structures and network constraints has been extensively explored using interval analysis. Managing decision-making under strict constraints in dynamic environments has been tackled by Savku [18] through advanced stochastic control methodologies for constrained differential games.

While these pioneering works provide elegant mathematical frameworks for Nash-type equilibria, interval games, and standard variational systems, the formulation of BE under jointly convex constraints via VI operators remains less explored. To bridge this gap, this paper establishes a novel framework.

The applicability of BE has gradually expanded beyond classical game theory. It has been employed in the analysis of economic cooperation, environmental management, ecological systems, smart energy networks, transportation, logistics, political science, and social network analysis, where preserving the collective performance of all participants is often more desirable than pursuing purely selfish strategies.

Despite these developments, the mathematical formulation of BE remains considerably less developed than that of Nash equilibrium. In particular, variational inequality theory has become one of the most successful analytical and computational frameworks for Nash equilibrium and generalized Nash equilibrium problems, yet an analogous formulation for BE has not been systematically established. Bridging this gap is the primary objective of the present work.

The main contributions of this paper are summarized as follows:

(i) We derive a VI formulation for classical Berge equilibrium problems (BEPs) and establish its exact equivalence under a pairwise symmetry condition.
(ii) We extend this framework to BEPs with shared (jointly convex) constraints, establishing the VI formulation as a rigorous sufficient condition.
(iii) We establish that the proposed VI formulation provides a rigorous sufficient condition for generalized BE under shared constraints.

The obtained theoretical results are presented in Theorems 4 and 5, where the existence of BE is proved through the proposed VI framework.

The remainder of this paper is organized as follows. Section 2 introduces the formulation of BEPs together with the necessary definitions. Section 3 briefly reviews the basic theory of VI. Section 4 summarizes the variational inequality approach for Nash equilibrium problems. Finally, Section 5 develops the proposed VI formulations for BEPs and establishes the corresponding theoretical results.

2. BEP and different definitions

This section introduces the notation and equilibrium concepts that will be used throughout the paper. We first describe the strategic game under consideration and then recall the principal notions of BE appearing in the literature. These concepts provide the mathematical background for the VI formulations developed in the subsequent sections.

Consider the following non-cooperative game in normal form:

$$G = (X_k, f_k)_{k\in I}. \tag{1}$$

Here, $I = \{1, 2, \ldots, N\}$ denotes the set of players and

$$X = \prod_{k\in I} X_k$$

is the strategy space of the game, where $X_k \subset E_k$ represents the strategy set of player k, E_k is a vector space, and $f_k : X \to \mathbb{R}$ denotes the payoff function of player k. Furthermore,

$$X_{-k} = \prod_{i\in I\setminus\{k\}} X_i,$$

and for every strategy profile $x = (x_1, \ldots, x_N) \in X$, we write

$$x_{-k} = (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_N),$$

so that $x = (x_{-k}, x_k) \in X_{-k} \times X_k = X$.

Each player seeks to maximize his or her own payoff function. Since several notions of BE are formulated in terms of coalitions, we introduce the corresponding notation.

Let Σ denote the family of all coalitions. For every coalition $R \in \Sigma$, its complementary coalition is defined by

$$-R = \{\, i \in I \mid i \notin R \,\}.$$

When R consists of a single player $\{k\}$, we simply write $-k$ instead of $-R$. The strategy set associated with coalition R is

$$X_R = \prod_{k\in R} X_k.$$

If $\{P_k\}_{k=1}^{s}$ is a partition of the player set I, then every strategy profile can be represented as

$$x = (x_{P_1}, x_{P_2}, \ldots, x_{P_s}),$$

where $x_{P_k} \in \prod_{j\in P_k} X_j$.

Several formulations of BE have been proposed in the literature. We first recall the classical definition introduced in [8], which serves as the basis for the developments presented later in this paper.

Definition 1 ([8]). *A strategy $\bar{x} \in X$ is a simple BE of the game (1) if*

$$f_k(\bar{x}) \geq f_k(x_{-k}, \bar{x}_k) \tag{2}$$

for every player $k \in I$ and every $x_{-k} \in X_{-k}$.

The above definition captures the central idea of BE: instead of optimizing their own objectives, the remaining players jointly choose strategies that support player k. Consequently, the equilibrium is based on mutual support rather than unilateral optimality, which fundamentally distinguishes it from the Nash equilibrium concept.

Although the subsequent theoretical developments mainly rely on the classical BE and its shared-constraint extension, we also recall the following generalized definitions for completeness.

