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Research article

Global error bounds for TCP and VTCP

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Abstract: This paper investigated conditionally computable global error bounds for tensor complementarity problems (TCP) and vertical tensor complementarity problems (VTCP). Existing global error bounds for TCP and VTCP rely on an unknown exact solution vector, which cannot be numerically evaluated in practice and thus limits their applicability. To address this limitation, we established conditionally computable global error bounds under structured tensor assumptions. The derived bounds only require an arbitrary feasible test vector and are independent of the unknown true solution. Moreover, our theoretical results generalized several classic error bound theorems for linear complementarity problems (LCP), such as the Mathias–Pang and Chen–Xiang estimates. Numerical examples were presented to verify the tightness and computational efficiency of our proposed bounds.

Keywords: global error bound; tensor complementarity problem; vertical tensor complementarity problem
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1. Introduction

The tensor complementarity problem (TCP), abbreviated as TCP$(\mathcal{H}, \mathbf{b})$, is to find a vector $\mathbf{u} \in \mathbb{R}^n$ such that

$$\mathbf{u} \geq \mathbf{0}, \mathcal{H}\mathbf{u}^{m-1} + \mathbf{b} \geq \mathbf{0}, \mathbf{u}^\top(\mathcal{H}\mathbf{u}^{m-1} + \mathbf{b}) = 0, \tag{1}$$

where $\mathcal{H} \in \mathbb{R}^{[m,n]}$, $\mathbf{b} \in \mathbb{R}^n$. This high-order complementarity framework was originally proposed by Song and Qi [1]. It is also a generalization of the classical linear complementarity problem (LCP) [2]. A wide range of practical optimization models can be equivalently transformed into the TCP, including multi-agent non-cooperative game equilibria, hypergraph clustering tasks, and traffic network equilibrium assignments. Readers may consult the comprehensive overview provided in [3] for more application backgrounds. A substantial body of theoretical work has discussed the existence and uniqueness of TCP solutions in existing publications. As one representative achievement, Bai et al. demonstrated in [4] that the solution set of TCP$(\mathcal{H}, \mathbf{b})$ is both nonempty and compact if $\mathcal{H}$ is a P-tensor; further, strong P-tensors guarantee that TCP admits exactly one solution for any constant vector $\mathbf{b}$. Later, Dai and Wu defined a special family of power Lipschitz tensors, named SPL-tensors, and proved that the TCP with an SPL-tensor also has the property of global uniqueness and solvability [5].

As an extension, the vertical tensor complementarity problem VTCP($\overline{\mathcal{H}}, \overline{\mathbf{b}}$) is defined to find $\mathbf{u} \in \mathbb{R}^n$ such that

$$\mathbf{b}_1 + \mathcal{H}_1\mathbf{u}^{m_1-1} \geq \mathbf{0}, \mathbf{b}_2 + \mathcal{H}_2\mathbf{u}^{m_2-1} \geq \mathbf{0}, (\mathbf{b}_1 + \mathcal{H}_1\mathbf{u}^{m_1-1})^\top(\mathbf{b}_2 + \mathcal{H}_2\mathbf{u}^{m_2-1}) = 0, \tag{2}$$

where $\overline{\mathcal{H}} = \{\mathcal{H}_1, \mathcal{H}_2\}$ with $\mathcal{H}_1 \in \mathbb{R}^{[m_1,n]}, \mathcal{H}_2 \in \mathbb{R}^{[m_2,n]}$, and $\overline{\mathbf{b}} = \{\mathbf{b}_1, \mathbf{b}_2\}$; see [6–12]. Obviously, the VTCP($\overline{\mathcal{H}}, \overline{\mathbf{b}}$) is a generalized form of the TCP($\mathcal{H}, \mathbf{b}$). For generalized multilinear Nash games, all equilibrium points can be recovered via solving the corresponding VTCP system [13]. For the VTCP($\overline{\mathcal{H}}, \overline{\mathbf{b}}$), Li *et al.* in [8] has shown that its solution set is bounded if $\overline{\mathcal{H}}$ is a VR_0-tensor, and it has a solution if $\overline{\mathcal{H}}$ is a VP-tensor.

One of important issues for the above tensor-type complementarity problems is to study the related error bounds. There exist some results on the error bounds in the literature. For example, for the LCP with a P-matrix, its global error bounds and perturbation bounds were established [14–16]. For the TCP, Zheng *et al.* extended the Mathias-Pang global error bound for the LCP [14] to the TCP [17], and Li *et al.* extended the Chen-Xiang perturbation error bounds for the LCP [15] to the TCP [18]. Global error bounds for the VTCP were proposed in [8].

All prior global error bounds for TCP($\mathcal{H}, \mathbf{b}$) and VTCP($\overline{\mathcal{H}}, \overline{\mathbf{b}}$) [8, 17] involve an unknown optimal vector $\mathbf{u}_* \in SOL$ (where SOL denotes the solution set of TCP or VTCP). However, $\mathbf{u}_*$ remains unknown, and the corresponding global error bounds cannot be numerically evaluated, which restricts the practical value of existing error bounds. Motivated above, we develop conditionally computable global error bounds which depend only on an arbitrary trial vector, without requiring the unknown exact solution of the problem.

Based on these considerations, in this paper, we further discuss the global error bounds for TCP($\mathcal{H}, \mathbf{b}$) and VTCP($\overline{\mathcal{H}}, \overline{\mathbf{b}}$). The presented results regarding the global error bounds of two tensor complementarity problems can be computed by a given vector $\mathbf{u}$, and do not depend on their respective solutions. Not only that, these proposed results also cover some classical results, e.g., Mathias-Pang global error bounds [14] and Chen-Xiang global error bounds [15, 16]. To further confirm the efficiency and feasibility of the results of the global error bounds, some numerical examples are provided as well.

The remainder of this paper is organized as follows. Section 2 collects fundamental tensor notations, definitions, and auxiliary lemmas required for subsequent derivations. Section 3 establishes two families of computable global error estimates for TCP, under the assumptions that $\mathcal{H}$ is a strong P-tensor or an SPL-tensor separately. Inspired by the analysis for TCP, Section 4 introduces strong VP-tensors and $VSPL$-tensors and extends the error estimation framework to VTCP. Concluding remarks are summarized in Section 5.

2. Preliminaries

This section establishes basic notations, structured tensor definitions, and auxiliary inequalities that support the main theorems in later sections. For a vector $\mathbf{u} \in \mathbb{R}^n$, $\mathbf{u}_i$ represents its i-th component. The symbol $\mathbb{R}^{[m,n]}$ collects all m-th order n-dimensional real tensors, and the index set $[n] = \{1, 2, \ldots, n\}$.

For two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, we define component-wise minimum and maximum operators:

$$(\mathbf{u} \wedge \mathbf{v})_i = \min\{\mathbf{u}_i, \mathbf{v}_i\}, \quad (\mathbf{u} \vee \mathbf{v})_i = \max\{\mathbf{u}_i, \mathbf{v}_i\}, \quad \forall i \in [n].$$

The vector $\mathbf{u}^{[m]} \in \mathbb{R}^n$ is defined by component-wise power: $(\mathbf{u}^{[m]})_i = \mathbf{u}_i^m$. Nonnegative and positive orthants are denoted as $\mathbb{R}^n_+$ and $\mathbb{R}^n_{++}$, respectively. We use $\mathcal{I}$ to represent the identity tensor of order m and I for identity matrix. A tensor $\mathcal{H} \in \mathbb{R}^{[m,n]}$ is partially symmetric if symmetry holds over its last $m-1$ modes [19].

We define

$$\left(\mathcal{H}\mathbf{u}^{m-1}\right)_i = \sum_{i_2,\dots,i_m=1}^{n} h_{ii_2\dots i_m}\mathbf{u}_{i_2}\dots\mathbf{u}_{i_m}, i \in [n],$$

$$\mathcal{H}(\mathbf{u}^{m-1} - \mathbf{v}^{m-1}) = \mathcal{H}\mathbf{u}^{m-1} - \mathcal{H}\mathbf{v}^{m-1},$$

then

$$\left(\mathcal{H}(\mathbf{u}^{m-1} - \mathbf{v}^{m-1})\right)_i = \sum_{i_2,\dots,i_m=1}^{n} h_{ii_2\dots i_m}(\mathbf{u}_{i_2}\dots\mathbf{u}_{i_m} - \mathbf{v}_{i_2}\dots\ \mathbf{v}_{i_m}).$$

Also,

$$\|\mathcal{H}\|_\infty := \max_{i\in[n]} \sum_{i_2,\dots,i_m=1}^{n} \left|h_{ii_2,\dots,i_m}\right|,$$

then from [18], we have

$$\left\|\mathcal{H}\mathbf{u}^{m-1}\right\|_\infty \le \|\mathcal{H}\|_\infty \left\|\mathbf{u}^{m-1}\right\|_\infty \text{ for any } \mathcal{H} \in \mathbb{R}^{[m,n]} \text{ and } \mathbf{u} \in \mathbb{R}^n,$$

where

$$\mathbf{u}^{m-1} := (\mathbf{u}_{i_1}\cdots\mathbf{u}_{i_{m-1}}) \in \mathbb{R}^{[m-1,n]} \text{ with } i_1,\cdots,i_{m-1} \in [n].$$

2.1. Definitions and auxiliary results for TCP

Definition 2.1. [1, 4, 5] *Let $\mathcal{H} \in \mathbb{R}^{[m,n]}$. We call $\mathcal{H}$ a P-tensor if for every nonzero vector $\boldsymbol{u} \in \mathbb{R}^n$, there exists at least one index $i \in [n]$ such that*

$$u_i(\mathcal{A}\boldsymbol{u}^{m-1})_i > 0.$$

Define the tensor mapping $F(\boldsymbol{u}) = \mathcal{H}\boldsymbol{u}^{m-1}$. Then, $\mathcal{H}$ is a strong P-tensor if for any distinct $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$,

$$\max_{i\in[n]}(\boldsymbol{u} - \boldsymbol{v})_i(F(\boldsymbol{u}) - F(\boldsymbol{v}))_i > 0.$$

If there exists a P-matrix J satisfying

$$\mathcal{H}\boldsymbol{u}^{m-1} = J\boldsymbol{u}^{[m-1]},$$

then $\mathcal{H}$ is termed an SPL-tensor.

