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Research article

Budget-constrained semi-dynamic optimal reinsurance strategies

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Abstract: In this paper, we study a budget-constrained reinsurance problem under a semi-dynamic framework, where an insurer faces a random number of claims and seeks to maximize the expected utility of terminal wealth. The occurrence times of claims are incorporated through discounting, allowing the model to capture the time value. For independent claims, we prove the optimality of stop-loss reinsurance contracts for a risk-averse insurer under a premium budget constraint. In the risk-neutral case, we derive pairwise structural properties of the deductible allocation in terms of the claim severities and occurrence times. We find that assigning the larger of two deductible levels to a stochastically larger claim whose arrival time is smaller in the Laplace-transform order is weakly preferable. Additionally, we establish a comparison among deductible allocations via the majorization order. When claims are dependent but the exact dependence structure is unknown, we characterize the worst-case dependence structure as comonotonicity and also establish the optimality of stop-loss reinsurance contracts under this worst-case scenario. Numerical examples and comparative statics based on Monte Carlo simulation are provided to illustrate the theoretical results.

Keywords: budget constraint; optimal reinsurance; semi-dynamic; stop-loss; stochastic order; comonotonicity
Mathematics Subject Classification: 91G05, 91G80

1. Introduction

Due to significant interest from both actuaries and academia, the optimal reinsurance decision problem has long been one of the most popular research topics in the context of actuarial science. On the one hand, reinsurance is an effective tool for risk management for insurers, making them constantly seek better reinsurance strategies for controlling liabilities. On the other hand, optimal reinsurance involves formulating optimization problems and deriving their solutions. Due to the intertwined nature of the optimal reinsurance problem, many reinsurance models have been proposed in the literature. The seminal work [1] demonstrated that stop-loss contracts are optimal for policyholders subject to the classical expected value premium principle for minimizing the variance of the retained loss. Using the same premium principle, [2] showed the optimality of stop-loss reinsurance for the maximization of the expected utility of the terminal wealth of a risk-averse insurer. Following these two seminal works, considerable effort has been made to solve (re)insurance problems under various model settings with different objective functions and constraints, thus leading to a rich class of theoretical results and contributing to a better understanding of this research area. Subsequent studies have extended optimal reinsurance design in several directions, including VaR- and CTE-based criteria [3, 4], general risk measures [5–7], variance-related premium principles [8].

The notions of stochastic orders, particularly the convex order and the increasing convex and concave orders,[1] have been widely used in the literature to compare risky alternatives in decision analyses according to different criteria. If a group of risks have the same means, then the convex order can be adopted to compare risks by their variations, which helps the decision-maker to select the least risky one, that is, the one with the smallest variation. For example, with the aid of the convex ordering technique, [9] solved the optimal reinsurance decision problem by minimizing measures of the dispersion of the ceded losses. Follow-up works on the application of convex order can be found in [10–12]. Additionally, the increasing convex and concave orders are commonly used in economics and actuarial science to compare the means of different sets of portfolios or risks and their variations for investors' profit-seeking purposes and insurers' risk-averse attitudes, respectively. This strand of literature and applications includes [13–16]. As will be seen later, these orders play a significant role in proving our main results.

In static reinsurance problems, one often assumes a fixed number of claims n within a single period. However, this assumption is restrictive in practice, as the number of claims over a time horizon $[0, t]$ is typically random. This motivates extending the model by treating the number of claims as a random variable. Specifically, we denote the random number of claims occurring in $[0, t]$ by N, where $t > 0$. Conditional on $N = n$, we consider a collection of labeled claims $(X_1, \ldots, X_n)$ with the corresponding occurrence times denoted by $(T_1, \ldots, T_n)$. Here, the index i only serves as a label and does not indicate the chronological order. When the number of claims N is modeled as an integer-valued random variable, a simplifying assumption is considered, which is that N is independent of the claim severities. Under this framework, [17] studied the optimal reinsurance arrangements by minimizing the variance or the linear combination of the squared expectation and variance of the aggregate retained loss.

Another major stream of reinsurance literature considers fully dynamic models by introducing surplus processes in continuous time. In this setting, reinsurance decisions are dynamically made according to the evolution of the insurer's surplus, and the objective is typically to either minimize ruin probabilities, maximize the expected utility of terminal wealth, or optimize a mean–variance criterion. Dynamic treatments of optimal reinsurance problems can be traced back to [18, 19]. Under

[1]The increasing convex order is also known as the stop-loss order in actuarial science, and it can be equivalently characterized by comparing their corresponding stop-loss premiums among risks.

such surplus dynamics, the Hamilton–Jacobi–Bellman (HJB) approach is commonly employed. Within this stochastic-control framework, subsequent studies have considered various objectives, including ruin-probability minimization [20], joint reinsurance and dividend optimization [21], optimal dividend and capital-management policies [22–24], mean–variance investment–reinsurance problems [25, 26]. For broader background on insurance risk and ruin theory, see [27, 28].

However, there are few studies on optimal reinsurance problems considering both the random number of claims and occurrence times of claims. Classical collective risk models, such as that of [17], consider aggregate losses but ignore the occurrence times of claims. In practical situations, the occurrence time of each claim affects its present value and therefore influences reinsurance decisions. This is particularly relevant for liability insurance, where the claim settlement may be delayed. Motivated by this, we extend the classical collective risk model by incorporating claim occurrence times through discounting and refer to this framework as a semi-dynamic claim model. The model remains static in the sense that reinsurance decisions are made at the portfolio level rather than dynamically through surplus evolution, while it retains dynamic features through discounted claim occurrences. Compared with fully dynamic continuous-time models based on surplus processes, this framework preserves tractability and avoids unnecessary assumptions on the underlying arrival process. Under this semi-dynamic setting, we study the optimal reinsurance design for an insurer with initial wealth facing a random number of claims over $[0, t]$. The insurer is assumed to have a limited budget for purchasing reinsurance, and the objective is to maximize the expected utility of the insurer's terminal wealth under different assumptions on N, T_i, and X_i.

By extending the classical collective risk framework to a semi-dynamic setting with discounted claim occurrence times, we revisit and enrich the study of optimal reinsurance design under a random number of claims. This paper contributes to the existing literature in the following aspects.

- We identify the optimal reinsurance contract structure on a claim-by-claim basis, reflecting a tailor-made contracting feature under the proposed semi-dynamic claim model. This differs from [17], where a single aggregate contract is considered and discounting effects are not incorporated.
- Unlike HJB-based approaches for surplus-process models, our analysis relies on stochastic-ordering arguments. Under the stated assumptions, we prove that the optimal contract takes a stop-loss form. When the insurer is risk-neutral, we establish conditional pairwise comparisons of deductible allocations according to the claim severities and arrival times.
- When the claims are dependent, while remaining independent of occurrence times and the claim number, we characterize the worst-case dependence structure as comonotonic and establish the robust optimality of stop-loss reinsurance contracts under this worst-case scenario. This contributes to the literature on robust optimal reinsurance; for example, see [29, 30].

The remainder of the paper is organized as follows: Section 2 revisits definitions and notions used in our present work; Section 3 formulates two classes of optimal reinsurance decision-making problems with respect to the expected utility of the terminal wealth for an insurer confronted with a random number of losses and a fixed premium budget; in Section 4, the optimal reinsurance decision problem is investigated when the claim severities are independent; for the case of generally dependent claim severities, the optimization problem is studied in Section 5; and numerical examples and comparative studies are provided in Section 6 and Section 7 concludes the paper.

2. Preliminaries

We recall definitions and properties of several stochastic orders and the concept of comonotonicity, which will be used in this paper. Throughout the paper, all expectations are assumed to exist whenever they appear, and the terms *increasing* and *decreasing* mean *non-decreasing* and *non-increasing*, respectively. For non-negative random loss variables X and Y with finite means defined on the common atomless probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we denote their cumulative distribution functions (c.d.f.'s) by F_X and F_Y, and survival functions (s.f.'s) by $\bar{F}_X$ and $\bar{F}_Y$, respectively. Let $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}_+ = [0, +\infty)$ and $\overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{\infty\}$. We use '$\leq_{\text{a.s.}}$' to denote that the left-hand side of the inequality is smaller than the right-hand side almost surely, '$\stackrel{\text{sgn}}{=}$' to denote that both sides of the equality have the same sign, and '$\stackrel{\text{d}}{=}$' to denote the equality in distribution.

2.1. Stochastic orders

Stochastic orders are very useful to compare the magnitude and variability of losses or claims in actuarial science. For example, if two risks are ordered in the usual stochastic order, then their Value-at-Risk (VaR) measures can be compared for any given confidence level.[2] Moreover, the convex order and related orders are commonly used in risk management to compare the expected utility functions or risk measures of different risks encountered by various kinds of investors.

Definition 2.1. *X is said to be smaller than Y in the usual stochastic (convex, concave, increasing convex, increasing concave, resp.) order, denoted by $X \leq_{\text{st}} Y$ ($X \leq_{\text{cx}} Y$, $X \leq_{\text{cv}} Y$, $X \leq_{\text{icx}} Y$, $X \leq_{\text{icv}} Y$, resp.), if $\mathbb{E}[\phi(X)] \leq \mathbb{E}[\phi(Y)]$ for all increasing (convex, concave, increasing convex, increasing concave, resp.) functions $\phi : \mathbb{R} \mapsto \mathbb{R}$.*

In finance and economics, $X \leq_{\text{st}} Y$ means that Y has first-order stochastic dominance over X, while $X \leq_{\text{icv}} Y$ means that Y has second-order stochastic dominance over X. The following implications always hold:

$$X \leq_{\text{st}} Y \Longrightarrow \begin{cases} X \leq_{\text{icx}} Y; \\ X \leq_{\text{icv}} Y, \end{cases}$$

and each of the latter two relations implies $\mathbb{E}[X] \leq \mathbb{E}[Y]$. We refer interested readers to the monographs by [13, 15] for a detailed treatment of the mentioned stochastic orders.

The following lemma gives a useful characterization of the usual stochastic order based on the coupling method, which also plays a key role in our proof.

Lemma 2.2. (Theorem 1.A.1 in [15]) *If $X \leq_{\text{st}} Y$, then there exist random variables $\widetilde{X}$ and $\widetilde{Y}$, defined on a common probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$, such that $\widetilde{X} \stackrel{\text{d}}{=} X$, $\widetilde{Y} \stackrel{\text{d}}{=} Y$ and $\widetilde{X}(\widetilde{u}) \leq \widetilde{Y}(\widetilde{u})$ for all $\widetilde{u} \in \widetilde{\Omega}$, (i.e., $\widetilde{X} \leq_{\text{a.s.}} \widetilde{Y}$).*

It is well known that $X \leq_{\text{cx}} Y$ is equivalent to $Y \leq_{\text{cv}} X$, and $X \leq_{\text{icx}} Y$ is equivalent to $-X \geq_{\text{icv}} -Y$. Additionally, it holds that $X \leq_{\text{cx}} Y$ if and only if $X \leq_{\text{icx}} Y$ and $\mathbb{E}[X] = \mathbb{E}[Y]$.