Definition 2 ([8]). *Let $R = \{R_k\}_{k\in M} \subset \Sigma$ be a partition of I, and let $S = \{S_k\}_{k\in M}$ be a family of subsets of I. A feasible strategy $\bar{x} \in X$ is called a BE point for (1) relative to R and S if*

$$f_{r_m}(\bar{x}) \geq f_{r_m}(x_{S_m}, \bar{x}_{-S_m}),$$

for every $m \in M$, $r_m \in R_m$, and $x_{S_m} \in X_{S_m}$.

Another important extension is the concept of strong BE, which strengthens the stability requirement by taking into account deviations of the remaining players.

Definition 3 ([8]). *A strategy $\bar{x} \in X$ is called a strong Berge equilibrium (SBE) of game (1) if*

$$\forall k \in I,\ \forall j \in -k,\ f_k(\bar{x}_k, y_{-k}) \leq f_j(\bar{x}), \quad \forall y_{-k} \in X_{-k}. \tag{3}$$

Intuitively, a SBE guarantees that no coalition of the remaining players can simultaneously improve the payoffs of all its members while player k keeps the equilibrium strategy. Hence, it represents a stronger stability concept than the classical BE.

Many practical applications involve common resource limitations shared by all participants. Such situations naturally motivate the introduction of BE under shared constraints.

Let $D \subseteq X = \prod_{k\in I} X_k$ be a nonempty, closed, and convex set representing the common feasible strategy space. For a fixed rival strategy vector x_{-k}, the feasible strategy set of player k is defined by

$$C_k(x_{-k}) = \{\, x_k \in X_k \mid (x_k, x_{-k}) \in D \,\}.$$

Definition 4 (BE with jointly convex constraints). *A feasible strategy profile $\bar{x} = (\bar{x}_k, \bar{x}_{-k}) \in D$ is called a BE with jointly convex constraints (or generalized BE with shared constraints) if, for every player $k \in I$,*

$$f_k(\bar{x}_k, \bar{x}_{-k}) \geq f_k(\bar{x}_k, y_{-k}), \quad \forall y_{-k} \in X_{-k} \text{ such that } (\bar{x}_k, y_{-k}) \in D. \tag{4}$$

Remark 1. *Unlike the classical BE, the rival players cannot choose their strategies independently. Every feasible response must satisfy the common constraint set D, thereby introducing coupling among the players' decisions.*

3. A Brief on variational inequality problems (VIPs)

VI theory provides a unified mathematical framework for analyzing a broad class of equilibrium models arising in optimization, game theory, economics, and engineering. Unlike classical optimization problems, where the objective is to optimize a single function, VI allow equilibrium conditions to be characterized directly through an operator acting on a feasible set. Owing to this flexibility, VI theory has become one of the principal analytical tools for studying equilibrium problems that cannot be naturally reformulated as standard optimization models.

Definition 5. *VIP* VI(F, K) *is to determine a vector* $x^* \in K \subseteq \mathbb{R}^n$*, such that*

$$F(x^*)^T(x - x^*) \geq 0, \qquad \forall x \in K, \tag{5}$$

where F *is a given continuous mapping from* K *into* $\mathbb{R}^n$*,* K *is a nonempty closed convex set, and the corresponding solution set is denoted by* SOL(K, F)*.*

The VI above admits several equivalent interpretations. Geometrically, the vector $F(x^*)$ belongs to the normal direction of the feasible set at the solution point x^*. Consequently, the VI can be expressed in terms of the normal cone as

$$-F(x^*) \in N(K, x^*). \tag{6}$$

Equivalently, the same condition may be written as the generalized inclusion

$$0 \in F(x^*) + N(K, x^*). \tag{7}$$

This inclusion represents a particular class of generalized equations and establishes a close connection between VI and several other mathematical models. In particular, systems of nonlinear equations, optimization problems, complementarity problems, fixed-point formulations, and many equilibrium models can all be studied within the same theoretical framework. For later use, we recall several classical existence and uniqueness results for VI(F, K).

Theorem 1 ([19] Existence under compactness and continuity)**.** *If* K *is a compact convex set and* $F(x)$ *is continuous on* K*, then the VIP problem admits at least one solution* x^**.*

The following theorem gives a sufficient condition guaranteeing uniqueness of the solution whenever the VI is solvable.

Theorem 2 ([19] Uniqueness)**.** *Suppose that* $F(x)$ *is strictly monotone on* K*. Then, the solution is unique, if one exists.*

A stronger monotonicity assumption leads simultaneously to both existence and uniqueness, even without requiring the feasible set to be compact.