The relationship between above structured tensors is clear: every even-order SPL-tensor is a strong P-tensor, and all strong P-tensors belong to the set of P-tensors.

Lemma 2.1. [4, 5] *If $\mathcal{H} \in \mathbb{R}^{[m,n]}$ is a strong P-tensor or an SPL-tensor, then TCP$(\mathcal{H}, \boldsymbol{b})$ has a unique solution for arbitrary $\boldsymbol{b} \in \mathbb{R}^n$.*

Let D be a nonnegative diagonal matrix $D = \mathrm{diag}(d)$ with $d \in [0, 1]^n$.

Lemma 2.2. [20] *A second-order tensor $H \in \mathbb{R}^{[2,n]}$ is a P-matrix if and only if $I - D + DH$ is nonsingular.*

For a strong P-tensor $\mathcal{H} \in \mathbb{R}^{[m,n]}$, let

$$C_1(\mathcal{H}) := \min_{\left\|\mathbf{u}^{m-1}-\mathbf{v}^{m-1}\right\|_\infty=1} \left\{ \max_{i\in[n]} \left(\mathcal{I}\left(\mathbf{u}^{m-1} - \mathbf{v}^{m-1}\right)\right)_i \left(\mathcal{H}\left(\mathbf{u}^{m-1} - \mathbf{v}^{m-1}\right)\right)_i \right\},$$

and it holds that $C_1(\mathcal{H}) > 0$ [18]. We define the two constants for the SPL-tensor $\mathcal{H}$:

$$\beta_1(\mathcal{H}) := \max_{d\in[0,1]^n} \|I - D + DJ\|_\infty \text{ and } \beta_2(\mathcal{H}) := \max_{d\in[0,1]^n} \left\|(I - D + DJ)^{-1}\right\|_\infty.$$

Lemma 2.3. [18] *For arbitrary $\boldsymbol{u} \in \mathbb{R}^n$ and strong P-tensor $\mathcal{H}$, the inequality*

$$C_1(\mathcal{H}) \|\mathbf{u}\|_\infty^{2m-2} \le \max_{i\in[n]} \left(\mathbf{u}^{[m-1]}\right)_i \left(\mathcal{H}\mathbf{u}^{m-1}\right)_i$$

holds true.

Lemma 2.4. [18] *Suppose $\mathcal{H}$ is an even-order tensor. Then, the TCP$(\mathcal{H}, \mathbf{b})$ is equivalent to find a vector $\mathbf{u}$ such that*

$$\mathbf{u}^{[m-1]} \wedge \left(\mathbf{b} + \mathcal{H}\mathbf{u}^{m-1}\right) = 0.$$

Lemma 2.5. [21] *Given real scalars $\{g_i\}_{i=1}^n, \{k_i\}_{i=1}^n$, there exist weights $\theta_i \in [0,1]$ with $\sum_{i=1}^n \theta_i = 1$ such that*

$$\min_{i\in[n]} g_i - \min_{i\in[n]} k_i = \sum_{i=1}^n \theta_i(g_i - k_i).$$

2.2. Some definitions and properties for VTCP

Definition 2.2. [8] *For $\overline{\mathcal{H}} = \{\mathcal{H}_1, \mathcal{H}_2\}$ with $\mathcal{H}_1 \in \mathbb{R}^{[m_1,n]}, \mathcal{H}_2 \in \mathbb{R}^{[m_2,n]}$, it is called*

1. *a VR_0-tensor if $\mathcal{H}_1\mathbf{u}^{m_1-1} \wedge \mathcal{H}_2\mathbf{u}^{m_2-1} = \mathbf{0} \Rightarrow \mathbf{u} = \mathbf{0}$;*
2. *a VP-tensor if for any $t \in [0,1]$, $\left(\mathcal{H}_1\mathbf{u}^{m_1-1} + t\mathbf{u}\right) \wedge \mathcal{H}_2\mathbf{u}^{m_2-1} \le \mathbf{0} \le \left(\mathcal{H}_1\mathbf{u}^{m_1-1} + t\mathbf{u}\right) \vee \mathcal{H}_2\mathbf{u}^{m_2-1} \Rightarrow \mathbf{u} = \mathbf{0}$.*

Definition 2.3. [22] *For $M = (M_0, M_1, \ldots, M_k)$ with $M_0, \ldots, M_k \in \mathbb{R}^{[2,n]}$, we say M has the row W-property if*

$$\min(M_0\mathbf{u}, M_1\mathbf{u}, \ldots, M_k\mathbf{u}) \le 0 \le \max(M_0\mathbf{u}, M_1\mathbf{u}, \ldots, M_k\mathbf{u}) \Rightarrow \mathbf{u} = 0.$$

Lemma 2.6. [23] *$M = (M_0, M_1)$ with $M_0, M_1 \in \mathbb{R}^{[2,n]}$ has the row W-property if and only if $(I - D)M_0 + DM_1$ is nonsingular, where D is a nonnegative diagonal matrix $D = \operatorname{diag}(d)$ with $d \in [0,1]^n$.*

Lemma 2.7. [8] *If $\overline{\mathcal{H}} = \{\mathcal{H}_1, \mathcal{H}_2\}$ is a VP-tensor with $\mathcal{H}_1 \in \mathbb{R}^{[m_1,n]}, \mathcal{H}_2 \in \mathbb{R}^{[m_2,n]}$, then the VTCP$(\overline{\mathcal{H}}, \overline{\mathbf{b}})$ has a solution for any $\mathbf{b}_1, \mathbf{b}_2 \in \mathbb{R}^n$.*

3. Global error bounds for TCP

We split the analysis of TCP error bounds into two separate cases based on the tensor structure: strong P-tensors and SPL-tensors.

Case 1: $\mathcal{H} \in \mathbb{R}^{[m,n]}$ **is a strong** P**-tensor**. For this case, the global error bounds of the form $\|\mathbf{u}^{m-1} - \mathbf{u}_*^{m-1}\|_\infty$ are presented. To achieve this goal, we define the residual function for the TCP below

$$\overline{\mathbf{v}} := \min\left\{\mathbf{u}, (\mathcal{H}\mathbf{u}^{m-1} + \mathbf{b})^{[\frac{1}{m-1}]}\right\},$$

from which we have

Theorem 3.1. *Let* $\mathcal{H} \in \mathbb{R}^{[m,n]}$ *be a strong P-tensor and m be an even number, and* $\mathbf{u}_*$ *be the unique solution of the TCP*$(\mathcal{H}, \mathbf{b})$*. Then, for any* $\mathbf{u} \in \mathbb{R}^n$*,*

$$\frac{\|\overline{\mathbf{v}}\|_\infty^{m-1}}{\max\{1, \|\mathcal{H}\|_\infty\}} \le \|\mathbf{u}^{m-1} - \mathbf{u}_*^{m-1}\|_\infty \le \frac{1 + \|\mathcal{H}\|_\infty}{C_1(\mathcal{H})}\|\overline{\mathbf{v}}\|_\infty^{m-1}. \tag{3}$$

Proof. We first prove the upper bound. Define

$$\alpha = \mathbf{u}^{[m-1]} - \overline{\mathbf{v}}^{[m-1]}, \quad \beta = \mathcal{H}\mathbf{u}^{m-1} + \mathbf{b} - \overline{\mathbf{v}}^{[m-1]}.$$

By the definition of the residual $\overline{\mathbf{v}}^{[m-1]} = \min\{\mathbf{u}^{[m-1]}, \mathcal{H}\mathbf{u}^{m-1} + \mathbf{b}\}$, we have

$$\alpha \ge 0, \quad \beta \ge 0, \quad \alpha^\top \beta = 0.$$

Let $\mathbf{u}_*$ be the unique solution of TCP$(\mathcal{H}, \mathbf{b})$. Denote

$$\omega = \mathcal{H}\mathbf{u}_*^{m-1} + \mathbf{b}.$$

From the TCP complementarity condition, we know

$$\mathbf{u}_*^{[m-1]} \ge 0, \quad \omega \ge 0, \quad (\mathbf{u}_*^{[m-1]})^\top \omega = 0.$$

For each index $i \in [n]$, we compute the product

$$\begin{aligned}
0 &\ge -\alpha_i\omega_i - (\mathbf{u}_*^{[m-1]})_i\beta_i \\
&= (\alpha - \mathbf{u}_*^{[m-1]})_i\,(\beta - \omega)_i \\
&= \left(\mathbf{u}^{[m-1]} - \overline{\mathbf{v}}^{[m-1]} - \mathbf{u}_*^{[m-1]}\right)_i\left(\mathcal{H}\mathbf{u}^{m-1} + \mathbf{b} - \overline{\mathbf{v}}^{[m-1]} - \mathcal{H}\mathbf{u}_*^{m-1} - \mathbf{b}\right)_i \\
&= \left((\mathbf{u}^{[m-1]} - \mathbf{u}_*^{[m-1]}) - \overline{\mathbf{v}}^{[m-1]}\right)_i\left(\mathcal{H}(\mathbf{u}^{m-1} - \mathbf{u}_*^{m-1}) - \overline{\mathbf{v}}^{[m-1]}\right)_i.
\end{aligned}$$

Expand the product:

$$\begin{aligned}
0 \ge\ & (\mathbf{u}^{[m-1]} - \mathbf{u}_*^{[m-1]})_i \cdot \left(\mathcal{H}(\mathbf{u}^{m-1} - \mathbf{u}_*^{m-1})\right)_i - (\mathbf{u}^{[m-1]} - \mathbf{u}_*^{[m-1]})_i\overline{\mathbf{v}}_i^{[m-1]} \\
& - \overline{\mathbf{v}}_i^{[m-1]}\left(\mathcal{H}(\mathbf{u}^{m-1} - \mathbf{u}_*^{m-1})\right)_i + \overline{\mathbf{v}}_i^{2[m-1]}.
\end{aligned}$$