Recall that the Laplace transform of a non-negative random variable X is defined as follows:

$$L_X(t) = \mathbb{E}[e^{-tX}], \quad t \geq 0.$$

[2] The VaR of a nonnegative random variable Z at a confidence level $\alpha \in (0,1)$ is defined as $\text{VaR}_\alpha(Z) := F_Z^{-1}(\alpha) = \inf\{z \in \mathbb{R}_+ | \mathbb{P}(Z \leq z) \geq \alpha\}$.

Definition 2.3. *Given two non-negative random variables X and Y, X is said to be smaller than Y in the Laplace transform order, denoted by $X \leq_{\rm lt} Y$, if $L_X(t) \geq L_Y(t)$ for $t \geq 0$.*

The Laplace transform order compares random variables according to both their "location" and their "spread". Compared to the usual stochastic order and the increasing concave order, the following implication is well known:

$$X \leq_{\rm st} Y \Longrightarrow X \leq_{\rm icv} Y \Longrightarrow X \leq_{\rm lt} Y.$$

Hence, the Laplace transform order is a weaker order, but it exhibits stronger analytical properties in some research problems, especially in convolution/mixture calculations. In actuarial science, the Laplace transform order has been widely applied in various research directions, such as those in [31–33], to name a few. For more details, interested readers can refer to [15].

2.2. Comonotonicity

A subset $A \subseteq \mathbb{R}^n$ is said to be *comonotonic* if, for any two vectors $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_n)$ belonging to A, either $x_i \leq y_i$ or $x_i \geq y_i$ for all $i = 1, \ldots, n$. A random vector $\boldsymbol{X} = (X_1, \ldots, X_n)$ in $\mathbb{R}^n$ is said to be comonotonic if there exists a comonotonic subset A of $\mathbb{R}^n$ such that $\mathbb{P}(\boldsymbol{X} \in A) = 1$. The comonotonicity of a random vector depicts the strongest positive dependence structure among its coordinates, which is commonly adopted to model risks/losses in actuarial science in the most conservative sense when their exact dependence structure is unknown.

Let $F_1, \ldots, F_n$ be n univariate distribution functions for $X_1, \ldots, X_n$. We use $\mathcal{R}(F_1, \ldots, F_n)$ to denote the class of all the n-dimensional random vectors whose marginal distributions are given by $F_1, \ldots, F_n$. The following lemma is well known, stating that the sum of comonotonic random vectors is maximal in the sense of the convex order. Given $X_1, X_2, \ldots$ with marginal $F_1, F_2, \ldots$, we say $\{X_i^c\}_{i\geq 1}$ is the sequence of comonotonic counterparts of $\{X_i\}_{i\geq 1}$ if $(X_1^c, \ldots, X_n^c) \in \mathcal{R}(F_1, \ldots, F_n)$ is comonotonic, for all $n = 1, 2, \ldots$

Lemma 2.4. (Theorem 6 in [34]) *If $(X_1^c, \ldots, X_n^c) \in \mathcal{R}(F_1, \ldots, F_n)$ is comonotonic, then $\sum_{i=1}^n X_i \leq_{\rm cx} \sum_{i=1}^n X_i^c$.*

We extend Lemma 2.4 to the case of a sequence of variables when the number of variables is random.

Lemma 2.5. *Let $\{X_i\}_{i\geq 1}$ be a sequence of random variables, and let $\{X_i^c\}_{i\geq 1}$ be a sequence of random variables with X_i^c being the comonotonic counterpart of X_i for $i = 1, 2, \ldots$, and N be a random variable taking values in $\{0, 1, 2, \ldots\}$, which is independent of $\{X_i\}_{i\geq 1}$ and $\{X_i^c\}_{i\geq 1}$. Then,*

$$\sum_{i=1}^N X_i \leq_{\rm cx} \sum_{i=1}^N X_i^c.$$

Proof. Note that Lemma 2.4 holds for any fixed $N = n$. It is straightforward to have, for any convex function ϕ,

$$\mathbb{E}\left[\phi\left(\sum_{i=1}^N X_i\right)\right] = \sum_{n=0}^{\infty} \mathbb{P}(N = n)\mathbb{E}\left[\phi\left(\sum_{i=1}^n X_i\right)\right]$$

$$\leq \sum_{n=0}^{\infty} \mathbb{P}(N=n)\mathbb{E}\left[\phi\left(\sum_{i=1}^{n} X_i^c\right)\right] = \mathbb{E}\left[\phi\left(\sum_{i=1}^{N} X_i^c\right)\right],$$

which yields the desired result. □

The following lemma presents a useful convex ordering result for two different sequences of increasing functions imposed on a set of comonotonic random variables.

Lemma 2.6. (Theorem 3.4 in [11]) *Let $(X_1, \ldots, X_n)$ be a comonotonic random vector, and f_i, g_i be increasing functions such that $f_i(X_i) \leq_{\mathrm{cx}} g_i(X_i)$, for $i = 1, \ldots, n$. Then, $\sum_{k=1}^{n} f_k(X_k) \leq_{\mathrm{cx}} \sum_{k=1}^{n} g_k(X_k)$.*

We extend Lemma 2.6 to the case of a sequence of comonotonic variables when the number of variables is random.

Lemma 2.7. *Let $\{X_i\}_{i\geq 1}$ be a sequence of random variables, N be a random variable taking values in $\{0, 1, 2, \ldots\}$ which is independent of $\{X_i\}_{i\geq 1}$, and $\{f_i\}_{i\geq 1}$, $\{g_i\}_{i\geq 1}$ be two sequences of increasing functions such that $f_i(X_i) \leq_{\mathrm{cx}} g_i(X_i)$ for $i = 1, 2, \ldots$ If the random vector $(X_1, \ldots, X_n)$ is comonotonic for any $n = 2, 3, \ldots$, then*

$$\sum_{i=1}^{N} f_i(X_i) \leq_{\mathrm{cx}} \sum_{i=1}^{N} g_i(X_i).$$

The proof of Lemma 2.7 follows the same argument as that of Lemma 2.5, hence we omit it. Both Lemma 2.5 and Lemma 2.7 are important in deriving the main theoretical results in subsequent sections.

3. Problem formulation

The insurer faces a loss, denoted by X, and considers purchasing a reinsurance contract to cede part of the loss to the reinsurer. Throughout the paper, we assume that the claim number and all claim severities have finite second moments. In a typical reinsurance contract, the loss X is split into $R(X)$ and $X - R(X)$, where $R(X)$ represents the portion of loss ceded to a reinsurer while $X - R(X)$ is the residual loss retained by the insurer. Any ceded loss function $R(\cdot)$ (also known as the *indemnity function*) is assumed to belong to the following admissible set:

$$\mathcal{L} := \{R : \mathbb{R}_+ \mapsto \mathbb{R}_+ | R(0) = 0, 0 \leq R(x_2) - R(x_1) \leq x_2 - x_1, \forall\, 0 \leq x_1 \leq x_2\}.$$

Functions in $\mathcal{L}$ satisfy the following desirable properties: (i) the ceded loss is increasing, and the compensation from the reinsurer should be no greater than the size of the total loss; and (ii) the increment of a ceded loss cannot exceed the increment of the total loss. The conditions imposed on the admissible set $\mathcal{L}$ are referred to as *incentive-compatibility* conditions, which are motivated by [35]. Recently, incentive compatibility conditions have been adopted in many studies on optimal reinsurance design; for example, see [7, 36–38].

The following result states that for any admissible ceded loss function $R \in \mathcal{L}$, there always exists a stop-loss contract such that the retained loss of the insurer is the smallest in the sense of the convex order.

Lemma 3.1. (Lemma 3 in [9]) *Suppose that for any non-negative random variable X and $R \in \mathcal{L}$, there exists some $d \in \mathbb{R}_+$ such that $\mathbb{E}[R(X)] = \mathbb{E}[(X-d)_+]$. Then, it must hold that $X - (X-d)_+ \leq_{\mathrm{cx}} X - R(X)$.*

The above lemma applies whenever a finite deductible d can be chosen to match the expected ceded loss. It is clear to see that the above result also holds for $d = \infty$, which corresponds to the no-reinsurance case.

For a risk-averse insurer, the total risk exposure within $[0, t]$ consists of two parts: the retained loss and the premium paid for the reinsurance contract. Let N be a random variable representing the number of claims occurring during $[0, t]$. Each claim X_i is associated with an occurrence time T_i, which indicates when the loss is realized. The insurer considers purchasing an individualized reinsurance contract $R_i(X_i)$ for each claim X_i. Consequently, the aggregate discounted retained loss is given by the following:

$$\sum_{i=1}^{N}(X_i - R_i(X_i))e^{-\delta T_i},$$

where $\delta > 0$ is a positive constant denoting force of interest (the continuously compounded interest rate). This formulation differs from the classical static optimal reinsurance models [1, 2], where the occurrence times of claims are not incorporated in risk management and only the aggregate loss is considered. In our setting, the occurrence times affect both the discounted retained losses and the premium valuation, which introduces an intermediate structure between static and fully dynamic models.

We refer to this framework as a semi-dynamic model as all reinsurance decisions are made at the initial time, whereas realized claim losses are discounted according to their random occurrence times. Moreover, we allow the insurer to adopt individualized reinsurance contracts for different claims, in contrast to the single-contract framework studied in [17]. Additionally, related discount effects have been considered in [39–41], where the optimal allocation of upper limits and deductibles was studied for a fixed number of claims. Our framework accommodates a random number of claims and seeks to determine the optimal contract form itself, which requires new analytical arguments to establish optimality.

Following the expected value premium principle $\pi(\cdot) = (1 + \theta)\mathbb{E}[\cdot]$ with $\theta > 0$ being the premium loading factor, the total reinsurance premium paid at time 0 is given by the following:

$$(1+\theta)\mathbb{E}\left[\sum_{i=1}^{N} R_i(X_i)e^{-\delta T_i}\right].$$

When deciding the optimal reinsurance arrangements, the insurer has an initial budget, denoted by $\Pi_0 > 0$. It is natural to assume that, throughout this work,

$$0 < \Pi_0 \leq (1+\theta)\mathbb{E}\left[\sum_{i=1}^{N} X_i e^{-\delta T_i}\right]. \tag{3.1}$$

The available budget does not exceed the premium required for full reinsurance of all claims.

We intend to study the optimal reinsurance contract per claim to maximize the expected utility of the insurer's terminal wealth. The insurer is endowed with an initial wealth $0 < w < \infty$ and has a premium budget. The risk-averse attitude of the insurer is reflected by a utility function u, which is increasing and concave. We state the problem of interest as follows.