Theorem 3 ([19] Existence and uniqueness)**.** *Assume that* $F(x)$ *is strongly monotone. Then, there exists precisely one solution* x^* *to* VI(F, K)*.*

4. VI reformulation of Nash equilibrium problems

Before introducing a VI formulation for BEPs, it is useful to briefly review the corresponding framework for Nash equilibrium problems. VI have become one of the standard mathematical tools for analyzing Nash and generalized Nash equilibrium problems because they provide a unified theoretical framework together with well-developed existence theory and efficient numerical algorithms ([20, 21]). The purpose of this section is therefore to summarize the classical VI formulation for Nash equilibrium and to motivate the developments presented in Section 5.

Consider a non-cooperative game with N players, where player k controls the decision variable $x_k \in \mathbb{R}^{n_k}$. The joint strategy vector is denoted by

$$x = (x_1, \ldots, x_N) \in \mathbb{R}^n, \qquad n = n_1 + \cdots + n_N,$$

and is sometimes written as $x = (x_k, x_{-k})$, where x_{-k} represents the strategies of all players except player k.

We consider the generalized Nash equilibrium problem (GNEP), in which every player solves an optimization problem while taking the strategies of the remaining players as fixed. For player k, this problem is given by

$$P_k(x_{-k}): \qquad \inf_{x_k} f_k(x_k, x_{-k})$$

subject to

$$x_k \in C_k(x_{-k}).$$

More specifically, let $g_k : \mathbb{R}^n \to \mathbb{R}^{m_k}$ be given, and define

$$C_k(x_{-k}) = \{x_k \in \mathbb{R}^{n_k} \mid g_k(x_k, x_{-k}) \leq 0\},$$

where each component $g_k^i,\ i = 1, \ldots, m_k$, is continuously differentiable and convex with respect to the variable x_k.

The GNEP consists of finding a strategy profile x^* such that each player's strategy is optimal with respect to the equilibrium strategies of the remaining players, namely,

$$f_k(x_k^*, x_{-k}^*) \leq f_k(y_k, x_{-k}^*), \qquad \forall y_k \in C_k(x_{-k}^*).$$

Such a point is called a generalized Nash equilibrium (GNE). When the feasible sets are independent of the rivals' decisions, that is, $C_k(x_{-k}) = C_k$, the GNEP reduces to the classical Nash equilibrium problem (NEP).

For the classical NEP with fixed feasible sets, we adopt the following standard assumptions:

(A1) $f_k : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable and, for every fixed x_{-k}, the function $f_k(\cdot, x_{-k})$ is convex.

(A2) C_k is a nonempty, closed, and convex set.

Under assumptions (A1)–(A2), the NEP admits an equivalent variational inequality formulation (see [22]):

$$\text{find } x^* \in C = \prod_{k=1}^{N} C_k, \text{ such that } F(x^*)^T(x - x^*) \geq 0, \quad \forall x \in C,$$

where $F(x) = (\nabla_{x_k} f_k(x))_{k=1}^N$.

Solutions obtained through this formulation are commonly referred to as variational equilibria (or normalized equilibria). More importantly, the VI representation enables the use of a broad range of theoretical results and computational methods that have been extensively developed in the VI literature.

For general GNEPs, however, the feasible strategy sets depend explicitly on the equilibrium point itself. Consequently, the equilibrium conditions can no longer be represented by an ordinary VI and are naturally formulated as a quasi-variational inequality (QVI). Although this formulation is mathematically equivalent, QVIs are considerably more difficult to analyze and solve algorithmically.

An important intermediate class consists of jointly convex GNEPs, also known as GNEPs with shared constraints. In this setting,

$$C_k(x_{-k}) = \{x_k \in \mathbb{R}^{n_k} \mid (x_k, x_{-k}) \in D\},$$

where $D \subseteq \mathbb{R}^n$ is a nonempty closed convex set shared by all players.

The problem is jointly convex whenever $g_1 = g_2 = \cdots = g_N = g$, and the common constraint function g is convex in the joint decision vector x. In this case, the equilibrium conditions can still be represented by the standard VI $\mathrm{VI}(F, C)$, where $C = \{x \in D \mid g(x) \leq 0\}$.

Proposition 1 ([22]). *Let a jointly convex GNEP be given and suppose that (A1)–(A2) hold. Then every solution of* $\mathrm{VI}(F, C)$ *is also a solution of the GNEP.*

This result shows that introducing shared constraints does not destroy the VI structure. Consequently, jointly convex GNEPs remain attractive from both theoretical and computational viewpoints.

When the feasible strategy set of each player depends explicitly on the strategies of the remaining players through coupling constraints, the corresponding equilibrium conditions must be formulated as a QVI.