By $\mathbf{u}^{[m-1]} - \mathbf{u}_*^{[m-1]} = \mathcal{I}(\mathbf{u}^{m-1} - \mathbf{u}_*^{m-1})$ and dropping the nonnegative term $\overline{\mathbf{v}}_i^{2[m-1]}$ yields

$$(\mathcal{I}(\mathbf{u}^{m-1} - \mathbf{u}_*^{m-1}))_i\left(\mathcal{H}(\mathbf{u}^{m-1} - \mathbf{u}_*^{m-1})\right)_i \le \overline{\mathbf{v}}_i^{[m-1]}\left(\mathcal{H}(\mathbf{u}^{m-1} - \mathbf{u}_*^{m-1})\right)_i + (\mathcal{I}(\mathbf{u}^{m-1} - \mathbf{u}_*^{m-1}))_i\overline{\mathbf{v}}_i^{[m-1]}.$$

Take maximum over $i \in [n]$ on both sides:

$$\max_{i\in[n]} (\mathcal{I}(\mathbf{u}^{m-1} - \mathbf{u}_*^{m-1}))_i\left(\mathcal{H}(\mathbf{u}^{m-1} - \mathbf{u}_*^{m-1})\right)_i \le \|\overline{\mathbf{v}}\|_\infty^{[m-1]}\left\|\mathcal{H}(\mathbf{u}^{m-1} - \mathbf{u}_*^{m-1})\right\|_\infty + \|\overline{\mathbf{v}}\|_\infty^{[m-1]}\left\|\mathcal{I}(\mathbf{u}^{m-1} - \mathbf{u}_*^{m-1})\right\|_\infty.$$

Since $\mathcal{H}$ is a strong P-tensor, $C_1(\mathcal{H}) > 0$. By Lemma 2.3,

$$\begin{aligned} C_1(\mathcal{H})\|\mathbf{u}^{m-1} - \mathbf{u}_*^{m-1}\|_\infty^2 &\le \max_{i\in[n]}(\mathcal{I}(\mathbf{u}^{m-1} - \mathbf{u}_*^{m-1}))_i(\mathcal{H}(\mathbf{u}^{m-1} - \mathbf{u}_*^{m-1}))_i \\ &\le \|\overline{\mathbf{v}}\|_\infty^{m-1}\|\mathcal{H}(\mathbf{u}^{m-1} - \mathbf{u}_*^{m-1})\|_\infty + \|\overline{\mathbf{v}}\|_\infty^{m-1}\|\mathbf{u}^{m-1} - \mathbf{u}_*^{m-1}\|_\infty \\ &\le \|\overline{\mathbf{v}}\|_\infty^{m-1}\|\mathcal{H}\|_\infty\|\mathbf{u}^{m-1} - \mathbf{u}_*^{m-1}\|_\infty + \|\overline{\mathbf{v}}\|_\infty^{m-1}\|\mathbf{u}^{m-1} - \mathbf{u}_*^{m-1}\|_\infty. \end{aligned}$$

Apply the tensor norm bound

$$\left\|\mathcal{H}(\mathbf{u}^{m-1} - \mathbf{u}_*^{m-1})\right\|_\infty \le \|\mathcal{H}\|_\infty \left\|\mathbf{u}^{m-1} - \mathbf{u}_*^{m-1}\right\|_\infty.$$

Thus,

$$\begin{aligned} C_1(\mathcal{H})\|\mathbf{u}^{m-1} - \mathbf{u}_*^{m-1}\|_\infty^2 &\le \|\overline{\mathbf{v}}\|_\infty^{m-1}\|\mathcal{H}\|_\infty\|\mathbf{u}^{m-1} - \mathbf{u}_*^{m-1}\|_\infty + \|\overline{\mathbf{v}}\|_\infty^{m-1}\|\mathbf{u}^{m-1} - \mathbf{u}_*^{m-1}\|_\infty \\ &= \|\overline{\mathbf{v}}\|_\infty^{[m-1]}(1 + \|\mathcal{H}\|_\infty)\|\mathbf{u}^{m-1} - \mathbf{u}_*^{m-1}\|_\infty. \end{aligned}$$

Canceling $\|\mathbf{u}^{m-1} - \mathbf{u}_*^{m-1}\|_\infty$, we have

$$\|\mathbf{u}^{m-1} - \mathbf{u}_*^{m-1}\|_\infty \le \frac{1 + \|\mathcal{H}\|_\infty}{C_1(\mathcal{H})}\|\overline{\mathbf{v}}\|_\infty^{[m-1]}.$$

We now prove the lower bound

$$\frac{\|\overline{\mathbf{v}}\|_\infty^{[m-1]}}{\max\{1, \|\mathcal{H}\|_\infty\}} \le \|\mathbf{u}^{m-1} - \mathbf{u}_*^{m-1}\|_\infty.$$

Case 1: $\overline{\mathbf{v}}_i > 0$ for all $i \in [n]$. From residual definition $\overline{\mathbf{v}}_i^{[m-1]} = \min\{\mathbf{u}_i^{[m-1]}, (\mathcal{H}\mathbf{u}^{m-1} + \mathbf{b})_i\}$, we have

$$0 < \overline{\mathbf{v}}_i^{[m-1]} \le \mathbf{u}_i^{[m-1]}, \qquad 0 < \overline{\mathbf{v}}_i^{[m-1]} \le (\mathcal{H}\mathbf{u}^{m-1} + \mathbf{b})_i, \quad \forall i \in [n].$$

Because $\mathbf{u}_*$ solves TCP$(\mathcal{H}, \mathbf{b})$, for each i, either $(\mathbf{u}_*)_i = 0$ or $(\mathcal{H}\mathbf{u}_*^{m-1} + \mathbf{b})_i = 0$.

(a) If $(\mathbf{u}_*)_i = 0$:

$$|\overline{\mathbf{v}}_i|^{[m-1]} = \overline{\mathbf{v}}_i^{[m-1]} \le \mathbf{u}_i^{[m-1]} - (\mathbf{u}_*)_i^{[m-1]} \le \|\mathbf{u}^{m-1} - \mathbf{u}_*^{m-1}\|_\infty.$$

(b) If $(\mathcal{H}\mathbf{u}_*^{m-1} + \mathbf{b})_i = 0$:

$$\begin{aligned} |\overline{\mathbf{v}}_i|^{[m-1]} = \overline{\mathbf{v}}_i^{[m-1]} &\le (\mathcal{H}\mathbf{u}^{m-1} + \mathbf{b})_i - (\mathcal{H}\mathbf{u}_*^{m-1} + \mathbf{b})_i \\ &= \left(\mathcal{H}(\mathbf{u}^{m-1} - \mathbf{u}_*^{m-1})\right)_i \\ &\le \|\mathcal{H}\|_\infty\|\mathbf{u}^{m-1} - \mathbf{u}_*^{m-1}\|_\infty. \end{aligned}$$

Combining subcases (a) and (b):

$$\overline{\mathbf{v}}_i^{[m-1]} \le \max\{1, \|\mathcal{H}\|_\infty\} \cdot \|\mathbf{u}^{m-1} - \mathbf{u}_*^{m-1}\|_\infty, \quad \forall i \in [n].$$

Taking maximum over i gives the lower bound inequality for Case 1.
Case 2: $\overline{\mathbf{v}}_i \le 0$ for all $i \in [n]$. Then, $\overline{\mathbf{v}}_i^{[m-1]} = \mathbf{u}_i^{[m-1]}$ or $\overline{\mathbf{v}}_i^{[m-1]} = (\mathcal{H}\mathbf{u}^{m-1} + \mathbf{b})_i$. Recall $\mathbf{u}_*$ satisfies $\mathbf{u}_* \ge 0$, $\mathcal{H}\mathbf{u}_*^{m-1} + \mathbf{b} \ge 0$.

(a) If $\overline{\mathbf{v}}_i^{[m-1]} = \mathbf{u}_i^{[m-1]} \leq 0$:

$$|\overline{\mathbf{v}}_i|^{[m-1]} = -\overline{\mathbf{v}}_i^{[m-1]} = -\mathbf{u}_i^{[m-1]} \leq -\mathbf{u}_i^{[m-1]} + (\mathbf{u}_*)_i^{[m-1]} \leq \|\mathbf{u}^{m-1} - \mathbf{u}_*^{m-1}\|_\infty.$$

(b) If $\overline{\mathbf{v}}_i^{[m-1]} = (\mathcal{H}\mathbf{u}^{m-1} + \mathbf{b})_i \leq 0$:

$$\begin{aligned}
|\overline{\mathbf{v}}_i|^{[m-1]} = -\overline{\mathbf{v}}_i^{[m-1]} &= -(\mathcal{H}\mathbf{u}^{m-1} + \mathbf{b})_i \\
&\leq -(\mathcal{H}\mathbf{u}^{m-1} + \mathbf{b})_i + (\mathcal{H}\mathbf{u}_*^{m-1} + \mathbf{b})_i \\
&= \left(\mathcal{H}(\mathbf{u}_*^{m-1} - \mathbf{u}^{m-1})\right)_i \\
&\leq \|\mathcal{H}\|_\infty \|\mathbf{u}^{m-1} - \mathbf{u}_*^{m-1}\|_\infty.
\end{aligned}$$

Again we obtain

$$|\overline{\mathbf{v}}_i|^{[m-1]} \leq \max\{1, \|\mathcal{H}\|_\infty\} \|\mathbf{u}^{m-1} - \mathbf{u}_*^{m-1}\|_\infty, \quad \forall i \in [n].$$

Taking $\|\cdot\|_\infty$ yields the desired lower bound. Combining upper bound and lower bound completes the proof.

Remark 3.1. *It is not difficult to find that the global error bounds in Theorem 3.1 do not depend on the solution of the TCP($\mathcal{H}, \mathbf{b}$), which can be computed by a given vector $\mathbf{u} \in \mathbb{R}^n$. In addition, when $m = 2$, the strong P-tensor reduces to the P-matrix, and Theorem 3.1 reduces to Lemma 2 [14] by Mathias and Pang, which is the well-known global error bounds of the LCP.*

For $\mathcal{H} \in \mathbb{R}^{[m,n]}$, there exists a corresponding partially symmetric tensor $\widehat{\mathcal{H}} \in \mathbb{R}^{[m,n]}$ such that $\mathcal{H}\mathbf{u}^{m-1} = \widehat{\mathcal{H}}\mathbf{u}^{m-1}$, and $\|\widehat{\mathcal{H}}\|_\infty \leq \|\mathcal{H}\|_\infty$ [24, 25]. Based on this fact, together with the proof of Theorem 3.1, we have

Theorem 3.2. *Let $\mathcal{H} \in \mathbb{R}^{[m,n]}$ be a strong P-tensor, and let $\mathbf{u}_*$ be the unique solution of the TCP($\mathcal{H}, \mathbf{b}$). Then, for any $\mathbf{u} \in \mathbb{R}^n$,*

$$\frac{\|\overline{\mathbf{v}}\|_\infty^{m-1}}{\max\left\{1, \|\widehat{\mathcal{H}}\|_\infty\right\}} \leq \|\mathbf{u}^{m-1} - \mathbf{u}_*^{m-1}\|_\infty \leq \frac{1 + \|\widehat{\mathcal{H}}\|_\infty}{C_1(\widehat{\mathcal{H}})} \|\overline{\mathbf{v}}\|_\infty^{m-1}.$$

Comparing Theorem 3.1 and Theorem 3.2, it is easy to see that the latter is generally tighter than the former. To illustrate this, here we provide the following example; see Example 3.1.