Problem 1. *Suppose the non-negative claim severities $\{X_i\}_{i\geq 1}$ are independent and their associated reinsurance policy is denoted by $\boldsymbol{R} = \{R_i\}_{i\geq 1}$, consisting of the reinsurance contract R_i, such that $\boldsymbol{R} \in \mathcal{L}$,*[3] *we intend to determine the optimal reinsurance policy $\boldsymbol{R}^* = (R_i^*)_{i\geq 1}$ that solves*

$$\begin{cases} \max\limits_{\boldsymbol{R}\in\mathcal{L}} \mathbb{E}\left[u\left(w - \sum\limits_{i=1}^{N}(X_i - R_i(X_i))e^{-\delta T_i} - \Pi_{\boldsymbol{R}_N}\right)\right] \\ \text{s.t.} \quad \Pi_{\boldsymbol{R}_N} \leq \Pi_0, \\ \textit{where } X_i\textit{'s are independent of } T_i\textit{'s and } N, \end{cases}$$

and $\Pi_{\boldsymbol{R}_N}$ denotes the aggregate reinsurance premium:

$$\Pi_{\boldsymbol{R}_N} = (1+\theta)\mathbb{E}\left[\sum_{i=1}^{N} R_i(X_i)e^{-\delta T_i}\right]. \tag{3.2}$$

In some insurance scenarios, the claims may be (highly) interdependent due to external common shocks affecting claim severities. Though copulas are often implemented to capture the dependence structure among claims, a single fixed finite-dimensional copula is not directly suited to the present setting with a random number of claims. Therefore, we adopt a worst-case approach, and identify which dependence structure returns the least favorable outcome for the insurer. To this end, we next formulate a robust version of Problem 1. First, we shall determine the worst dependence structure that minimizes the insurer's expected utility over an admissible set of dependence structures. Since the exact dimension of the claim vector may not be fixed *a priori* due to the random number of claims, we represent the claims by an infinite sequence to accommodate the random claim number represented as below.

To formalize the uncertainty in the dependence structure, we extend the finite-dimensional notation introduced in Section 2.2. Let

$$\mathcal{R}(F_1, F_2, \ldots) := \{\boldsymbol{X} = (X_1, X_2, \ldots) : X_i \sim F_i, i \geq 1\}$$

denote the class of all claim sequences with marginal distributions $F_1, F_2, \ldots$, while their dependence structure is left unspecified.

Problem 2. *For marginal distributions $F_1, F_2, \ldots$, we consider the following robust optimal reinsurance problem:*

$$\begin{cases} \max\limits_{\boldsymbol{R}\in\mathcal{L}} \min\limits_{\boldsymbol{X}\in\mathcal{R}(F_1,F_2,\ldots)} \mathbb{E}\left[u\left(w - \sum\limits_{i=1}^{N}(X_i - R_i(X_i))e^{-\delta T_i} - \Pi_{\boldsymbol{R}_N}\right)\right] \\ \qquad \text{s.t.} \quad \Pi_{\boldsymbol{R}_N} \leq \Pi_0, \\ \textit{where } X_i\textit{'s are independent of } T_i\textit{'s and } N, \end{cases}$$

and u, w, R_i, T_i, δ, Π_0, and $\Pi_{\boldsymbol{R}_N}$ are the same as those in Problem 1.

In the subsequent sections, we shall solve Problems 1 and 2 by identifying that stop-loss contracts are optimal. Additionally, we investigate qualitative properties of the deductible allocations, including pairwise ordering relations and the dispersion of deductible components.

[3] With a slight abuse of notation, $\boldsymbol{R} \in \mathcal{L}$ means that every element $R_i \in \mathcal{L}$ for $i = 1, 2, \ldots$, here and afterward.

4. Independent claim severities

Throughout this section, the claim number N is an integer-valued random variable, independent of the claim sequence $\{X_i\}_{i\geq 1}$ and the occurrence time sequence $\{T_i\}_{i\geq 1}$. The non-negative claim severities $\{X_i\}_{i\geq 1}$ are independent of both $\{T_i\}_{i\geq 1}$ and N, and they are independent but not necessarily identically distributed. Moreover, the occurrence times are assumed to be mutually independent. The optimal reinsurance contract can be characterized.

Theorem 4.1. *Suppose that N is an integer-valued random variable independent of both the non-negative claim severities $\{X_i\}_{i\geq 1}$ and the occurrence times $\{T_i\}_{i\geq 1}$. Assume that $\{X_i\}_{i\geq 1}$ consists of mutually independent random variables, $\{T_i\}_{i\geq 1}$ consists of mutually independent random variables, and the two sequences are independent of each other. Any admissible ceded loss function $R_i(\cdot)$ is suboptimal to the stop-loss contract $R_{d_i}(\cdot) = (\cdot - d_i)_+$ with $d_i \in \overline{\mathbb{R}}_+$ satisfying $\mathbb{E}[R_{d_i}(X_i)] = \mathbb{E}[R_i(X_i)]$ for $i = 1, 2, \ldots$ in Problem 1. The reinsurance premium $\Pi_{\boldsymbol{R}_N^*}$ is defined as follows:*

$$\Pi_{\boldsymbol{R}_N^*} = (1+\theta)\mathbb{E}\left[\sum_{i=1}^{N} R_{d_i}(X_i)e^{-\delta T_i}\right].$$

Proof. For any admissible ceded loss function $R_i(\cdot)$, we choose the corresponding stop-loss function $R_{d_i}(X_i) = (X_i - d_i)_+$ where $d_i \in \overline{\mathbb{R}}_+$ satisfies $\mathbb{E}[R_i(X_i)] = \mathbb{E}[(X_i - d_i)_+]$ for $i = 1, 2, \ldots$ If $d_i = \infty$ for some i, then $\mathbb{E}[R_i(X_i)] = 0$. Since $R_i(X_i) \geq 0$, this implies $R_i(X_i) = 0$ a.s., and hence $R_{d_i}(X_i) = 0 = R_i(X_i)$ a.s.; therefore, the desired comparison holds with equality. Thus, in applying Lemma 3.1 below, it suffices to consider $d_i < \infty$ for all i. In light of the independence assumption between N and X_i's (also X_i's and T_i's), we can see that

$$\begin{aligned}
\Pi_{\boldsymbol{R}_N} &= (1+\theta)\mathbb{E}\left\{\mathbb{E}\left[\sum_{i=1}^{N} R_i(X_i)e^{-\delta T_i}\middle| N\right]\right\} \\
&= (1+\theta)\sum_{n=0}^{\infty}\mathbb{P}(N=n)\mathbb{E}\left[\sum_{i=1}^{n} R_i(X_i)e^{-\delta T_i}\right] \\
&= (1+\theta)\sum_{n=0}^{\infty}\mathbb{P}(N=n)\sum_{i=1}^{n}\mathbb{E}[R_i(X_i)]\mathbb{E}[e^{-\delta T_i}] \\
&= (1+\theta)\sum_{n=0}^{\infty}\mathbb{P}(N=n)\sum_{i=1}^{n}\mathbb{E}[R_{d_i}(X_i)]\mathbb{E}[e^{-\delta T_i}] \\
&= (1+\theta)\mathbb{E}\left\{\mathbb{E}\left[\sum_{i=1}^{N} R_{d_i}(X_i)e^{-\delta T_i}\middle| N\right]\right\} = \Pi_{\boldsymbol{R}_N^*}.
\end{aligned}$$

Hence, the budget constraint is not affected when $\boldsymbol{R}_N$ is replaced by $\boldsymbol{R}_N^*$. Upon using Lemma 3.1 and the convexity of $-u(a-x)$ in x for any constant a, one has the following:

$$\mathbb{E}\left[-u\left(w - \sum_{i=1}^{N}(X_i - R_i(X_i))e^{-\delta T_i} - \Pi_{\boldsymbol{R}_N}\right)\right]$$

$$
\begin{aligned}
&= \mathbb{E}\left[-u\left(w-\sum_{i=1}^{N}(X_i-R_i(X_i))e^{-\delta T_i}-\Pi_{\boldsymbol{R}_N^*}\right)\right]\\
&= \mathbb{E}\left\{\mathbb{E}\left[-u\left(w-\sum_{i=1}^{N}(X_i-R_i(X_i))e^{-\delta T_i}-\Pi_{\boldsymbol{R}_N^*}\right)\middle|N\right]\right\}\\
&= \sum_{n=0}^{\infty}\mathbb{P}(N=n)\mathbb{E}\left[-u\left(w-\sum_{i=1}^{n}(X_i-R_i(X_i))e^{-\delta T_i}-\Pi_{\boldsymbol{R}_N^*}\right)\right]\\
&\geq \sum_{n=0}^{\infty}\mathbb{P}(N=n)\mathbb{E}\left[-u\left(w-\sum_{i=1}^{n}(X_i-R_{d_i}(X_i))e^{-\delta T_i}-\Pi_{\boldsymbol{R}_N^*}\right)\right]\\
&= \mathbb{E}\left\{\mathbb{E}\left[-u\left(w-\sum_{i=1}^{N}(X_i-R_{d_i}(X_i))e^{-\delta T_i}-\Pi_{\boldsymbol{R}_N^*}\right)\middle|N\right]\right\}\\
&= \mathbb{E}\left[-u\left(w-\sum_{i=1}^{N}(X_i-R_{d_i}(X_i))e^{-\delta T_i}-\Pi_{\boldsymbol{R}_N^*}\right)\right],
\end{aligned}
$$

where the inequality is based on the independence between T_i's and X_i's by applying Theorem 3.A.12(b) in [15] that the convex order is closed under mixtures. The desired result follows. □

Theorem 4.1 implies that stop-loss reinsurance is optimal to maximize the expected utility of terminal wealth under a premium budget, provided that X_i's are independent but not necessarily identically distributed, and the occurrence times T_i's are independent. The next corollary shows that the stop-loss reinsurance contracts minimize the insurer's total exposure under the mean-variance preference.

Corollary 4.2. *Under the setup of Theorem 4.1, we additionally assume that N is bound almost surely; then, any admissible ceded loss function $R_i(\cdot)$ is suboptimal to the stop-loss contract $R_{d_i}(\cdot) = (\cdot - d_i)_+$ with $d_i \in \overline{\mathbb{R}}_+$ satisfying $\mathbb{E}[R_{d_i}(X_i)] = \mathbb{E}[R_i(X_i)]$ of the following problem:*

$$
\begin{cases}
\min\limits_{\boldsymbol{R}\in\mathcal{L}} \left\{\mathbb{E}\left[\sum_{i=1}^{N}(X_i-R_i(X_i))e^{-\delta T_i}+\Pi_{\boldsymbol{R}_N}\right]+\beta\mathrm{Var}\left[\sum_{i=1}^{N}(X_i-R_i(X_i))e^{-\delta T_i}+\Pi_{\boldsymbol{R}_N}\right]\right\}\\
\text{s.t. } \Pi_{\boldsymbol{R}_N}\leq \Pi_0,
\end{cases}
$$

where $\beta \geq 0$ is the weight assigned to the variability of the retained loss of the insurer.