Proposition 2 ([22]). *Let a GNEP be given. Then a point x is a GNE if and only if it is a solution of the QVI* $\mathrm{QVI}(C(x), F(x))$*, where*

$$C(x) = \prod_{k=1}^{N} C_k(x_{-k}), \qquad F(x) = (\nabla_{x_k} f_k(x))_{k=1}^N.$$

The above discussion illustrates how VI theory provides a natural mathematical framework for modeling both classical Nash equilibrium problems and important classes of GNEP. This observation serves as the principal motivation for the present work. Although BE is based on a fundamentally different behavioral principle, the next section demonstrates that an analogous VI formulation can also be established for both the classical BEP and its extension with shared constraints.

5. VI formulations of BEPs

The previous section reviewed the classical VI formulation of NEPs, which serves as the methodological foundation for many equilibrium models. Motivated by this framework, we now investigate whether an analogous VI formulation can also be established for BEPs. Although BE is based on the principle of mutual support rather than individual optimality, we show that it likewise admits a natural VI characterization. This characterization forms the basis for the existence results established in the remainder of this section.

5.1. Classical BEP and VIPs

In this subsection, we consider a VIs approach to classic BEPs. We formulate the BEP for each player k ($k = 1, ..., N$) and try to solve the optimization problem

$$P_k : \quad \max_{x_{-k} \in X_{-k}} f_k(x_{-k}, x_k).$$

For each player $k \in I$, the strategy set $X_k \subset \mathbb{R}^{n_k}$ is nonempty, compact, and convex, and the joint strategy space is given exactly by $X = X_1 \times X_2 \times \cdots \times X_n$. The payoff function of each player k, $f_k : X \to \mathbb{R}$, is continuously differentiable on X and continuously concave with respect to the strategy profile $x_{-k} \in X_{-k} = \prod_{j \neq k} X_j$.

Let the mapping $F : X \to \mathbb{R}^M$, where $M = \sum_{k=1}^{n} n_k$, define the Berge operator given by:

$$F(x) = \begin{pmatrix} -\sum_{j \neq 1} \nabla_{x_1} f_j(x) \\ -\sum_{j \neq 2} \nabla_{x_2} f_j(x) \\ \vdots \\ -\sum_{j \neq n} \nabla_{x_n} f_j(x) \end{pmatrix} \tag{8}$$

Theorem 4. *Let $I = \{1, 2, \ldots, N\}$ be the set of players. Let $X = \prod_{k \in I} X_k$, where each strategy set $X_k \subset \mathbb{R}^{n_k}$ is nonempty, convex, and compact. For each player $k \in I$, assume that the payoff function $f_k : X \to \mathbb{R}$ is continuously differentiable on X and continuously concave with respect to $x_{-k} \in X_{-k}$.*

Furthermore, assume the pairwise symmetry condition holds: for every player $j \in I$, the partial gradients of all rival payoff functions with respect to x_j are identical, that is,

$$\nabla_{x_j} f_k(x) = \nabla_{x_j} f_m(x), \quad \forall k, m \in I \setminus \{j\}, \quad \forall x \in X. \tag{S1}$$

Then, a strategy profile $x^* \in X$ is a classical BE if and only if x^* satisfies the VI $\mathrm{VI}(F, X)$*:*

$$F(x^*)^T (y - x^*) \geq 0, \quad \forall y \in X, \tag{9}$$

where $F(x) = (F_1(x), F_2(x), \ldots, F_N(x))^T$ with $F_j(x) = -\sum_{k \neq j} \nabla_{x_j} f_k(x)$.

Proof. Necessity: Let $x^* = (x_1^*, \ldots, x_N^*) \in X$ be a classical BE. By definition, for each player $k \in I$, given x_k^*, the rival strategy profile x_{-k}^* maximizes player k's payoff function over X_{-k}:

$$x_{-k}^* \in \arg\max_{y_{-k} \in X_{-k}} f_k(y_{-k}, x_k^*).$$

Since X_{-k} is convex and $f_k(\cdot, x_k^*)$ is continuously differentiable and concave on X_{-k}, this optimization problem is equivalently characterized by the first-order necessary and sufficient condition:

$$\nabla_{x_{-k}} f_k(x^*)^T (y_{-k} - x_{-k}^*) \leq 0, \quad \forall y_{-k} \in X_{-k}.$$

Multiplying by -1 gives:

$$-\nabla_{x_{-k}} f_k(x^*)^T (y_{-k} - x_{-k}^*) \geq 0, \quad \forall y_{-k} \in X_{-k}.$$