Example 3.1. *Let $\mathcal{H} = (h_{i_1 i_2 i_3 i_4})$ such that*

$$h_{1111} = h_{2222} = 1, h_{2211} = 2, h_{2112} = h_{2121} = -1$$

and $h_{i_1 i_2 i_3 i_4} = 0$, otherwise.

Consider $\mathbf{u} = \mathbf{b} = (1, 1)^\top$. Then, $\mathcal{H}$ satisfies the conditions of Theorems 3.1 and 3.2. By the calculation, we have

$$\overline{\boldsymbol{v}} := \min\left\{\boldsymbol{u}, (\mathcal{H}\boldsymbol{u}^{m-1} + \boldsymbol{b})^{[\frac{1}{m-1}]}\right\} = (1, 1)^\top,$$

$$\|\mathcal{H}\|_\infty = 5, \ \|\widehat{\mathcal{H}}\|_\infty = 1,$$

$$C_1(\mathcal{H}) = C_1\left(\widehat{\mathcal{H}}\right) = \min_{\|\boldsymbol{u}^{m-1} - \boldsymbol{v}^{m-1}\|_\infty = 1} \left\{\max_{i \in [n]} \left(\mathcal{I}\left(\boldsymbol{u}^{m-1} - \boldsymbol{v}^{m-1}\right)\right)_i \left(\mathcal{H}\left(\boldsymbol{u}^{m-1} - \boldsymbol{v}^{m-1}\right)\right)_i\right\}$$

$$= \min_{\|\boldsymbol{u}^{m-1}-\boldsymbol{v}^{m-1}\|_\infty=1} \left\{ \max_{i\in[n]} \{\boldsymbol{u}_{iiii} - \boldsymbol{v}_{iiii}\} \right\} = 1.$$

Then, from Theorem 3.1, we have

$$0.2 \le \|\mathbf{u}^3 - \mathbf{u}_*^3\|_\infty \le 6.$$

From Theorem 3.2, we have

$$1 \le \|\mathbf{u}^3 - \mathbf{u}_*^3\|_\infty \le 2.$$

Case 2: $\mathcal{H} \in \mathbb{R}^{[m,n]}$ **is an** SPL**-tensor**. For this case, we consider the global error bounds of the form $\|\mathbf{u}^{[m-1]} - \mathbf{u}_*^{[m-1]}\|_\infty$. Such a case, we have the following.

Theorem 3.3. *Let* $\mathcal{H} \in \mathbb{R}^{[m,n]}$ *be a SPL-tensor, and let* $\mathbf{u}_*$ *be the unique solution of the TCP*($\mathcal{H}, \mathbf{b}$). *Then, for any* $\mathbf{u} \in \mathbb{R}^n$,

$$\frac{\|\overline{\mathbf{v}}\|_\infty^{m-1}}{\beta_1(\mathcal{H})} \le \|\mathbf{u}^{[m-1]} - \mathbf{u}_*^{[m-1]}\|_\infty \le \beta_2(\mathcal{H})\, \|\overline{\mathbf{v}}\|_\infty^{m-1}.$$

Proof. Let $\alpha = \mathcal{I}\mathbf{u}^{m-1} - \mathcal{I}\overline{\mathbf{v}}^{m-1}, \beta = \mathcal{H}\mathbf{u}^{m-1} + \mathbf{b} - \mathcal{I}\overline{\mathbf{v}}^{m-1}$, then

$$\alpha \ge 0,\ \beta \ge 0,\ \alpha^\top\beta = 0.$$

Let $\omega = \mathcal{H}\mathbf{u}_*^{m-1} + \mathbf{b}$, then $\mathbf{u}_*^\top\omega = 0$. By Lemmas 2.4 and 2.5, we have

$$\begin{aligned}
&\min\left\{\mathcal{I}\mathbf{u}^{m-1} - \mathcal{I}\overline{\mathbf{v}}^{m-1}, \mathcal{H}\mathbf{u}^{m-1} + \mathbf{b} - \mathcal{I}\overline{\mathbf{v}}^{m-1}\right\} - \min\left\{\mathbf{u}_*^{[m-1]}, \mathcal{H}\mathbf{u}_*^{m-1} + \mathbf{b}\right\}\\
&= \min\left\{\mathbf{u}^{[m-1]} - \overline{\mathbf{v}}^{[m-1]}, \mathcal{H}\mathbf{u}^{m-1} + \mathbf{b} - \overline{\mathbf{v}}^{[m-1]}\right\} - \min\left\{\mathbf{u}_*^{[m-1]}, \mathcal{H}\mathbf{u}_*^{m-1} + \mathbf{b}\right\}\\
&= (I - D)(\mathbf{u}^{[m-1]} - \overline{\mathbf{v}}^{[m-1]} - \mathbf{u}_*^{[m-1]}) + D(\mathcal{H}\mathbf{u}^{m-1} - \overline{\mathbf{v}}^{[m-1]} - \mathcal{H}\mathbf{u}_*^{m-1}) = 0,
\end{aligned}$$

where $D = \mathrm{diag}(d)$ with $d \in [0,1]^n$. Since $\mathcal{H}$ is an SPL-tensor, by $\mathcal{H}\mathbf{u}^{m-1} = J\mathbf{u}^{[m-1]}$, we have

$$(I - D + DJ)(\mathbf{u}^{[m-1]} - \mathbf{u}_*^{[m-1]}) = \overline{\mathbf{v}}^{[m-1]}. \tag{4}$$

Therefore,

$$\|I - D + DJ\|_\infty \|\mathbf{u}^{[m-1]} - \mathbf{u}_*^{[m-1]}\|_\infty \ge \|\overline{\mathbf{v}}\|_\infty^{m-1},$$

which implies

$$\|\mathbf{u}^{[m-1]} - \mathbf{u}_*^{[m-1]}\|_\infty \ge \frac{\|\overline{\mathbf{v}}\|_\infty^{m-1}}{\|I - D + DJ\|_\infty}.$$

By Eq (4) again, we have

$$\mathbf{u}^{[m-1]} - \mathbf{u}_*^{[m-1]} = (I - D + DJ)^{-1}\overline{\mathbf{v}}^{[m-1]},$$

from which one readily gains

$$\|\mathbf{u}^{[m-1]} - \mathbf{u}_*^{[m-1]}\|_\infty \le \|(I - D + DJ)^{-1}\|_\infty \|\overline{\mathbf{v}}\|_\infty^{m-1}.$$

The results in Theorem 3.3 hold.

Remark 3.2. *Clearly, when $m = 2$, the SPL-tensor reduces to the P-matrix, and the global error bounds in Theorem 3.3 reduce to the well-known global error bounds of the LCP given by Chen and Xiang; see page 516 in [15].*

The following example is given to show the efficiency of Theorem 3.3.

Example 3.2. *Let $\mathcal{H} = (h_{i_1 i_2 i_3 i_4}) \in \mathbb{R}^{[4,2]}$ with*

$$h_{1111} = 3, h_{2222} = 2, h_{2211} = 2, h_{2112} = h_{2121} = -1,$$

and $h_{i_1 i_2 i_3 i_4} = 0$, otherwise.

From Example 3.1 in [5], *$\mathcal{H}$ is a SPL-tensor. Consider $\mathbf{u} = \mathbf{b} = (1,1)^\top$, and by simple calculation, we have*

$$J = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}.$$

Let

$$D = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix},$$

and we have

$$\begin{aligned} \beta_1(\mathcal{H}) &= \max_{d\in[0,1]^n} \|I - D + DJ\|_\infty = \max_{d\in[0,1]^n} \left\| \begin{bmatrix} 1+2d_1 & 0 \\ 0 & 1+d_2 \end{bmatrix} \right\|_\infty \\ &= \max_{d\in[0,1]^n} \{1 + 2d_1\} = 3. \end{aligned}$$

$$\beta_2(\mathcal{H}) := \max_{d\in[0,1]^n} \left\| (I - D + DJ)^{-1} \right\|_\infty = \max_{d\in[0,1]^n} \left\| \begin{bmatrix} 1+2d_1 & 0 \\ 0 & 1+d_2 \end{bmatrix}^{-1} \right\|_\infty = 1.$$

From Theorem 3.3, we have

$$\frac{1}{3} \le \|\mathbf{u}^{[3]} - \mathbf{u}_*^{[3]}\|_\infty \le 1.$$

4. Global error bounds of VTCP

In this section, we establish error bounds for VTCP under two tensor categories: strong VP-tensors and $VSPL$-tensors.

Case 1: $\overline{\mathcal{H}}$ is a strong VP-tensor. First, inspired by the strong P-tensor, the strong VP-tensor is introduced (concretely see Definition 4.1), and some of its important properties are established.

Definition 4.1. *Let $\overline{\mathcal{H}} = \{\mathcal{H}_1, \mathcal{H}_2\}$ with $\mathcal{H}_1 \in \mathbb{R}^{[m_1,n]}, \mathcal{H}_2 \in \mathbb{R}^{[m_2,n]}$, $G(\mathbf{u}) = \mathcal{H}_1\mathbf{u}^{m_1-1}$, $F(\mathbf{u}) = \mathcal{H}_2\mathbf{u}^{m_2-1}$. Then, $\overline{\mathcal{H}}$ is called a strong VP-tensor if and only if*

$$\max_{i\in[n]} (G(\mathbf{u}) - G(\mathbf{v}))_i (F(\mathbf{u}) - F(\mathbf{v}))_i > 0,$$

for any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and $\mathbf{u} \neq \mathbf{v}$.