Proof. For any arbitrary given sequence of ceded loss functions $R_i(\cdot)$ for $i = 1, 2, \ldots$, we choose corresponding stop-loss functions $R_{d_i}(\cdot) = (\cdot - d_i)_+$ such that $\mathbb{E}[R_{d_i}(X_i)] = \mathbb{E}[R_i(X_i)]$. In light of the proof of Theorem 4.1, we have $\Pi_{\boldsymbol{R}_N} = \Pi_{\boldsymbol{R}_N^*}$, and the budget constraint is not affected. Thus, we have the following:

$$
\mathbb{E}\left[\sum_{i=1}^{N}(X_i-R_i(X_i))e^{-\delta T_i}+\Pi_{\boldsymbol{R}_N}\right]=\mathbb{E}\left[\sum_{i=1}^{N}(X_i-R_{d_i}(X_i))e^{-\delta T_i}+\Pi_{\boldsymbol{R}_N^*}\right].
$$

Thus, it remains to show that the variance term is minimized by the stop-loss contracts $R_{d_i}(\cdot) = (\cdot - d_i)_+$ for $i = 1, 2, \ldots$

According to Theorem 4.1, the established convex ordering implies that

$$\mathbb{E}\left[\left((X_i - R_i(X_i))e^{-\delta T_i}\right)^2\right] \geq \mathbb{E}\left[\left((X_i - R_{d_i}(X_i))e^{-\delta T_i}\right)^2\right].$$

For any fixed $N = n$, note that

$$\begin{aligned}
&\mathrm{Var}\left[\sum_{i=1}^{n}(X_i - R_i(X_i))e^{-\delta T_i} + \Pi_{\boldsymbol{R}_n}\right] \\
= &\sum_{i=1}^{n}\mathrm{Var}\left[(X_i - R_i(X_i))e^{-\delta T_i}\right] \\
= &\sum_{i=1}^{n}\left\{\mathbb{E}\left[\left(X_i - R_i(X_i))e^{-\delta T_i}\right)^2\right] - \left[\mathbb{E}(X_i - R_i(X_i))\mathbb{E}(e^{-\delta T_i})\right]^2\right\} \\
\geq &\sum_{i=1}^{n}\left\{\mathbb{E}\left[\left(X_i - R_{d_i}(X_i))e^{-\delta T_i}\right)^2\right] - \left[\mathbb{E}(X_i - R_{d_i}(X_i))\mathbb{E}(e^{-\delta T_i})\right]^2\right\} \\
= &\mathrm{Var}\left[\sum_{i=1}^{n}(X_i - R_{d_i}(X_i))e^{-\delta T_i}\right] \\
= &\mathrm{Var}\left[\sum_{i=1}^{n}(X_i - R_{d_i}(X_i))e^{-\delta T_i} + \Pi_{\boldsymbol{R}_n^*}\right].
\end{aligned}$$

By the law of total variance, we have

$$\begin{aligned}
&\mathrm{Var}\left[\sum_{i=1}^{N}(X_i - R_i(X_i))e^{-\delta T_i} + \Pi_{\boldsymbol{R}_N}\right] \\
= &\mathbb{E}\left[\mathrm{Var}\left[\sum_{i=1}^{N}(X_i - R_i(X_i))e^{-\delta T_i} + \Pi_{\boldsymbol{R}_N}\middle|N\right]\right] \\
&+\mathrm{Var}\left[\mathbb{E}\left[\sum_{i=1}^{N}(X_i - R_i(X_i))e^{-\delta T_i} + \Pi_{\boldsymbol{R}_N}\middle|N\right]\right] \\
\geq &\mathbb{E}\left[\mathrm{Var}\left[\sum_{i=1}^{N}(X_i - R_{d_i}(X_i))e^{-\delta T_i} + \Pi_{\boldsymbol{R}_N^*}\middle|N\right]\right] \\
&+\mathrm{Var}\left[\mathbb{E}\left[\sum_{i=1}^{N}(X_i - R_{d_i}(X_i))e^{-\delta T_i} + \Pi_{\boldsymbol{R}_N^*}\middle|N\right]\right] \\
= &\mathrm{Var}\left[\sum_{i=1}^{N}(X_i - R_{d_i}(X_i))e^{-\delta T_i} + \Pi_{\boldsymbol{R}_N^*}\right],
\end{aligned}$$

which implies the desired result. □

In Corollary 4.2, if the insurer chooses $R_i(\cdot) = R(\cdot)$ and adopts an equality constraint $\Pi_{\boldsymbol{R}_N} = \Pi_0$ (letting $\delta = 0$ in $\Pi_{\boldsymbol{R}_N}$), and if an additional constraint is imposed to ensure the standard deviation of the aggregate ceded loss $\sum_{i=1}^{N} R(X_i)$ does not exceed a certain threshold, then the minimization problem can be reduced to the one in [17] with $\delta = 0$ and $\beta = 1$. However, the model considered in Corollary 4.2 is, in general, very different from that of [17].

Remark 4.3. *The mean-variance preference is an important concept in economics and finance, which originates from Markowitz's portfolio selection. Recent literature in actuarial science that employs this preference includes [42–44], to name a few. Suppose $0 < w < \frac{1}{2\eta}$ and N is bounded almost surely as in Corollary 4.2. If the insurer adopts the quadratic utility function $u_q(x) = x - \eta x^2$, for $x < \frac{1}{2\eta}$ with $\eta > 0$, then it can be checked that any admissible ceded loss function is suboptimal to the stop-loss contract $R_{d_i}(\cdot) = (\cdot - d_i)_+$ with $d_i \in \overline{\mathbb{R}}_+$ satisfying $\mathbb{E}[R_{d_i}(X_i)] = \mathbb{E}[R_i(X_i)]$ for $i = 1, 2, \ldots$, of the following problem:*

$$\begin{cases} \max\limits_{\boldsymbol{R}\in\mathcal{L}} \mathbb{E}\left[u_q\left(w - \sum_{i=1}^{N}(X_i - R_i(X_i))e^{-\delta T_i} - \Pi_{\boldsymbol{R}_N}\right)\right] \\ \text{s.t.} \quad \Pi_{\boldsymbol{R}_N} \le \Pi_0, \end{cases}$$

since the optimization problem here and that in Corollary 4.2 reduce to minimizing the same variance part.

According to Theorem 4.1, under the setting that occurrence times T_i's are mutually independent and claims X_i's are independent but not necessarily identically distributed, Problem 1 boils down to solving the following parameter optimization problem:

$$\begin{cases} \max\limits_{d_1,d_2,\ldots\in\overline{\mathbb{R}}_+} \mathbb{E}\left[u\left(w - \sum_{i=1}^{N}(X_i - (X_i - d_i)_+)e^{-\delta T_i} - (1+\theta)\mathbb{E}\left[\sum_{i=1}^{N}(X_i - d_i)_+ e^{-\delta T_i}\right]\right)\right] \\ \text{s.t.} \quad (1+\theta)\mathbb{E}\left[\sum_{i=1}^{N}(X_i - d_i)_+ e^{-\delta T_i}\right] \le \Pi_0. \end{cases} \tag{4.1}$$

If (4.1) admits an optimizer, then we let $\{d_i^*\}_{i\ge1}$ be an optimal sequence of deductibles with every $d_i^* \in \overline{\mathbb{R}}_+$. For a general concave function u, it is interesting to investigate the ordering among $\{d_i^*\}_{i\ge1}$. However, we are unable to establish sufficient conditions to support a certain ordering to hold true. When the insurer is risk-neutral (i.e., $u(x) = x$), we restate the risk-neutral version of (4.1). Let $\boldsymbol{d} =: \{d_1, d_2, \ldots\}$ be a deductible vector and denote

$$\Xi(\boldsymbol{d}) := \mathbb{E}\left[w - \sum_{i=1}^{N}(X_i - (X_i - d_i)_+)e^{-\delta T_i}\right] - \Pi(\boldsymbol{d}),$$

where we define

$$\Pi(\boldsymbol{d}) := (1+\theta)\mathbb{E}\left[\sum_{i=1}^{N}(X_i - d_i)_+ e^{-\delta T_i}\right]$$

to address the premium's connection to $\boldsymbol{d}$. Thus, the problem is formulated by the following:

$$\begin{cases} \max\limits_{d_1,d_2\ldots\in\overline{\mathbb{R}}_+} \Xi(\boldsymbol{d}) \\ \text{s.t.} \ \Pi(\boldsymbol{d}) \le \Pi_0. \end{cases} \tag{4.2}$$

Next, we investigate the allocation of deductible levels across heterogeneous claims in Problem (4.2). Specifically, we examine how the stochastic characteristics of two claims determine which deductible vector is weakly preferable.

Theorem 4.4. *If one claim has an earlier occurrence time than another claim in the Laplace-transform order, while its claim severity is larger in the usual stochastic order, then it is weakly preferable to assign the larger deductible to the former claim.*

Proof. We assume $i < j$ without the loss of generality. Since the comparison is only meaningful when both claims are present, we assume the value of N is at least larger than j. Under this setup, let $\boldsymbol{d} = (d_1, \ldots, d_i, \ldots, d_j, \ldots)$ be the deductible vector that assigns the larger deductible to claim i, where $d_i \geq d_j$ and $d_i, d_j < \infty$. Let $\tau_{ij}(\boldsymbol{d}) = (d_1, \ldots, d_j, \ldots, d_i, \ldots)$, which is obtained by only interchanging the i-th and j-th components. First, we show that

$$\Xi(\boldsymbol{d}) \geq \Xi(\tau_{ij}(\boldsymbol{d})). \tag{4.3}$$

The comparison can be studied via conditioning on $\{N = n\}$, and then aggregated with weights $\mathbb{P}(N = n)$. By definition, we have the following:

$$\begin{aligned}
\Xi(\boldsymbol{d}) &= \mathbb{E}\left[w - \sum_{r=1}^{N}(X_r - (X_r - d_r)_+)e^{-\delta T_r} - \Pi(\boldsymbol{d})\right] \\
&= \mathbb{E}\left[w - \sum_{r=1}^{N}(X_r - (X_r - d_r)_+)e^{-\delta T_r}\right] - (1+\theta)\mathbb{E}\left[\sum_{r=1}^{N}(X_r - d_r)_+ e^{-\delta T_r}\right] \\
&= \sum_{n=0}^{\infty}\mathbb{P}(N = n)\left\{\mathbb{E}\left[w - \sum_{r=1}^{n}(X_r - (X_r - d_r)_+)e^{-\delta T_r}\middle| N = n\right]\right. \\
&\qquad \left. -(1+\theta)\mathbb{E}\left[\sum_{r=1}^{n}(X_r - d_r)_+ e^{-\delta T_r}\middle| N = n\right]\right\} \\
&:= \sum_{n=0}^{\infty}\mathbb{P}(N = n)\Xi_n(\boldsymbol{d}),
\end{aligned}$$

where we define $\mathbb{E}_n[\cdot] = \mathbb{E}[\cdot | N = n]$ and

$$\begin{aligned}
\Xi_n(\boldsymbol{d}) &:= \mathbb{E}_n\left\{w - \sum_{r\neq i,j}^{n}(X_r - (X_r - d_r)_+)\,e^{-\delta T_r} - (1+\theta)\mathbb{E}\left[\sum_{r\neq i,j}^{n}(X_r - d_r)_+ e^{-\delta T_r}\right]\right. \\
&\qquad - (X_i - (X_i - d_i)_+)\,e^{-\delta T_i} - \left(X_j - (X_j - d_j)_+\right)e^{-\delta T_j} \\
&\qquad \left. -(1+\theta)\mathbb{E}\left[(X_i - d_i)_+ e^{-\delta T_i} + (X_j - d_j)_+ e^{-\delta T_j}\right]\right\}.
\end{aligned}$$