Since $X_{-k} = \prod_{j \neq k} X_j$ is a Cartesian product, choosing $y_r = x_r^*$ for all $r \neq j, k$ decomposes this inequality component-wise for each $j \in I \setminus \{k\}$:

$$\langle -\nabla_{x_j} f_k(x^*), y_j - x_j^* \rangle \geq 0, \quad \forall y_j \in X_j.$$

By the symmetry condition (S1), for any fixed $j \in I$, all rival gradients $\nabla_{x_j} f_k(x^*)$ ($k \neq j$) are identical. Thus, each gradient equals the average of all rival gradients:

$$\nabla_{x_j} f_k(x^*) = \frac{1}{N-1} \sum_{r \neq j} \nabla_{x_j} f_r(x^*) = -\frac{1}{N-1} F_j(x^*).$$

Substituting this relation into the component-wise inequality yields:

$$\left\langle \frac{1}{N-1} F_j(x^*), y_j - x_j^* \right\rangle \geq 0 \iff \langle F_j(x^*), y_j - x_j^* \rangle \geq 0, \quad \forall y_j \in X_j.$$

Since this holds independently for every $j = 1, \ldots, N$, summing over all $j \in I$ gives:

$$\sum_{j=1}^{N} \langle F_j(x^*), y_j - x_j^* \rangle \geq 0, \quad \forall y \in X,$$

which is precisely $F(x^*)^T (y - x^*) \geq 0$ for all $y \in X$.

Sufficiency: Conversely, suppose $x^* \in X$ satisfies VI(F, X):

$$\sum_{j=1}^{N} \left\langle -\sum_{k \neq j} \nabla_{x_j} f_k(x^*), y_j - x_j^* \right\rangle \geq 0, \quad \forall y \in X. \tag{$\star$}$$

Fix an arbitrary player $m \in I$. Since the total strategy space $X = \prod_{j=1}^{N} X_j$ is Cartesian (uncoupled), we can isolate the components $j \neq m$.

Specifically, for any fixed $j \in I \setminus \{m\}$, choose a test strategy profile $y \in X$ such that $y_r = x_r^*$ for all $r \neq j$. Substituting this y into ($\star$) cancels all terms except the j-th term:

$$\left\langle -\sum_{k \neq j} \nabla_{x_j} f_k(x^*), y_j - x_j^* \right\rangle \geq 0, \quad \forall y_j \in X_j.$$

By condition (S1), for any $k \in I \setminus \{j\}$, the partial gradient $\nabla_{x_j} f_k(x^*)$ is identical to $\nabla_{x_j} f_m(x^*)$ (since $m \in I \setminus \{j\}$). Thus, the inner sum of $(N-1)$ identical terms simplifies to:

$$-\sum_{k \neq j} \nabla_{x_j} f_k(x^*) = -(N-1) \nabla_{x_j} f_m(x^*).$$

Substituting this back into the inequality and dividing by the positive constant $(N-1) > 0$ yields:

$$\langle -\nabla_{x_j} f_m(x^*), y_j - x_j^* \rangle \geq 0 \implies \nabla_{x_j} f_m(x^*)^T (y_j - x_j^*) \leq 0, \quad \forall y_j \in X_j.$$

Since this holds for every $j \in I \setminus \{m\}$, summing these inequalities across all $j \neq m$ gives:

$$\nabla_{x_{-m}} f_m(x^*)^T (y_{-m} - x_{-m}^*) \leq 0, \quad \forall y_{-m} \in X_{-m}.$$

Since $f_m(\cdot, x_m^*)$ is continuously concave on the convex set X_{-m}, this first-order condition guarantees global maximality:

$$f_m(x_{-m}^*, x_m^*) \geq f_m(y_{-m}, x_m^*), \quad \forall y_{-m} \in X_{-m}.$$

Since $m \in I$ was chosen arbitrarily, x^* satisfies the definition of a classical BE for all players. □

Remark 2. *It should be emphasized that the contribution of Theorem 4 is primarily formal and theoretical. The pairwise symmetry condition (S1) is essential to establish the exact mathematical equivalence between the classical BEP and the single VI* VI(F, X). *However, condition (S1) imposes a strong structural constraint on the payoff functions, thereby limiting the range of games to which this direct VI formulation can be applied. In particular, it is well suited for symmetric games, common-pool resource models, and infrastructure routing problems where a single player's action exerts identical marginal impacts on all remaining participants.*

The condition (S1) is primarily applied in the following three key domains and problems:

1. Game theory and cooperative modeling (BE):
   It is used to simplify the process of finding an altruistic equilibrium (BE) where players maximize each other's payoffs in a multi-player game by reformulating it into a VI problem VI(F, X).
2. Common-pool resources and environmental externalities:
   It is applied in economic models where one participant's action (e.g., overfishing, natural resource extraction, pollution) exerts an identical negative impact on all other competitors through a shared environment.
3. Network congestion and infrastructure routing:
   It is used to mathematically model processes in transportation and communication networks where the load introduced by a single user (e.g., vehicles, data traffic) affects the delays and costs for all other network users in equal proportions.