Theorem 4.1. *If $\overline{\mathcal{H}}$ is a strong VP-tensor, then $\overline{\mathcal{H}}$ is a VP-tensor.*

Proof. Let $G(\mathbf{u}) = \mathcal{H}_1\mathbf{u}^{m_1-1}$, $G(\mathbf{v}) = \mathbf{0}$, $\mathbf{0} \neq \mathbf{u} \in \mathbb{R}^n$, $\mathbf{v} = \mathbf{0}$. If $\overline{\mathcal{H}}$ is a strong VP-tensor, from Definition 4.1, there exists at least one index $i \in [n]$ satisfying

$$(\mathcal{H}_1\mathbf{u}^{m_1-1})_i(\mathcal{H}_2\mathbf{u}^{m_2-1})_i > 0,$$

which implies that $\overline{\mathcal{H}}$ is a VP-tensor.

The definition of the strong VP-tensor and the uniqueness for VTCP with $\mathcal{H}_1 \in \mathbb{R}^{[m,n]}, \mathcal{H}_2 \in \mathbb{R}^{[m,n]}$ were obtained in [26]. Based on Theorem 4.1, Lemma 4.1 in [26], and Lemma 2.7, we can obtain a sufficient condition for the existence and uniqueness of the solution for VTCP with $\mathcal{H}_1 \in \mathbb{R}^{[m_1,n]}, \mathcal{H}_2 \in \mathbb{R}^{[m_2,n]}$ directly.

Theorem 4.2. *If $\overline{\mathcal{H}}$ is a strong VP-tensor, then the VTCP has a unique solution for all $\overline{\boldsymbol{b}}$.*

Define a constant for the strong VP-tensor, i.e.,

$$C_2(\overline{\mathcal{H}}) = \min_{\|\mathbf{u}^{m_1-1}-\mathbf{v}^{m_1-1}\|_\infty\|\mathbf{u}^{m_2-1}-\mathbf{v}^{m_2-1}\|_\infty=1} \max_{i\in[n]}(\mathcal{H}_1(\mathbf{u}^{m_1-1} - \mathbf{v}^{m_1-1}))_i(\mathcal{H}_2(\mathbf{u}^{m_2-1} - \mathbf{v}^{m_2-1}))_i,$$

some properties of the strong VP-tensor are proposed as follows.

Theorem 4.3. *Let $\overline{\mathcal{H}}$ be a strong VP-tensor if and only if $C_2(\overline{\mathcal{H}}) > 0$.*

Proof. ($\Rightarrow$) If $\overline{\mathcal{H}}$ is a strong VP-tensor, then for any different $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, there exists an $i \in [n]$ such that

$$(\mathcal{H}_1(\mathbf{u}^{m_1-1} - \mathbf{v}^{m_1-1}))_i(\mathcal{H}_2(\mathbf{u}^{m_2-1} - \mathbf{v}^{m_2-1}))_i > 0.$$

So, $C_2(\overline{\mathcal{H}}) > 0$.

($\Leftarrow$) If $C_2(\overline{\mathcal{H}}) > 0$, then

$$\max_{i\in[n]}\left(\mathcal{H}_1\left(\frac{\mathbf{u}^{m_1-1} - \mathbf{v}^{m_1-1}}{\|\mathbf{u}^{m_1-1} - \mathbf{v}^{m_1-1}\|_\infty}\right)\right)_i\left(\mathcal{H}_2\left(\frac{\mathbf{u}^{m_2-1} - \mathbf{v}^{m_2-1}}{\|\mathbf{u}^{m_2-1} - \mathbf{v}^{m_2-1}\|_\infty}\right)\right)_i \geq C_2(\overline{\mathcal{H}}) > 0.$$

This implies that there exists an $i \in [n]$ such that

$$(\mathcal{H}_1(\mathbf{u}^{m_1-1} - \mathbf{v}^{m_1-1}))_i(\mathcal{H}_2(\mathbf{u}^{m_2-1} - \mathbf{v}^{m_2-1}))_i > 0.$$

Hence, $\overline{\mathcal{H}}$ is a strong VP-tensor.

Theorem 4.4. *Let $\overline{\mathcal{H}}$ be a strong VP-tensor. Then, for any different $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$,*

$$C_2(\overline{\mathcal{H}})\|\mathbf{u}^{m_1-1} - \mathbf{v}^{m_1-1}\|_\infty\|\mathbf{u}^{m_2-1} - \mathbf{v}^{m_2-1}\|_\infty \leq \max_{i\in[n]}(\mathcal{H}_1(\mathbf{u}^{m_1-1} - \mathbf{v}^{m_1-1}))_i(\mathcal{H}_2(\mathbf{u}^{m_2-1} - \mathbf{v}^{m_2-1}))_i.$$

Proof. If $\overline{\mathcal{H}}$ is a strong VP-tensor, then for any different $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$,

$$\max_{i\in[n]}\left(\mathcal{H}_1\left(\frac{\mathbf{u}^{m_1-1} - \mathbf{v}^{m_1-1}}{\|\mathbf{u}^{m_1-1} - \mathbf{v}^{m_1-1}\|_\infty}\right)\right)_i\left(\mathcal{H}_2\left(\frac{\mathbf{u}^{m_2-1} - \mathbf{v}^{m_2-1}}{\|\mathbf{u}^{m_2-1} - \mathbf{v}^{m_2-1}\|_\infty}\right)\right)_i \geq C_2(\overline{\mathcal{H}}) > 0,$$

which implies that the result in Theorem 4.4 holds.

Next, the upper bound of global error bounds of the VTCP$(\overline{\mathcal{H}}, \overline{\mathbf{b}})$ of the form $\|\mathbf{u}^{m-1} - \mathbf{u}_*^{m-1}\|_\infty$ can be given as well. For this goal, we introduce the residual function for the VTCP below

$$\underline{\mathbf{v}} = \min\left\{\left(\mathcal{H}_1\mathbf{u}^{m_1-1} + \mathbf{b}_1\right)^{[\frac{1}{m_1-1}]}, \left(\mathcal{H}_2\mathbf{u}^{m_2-1} + \mathbf{b}_2\right)^{[\frac{1}{m_2-1}]}\right\}$$

and have the following.

Theorem 4.5. *Let $\overline{\mathcal{H}}$ be a strong VP-tensor, m_1 and m_2 be even numbers, and $\mathbf{u}_*$ be the solution of the VTCP$(\overline{\mathcal{H}}, \overline{\mathbf{b}})$. Then, for any $\mathbf{u} \in \mathbb{R}^n$,*

$$\|\mathbf{u}^{m_1-1} - \mathbf{u}_*^{m_1-1}\|_\infty \le \frac{\|\underline{\mathbf{v}}\|_\infty^{m_1-1}\|\mathcal{H}_2\|_\infty + \|\mathcal{H}_1\|_\infty\|\underline{\mathbf{v}}\|_\infty^{m_2-1}}{C_2(\overline{\mathcal{H}})},$$

or

$$\|\mathbf{u}^{m_2-1} - \mathbf{u}_*^{m_2-1}\|_\infty \le \frac{\|\underline{\mathbf{v}}\|_\infty^{m_1-1}\|\mathcal{H}_2\|_\infty + \|\mathcal{H}_1\|_\infty\|\underline{\mathbf{v}}\|_\infty^{m_2-1}}{C_2(\overline{\mathcal{H}})}.$$

Proof. Let $\mathbf{v}_1 = \mathcal{H}_1\mathbf{u}^{m_1-1} + \mathbf{b}_1 - \mathcal{I}_1\underline{\mathbf{v}}^{m_1-1}$, $\mathbf{v}_2 = \mathcal{H}_2\mathbf{u}^{m_2-1} + \mathbf{b}_2 - \mathcal{I}_2\underline{\mathbf{v}}^{m_2-1}$, then

$$\mathbf{v}_1 \ge 0,\ \mathbf{v}_2 \ge 0,\ \mathbf{v}_1^\top\mathbf{v}_2 = 0.$$

Taking $\omega_1 = \mathcal{H}_1\mathbf{u}_*^{m_1-1} + \mathbf{b}_1$, $\omega_2 = \mathcal{H}_2\mathbf{u}_*^{m_2-1} + \mathbf{b}_2$, by (2), we have

$$\omega_1 \ge 0,\ \omega_2 \ge 0,\ \omega_1^\top\omega_2 = 0.$$

Further, for any $i \in [n]$, we obtain

$$\begin{aligned}
&0 \ge (\mathbf{v}_1 - \omega_1)_i(\mathbf{v}_2 - \omega_2)_i\\
&= (\mathcal{H}_1\mathbf{u}^{m_1-1} - \mathcal{I}_1\underline{\mathbf{v}}^{m_1-1} - \mathcal{H}_1\mathbf{u}_*^{m_1-1})_i(\mathcal{H}_2\mathbf{u}^{m_2-1} - \mathcal{I}_2\underline{\mathbf{v}}^{m_2-1} - \mathcal{H}_2\mathbf{u}_*^{m_2-1})_i\\
&= (\mathcal{H}_1(\mathbf{u}^{m_1-1} - \mathbf{u}_*^{m_1-1}))_i(\mathcal{H}_2(\mathbf{u}^{m_2-1} - \mathbf{u}_*^{m_2-1}))_i - \underline{v}_i^{m_1-1}(\mathcal{H}_2(\mathbf{u}^{m_2-1} - \mathbf{u}_*^{m_2-1}))_i\\
&- (\mathcal{H}_1(\mathbf{u}^{m_1-1} - \mathbf{u}_*^{m_1-1}))_i\underline{v}_i^{m_2-1} + \underline{v}_i^{m_1+m_2-2}\\
&\ge (\mathcal{H}_1(\mathbf{u}^{m_1-1} - \mathbf{u}_*^{m_1-1}))_i(\mathcal{H}_2(\mathbf{u}^{m_2-1} - \mathbf{u}_*^{m_2-1}))_i - \underline{v}_i^{m_1-1}(\mathcal{H}_2(\mathbf{u}^{m_2-1} - \mathbf{u}_*^{m_2-1}))_i\\
&- (\mathcal{H}_1(\mathbf{u}^{m_1-1} - \mathbf{u}_*^{m_1-1}))_i\underline{v}_i^{m_2-1}.
\end{aligned}$$