It remains to prove that

$$\begin{aligned}
&\mathbb{E}\Big[[X_i - (X_i - d_i)_+]\,e^{-\delta T_i} + (1+\theta)\mathbb{E}\left[(X_i - d_i)_+ e^{-\delta T_i}\right] \\
&\quad + \left[X_j - (X_j - d_j)_+\right]e^{-\delta T_j} + (1+\theta)\mathbb{E}\left[(X_j - d_j)_+ e^{-\delta T_j}\right]\Big] \\
&\leq \mathbb{E}\Big[\left[X_i - (X_i - d_j)_+\right]e^{-\delta T_i} + (1+\theta)\mathbb{E}\left[(X_i - d_j)_+ e^{-\delta T_i}\right] \\
&\qquad + \left[X_j - (X_j - d_i)_+\right]e^{-\delta T_j} + (1+\theta)\mathbb{E}\left[(X_j - d_i)_+ e^{-\delta T_j}\right]\Big].
\end{aligned} \tag{4.4}$$

Since N is independent of the claim severities and occurrence times, the conditional distributions of X_i, X_j, T_i, and T_j given $N = n$ coincide with their unconditional distributions.

Next, we compare the expectations of both sides in (4.4). By applying $X_i \geq_{\rm st} X_j$, we know from Lemma 2.2 that there exist two random variables X_i^c and X_j^c defined on a common measure space such that $X_i \stackrel{\rm d}{=} X_i^c$, $X_j \stackrel{\rm d}{=} X_j^c$, and $X_i^c \geq_{\rm a.s.} X_j^c$. Note that $T_i \leq_{\rm lt} T_j$ implies that for $\delta \geq 0$, $\mathbb{E}[e^{-\delta T_i}] \geq \mathbb{E}[e^{-\delta T_j}]$. It holds that

$$\begin{aligned}
&\text{R.H.S. in (4.4)} - \text{L.H.S. in (4.4)}\\
= {} & \theta\mathbb{E}\left[(X_i - d_j)_+ e^{-\delta T_i} + (X_j - d_i)_+ e^{-\delta T_j}\right] - \theta\mathbb{E}\left[(X_i - d_i)_+ e^{-\delta T_i} + (X_j - d_j)_+ e^{-\delta T_j}\right]\\
= {} & \theta\mathbb{E}\left[(X_i^c - d_j)_+ e^{-\delta T_i} + (X_j^c - d_i)_+ e^{-\delta T_j}\right] - \theta\mathbb{E}\left[(X_i^c - d_i)_+ e^{-\delta T_i} + (X_j^c - d_j)_+ e^{-\delta T_j}\right]\\
= {} & \theta\mathbb{E}\left[e^{-\delta T_i}\right]\mathbb{E}\left[(X_i^c - d_j)_+ - (X_i^c - d_i)_+\right] + \theta\mathbb{E}\left[e^{-\delta T_j}\right]\mathbb{E}\left[(X_j^c - d_i)_+ - (X_j^c - d_j)_+\right]\\
= {} & \theta\left(\mathbb{E}[e^{-\delta T_i}] - \mathbb{E}[e^{-\delta T_j}]\right)\mathbb{E}[(X_i^c - d_j)_+ - (X_i^c - d_i)_+]\\
& +\theta\mathbb{E}[e^{-\delta T_j}]\mathbb{E}\left[(X_i^c - d_j)_+ - (X_i^c - d_i)_+ + (X_j^c - d_i)_+ - (X_j^c - d_j)_+\right]\\
\geq {} & 0.
\end{aligned}$$

By the same calculation, we can also derive that

$$\Pi(\boldsymbol{d}) \leq \Pi(\tau_{ij}(\boldsymbol{d})).$$

It means that whenever $\tau_{ij}(\boldsymbol{d})$ satisfies the premium budget constraint, $\boldsymbol{d}$ also satisfies it. Therefore, among the two allocations, assigning the larger deductible to claim i yields a weakly larger objective value without requiring a larger reinsurance premium, and is thus weakly preferable.

Next, we consider the case involving an infinite deductible. Suppose that $d_i = \infty$ and $d_j < \infty$. Since $(X_i - d_i)_+ = 0$, we have the following:

$$\begin{aligned}
\Pi_n(\tau_{ij}(\boldsymbol{d})) - \Pi_n(\boldsymbol{d}) &= (1+\theta)\left\{\mathbb{E}\left[(X_i - d_j)_+ e^{-\delta T_i}\right] - \mathbb{E}\left[(X_j - d_j)_+ e^{-\delta T_j}\right]\right\}\\
&= (1+\theta)\left\{\mathbb{E}[e^{-\delta T_i}]\mathbb{E}[(X_i - d_j)_+] - \mathbb{E}[e^{-\delta T_j}]\mathbb{E}[(X_j - d_j)_+]\right\}\\
&\geq 0.
\end{aligned}$$

It follows that both $\Pi(\boldsymbol{d}) \leq \Pi(\tau_{ij}(\boldsymbol{d}))$ and $\Xi(\boldsymbol{d}) \geq \Xi(\tau_{ij}(\boldsymbol{d}))$ hold. If $d_i = d_j = \infty$, then $\tau_{ij}(\boldsymbol{d}) = \boldsymbol{d}$, and both inequalities hold trivially.

Therefore, we have shown that whenever $d_i \geq d_j$, $\Xi(\boldsymbol{d}) \geq \Xi(\tau_{ij}(\boldsymbol{d}))$ and $\Pi(\boldsymbol{d}) \leq \Pi(\tau_{ij}(\boldsymbol{d}))$. Hence, the allocation assigning the larger deductible to claim i yields a weakly larger objective value and requires a weakly smaller reinsurance premium than the alternative allocation. Therefore, it is weakly preferable. This completes the proof.

□

Theorem 4.4 shows that, a stochastically larger claim whose occurrence time is smaller in the Laplace-transform order is weakly better assigned the larger deductible. Next, we compare two general deductible vectors to investigate how the dispersion of the deductible components affects the insurer's expected terminal wealth, when both deductible components under comparison are present. To formalize the dispersion of a deductible vector, we first introduce the concept of the majorization order.

Definition 4.5. *Given any two vectors, $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$, $\boldsymbol{b}$ is said to be majorized by $\boldsymbol{a}$, denoted by $\boldsymbol{b} \overset{\mathrm{m}}{\leq} \boldsymbol{a}$, if $\sum_{i=1}^n b_{[i]} = \sum_{i=1}^n a_{[i]}$ and $\sum_{i=1}^m b_{[i]} \leq \sum_{i=1}^m a_{[i]}$, for $m = 1, \ldots, n-1$, where the notation $a_{[i]}$ is the i-th largest element of $\boldsymbol{a}$.*

If $\boldsymbol{b} \overset{\mathrm{m}}{\leq} \boldsymbol{a}$, then the coordinates of $\boldsymbol{a}$ are more spread out than the coordinates of $\boldsymbol{b}$. For further discussion on the majorization order and its variants and applications, we refer to [45].

Theorem 4.6. *If one claim has an earlier occurrence time than another claim in the Laplace-transform order, while its claim severity is larger in the usual stochastic order, then the less dispersed allocation of deductibles between these two claims is weakly preferred.*

Proof. We assume $i < j$ and the value of N is at least larger than j. Consider two deductible vectors, $\boldsymbol{d}$ and $\widetilde{\boldsymbol{d}}$, where $\widetilde{\boldsymbol{d}}$ differs from $\boldsymbol{d}$ only in the i-th and j-th elements in the sense that $d_j \leq d_i$, $\widetilde{d}_j \leq \widetilde{d}_i$, and $(\widetilde{d}_i, \widetilde{d}_j) \overset{m}{\leq} (d_i, d_j)$. It is obvious that $d_j \leq \widetilde{d}_j \leq \widetilde{d}_i \leq d_i$, $d_i + d_j = \widetilde{d}_i + \widetilde{d}_j$, and $d_r = \widetilde{d}_r$ for all $r \neq i, j$. Using a similar decomposition to that in the proof of Theorem 4.4, we study the difference $\Xi(\widetilde{\boldsymbol{d}}) - \Xi(\boldsymbol{d})$. To this end, after some rearrangement, it suffices to prove the following:

$$\begin{aligned}
&\mathbb{E}\left[\left[X_i - (X_i - \widetilde{d}_i)_+\right] e^{-\delta T_i} + (1+\theta)\mathbb{E}\left[(X_i - \widetilde{d}_i)_+ e^{-\delta T_i}\right]\right.\\
&\qquad \left. + \left[X_j - (X_j - \widetilde{d}_j)_+\right] e^{-\delta T_j} + (1+\theta)\mathbb{E}\left[(X_j - \widetilde{d}_j)_+ e^{-\delta T_j}\right]\right]\\
&\leq \mathbb{E}\left[\left[X_i - (X_i - d_i)_+\right] e^{-\delta T_i} + (1+\theta)\mathbb{E}\left[(X_i - d_i)_+ e^{-\delta T_i}\right]\right.\\
&\qquad \left. + \left[X_j - (X_j - d_j)_+\right] e^{-\delta T_j} + (1+\theta)\mathbb{E}\left[(X_j - d_j)_+ e^{-\delta T_j}\right]\right].
\end{aligned} \tag{4.5}$$