In short, the (S1) condition is applied to simplify the mathematical analysis of symmetric systems where one participant's decision impacts all other participants equally.

5.2. BEP with shared constraints and VIPs

The BE model with shared convex constraints is applicable in real-world scenarios where participants share limited resources and must ensure system stability or mutual survival. This model is particularly vital when the collective good or system security takes precedence over individual profit-maximization.

Now, let us consider the BE models with shared constraints that may arise in certain fields:

1. Environmental and ecological management: When multiple nations or industries share a common ecosystem (e.g., a shared river or the atmosphere) with a total cap on CO_2 emissions or maximum allowable pollutant levels. If one player over-pollutes, the entire system collapses, harming everyone. Therefore, players limit their own production to ensure others have the opportunity to operate in a clean environment.
2. Smart grids and energy management: As renewable energy sources integrate into a unified power grid with limitations on transmission line capacities and voltage stability requirements. If all households or factories feed solar energy into the grid simultaneously, the system may crash due to overload. Here, players adjust their energy trading to maintain grid stability.
3. Transportation and logistics networks: In urban traffic or freight systems with physical throughput capacity limits. In a Nash equilibrium, if every driver chooses their fastest route, gridlock occurs. In a BE model, drivers select routes that do not block overall flow.

4. Computer networking and cloud computing: In resource allocation for data centers and 5G networks involving server memory, CPU capacity, and bandwidth. When multiple users share a server, one application consuming all resources freezes other processes. BE optimizes resource distribution.
5. Supply chain management: In collaboration between manufacturers, suppliers, and distributors facing raw material shortages or total warehouse capacity constraints. BE creates long-term stability by supporting the profitability of partners.

Let $G = (X_k, f_k, X)_{k\in I}$ be a generalized N-player strategic game with shared constraints. The payoff function of each player k, $f_k : X \to \mathbb{R}$, is continuously differentiable on X and continuously concave with respect to the rival strategy profile x_{-k}.

Under shared constraints, we consider the following result. Note that this result establishes sufficiency, but not necessity.

Theorem 5. *Let $I = \{1, 2, \ldots, N\}$ be the set of players. Let $X \subset \mathbb{R}^n$ ($n = \sum_{k=1}^{N} n_k$) be a nonempty, closed, and convex shared set. For each player $k \in I$, let $f_k : X \to \mathbb{R}$ be continuously differentiable on X and continuously concave with respect to x_{-k} over the set (slice)*

$$K_k(x_k^*) = \{y_{-k} \in \mathbb{R}^{n_{-k}} \mid (x_k^*, y_{-k}) \in X\}.$$

Assume that the pairwise symmetry condition holds:

$$\nabla_{x_j} f_k(x) = \nabla_{x_j} f_m(x), \quad \forall k, m \in I \setminus \{j\}, \quad \forall x \in X.$$

If a strategy profile $x^* \in X$ satisfies the VI $\mathrm{VI}(F, X)$*:*

$$F(x^*)^T(y - x^*) \geq 0, \quad \forall y \in X,$$

where $F(x) = (F_1(x), F_2(x), \ldots, F_N(x))^T$ with $F_j(x) = -\sum_{k\neq j} \nabla_{x_j} f_k(x)$, then x^* is a GBE of the game with shared constraints.

Proof. Suppose that $x^* \in X$ satisfies the VI $\mathrm{VI}(F, X)$, that is,

$$\sum_{j=1}^{N} F_j(x^*)^T(y_j - x_j^*) \geq 0, \quad \forall y = (y_1, y_2, \ldots, y_N) \in X. \tag{10}$$

To show that x^* is a GBE, choose an arbitrary player $k \in I$ and consider any $y_{-k} \in K_k(x_k^*)$. Construct the joint profile $\tilde{y} = (x_k^*, y_{-k})$. By the definition of the slice $K_k(x_k^*)$, we have $\tilde{y} \in X$.