Since $\overline{\mathcal{H}}$ is a strong VP-tensor, we have $C_2(\overline{\mathcal{H}}) > 0$ and

$$\begin{aligned}
&C_2(\overline{\mathcal{H}})\|\mathbf{u}^{m_1-1} - \mathbf{u}_*^{m_1-1}\|_\infty\|\mathbf{u}^{m_2-1} - \mathbf{u}_*^{m_2-1}\|_\infty\\
&\le \max_{i\in[n]}(\mathcal{H}_1(\mathbf{u}^{m_1-1} - \mathbf{u}_*^{m_1-1}))_i(\mathcal{H}_2(\mathbf{u}^{m_2-1} - \mathbf{u}_*^{m_2-1}))_i\\
&\le \max_{i\in[n]}(\underline{v}_i^{m_1-1}(\mathcal{H}_2(\mathbf{u}^{m_2-1} - \mathbf{u}_*^{m_2-1}))_i + (\mathcal{H}_1(\mathbf{u}^{m_1-1} - \mathbf{u}_*^{m_1-1}))_i\underline{v}_i^{m_2-1})\\
&\le \|\underline{\mathbf{v}}\|_\infty^{m_1-1}\|\mathcal{H}_2\|_\infty\|\mathbf{u}^{m_2-1} - \mathbf{u}_*^{m_2-1}\|_\infty + \|\mathcal{H}_1\|_\infty\|\mathbf{u}^{m_1-1} - \mathbf{u}_*^{m_1-1}\|_\infty\|\underline{\mathbf{v}}\|_\infty^{m_2-1}.
\end{aligned}$$

Thereupon, we have the following two cases:
Case 1: when $\|\mathbf{u}^{m_1-1} - \mathbf{u}_*^{m_1-1}\|_\infty \le \|\mathbf{u}^{m_2-1} - \mathbf{u}_*^{m_2-1}\|_\infty$, we have

$$C_2(\overline{\mathcal{H}})\|\mathbf{u}^{m_1-1} - \mathbf{u}_*^{m_1-1}\|_\infty\|\mathbf{u}^{m_2-1} - \mathbf{u}_*^{m_2-1}\|_\infty$$

$$\le \|\underline{\mathbf{v}}\|_\infty^{m_1-1}\|\mathcal{H}_2\|_\infty\|\mathbf{u}^{m_2-1}-\mathbf{u}_*^{m_2-1}\|_\infty+\|\mathcal{H}_1\|_\infty\|\mathbf{u}^{m_1-1}-\mathbf{u}_*^{m_1-1}\|_\infty\|\underline{\mathbf{v}}\|_\infty^{m_2-1}$$
$$\le \|\underline{\mathbf{v}}\|_\infty^{m_1-1}\|\mathcal{H}_2\|_\infty\|\mathbf{u}^{m_2-1}-\mathbf{u}_*^{m_2-1}\|_\infty+\|\mathcal{H}_1\|_\infty\|\mathbf{u}^{m_2-1}-\mathbf{u}_*^{m_2-1}\|_\infty\|\underline{\mathbf{v}}\|_\infty^{m_2-1},$$

which implies that

$$\|\mathbf{u}^{m_1-1}-\mathbf{u}_*^{m_1-1}\|_\infty \le \frac{\|\underline{\mathbf{v}}\|_\infty^{m_1-1}\|\mathcal{H}_2\|_\infty+\|\mathcal{H}_1\|_\infty\|\underline{\mathbf{v}}\|_\infty^{m_2-1}}{C_2(\overline{\mathcal{H}})}.$$

Case 2: when $\|\mathbf{u}^{m_1-1}-\mathbf{u}_*^{m_1-1}\|_\infty > \|\mathbf{u}^{m_2-1}-\mathbf{u}_*^{m_2-1}\|_\infty$, we have

$$C_2(\overline{\mathcal{H}})\|\mathbf{u}^{m_1-1}-\mathbf{u}_*^{m_1-1}\|_\infty\|\mathbf{u}^{m_2-1}-\mathbf{u}_*^{m_2-1}\|_\infty$$
$$\le \|\underline{\mathbf{v}}\|_\infty^{m_1-1}\|\mathcal{H}_2\|_\infty\|\mathbf{u}^{m_2-1}-\mathbf{u}_*^{m_2-1}\|_\infty+\|\mathcal{H}_1\|_\infty\|\mathbf{u}^{m_1-1}-\mathbf{u}_*^{m_1-1}\|_\infty\|\underline{\mathbf{v}}\|_\infty^{m_2-1}$$
$$\le \|\underline{\mathbf{v}}\|_\infty^{m_1-1}\|\mathcal{H}_2\|_\infty\|\mathbf{u}^{m_1-1}-\mathbf{u}_*^{m_1-1}\|_\infty+\|\mathcal{H}_1\|_\infty\|\mathbf{u}^{m_1-1}-\mathbf{u}_*^{m_1-1}\|_\infty\|\underline{\mathbf{v}}\|_\infty^{m_2-1},$$

which implies that

$$\|\mathbf{u}^{m_2-1}-\mathbf{u}_*^{m_2-1}\|_\infty \le \frac{\|\underline{\mathbf{v}}\|_\infty^{m_1-1}\|\mathcal{H}_2\|_\infty+\|\mathcal{H}_1\|_\infty\|\underline{\mathbf{v}}\|_\infty^{m_2-1}}{C_2(\overline{\mathcal{H}})}.$$

Combining Case 1 and Case 2 completes the proof.

Finally, we present some results on the lower bounds of the global error bound of the VTCP($\overline{\mathcal{H}}, \overline{\mathbf{b}}$).

Theorem 4.6. *Let $\overline{\mathcal{H}}$ be a strong VP-tensor, m_1 and m_2 be even numbers, and $\mathbf{u}_*$ be the solution of the VTCP($\overline{\mathcal{H}}, \overline{\mathbf{b}}$). Then, for any $\mathbf{u} \in \mathbb{R}^n$,*

$$\|\mathbf{u}^{m_1-1}-\mathbf{u}_*^{m_1-1}\|_\infty \ge \frac{\|\underline{\mathbf{v}}\|_\infty^{m_1-1}}{\|\mathcal{H}_1\|_\infty},$$

or

$$\|\mathbf{u}^{m_2-1}-\mathbf{u}_*^{m_2-1}\|_\infty \ge \frac{\|\underline{\mathbf{v}}\|_\infty^{m_2-1}}{\|\mathcal{H}_2\|_\infty}.$$

Proof. Considering two cases: $\underline{\mathbf{v}}_i > 0$ and $\underline{\mathbf{v}}_i \le 0$.

If $\underline{\mathbf{v}}_i > 0$ for any $i \in [n]$, then we have

$$0 < \underline{\mathbf{v}}_i^{m_1-1} \le (\mathcal{H}_1\mathbf{u}^{m_1-1}+\mathbf{b}_1)_i, \text{ and } 0 < \underline{\mathbf{v}}_i^{m_2-1} \le (\mathcal{H}_2\mathbf{u}^{m_2-1}+\mathbf{b}_2)_i.$$

Since $\mathbf{u}_*$ is a solution of VTCP($\overline{\mathcal{H}}, \overline{\mathbf{b}}$), clearly,

$$(\mathcal{H}_1\mathbf{u}_*^{m_1-1}+\mathbf{b}_1)_i = 0 \text{ or } (\mathcal{H}_2\mathbf{u}_*^{m_2-1}+\mathbf{b}_2)_i = 0.$$

(a) If $(\mathcal{H}_1\mathbf{u}_*^{m_1-1}+\mathbf{b}_1)_i = 0$, then

$$|\underline{\mathbf{v}}_i|^{m_1-1} = \underline{\mathbf{v}}_i^{m_1-1} \le (\mathcal{H}_1\mathbf{u}^{m_1-1}+\mathbf{b}_1)_i - (\mathcal{H}_1\mathbf{u}_*^{m_1-1}+\mathbf{b}_1)_i$$
$$\le (\mathcal{H}_1\left(\mathbf{u}^{m_1-1}-\mathbf{u}_*^{m_1-1}\right))_i$$
$$\le \|\mathcal{H}_1\|_\infty\|\mathbf{u}^{m_1-1}-\mathbf{u}_*^{m_1-1}\|_\infty.$$

So,

$$\|\mathbf{u}^{m_1-1}-\mathbf{u}_*^{m_1-1}\|_\infty \ge \frac{\|\underline{\mathbf{v}}\|_\infty^{m_1-1}}{\|\mathcal{H}_1\|_\infty}.$$

(b) Similarly, if $(\mathcal{H}_2\mathbf{u}_*^{m_2-1}+\mathbf{b}_2)_i=0$, then

$$\|\mathbf{u}^{m_2-1}-\mathbf{u}_*^{m_2-1}\|_\infty \geq \frac{\|\underline{\mathbf{v}}\|_\infty^{m_2-1}}{\|\mathcal{H}_2\|_\infty}.$$

If $\underline{\mathbf{v}}_i \leq 0$ for any $i\in[n]$, then we have

$$0<\underline{\mathbf{v}}_i^{m_1-1}=(\mathcal{H}_1\mathbf{u}^{m_1-1}+\mathbf{b}_1)_i,\ \text{or}\ 0<\underline{\mathbf{v}}_i^{m_2-1}=(\mathcal{H}_2\mathbf{u}^{m_2-1}+\mathbf{b}_2)_i.$$

Since $\mathbf{u}_*$ is a solution of VTCP$(\overline{\mathcal{H}},\overline{\mathbf{b}})$, clearly,

$$(\mathcal{H}_1\mathbf{u}_*^{m_1-1}+\mathbf{b}_1)_i=0 \text{ and } (\mathcal{H}_2\mathbf{u}_*^{m_2-1}+\mathbf{b}_2)_i=0.$$

(a) If $\underline{\mathbf{v}}_i^{m_1-1}=(\mathcal{H}_1\mathbf{u}^{m_1-1}+\mathbf{b}_1)_i$, then

$$\begin{aligned}|\underline{\mathbf{v}}_i|^{m_1-1}=-\underline{\mathbf{v}}_i^{m_1-1} &\leq -(\mathcal{H}_1\mathbf{u}^{m_1-1}+\mathbf{b}_1)_i+(\mathcal{H}_1\mathbf{u}_*^{m_1-1}+\mathbf{b}_1)_i\\ &\leq \|\mathcal{H}_1\|_\infty\|\mathbf{u}^{m_1-1}-\mathbf{u}_*^{m_1-1}\|_\infty,\end{aligned}$$

thus,

$$\|\mathbf{u}^{m_1-1}-\mathbf{u}_*^{m_1-1}\|_\infty \geq \frac{\|\underline{\mathbf{v}}\|_\infty^{m_1-1}}{\|\mathcal{H}_1\|_\infty}.$$

(b) Similarly, if $\underline{\mathbf{v}}_i^{m_2-1}=(\mathcal{H}_2\mathbf{u}^{m_2-1}+\mathbf{b}_2)_i$, then

$$\|\mathbf{u}^{m_2-1}-\mathbf{u}_*^{m_2-1}\|_\infty \geq \frac{\|\underline{\mathbf{v}}\|_\infty^{m_2-1}}{\|\mathcal{H}_2\|_\infty}.$$

The proof of Theorem 4.6 is completed.