As in the proof of Theorem 4.4, the expectations can be written in unconditional form because N is independent of the claim severities and occurrence times. Then, by applying $T_i \geq_{\mathrm{lt}} T_j$, we have, for $\delta \geq 0$, $\mathbb{E}[e^{-\delta T_i}] \leq \mathbb{E}[e^{-\delta T_j}]$. Moreover, $X_i \leq_{\mathrm{st}} X_j$ implies that $\bar{F}_{X_i}(t) \leq \bar{F}_{X_j}(t)$ for $t \in \mathbb{R}$. It holds that

$$\begin{aligned}
&\text{R.H.S. in (4.5)} - \text{L.H.S. in (4.5)}\\
&= \theta \cdot \mathbb{E}\left[(X_i - d_i)_+ e^{-\delta T_i} - (X_i - \widetilde{d}_i)_+ e^{-\delta T_i} + (X_j - d_j)_+ e^{-\delta T_j} - (X_j - \widetilde{d}_j)_+ e^{-\delta T_j}\right]\\
&= \theta\left(\mathbb{E}[e^{-\delta T_i}] - \mathbb{E}[e^{-\delta T_j}]\right)\mathbb{E}[(X_i - d_i)_+ - (X_i - \widetilde{d}_i)_+]\\
&\quad + \theta\mathbb{E}[e^{-\delta T_j}]\mathbb{E}[(X_i - d_i)_+ - (X_i - \widetilde{d}_i)_+ + (X_j - d_j)_+ - (X_j - \widetilde{d}_j)_+]\\
&\geq 0,
\end{aligned}$$

where

$$\begin{aligned}
&\mathbb{E}[(X_i - d_i)_+ - (X_i - \widetilde{d}_i)_+ + (X_j - d_j)_+ - (X_j - \widetilde{d}_j)_+]\\
&= \int_{d_i}^{\infty} \bar{F}_{X_i}(t)dt - \int_{\widetilde{d}_i}^{\infty} \bar{F}_{X_i}(t)dt + \int_{d_j}^{\infty} \bar{F}_{X_j}(t)dt - \int_{\widetilde{d}_j}^{\infty} \bar{F}_{X_j}(t)dt\\
&= -\int_{\widetilde{d}_i}^{d_i} \bar{F}_{X_i}(t)dt + \int_{d_j}^{\widetilde{d}_j} \bar{F}_{X_j}(t)dt \geq -\int_{\widetilde{d}_i}^{d_i} \bar{F}_{X_j}(t)dt + \int_{d_j}^{\widetilde{d}_j} \bar{F}_{X_j}(t)dt \geq 0.
\end{aligned}$$

The preceding calculation proves $\Xi(\widetilde{\boldsymbol{d}}) \geq \Xi(\boldsymbol{d})$. The same argument also yields $\Pi(\widetilde{\boldsymbol{d}}) \leq \Pi(\boldsymbol{d})$. Therefore, the less dispersed deductible vector $\widetilde{\boldsymbol{d}}$ yields a weakly larger objective value and a weakly smaller reinsurance premium than $\boldsymbol{d}$. Hence, $\widetilde{\boldsymbol{d}}$ is weakly preferable to $\boldsymbol{d}$. This completes the proof. □

5. Dependent claim severities

In this section, we consider the case where the claim severities $\{X_i\}_{i\geq 1}$ may be dependent, but their exact dependence structure is unspecified. Other model assumptions remain the same as in Section 4, where we assume the claim number N is an integer-valued random variable, independent of the claim severities and the occurrence times; claim severities are independent of the occurrence times; and the occurrence time sequence $\{T_i\}_{i\geq 1}$ is assumed to be mutually independent. We refer to the corresponding solution as the budget-constrained robust optimal reinsurance contract for the insurer.

Theorem 5.1. *Suppose that the non-negative claim severities $\{X_i\}_{i\geq 1}$ may be dependent, and let $\{X_i^c\}_{i\geq 1}$ denote their comonotonic counterparts. Assume that $\{X_i\}_{i\geq 1}$ are independent of N and $\{T_i\}_{i\geq 1}$, and that the occurrence times $\{T_i\}_{i\geq 1}$ are mutually independent. Then, any admissible ceded loss function $R_i(\cdot)$ is suboptimal to the stop-loss contract $R_{d_i}(\cdot) = (\cdot - d_i)_+$ in Problem 2, where $i = 1, 2, \ldots$, and $d_i \in \overline{\mathbb{R}}_+$ is chosen such that $\mathbb{E}[R_i(X_i^c)] = \mathbb{E}[(X_i^c - d_i)_+]$. The reinsurance premium $\Pi_{\boldsymbol{R}_N^*}$ is given as follows:*

$$\Pi_{\boldsymbol{R}_N^*} = (1+\theta)\mathbb{E}\left[\sum_{i=1}^{N}(X_i^c - d_i)_+ e^{-\delta T_i}\right].$$

Proof. The proof is divided into the following two steps.
Step 1: First, we show that the solution of the inner optimization problem in Problem 2 is characterized by comonotonicity. For $i = 1, 2, \ldots$, we note that $R_i(X_i^c)$ and $R_i(X_i)$ have the same distribution, and for any feasible reinsurance policy $\boldsymbol{R}$, we have the following:

$$\Pi_{\boldsymbol{R}_N} = (1+\theta)\mathbb{E}\left[\sum_{i=1}^{N} R_i(X_i)e^{-\delta T_i}\right] = (1+\theta)\mathbb{E}\left[\sum_{i=1}^{N} R_i(X_i^c)e^{-\delta T_i}\right] \leq \Pi_0.$$

It is easy to observe that both $(X_1 - R_1(X_1), X_2 - R_2(X_2), \ldots)$ and $(X_1^c - R_1(X_1^c), X_2^c - R_2(X_2^c), \ldots)$ have the same marginal distributions. Based on the increasing property of $x - R(x)$ in $x \in \overline{\mathbb{R}}_+$, $(X_1^c - R_1(X_1^c), X_2^c - R_2(X_2^c), \ldots)$ are also comonotonic. Since the sum of comonotonic vectors is maximal in terms of the convex order, for any convex function ϕ, we have the following:

$$\begin{aligned}
&\mathbb{E}\left[\phi\left(\sum_{i=1}^{N}(X_i - R_i(X_i))e^{-\delta T_i}\right)\right] \\
= &\sum_{n=0}^{\infty}\mathbb{P}(N=n)\mathbb{E}\left[\phi\left(\sum_{i=1}^{n}(X_i - R_i(X_i))e^{-\delta T_i}\right)\right] \\
= &\sum_{n=0}^{\infty}\mathbb{P}(N=n)\mathbb{E}\left[\mathbb{E}\left[\phi\left(\sum_{i=1}^{n}(X_i - R_i(X_i))e^{-\delta T_i}\right)\middle| T_1, \ldots, T_n\right]\right] \\
\leq &\sum_{n=0}^{\infty}\mathbb{P}(N=n)\mathbb{E}\left[\mathbb{E}\left[\phi\left(\sum_{i=1}^{n}(X_i^c - R_i(X_i^c))e^{-\delta T_i}\right)\middle| T_1, \ldots, T_n\right]\right] \\
= &\sum_{n=0}^{\infty}\mathbb{P}(N=n)\mathbb{E}\left[\phi\left(\sum_{i=1}^{n}(X_i^c - R_i(X_i^c))e^{-\delta T_i}\right)\right]
\end{aligned}$$

$$= \mathbb{E}\left[\phi\left(\sum_{i=1}^{N}(X_i^c - R_i(X_i^c))e^{-\delta T_i}\right)\right],$$

where the inequality is true due to Lemma 2.5. Hence, we arrive at the following:

$$\sum_{i=1}^{N}(X_i - R_i(X_i))e^{-\delta T_i} \leq_{\mathrm{cx}} \sum_{i=1}^{N}(X_i^c - R_i(X_i^c))e^{-\delta T_i}.$$

Furthermore, it can be derived that

$$\begin{aligned}
& \mathbb{E}\left[-u\left(w - \sum_{i=1}^{N}(X_i - R_i(X_i))e^{-\delta T_i} - \Pi_{\boldsymbol{R}_N}\right)\right] \\
= & \sum_{n=0}^{\infty}\mathbb{P}(N=n)\mathbb{E}\left[-u\left(w - \sum_{i=1}^{n}(X_i - R_i(X_i))e^{-\delta T_i} - \Pi_{\boldsymbol{R}_N}\right)\right] \\
= & \sum_{n=0}^{\infty}\mathbb{P}(N=n)\mathbb{E}\left[\mathbb{E}\left[-u\left(w - \sum_{i=1}^{n}(X_i - R_i(X_i))e^{-\delta T_i} - \Pi_{\boldsymbol{R}_N}\right)\middle| T_1, \ldots, T_n\right]\right] \\
\geq & \sum_{n=0}^{\infty}\mathbb{P}(N=n)\mathbb{E}\left[\mathbb{E}\left[-u\left(w - \sum_{i=1}^{n}(X_i^c - R_i(X_i^c))e^{-\delta T_i} - \Pi_{\boldsymbol{R}_N}\right)\middle| T_1, \ldots, T_n\right]\right] \\
= & \sum_{n=0}^{\infty}\mathbb{P}(N=n)\mathbb{E}\left[-u\left(w - \sum_{i=1}^{n}(X_i^c - R_i(X_i^c))e^{-\delta T_i} - \Pi_{\boldsymbol{R}_N}\right)\right] \\
= & \mathbb{E}\left[-u\left(w - \sum_{i=1}^{N}(X_i^c - R_i(X_i^c))e^{-\delta T_i} - \Pi_{\boldsymbol{R}_N}\right)\right],
\end{aligned}$$

which is equal to

$$\begin{aligned}
& \max_{(X_1,X_2,\ldots)\in\mathcal{R}(F_1,F_2,\ldots)} \mathbb{E}\left[-u\left(w - \sum_{i=1}^{N}(X_i - R_i(X_i))e^{-\delta T_i} - \Pi_{\boldsymbol{R}_N}\right)\right] \\
& \qquad = \mathbb{E}\left[-u\left(w - \sum_{i=1}^{N}(X_i^c - R_i(X_i^c))e^{-\delta T_i} - \Pi_{\boldsymbol{R}_N}\right)\right].
\end{aligned}$$

In other words, we have the following:

$$\begin{aligned}
& \min_{(X_1,X_2,\ldots)\in\mathcal{R}(F_1,F_2,\ldots)} \mathbb{E}\left[u\left(w - \sum_{i=1}^{N}(X_i - R_i(X_i))e^{-\delta T_i} - \Pi_{\boldsymbol{R}_N}\right)\right] \\
& \qquad = \mathbb{E}\left[u\left(w - \sum_{i=1}^{N}(X_i^c - R_i(X_i^c))e^{-\delta T_i} - \Pi_{\boldsymbol{R}_N}\right)\right].
\end{aligned}$$

Step 2: According to Step 1, the original problem boils down to finding the solution of

$$\max_{\boldsymbol{R}\in\mathcal{L}} \mathbb{E}\left[u\left(w - \sum_{i=1}^{N}(X_i^c - R_i(X_i^c))e^{-\delta T_i} - \Pi_{\boldsymbol{R}_N}\right)\right]. \tag{5.1}$$

Observe that, for $i = 1, 2, \ldots$, and any arbitrary ceded loss function $R_i(\cdot)$, there always exists a sequence of stop-loss functions $R_{d_i}(\cdot) = (\cdot - d_i)_+$ such that $\mathbb{E}[R_i(X_i^c)] = \mathbb{E}[(X_i^c - d_i)_+]$. If $d_i = \infty$ for some i, then $\mathbb{E}[R_i(X_i^c)] = 0$, and hence $R_i(X_i^c) = 0$ a.s. For $d_i < \infty$ for all i, Lemma 3.1 directly applies. It is easy to check that

$$\Pi_{\boldsymbol{R}_N} = (1+\theta)\mathbb{E}\left[\sum_{i=1}^{N} R_i(X_i^c)e^{-\delta T_i}\right] = (1+\theta)\mathbb{E}\left[\sum_{i=1}^{N} R_{d_i}(X_i^c)e^{-\delta T_i}\right] = \Pi_{\boldsymbol{R}_N^*} \le \Pi_0.$$