Substituting $\tilde{y}$ into inequality (10), the k-th component term vanishes since $\tilde{y}_k - x_k^* = x_k^* - x_k^* = 0$, which reduces the inequality to:

$$\sum_{j\neq k} F_j(x^*)^T(y_j - x_j^*) \geq 0, \quad \forall y_{-k} \in K_k(x_k^*). \tag{11}$$

From the definition of the Berge operator components, we have $F_j(x^*) = -\sum_{m\neq j} \nabla_{x_j} f_m(x^*)$. Under the pairwise symmetry condition, for any $m \neq j$, the partial derivatives $\nabla_{x_j} f_m(x^*)$ are identical to $\nabla_{x_j} f_k(x^*)$ (since $k \in I \setminus \{j\}$). Thus, the inner sum simplifies to:

$$F_j(x^*) = -(N-1)\nabla_{x_j} f_k(x^*), \quad \forall j \neq k.$$

Substituting this relation into inequality (11) gives:

$$-(N-1)\sum_{j\neq k}\nabla_{x_j}f_k(x^*)^T(y_j-x_j^*)\geq 0.$$

Dividing both sides by the positive constant $(N-1)>0$ and reversing the inequality yields:

$$\sum_{j\neq k}\nabla_{x_j}f_k(x^*)^T(y_j-x_j^*)\leq 0,\quad \forall y_{-k}\in K_k(x_k^*),$$

which can be compactly written using the gradient with respect to y_{-k} as:

$$\nabla_{y_{-k}}f_k(x_k^*,x_{-k}^*)^T(y_{-k}-x_{-k}^*)\leq 0,\quad \forall y_{-k}\in K_k(x_k^*).$$

Since the function $f_k(x_k^*,\cdot)$ is continuously concave over the convex set $K_k(x_k^*)$, this first-order VI condition is both necessary and sufficient to guarantee global maximality over the slice:

$$x_{-k}^*\in\arg\max_{y_{-k}\in K_k(x_k^*)}f_k(x_k^*,y_{-k}).$$

Since player $k\in I$ was chosen arbitrarily, x^* satisfies the definition of a GBE for all players. □

To illustrate the transition from the BEP to a variational inequality, the following simple examples are presented, which can be easily solved.

Example 1. *Consider a 3-player non-cooperative game in normal form with strategy spaces $X_1=X_2=X_3=[0,1]$ (hence, $X=[0,1]^3$). The payoff functions $f_i:X\to\mathbb{R}$ for each player $i\in\{1,2,3\}$ are defined by:*

$$\begin{aligned}f_1(x_1,x_2,x_3)&=-(x_2-1)^2-(x_3-1)^2,\\ f_2(x_1,x_2,x_3)&=-(x_1-1)^2-(x_3-1)^2,\\ f_3(x_1,x_2,x_3)&=-(x_1-1)^2-(x_2-1)^2.\end{aligned}$$

First, we verify condition (S1). Computing the partial derivatives yields:

$$\begin{aligned}\nabla_{x_1}f_2(x)&=-2(x_1-1)=\nabla_{x_1}f_3(x),\\ \nabla_{x_2}f_1(x)&=-2(x_2-1)=\nabla_{x_2}f_3(x),\\ \nabla_{x_3}f_1(x)&=-2(x_3-1)=\nabla_{x_3}f_2(x).\end{aligned}$$

Thus, condition (S1) holds globally on X. The associated Berge operator $F(x)$ defined in (8) is:

$$F(x)=\begin{pmatrix}-\nabla_{x_1}f_2(x)-\nabla_{x_1}f_3(x)\\ -\nabla_{x_2}f_1(x)-\nabla_{x_2}f_3(x)\\ -\nabla_{x_3}f_1(x)-\nabla_{x_3}f_2(x)\end{pmatrix}=\begin{pmatrix}4(x_1-1)\\ 4(x_2-1)\\ 4(x_3-1)\end{pmatrix}.$$

Solving the VI $\mathrm{VI}(F,X)$*, we seek $x^*\in[0,1]^3$ such that:*

$$F(x^*)^T(y-x^*)=4(x_1^*-1)(y_1-x_1^*)+4(x_2^*-1)(y_2-x_2^*)+4(x_3^*-1)(y_3-x_3^*)\geq 0,\quad \forall y\in[0,1]^3.$$

The unique solution to this VI is $x^*=(1,1,1)^T$.