Further, when $m_1=m_2=m$ are even numbers, for $\overline{\mathcal{H}}$ being a strong VP-tensor, from Theorems 4.5 and 4.6, one obtains

Theorem 4.7. *Let $\overline{\mathcal{H}}$ be a strong VP-tensor, $m_1=m_2=m$ be even numbers, and $\mathbf{u}_*$ be the solution of the VTCP$(\overline{\mathcal{H}},\overline{\mathbf{b}})$. Then, for any $\mathbf{u}\in\mathbb{R}^n$,*

$$\frac{\|\underline{\mathbf{v}}\|_\infty^{m-1}}{\max\{\|\mathcal{H}_1\|_\infty,\|\mathcal{H}_2\|_\infty\}} \leq \|\mathbf{u}^{m-1}-\mathbf{u}_*^{m-1}\|_\infty \leq \frac{\|\underline{\mathbf{v}}\|_\infty^{m-1}\|\mathcal{H}_2\|_\infty+\|\mathcal{H}_1\|_\infty\|\underline{\mathbf{v}}\|_\infty^{m-1}}{C_2(\overline{\mathcal{H}})}.$$

Remark 4.1. *The global error bounds in Theorems 4.5, 4.6, and 4.7 can be computed by a given vector $\mathbf{u}\in\mathbb{R}^n$ and do not depend on the solution of the VTCP$(\overline{\mathcal{H}},\overline{\mathbf{b}})$.*

The following example is presented to verify the properties stated in Theorem 4.7.

Example 4.1. *Let $\mathcal{H}_1=(a_{i_1i_2i_3i_4})$, $\mathcal{H}_2=(b_{i_1i_2i_3i_4})\in\mathbb{R}^{[4,2]}$, in which*

$$a_{1111}=a_{2222}=1, b_{1111}=b_{2222}=8,$$

and $a_{i_1i_2i_3i_4}=b_{i_1i_2i_3i_4}=0$, otherwise.

Consider $\mathbf{b}_1 = \mathbf{b}_2 = (-1,-1)^\top$, $\mathbf{u} = (0,0)^\top$. *It is easy to know that* $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *satisfy the conditions of Theorem 4.7. By Theorem 4.7, we have*

$$\underline{\boldsymbol{v}} = \min\left\{\left(\mathcal{H}_1\boldsymbol{u}^{m_1-1}+\boldsymbol{b}_1\right)^{\left[\frac{1}{m_1-1}\right]}, \left(\mathcal{H}_2\boldsymbol{u}^{m_2-1}+\boldsymbol{b}_2\right)^{\left[\frac{1}{m_2-1}\right]}\right\} = (-1,-1)^\top,$$

$$\|\mathcal{H}_1\|_\infty = 1,\ \|\mathcal{H}_2\|_\infty = 8.$$

Then,

$$\frac{1}{8} \le \|\mathbf{u}^3 - \mathbf{u}_*^3\|_\infty \le \frac{9}{8}.$$

Case 2: $\overline{\mathcal{H}} \in \mathbb{R}^{[m,n]}$ **is a** $VSPL$**-tensor**. Similarly, for this case, we investigate the global error bounds of the VTCP of the form $\|\mathbf{u}^{[m-1]} - \mathbf{u}_*^{[m-1]}\|_\infty$. For this aim, a $VSPL$-tensor is introduced; see Definition 4.2.

Definition 4.2. *For* $\overline{\mathcal{H}} = \{\mathcal{H}_1, \mathcal{H}_2\}$ *with* $\mathcal{H}_1 \in \mathbb{R}^{[m_1,n]}, \mathcal{H}_2 \in \mathbb{R}^{[m_2,n]}$, $\overline{\mathcal{H}}$ *is called a VSPL-tensor if*

$$\mathcal{H}_1\mathbf{u}^{m_1-1} = J_1\mathbf{u}^{[m_1-1]}$$

and

$$\mathcal{H}_2\mathbf{u}^{m_2-1} = J_2\mathbf{u}^{[m_2-1]},$$

where $J = (J_1, J_2)$ *has the row* W*-property.*

Lemma 4.1. *If* $\overline{\mathcal{H}}$ *is a VSPL-tensor, then* $\overline{\mathcal{H}}$ *is a VP-tensor.*

Proof. If $\overline{\mathcal{H}}$ is a VSPL-tensor, from Definition 4.2, there at least exists an $i \in [n]$ such that

$$(J_1\mathbf{u}^{[m_1-1]})_i(J_2\mathbf{u}^{[m_2-1]})_i = (\mathcal{H}_1\mathbf{u}^{m_1-1})_i(\mathcal{H}_2\mathbf{u}^{m_2-1})_i > 0,$$

which implies that $\overline{\mathcal{H}}$ is a VP-tensor.

Gowda and Sznajder [22] showed that the VLCP(M,q) has a unique solution for any q if and only if M has the row W-property. Based on Definition 4.2, we can obtain the result directly.

Theorem 4.8. *Let* $\overline{\mathcal{H}} = \{\mathcal{H}_1, \mathcal{H}_2\}$ *be a* $VSPL$*-tensor. Then, the VTCP*$(\overline{\mathcal{H}}, \overline{\mathbf{b}})$ *has a unique solution for any* $\overline{\mathbf{b}}$.

Now, we give the global error bounds of the VTCP of the form $\|\mathbf{u}^{[m-1]} - \mathbf{u}_*^{[m-1]}\|_\infty$.

Theorem 4.9. *Let* $\overline{\mathcal{H}} = \{\mathcal{H}_1, \mathcal{H}_2\}$ *be a* $VSPL$*-tensor,* m_1 *and* m_2 *be even numbers, and* $\mathbf{u}_*$ *be the solution of the VTCP*$(\overline{\mathcal{H}}, \overline{\mathbf{b}})$. *Then, for any* $\mathbf{u} \in \mathbb{R}^n$,

$$\|\mathbf{u}^{[m_1-1]} - \mathbf{u}_*^{[m_1-1]}\|_\infty \ge \min_{d\in[0,1]^n} \frac{\|(I-D)\underline{\mathbf{v}}^{[m_1-1]} + D\underline{\mathbf{v}}^{[m_2-1]}\|_\infty^{m-1}}{\|(I-D)J_1\|_\infty + \|DJ_2\|_\infty},$$

or

$$\|\mathbf{u}^{[m_2-1]} - \mathbf{u}_*^{[m_2-1]}\|_\infty \ge \min_{d\in[0,1]^n} \frac{\|(I-D)\underline{\mathbf{v}}^{[m_1-1]} + D\underline{\mathbf{v}}^{[m_2-1]}\|_\infty^{m-1}}{\|(I-D)J_1\|_\infty + \|DJ_2\|_\infty},$$

where $D = \mathrm{diag}(d)$ *with* $d \in [0,1]^n$.

Proof. Let $\mathbf{v}_1 = \mathcal{H}_1\mathbf{u}^{m_1-1} + \mathbf{b}_1 - \mathcal{I}_1\underline{\mathbf{v}}^{m_1-1}$, $\mathbf{v}_2 = \mathcal{H}_2\mathbf{u}^{m_2-1} + \mathbf{b}_2 - \mathcal{I}_2\underline{\mathbf{v}}^{m_2-1}$, then

$$\mathbf{v}_1 \geq 0,\ \mathbf{v}_2 \geq 0,\ \mathbf{v}_1^\top \mathbf{v}_2 = 0.$$

Taking $\omega_1 = \mathcal{H}_1\mathbf{u}_*^{m_1-1} + \mathbf{b}_1$, $\omega_2 = \mathcal{H}_2\mathbf{u}_*^{m_2-1} + \mathbf{b}_2$, it is easy to see that

$$\omega_1 \geq 0,\ \omega_2 \geq 0,\ \omega_1^\top \omega_2 = 0.$$

Then,

$$\begin{aligned}
&\min\{\mathbf{v}_1, \mathbf{v}_2\} - \min\{\omega_1, \omega_2\} \\
&= \min\left\{\mathcal{H}_1\mathbf{u}^{m_1-1} + \mathbf{b}_1 - \mathcal{I}_1\underline{\mathbf{v}}^{m_1-1}, \mathcal{H}_2\mathbf{u}^{m_2-1} + \mathbf{b}_2 - \mathcal{I}_2\underline{\mathbf{v}}^{m_2-1}\right\} \\
&- \min\left\{\mathcal{H}_1\mathbf{u}_*^{m_1-1} + \mathbf{b}_1, \mathcal{H}_2\mathbf{u}_*^{m_2-1} + \mathbf{b}_2\right\} \\
&= (I-D)(\mathcal{H}_1\mathbf{u}^{m_1-1} + \mathbf{b}_1 - \mathcal{I}_1\underline{\mathbf{v}}^{m_1-1} - \mathcal{H}_1\mathbf{u}_*^{m_1-1} - \mathbf{b}_1) \\
&+ D(\mathcal{H}_2\mathbf{u}^{m_2-1} + \mathbf{b}_2 - \mathcal{I}_2\underline{\mathbf{v}}^{m_2-1} - \mathcal{H}_2\mathbf{u}_*^{m_2-1} - \mathbf{b}_2) = 0.
\end{aligned}$$