By conditioning on $T_i = t_i$ for $i = 1, 2, \ldots$, it is obvious that $(X_i^c - R_{d_i}(X_i^c))e^{-\delta t_i} \le_{\mathrm{cx}} (X_i^c - R_i(X_i^c))e^{-\delta t_i}$. As a result of Lemma 2.7, for any convex function ϕ, we have the following:

$$\begin{aligned}
& \mathbb{E}\left[\phi\left(\sum_{i=1}^{N}(X_i^c - R_i(X_i^c))e^{-\delta T_i}\right)\right] \\
= & \sum_{n=0}^{\infty}\mathbb{P}(N=n)\mathbb{E}\left[\phi\left(\sum_{i=1}^{n}(X_i^c - R_i(X_i^c))e^{-\delta T_i}\right)\right] \\
= & \sum_{n=0}^{\infty}\mathbb{P}(N=n)\mathbb{E}\left[\mathbb{E}\left[\phi\left(\sum_{i=1}^{n}(X_i^c - R_i(X_i^c))e^{-\delta T_i}\right)\middle| T_1, \ldots, T_n\right]\right] \\
\ge & \sum_{n=0}^{\infty}\mathbb{P}(N=n)\mathbb{E}\left[\mathbb{E}\left[\phi\left(\sum_{i=1}^{n}(X_i^c - R_{d_i}(X_i^c))e^{-\delta T_i}\right)\middle| T_1, \ldots, T_n\right]\right] \\
= & \sum_{n=0}^{\infty}\mathbb{P}(N=n)\mathbb{E}\left[\phi\left(\sum_{i=1}^{n}(X_i^c - R_{d_i}(X_i^c))e^{-\delta T_i}\right)\right] \\
= & \mathbb{E}\left[\phi\left(\sum_{i=1}^{N}(X_i^c - R_{d_i}(X_i^c))e^{-\delta T_i}\right)\right],
\end{aligned}$$

which implies that

$$\sum_{i=1}^{N}(X_i^c - R_{d_i}(X_i^c))e^{-\delta T_i} + \Pi_{\boldsymbol{R}_N^*} \le_{\mathrm{cx}} \sum_{i=1}^{N}(X_i^c - R_i(X_i^c))e^{-\delta T_i} + \Pi_{\boldsymbol{R}_N}. \tag{5.2}$$

Based on (5.2), for an increasing and concave function u, we have the following:

$$\begin{aligned}
& \mathbb{E}\left[-u\left(w - \sum_{i=1}^{N}(X_i^c - R_i(X_i^c))e^{-\delta T_i} - \Pi_{\boldsymbol{R}_N}\right)\right] \\
= & \mathbb{E}\left[-u\left(w - \sum_{i=1}^{N}(X_i^c - R_i(X_i^c))e^{-\delta T_i} - \Pi_{\boldsymbol{R}_N^*}\right)\right] \\
= & \sum_{n=0}^{\infty}\mathbb{P}(N=n)\mathbb{E}\left[-u\left(w - \sum_{i=1}^{n}(X_i^c - R_i(X_i^c))e^{-\delta T_i} - \Pi_{\boldsymbol{R}_N^*}\right)\right] \\
= & \sum_{n=0}^{\infty}\mathbb{P}(N=n)\mathbb{E}\left[\mathbb{E}\left[-u\left(w - \sum_{i=1}^{n}(X_i^c - R_i(X_i^c))e^{-\delta T_i} - \Pi_{\boldsymbol{R}_N^*}\right)\middle| T_1, \ldots, T_n\right]\right]
\end{aligned}$$

$$\geq \sum_{n=0}^{\infty} \mathbb{P}(N=n)\mathbb{E}\left[\mathbb{E}\left[-u\left(w-\sum_{i=1}^{n}(X_i^c - R_{d_i}(X_i^c))e^{-\delta T_i} - \Pi_{\boldsymbol{R}_N^*}\right)\middle| T_1,\ldots,T_n\right]\right]$$
$$= \mathbb{E}\left[-u\left(w-\sum_{i=1}^{N}(X_i^c - R_{d_i}(X_i^c))e^{-\delta T_i} - \Pi_{\boldsymbol{R}_N^*}\right)\right].$$

Thus,

$$\mathbb{E}\left[u\left(w-\sum_{i=1}^{N}(X_i^c - R_i(X_i^c))e^{-\delta T_i} - \Pi_{\boldsymbol{R}_N}\right)\right]$$
$$\leq \mathbb{E}\left[u\left(w-\sum_{i=1}^{N}(X_i^c - R_{d_i}(X_i^c))e^{-\delta T_i} - \Pi_{\boldsymbol{R}_N^*}\right)\right].$$

Combining Step 1 and Step 2 completes the proof. □

Theorem 5.1 states that if the exact dependence structure among the claim severities is unknown, the dependence that yields the worst-case evaluation can be identified as the comonotonicity representing the strongest positive dependence. Moreover, the stop-loss contract is further identified as the robust optimal reinsurance contract, which echoes the findings in the previous section.

Therefore, with the help of Theorem 5.1, the optimization Problem 2 can be reduced to the following:

$$\begin{cases} \max\limits_{d_1,d_2,\ldots\in\overline{\mathbb{R}}_+} \mathbb{E}\left[u\left(w-\sum\limits_{i=1}^{N}(X_i^c-(X_i^c-d_i)_+)e^{-\delta T_i}-(1+\theta)\mathbb{E}\left[\sum\limits_{i=1}^{N}(X_i^c-d_i)_+e^{-\delta T_i}\right]\right)\right] \\ \text{s.t.} \quad (1+\theta)\mathbb{E}\left[\sum\limits_{i=1}^{N}(X_i^c-d_i)_+e^{-\delta T_i}\right]\leq \Pi_0. \end{cases} \tag{5.3}$$

As with Problem (4.1), if an optimizer exists, then let $\boldsymbol{d}^* = (d_1^*, d_2^*, \ldots)$ denote a solution of (5.3). It remains difficult to establish a general ordering among its components. Moreover, when $u(x) = x$, the unrestricted problem again admits the degenerate no-reinsurance solution. Therefore, rather than characterizing the optimizers of (5.3), we investigate pairwise deductible allocations under the risk-neutral robust setting. Since $X_i^c \overset{d}{=} X_i$ for every i, and since under $u(x) = x$ the objective and premium contributions only depend on the marginal distributions of the individual claims, replacing the claim vector by its comonotonic counterpart does not affect the pairwise comparisons established in Theorem 4.4. Hence, the result of Theorem 4.4 carries over to the robust setting.

Theorem 5.2. *If one claim has an earlier occurrence time than another claim in the Laplace-transform order, while its claim severity is larger in the usual stochastic order, then it is weakly preferable to assign the larger deductible to the former claim under the worst-case dependence structure.*

Provided that both claims under comparison are present, an analogous majorization comparison to Theorem 4.6 also holds in the present robust setting among deductible vectors. Specifically, under the corresponding conditions with the claim severities replaced by their comonotonic counterparts, a less dispersed deductible vector is weakly preferred. The proof follows the same arguments as that of Theorem 4.6 and is therefore omitted.

6. Numerical examples

This section investigates the solutions for the optimal deductibles subject to a budget constraint. For illustration purposes, we present the numerical solutions of Problem 1. The insurer's risk-averse attitude is characterized by the utility function $u(x) = 1 - \exp(-x)$. We analyze the parameter sensitivities regarding the discount rate δ, safety loading θ, and budget constraint Π_0. We consider the time interval $[0, 3]$.

To facilitate the experiment, we assume N is a nonnegative bounded discrete variable, and the largest value of N is denoted by $N_{\max}$. The claim occurrence times $T_1, T_2, \ldots, T_{N_{\max}}$ are assumed to be independent. In addition, the claims $X_1, X_2, \ldots, X_{N_{\max}}$ are independent but not necessarily identically distributed. The claim sequence and the occurrence-time sequence are both independent of N, and the two sequences are independent of each other. By Theorem 4.1, we intend to compute the optimal deductible vector $(d_1^*, \ldots, d_{N_{\max}}^*)$ and examine the ordering of its components. We use a simulation approach with 10^7 sample paths.

In the following experiments, we consider two scenarios, $N_{\max} = 2$ and $N_{\max} = 3$, to show numerical solutions and the corresponding sensitivity results under the two-claim scenario and the three-claim scenario, respectively. All results are rounded to four decimal places.

6.1. Two-claim scenario

We assume that the probability mass function of N is given by the following:

$$\mathbb{P}(N = n) = \begin{cases} 0.2, & \text{if } n = 0; \\ 0.5, & \text{if } n = 1; \\ 0.3, & \text{if } n = 2. \end{cases}$$

When $N = 1$, we assume $X_1 \sim \text{Exp}(\lambda_1)$ and its occurrence time $T_1 \sim \text{Uniform}[0, 1]$. In addition, when $N = 2$, we assume the other claim $X_2 \sim \text{Exp}(\lambda_2)$ and its occurrence time $T_2 \sim \text{Uniform}[0, 2]$ where T_1 and T_2 are independent. We assume the rate parameters $\lambda_1 = 0.05$ and $\lambda_2 = 0.1$ in the numerical experiments. Hence, it suffices to determine the optimal deductibles d_1^* and d_2^* that maximize the expected utility of the insurer's terminal wealth, subject to a budget constraint on the premium.

First, we investigate the impact of the discount rate δ on optimal deductibles in Table 1. The results show that an increase in the value of δ lowers the optimal deductible for claim X_1. This indicates that the insurer purchases more reinsurance coverage for claim X_1 when future claim payments are more heavily discounted. For the parameter values considered, the premium constraint is numerically binding when $\delta < 1$, whereas the optimizer may choose an interior solution with a slack premium constraint when $\delta \geq 1$. Additionally, we observe that the optimal deductible associated with claim X_1 is larger than that associated with claim X_2, irrespective of whether the premium constraint is binding or not.

We report the optimal deductibles in Table 2 and explore the effect of the safety loading θ on them, subject to a fixed premium budget. A higher safety loading increases the cost of a given ceded-loss indemnity profile. Under the fixed premium budget, the optimal deductibles consequently increase. This makes sense because the insurer will be less willing to purchase reinsurance coverage if the reinsurance premium increases. It is also observed that the deductible obtained for claim X_1 is usually larger than that for claim X_2.

Table 1. Impact of δ on optimal deductible pair (d_1^*, d_2^*) with $w = 20$, $\theta = 0.5$, and $\Pi_0 = 15$.