Direct verification against Definition 1 shows that given $x_1^*=1$, the rival profile $(x_2^*,x_3^*)=(1,1)$ maximizes $f_1(\cdot,x_1^*)$ over $[0,1]^2$. By symmetry across all players, $x^*=(1,1,1)^T$ is the classical BE. This confirms that under condition (S1), the equivalence $\mathrm{VI}(F,X)\iff$ *BE in Theorem 4 holds strictly.*

Example 2. *Consider an $N = 3$ player game with strategy vectors $x_1, x_2, x_3 \in \mathbb{R}$ and joint feasible set:*

$$X = \left\{ (x_1, x_2, x_3) \in [0,2]^3 \;\middle|\; \begin{array}{l} (x_2-1)^2 + (x_3-1)^2 \le 1, \\ (x_1-1)^2 + (x_3-1)^2 \le 1, \\ (x_1-1)^2 + (x_2-1)^2 \le 1 \end{array} \right\}.$$

Let the payoff functions $f_k : X \to \mathbb{R}$ for $k \in \{1,2,3\}$ be defined as:

$$\begin{aligned} f_1(x_1, x_2, x_3) &= -(x_2-2)^2 - (x_3-2)^2 + x_1, \\ f_2(x_1, x_2, x_3) &= -(x_1-2)^2 - (x_3-2)^2 + x_2, \\ f_3(x_1, x_2, x_3) &= -(x_1-2)^2 - (x_2-2)^2 + x_3. \end{aligned}$$

For each player $j \in \{1,2,3\}$, the partial derivatives of rival payoffs with respect to x_j satisfy:

$$\nabla_{x_j} f_k(x) = -2(x_j - 2), \quad \forall k \neq j.$$

Thus, $\nabla_{x_j} f_k(x) = \nabla_{x_j} f_m(x)$ holds for all $k, m \neq j$, confirming condition (S1).

The Berge operator $F(x) = (F_1(x), F_2(x), F_3(x))^T$ is given by:

$$F_j(x) = -\sum_{k \neq j} \nabla_{x_j} f_k(x) = 4(x_j - 2).$$

Consider the point $x^* = \left(1 + \frac{1}{\sqrt{2}}, 1 + \frac{1}{\sqrt{2}}, 1 + \frac{1}{\sqrt{2}}\right)^T$, which belongs to the feasible set X. Evaluating the Berge operator at this point gives:

$$F(x^*) = \left(-4 + \frac{4}{\sqrt{2}}, -4 + \frac{4}{\sqrt{2}}, -4 + \frac{4}{\sqrt{2}}\right)^T.$$

One can verify that x^* satisfies the VI $\mathrm{VI}(F, X)$*:*

$$F(x^*)^T (y - x^*) \ge 0, \quad \forall y \in X.$$

Since x^* satisfies the VI $\mathrm{VI}(F, X)$*, by Theorem 5 (sufficiency), x^* is guaranteed to be a GBE of the game with shared constraints.*

6. Conclusion

In this paper, we establish a unified VI framework for both classical BEPs and BEPs with jointly convex (shared) constraints. Motivated by the successful application of VI theory in analyzing Nash and generalized Nash equilibrium problems, we prove that a similar mathematical structure can be effectively applied for BE, which are fundamentally governed by the principle of mutual support and cooperation.

The main contributions of this study are summarized as follows:

1. Mathematical formulation and equivalence: Under the pairwise symmetry condition, we prove the exact equivalence between the classical BEP and its corresponding VI formulation, establishing the analytical form of the Berge operator.

2. Extension to shared constraints: We extend this approach to GBE problems involving shared resource and environmental constraints. We establish that the VI formulation serves as a robust and rigorous sufficient condition for GBE under the pairwise symmetry condition, rather than an existence guarantee under the stated assumptions.
3. Verification through examples: We illustrate and confirm the validity of the proposed theoretical findings through concrete numerical examples, highlighting both the exact equivalence for classical problems and the sufficient condition for shared-constraint models.

Author contributions

Conceptualization: Battur Gompil; Methodology: Battur Gompil, Batbileg Sukhee, Mengkezhula Sagaarenchen, Ankhbayar Gelegbadam; Formal analysis: Battur Gompil, Mengkezhula Sagaarenchen; Investigation: Mengkezhula Sagaarenchen, Battur Gompil; Validation: Batbileg Sukhee, Mengkezhula Sagaarenchen; Visualization: Mengkezhula Sagaarenchen, Ankhbayar Gelegbadam; Writing – original draft: Battur Gompil; Writing – review and editing: Batbileg Sukhee, Battur Gompil; Supervision: Enkhbat Rentsen; Project administration: Batbileg Sukhee.

Use of Generative-AI tools declaration

The authors declare that generative AI tools (ChatGPT) were used for translating some sentences during the preparation of this manuscript. All translated content was carefully reviewed and edited by the authors, who take full responsibility for the accuracy, originality, and integrity of the work.

Acknowledgments

The research was supported by the Mongolian Foundation for Science and Technology (MFST) under Grant No. 2024/203.

Conflict of interest

The authors declare that they have no conflict of interest.

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