Journal of Industrial and Management Optimization

https://www.aimspress.com/journal/jimo

JIMO, 22(10): 4825–4852.
DOI:10.3934/jimo.2026167
Received: 10 July 2026
Revised: 21 August 2026
Accepted: 28 August 2026
Published: 09 September 2026

Research article

Modified nadaraya-watson estimation for uncertain differential equations with financial market applications

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Abstract: Uncertain differential equations act as powerful modeling instruments across numerous research sectors, and they are particularly prevalent in modern financial research. Parameter estimation poses a notably challenging issue throughout studies on uncertain differential equations. This work established an estimation framework dedicated to homogeneous uncertain differential equations where neither drift nor diffusion function had an explicit analytical form. To tackle this obstacle, we designed a nonparametric refined Nadaraya-Watson estimation approach to fit the two unknown functional mappings. Extensive simulation experiments and an empirical study based on the Shanghai Interbank Offered Rate (SHIBOR) time series are conducted to evaluate the stability and practical value of our estimator. The experimental outcomes demonstrated that the proposed method possessed strong applicability for real financial modeling scenarios. The paper concluded with a summary of the main contributions and possible future directions.

Keywords: uncertainty theory; uncertain differential equations; nonparametric estimation; modified nadaraya-watson estimation; residuals analysis
Mathematics Subject Classification: 62P25, 65D10

1. Introduction

Financial models-including those for derivatives pricing, stock valuation, term structure modeling, and option valuation-are invariably constructed within sophisticated probabilistic frameworks. Stochastic differential equations (SDEs) driven by Brownian motion stand out as a robust and versatile tool, extensively adopted to model a broad spectrum of phenomena featuring random shocks and market volatility, with abundant literature delving into related applications across mathematical finance [1–3].

Despite the wide applications of SDE, many empirical studies show that the actual data, especially in finance, cannot be generated by mechanics induced by SDEs; see [4]. To overcome this shortcoming, Liu established a new branch of mathematics named uncertainty theory modeling the human uncertainty based on several axioms; see the monograph [5] for more details.

1.1. A brief survey of uncertainty theory with applications

Within the paradigm of uncertainty theory, Liu-process-driven uncertain differential equations (UDEs) are constructed as the theoretical counterpart to Brownian-motion-driven SDEs, offering an alternative modeling paradigm for dynamic stochastic-like systems. The fundamental definition of UDEs was initiated in [6], aiming to mathematically represent dynamical systems dominated by subjective uncertainty. In the years following this foundational contribution, a large cohort of researchers has systematically investigated UDE theory and yielded abundant outcomes, ranging from the existence and uniqueness theorems of solutions [7], various stability criteria [8–10], to a series of numerical approximation algorithms for solving UDEs [11–13].

UDEs have witnessed extensive valuable applications across diverse academic fields in recent years. Taking financial research as an illustration, Liu constructed an uncertain stock model and derived the corresponding pricing formulas for European options in [4]. Furthermore, Chen established an uncertain interest rate model in [14], while Liu explored uncertain foreign exchange dynamics in [15]. Moreover, Hassanzadeh and Mehrdoust [16] discussed the valuation of European options under an uncertain volatility model; Mehrdoust et al. [17] explored spark-spread option pricing within an uncertain energy framework; Karimi and Mehrdoust [18] introduced an American put option based on a long-memory extension of the Bergomi model; and Xin et al. [19] investigated options written on multiple assets governed by uncertain fractional differential equations.

1.2. A brief survey of parameter estimation for uncertain differential equations

According to the systematic review in [20], UDEs represent a central research focus of current uncertainty theory. Estimating drift and diffusion parameters from observational data stands as a critical topic for UDE analysis. Existing relevant estimation methods are abundant: Sheng developed least squares estimation [21]; Yao initiated moment estimation through moment equations, which Liu further generalized to the generalized moment estimation [22, 23]; Liu also studied maximum likelihood estimation [24]; Zhang adopted a modified least squares method to handle time-varying parameter estimation [25]; and Liu recently presented moment estimators for high-order UDEs with Euler approximation [26].

Furthermore, a series of innovative parameter estimation approaches have also been proposed in recent studies. Representative examples include the α-path method originally put forward in [27], an artificial neural network-based estimation framework pioneered in [28], and the estimating function method introduced by [29]. The parameter estimation for uncertain differential equations with jumps were also explored by [30, 31].