Further,

$$(I-D)J_1(\mathbf{u}^{[m_1-1]} - \mathbf{u}_*^{[m_1-1]}) + DJ_2(\mathbf{u}^{[m_2-1]} - \mathbf{u}_*^{[m_2-1]}) = (I-D)\underline{\mathbf{v}}^{[m_1-1]} + D\underline{\mathbf{v}}^{[m_2-1]}, \tag{5}$$

where $J = (J_1, J_2)$ has the row W-property. It follows that

$$\|(I-D)J_1\|_\infty \|\mathbf{u}^{[m_1-1]} - \mathbf{u}_*^{[m_1-1]}\|_\infty + \|DJ_2\|_\infty \|\mathbf{u}^{[m_2-1]} - \mathbf{u}_*^{[m_2-1]}\|_\infty \geq \|(I-D)\underline{\mathbf{v}}^{[m_1-1]} + D\underline{\mathbf{v}}^{[m_2-1]}\|_\infty^{m-1}.$$

Next, we only consider two cases:

When $\|\mathbf{u}^{[m_1-1]} - \mathbf{u}_*^{[m_1-1]}\|_\infty \leq \|\mathbf{u}^{[m_2-1]} - \mathbf{u}_*^{[m_2-1]}\|_\infty$, we have

$$\|(I-D)J_1\|_\infty \|\mathbf{u}^{[m_2-1]} - \mathbf{u}_*^{[m_2-1]}\|_\infty + \|DJ_2\|_\infty \|\mathbf{u}^{[m_2-1]} - \mathbf{u}_*^{[m_2-1]}\|_\infty \geq \|(I-D)\underline{\mathbf{v}}^{[m_1-1]} + D\underline{\mathbf{v}}^{[m_2-1]}\|_\infty^{m-1},$$

which implies

$$\|\mathbf{u}^{[m_2-1]} - \mathbf{u}_*^{[m_2-1]}\|_\infty \geq \frac{\|(I-D)\underline{\mathbf{v}}^{[m_1-1]} + D\underline{\mathbf{v}}^{[m_2-1]}\|_\infty^{m-1}}{\|(I-D)J_1\|_\infty + \|DJ_2\|_\infty}.$$

When $\|\mathbf{u}^{[m_2-1]} - \mathbf{u}_*^{[m_2-1]}\|_\infty \leq \|\mathbf{u}^{[m_1-1]} - \mathbf{u}_*^{[m_1-1]}\|_\infty$, we have

$$\|(I-D)J_1\|_\infty \|\mathbf{u}^{[m_1-1]} - \mathbf{u}_*^{[m_1-1]}\|_\infty + \|DJ_2\|_\infty \|\mathbf{u}^{[m_1-1]} - \mathbf{u}_*^{[m_1-1]}\|_\infty \geq \|(I-D)\underline{\mathbf{v}}^{[m_1-1]} + D\underline{\mathbf{v}}^{[m_2-1]}\|_\infty^{m-1},$$

which implies

$$\|\mathbf{u}^{[m_1-1]} - \mathbf{u}_*^{[m_1-1]}\|_\infty \geq \frac{\|(I-D)\underline{\mathbf{v}}^{[m_1-1]} + D\underline{\mathbf{v}}^{[m_2-1]}\|_\infty^{m-1}}{\|(I-D)J_1\|_\infty + \|DJ_2\|_\infty}.$$

Particularly, when $m_1 = m_2 = m$ are even numbers, for $\overline{\mathcal{H}} = \{\mathcal{H}_1, \mathcal{H}_2\}$ being a $VSPL$-tensor, we have

Theorem 4.10. *Let $\overline{\mathcal{H}} = \{\mathcal{H}_1, \mathcal{H}_2\}$ be a VSPL-tensor, $m_1 = m_2 = m$ be even numbers, and $\mathbf{u}_*$ be the solution of the VTCP($\overline{\mathcal{H}}, \overline{\mathbf{b}}$). Then, for any $\mathbf{u} \in \mathbb{R}^n$,*

$$\frac{\|\underline{\mathbf{v}}\|_\infty^{m-1}}{\max\limits_{d\in[0,1]^n} \|(I-D)J_1 + DJ_2\|_\infty} \leq \|\mathbf{u}^{[m-1]} - \mathbf{u}_*^{[m-1]}\|_\infty \leq \max_{d\in[0,1]^n} \|((I-D)J_1 + DJ_2)^{-1}\|_\infty \|\underline{\mathbf{v}}\|_\infty^{m-1}.$$

Proof. If $m_1 = m_2 = m$, by Eq (5) again, we have

$$((I - D)J_1 + DJ_2)(\mathbf{u}^{[m-1]} - \mathbf{u}_*^{[m-1]}) = \underline{\mathbf{v}}^{[m-1]}. \tag{6}$$

From Eq (6), we have

$$\|(I - D)J_1 + DJ_2\|_\infty \|\mathbf{u}^{[m-1]} - \mathbf{u}_*^{[m-1]}\|_\infty \geq \|\underline{\mathbf{v}}\|_\infty^{m-1},$$

from which one obtains

$$\|\mathbf{u}^{[m-1]} - \mathbf{u}_*^{[m-1]}\|_\infty \geq \frac{\|\underline{\mathbf{v}}\|_\infty^{m-1}}{\|(I - D)J_1 + DJ_2\|_\infty}.$$

By using Eq (6) again, from Lemma 2.6, we have

$$\mathbf{u}^{[m-1]} - \mathbf{u}_*^{[m-1]} = ((I - D)J_1 + DJ_2)^{-1}\underline{\mathbf{v}}^{[m-1]},$$

from which easily leads to

$$\|\mathbf{u}^{[m-1]} - \mathbf{u}_*^{[m-1]}\|_\infty \leq \|((I - D)J_1 + DJ_2)^{-1}\|_\infty \|\underline{\mathbf{v}}\|_\infty^{m-1}.$$

Remark 4.2. *Apparently, when $m_1 = m_2 = 2$, a strong VP-tensor reduces exactly to a strong VP-matrix, a VSPL-tensor reduces to matrix pairs with row W-property for VLCP, and the global error bounds in Theorem 4.10 reduce to the global error bounds of the VLCP given by Zhang et al; see Theorem 3.1 and Remark 4.1 in [23].*

Below we provide a concrete example to interpret Theorem 4.10, as shown in Example 4.2.

Example 4.2. *Let $\mathcal{H}_1 = (a_{i_1i_2i_3i_4})$, $\mathcal{H}_2 = (b_{i_1i_2i_3i_4}) \in \mathbb{R}^{[4,2]}$, in which*

$$a_{1111} = 2, a_{2222} = 1, a_{2211} = 2, a_{2112} = a_{2121} = -1,$$

$$b_{1111} = 2, b_{2222} = 2, b_{2211} = 2, b_{2112} = a_{2121} = -1,$$

and $a_{i_1i_2i_3i_4} = b_{i_1i_2i_3i_4} = 0$, otherwise.

Consider $\mathbf{b}_1 = \mathbf{b}_2 = (1, 1)^\top$, $\mathbf{u} = (0, 0)^\top$. It turns out that $\mathcal{H}_1$ and $\mathcal{H}_2$ fulfill the hypotheses required by Theorem 4.10. Straightforward computation yields that

$$J_1 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, J_2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix},$$

$$\underline{\boldsymbol{v}} = \min\left\{\left(\mathcal{H}_1\boldsymbol{u}^{m_1-1} + \boldsymbol{b}_1\right)^{[\frac{1}{m_1-1}]}, \left(\mathcal{H}_2\boldsymbol{u}^{m_2-1} + \boldsymbol{b}_2\right)^{[\frac{1}{m_2-1}]}\right\} = (1, 1)^\top.$$

Let

$$D = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix},$$

then

$$\max_{d\in[0,1]^n} \|(I - D)J_1 + DJ_2\|_\infty = \max_{d\in[0,1]^n} \left\|\begin{bmatrix} 2(1 - d_1) + 2d_1 & 0 \\ 0 & 1 - d_1 + 2d_1 \end{bmatrix}\right\|_\infty = 2,$$

$$\max_{d\in[0,1]^n} \|((I - D)J_1 + DJ_2)^{-1}\|_\infty = \max_{d\in[0,1]^n} \left\|\begin{bmatrix} 2(1 - d_1) + 2d_1 & 0 \\ 0 & 1 - d_1 + 2d_1 \end{bmatrix}^{-1}\right\|_\infty = 1.$$

By Theorem 4.10, we have

$$\frac{1}{2} \leq \|\mathbf{u}^{[3]} - \mathbf{u}_*^{[3]}\|_\infty \leq 1.$$

5. Conclusions

This paper constructs novel residual mappings, which investigates computable global error estimates for TCP and VTCP under structured tensor assumptions. For standard TCP, we establish two families of error bounds. For VTCP, we introduce two new structured tensor classes (strong VP-tensors and $VSPL$-tensors) and extend the error estimation framework from TCP to VTCP correspondingly. All derived bounds only require arbitrary test vector for numerical evaluation and eliminate the dependence on unknown solution elements inside the solution set. Multiple numerical examples are provided to validate the tightness and computational feasibility of our theoretical results.

Table 1. Comparison of error bounds.

Bound type	Required input	Evaluable?	Depends on u_*?
Classical bounds [8, 17]	true solution u_*	No	Yes
Proposed bounds	trial vector u	Yes	No

It is worth emphasizing that our derived global error bounds are *conditionally computable*: given that the tensor satisfies the corresponding structural assumptions, our bounds can be numerically evaluated using an arbitrary trial vector and do not involve the unknown exact solution; see Table 1.

One practical limitation of the current work lies in the two structural constants $C_1(\mathcal{H})$ and $C_2(\overline{\mathcal{H}})$ which are defined via continuous constrained minimization problems. For high-dimensional tensors, solving these optimization problems brings nontrivial computational overhead, which restricts direct application of the proposed bounds. From a computational perspective, sampling-based approximation strategies and surrogate bounding techniques for minimax-type constrained optimization problems may serve as promising candidates to reduce the heavy cost of evaluating $C_1(\mathcal{H})$ and $C_2(\overline{\mathcal{H}})$. Developing such practical numerical approximations deserves further dedicated investigation in future work.

Author contributions

These authors contributed equally to this work.

Declaration of Use of Generative-AI Tools

The authors declare that they did not utilize any artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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