δ	d_1^*	d_2^*	$\Pi_{\boldsymbol{R}_N^*}$
0.001	15.3663	1.5047	15.0000
0.01	15.2210	1.4949	15.0000
0.03	14.9004	1.4746	15.0000
0.05	14.5831	1.4560	15.0000
0.1	13.8047	1.4156	15.0000
0.3	10.3265	1.2856	15.0000
0.5	8.3265	1.1422	15.0000
1	2.8622	0.5464	14.9974
1.1	2.6593	0.5499	14.4721
1.3	2.4059	0.5779	13.4273
1.5	2.2012	0.5983	12.4839
2	1.8534	0.7663	10.4868

Table 2. Impact of θ on optimal deductible pair (d_1^*, d_2^*) with $w = 20$, $\delta = 0.3$, and $\Pi_0 = 15$.

θ	d_1^*	d_2^*	$\Pi_{\boldsymbol{R}_N^*}$
0.001	1.6222	0.0995	14.9999
0.01	1.8263	0.1180	14.9999
0.03	2.2728	0.1619	15.0000
0.05	2.7106	0.2081	15.0000
0.1	3.7702	0.3286	15.0000
0.3	7.5978	0.8157	15.0000
0.5	10.9094	1.2855	15.0000
1	17.6182	2.4419	15.0000
1.1	18.7536	2.6764	15.0000
1.3	20.8586	3.1526	15.0000
1.5	22.7663	3.6399	15.0000
2	26.8439	4.8791	15.0000

Table 3 demonstrates the influence of the budget Π_0 on the optimal deductibles. As Π_0 increases, a decreasing trend is observed for both d_1^* and d_2^*. It indicates that when more financial resources are available, the insurer is willing to purchase more reinsurance coverage. It is observed again that the optimal deductible for claim X_1 is larger than that for claim X_2. Given the current parameters, the budget is exhausted in all experiment scenarios.

Table 3. Impact of Π_0 on optimal deductible pair (d_1^*, d_2^*) with $w = 20$, $\delta = 0.3$, and $\theta = 0.5$.

Π_0	d_1^*	d_2^*	$\Pi_{\boldsymbol{R}_N^*}$
1	68.1779	23.7454	1.0000
3	45.9697	13.0690	3.0000
5	35.4752	8.2862	5.0000
8	25.4155	4.4148	8.0000
10	20.3520	3.0324	10.0000
11	18.1417	2.5481	11.0000
13	14.2435	1.8183	13.0000
15	10.9094	1.2855	15.0000
18	6.6954	0.6954	18.0000

6.2. *Three-claim scenario*

Suppose that the probability mass function of N is given by the following:

$$\mathbb{P}(N = n) = \begin{cases} 0.2, & \text{if } n = 0; \\ 0.4, & \text{if } n = 1; \\ 0.3, & \text{if } n = 2; \\ 0.1, & \text{if } n = 3. \end{cases}$$

For $N \leq 2$, we adopt the same setting for the claim severities and their occurrence times as in Section 6.1. When $N = 3$, we assume the new claim $X_3 \sim \text{Exp}(\lambda_3)$ with $\lambda_3 = 0.2$ and its occurrence time $T_3 \sim \text{Uniform}[0, 3]$. We assume T_1, T_2, and T_3 are mutually independent. In this case, we compute the optimal deductibles (d_1^*, d_2^*, d_3^*) via a simulation and conduct a sensitivity analysis on parameters.

Table 4 displays the influence of the discount rate δ on the optimal deductibles. As δ increases, the optimal deductibles associated with claims X_1 and X_2 generally decrease, while the optimal deductible for claim X_3 first decreases and then increases. The ordering $d_1^* \geq d_2^* \geq d_3^*$ holds for all parameter settings considered in Table 4. For $\delta \leq 1$, the premium constraint is binding, and the available premium budget is exhausted. For $\delta > 1$, the insurer may choose an interior solution where the premium is not fully consumed. The discount rate affects both the present value of retained losses and the valuation of the reinsurance premium. Under the parameter settings in Table 4, d_1^* decreases as δ increases, while d_2^* generally decreases but turns upward as δ approaches 2, whereas d_3^* exhibits a non-monotone pattern, illustrating that the net effect may not be monotone and can vary across claims and parameter values. Figure 1 displays these patterns more clearly.

Table 4. Impact of δ on optimal deductible vector (d_1^*, d_2^*, d_3^*) with $w = 20$, $\theta = 0.5$, and $\Pi_0 = 15$.

δ	d_1^*	d_2^*	d_3^*	$\Pi_{\boldsymbol{R}_N^*}$
0.001	17.8144	3.0000	0.2761	15.0000
0.01	17.6602	2.9611	0.2736	15.0000
0.03	17.3177	2.8788	0.2690	15.0000
0.05	16.9758	2.8017	0.2657	15.0000
0.1	16.1256	2.6311	0.2619	15.0000
0.3	12.8968	2.1315	0.2877	15.0000
0.5	10.0125	1.7584	0.3318	15.0000
1	4.0357	0.9014	0.3052	15.0000
1.1	3.0078	0.6987	0.2563	14.9998
1.3	2.3890	0.6210	0.2658	14.1115
1.5	2.1852	0.6369	0.3320	13.0877
2	1.8286	0.6937	0.6482	10.9590

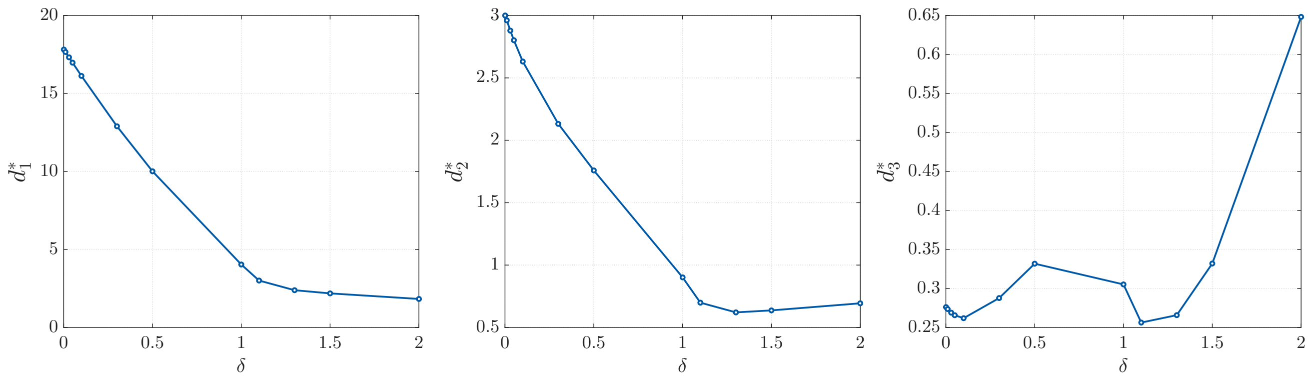

Figure 1. Optimal deductibles d_1^*, d_2^*, and d_3^* under different values of δ with reported values given in Table 4.

We investigate the influence of safety loading on optimal deductibles in Table 5. It is observed that a higher value of θ leads to larger optimal deductibles because the higher associated reinsurance premium lowers the demand of the insurer. In Table 5, we notice that the budget constraint is binding for all experiments considered, and the ordering $d_1^* \geq d_2^* \geq d_3^*$ holds in all experiments in Table 5.

Furthermore, we examine the influence of the budget constraint Π_0 on the optimal deductibles in Table 6. As the budget threshold increases, the optimal deductibles decrease for all claims. In the experiments with given parameters, the constraint is binding, so that the budget is exhausted in all experiments. In addition, the ordering $d_1^* \geq d_2^* \geq d_3^*$ holds in Table 6.

Table 5. Impact of θ on optimal deductible vector (d_1^*, d_2^*, d_3^*) with $w = 20$, $\delta = 0.3$, $\Pi_0 = 15$.

θ	d_1^*	d_2^*	d_3^*	$\Pi_{\boldsymbol{R}_N^*}$
0.001	3.2628	0.3338	0.0221	15.0000
0.01	3.4758	0.3642	0.0263	15.0000
0.03	3.9426	0.4336	0.0314	15.0000
0.05	4.4005	0.5039	0.0368	15.0000
0.1	5.5088	0.6806	0.0557	15.0000
0.3	9.4968	1.3954	0.1564	15.0000
0.5	12.8968	2.1315	0.2877	15.0000
1	19.5296	4.0678	0.6977	15.0000
1.1	21.6478	4.8382	0.8740	15.0000
1.3	22.6238	5.2142	0.9616	15.0000
1.5	24.4366	5.9449	1.1326	15.0000
2	28.3529	7.6133	1.5295	15.0000

Table 6. Impact of Π_0 on optimal deductible vector (d_1^*, d_2^*, d_3^*) with $w = 20$, $\delta = 0.3$, and $\theta = 0.5$.

Π_0	d_1^*	d_2^*	d_3^*	$\Pi_{\boldsymbol{R}_N^*}$
1	70.3654	27.5173	8.0942	1.0000
3	47.7288	16.5525	3.8365	3.0000
5	36.9566	11.4986	2.4949	5.0000
8	26.9665	7.0205	1.3880	8.0000
10	22.1431	5.0258	0.9181	10.0000
11	20.0324	4.2446	0.7377	11.0000
13	16.2422	3.0128	0.4671	13.0000
15	12.8968	2.1315	0.2877	15.0000
18	8.5605	1.2153	0.1280	18.0000

7. Conclusions

We revisited the optimal reinsurance decision problem for an insurer confronting a random number of claims during a fixed time period. In the case of independent claim severities, under the stated independence assumptions on claim occurrence times, we showed that stop-loss reinsurance contracts maximize the expected utility of the insurer's terminal wealth. When the insurer is risk-neutral, we derived pairwise structural properties of the deductible vectors. It was found that assigning the larger of two deductible levels to a stochastically larger claim whose arrival time is smaller in the Laplace-transform order is weakly preferable. We further obtained majorization-based comparisons among deductible allocations. These qualitative results continue to hold in the robust setting with comonotonic claim severities subject to corresponding conditions. Additionally, we provided numerical examples, based on Monte Carlo simulation, to examine the effects of the discount rate, safety loading, and premium budget.

Author contributions

Ka Chun Cheung contributed to conceptualization, supervision, funding acquisition, and project administration. Xin Song contributed to methodology, software, analysis, and writing of the original draft. Yaodi Yong contributed to methodology, software, formal analysis, writing of the original draft, funding acquisition and review and editing. Yiying Zhang contributed to conceptualization, supervision, writing of the original draft, review and editing, funding acquisition and project administration. All authors have read and approved the final manuscript.

Use of Generative-AI tools declaration

During the preparation of this work, the authors used generative-AI tools for language polishing and expression improvement. All AI-generated suggestions were reviewed and validated by the authors, who take full responsibility for the final manuscript.

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Conflict of interest

Ka Chun Cheung is an editorial board member for Journal of Industrial and Management Optimization and was not involved in the editorial review or the decision to publish this article. The authors report no potential competing interests.

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