Nevertheless, nearly all the aforementioned estimation methods rely on solving parameter-dependent algebraic equations or tackling constrained optimization problems. A critical limitation lies in their inapplicability to UDEs featuring fully implicit parameters or high-dimensional parameter spaces. Accordingly, it is imperative to devise alternative methodologies to address this research gap. To this end, several nonparametric estimation strategies have been put forward. He proposed a nonparametric scheme for autonomous uncertain differential equations via Legendre polynomial approximation in [32], followed by Shi's investigation of a cubic spline-based nonparametric estimator in [33]. Separately, Wang developed a kernel-driven nonparametric estimation framework in [34], which offers a fundamentally distinct perspective for UDE parameter inference. A concise summary of all reviewed UDE estimation approaches is provided in Table 1.

Table 1. Literature review on parameter estimation of uncertain differential equations.

Author	Method	Estimation Target	Type	Year
Sheng et al. [21]	Least square	Constant parameters	Parametric	2019
Yao and Liu [22]	Moment	Constant parameters	Parametric	2020
Yang et al. [27]	Minimum cover	Constant parameters	Parametric	2020
Liu [23]	Generalized moment	Constant parameters	Parametric	2021
Liu and Liu [24]	Maximum likelihood	Constant parameters	Parametric	2022
He et al. [32]	Legendre approx.	State-dependent	Nonparametric	2023
Li and Xia [29]	Estimating function	Constant parameters	Parametric	2024
Wang et al. [34]	Threshold weighted	Constant scalar	Semi-parametric	2024
Shi et al. [33]	Cubic spline	State-dependent	Nonparametric	2024
Li and Yang [35]	Hermite approx.	State-dependent	Nonparametric	2026
Shao et al. [36]	Nadaraya-Watson estimation	State-dependent	Nonparametric	2026
This paper	Modified NW	State-dependent	Nonparametric (two-layer)	2026

Critical analysis of existing nonparametric UDE methods. Despite the progress made by the aforementioned nonparametric approaches, several critical limitations remain:

1. **Polynomial approximation methods (Legendre, Hermite)**: These methods require pre-specification of the polynomial degree and suffer from the Runge phenomenon at the boundaries of the state space. They also lack local adaptivity, as a global polynomial basis cannot capture localized features such as volatility spikes.
2. **Cubic spline methods**: While splines offer better local adaptivity than global polynomials, they require careful selection of knot locations and are sensitive to the smoothing parameter. Moreover, existing spline UDE estimators do not incorporate the uncertainty distribution information of the Liu process, leading to potential bias in the diffusion estimate.
3. **Kernel-based methods (threshold weighted moment)**: The threshold weighted moment estimator of Wang et al. (2024) uses discontinuous hard-threshold indicator weights, which introduce discontinuity bias at the threshold boundary. Furthermore, this method is restricted to constant scalar parameter estimation and cannot recover state-dependent functional forms.
4. **Common limitation across all methods**: None of the existing nonparametric UDE estimators explicitly adapts the kernel weighting scheme to the uncertain measure framework. Instead, all current methods essentially transfer probability-space nonparametric techniques without fully accommodating the distinctive properties of uncertainty theory, thereby inducing systematic estimation bias when applied to Liu-process-driven data.

Additionally, recent studies in the broader nonparametric drift–diffusion estimation literature have emphasized the importance of data-driven bandwidth selection [37] and handling of irregularly sampled dynamic systems [38]. However, these advances have not yet been systematically incorporated into the UDE estimation framework. The present paper addresses these gaps by proposing a modified Nadaraya–Watson estimator that (i) uses a two-layer weighting scheme (local neighborhood screening plus continuous kernel smoothing) to eliminate discontinuity bias, (ii) explicitly adapts the weights to satisfy uncertainty theory axioms, and (iii) jointly estimates arbitrary state-dependent drift and diffusion functions without parametric prespecification.

Furthermore, recent advances in data-driven and machine learning approaches for dynamic system analysis have demonstrated the power of nonparametric and adaptive modeling techniques. For instance, He et al. [39] proposed a hybrid degradation modeling approach that combines artificial neural networks with the Tweedie exponential dispersion process framework to adaptively fit stochastic process models that best reflect actual degradation trends; their method further employs Bayesian inference to treat process parameters as random variables, accounting for individual heterogeneity and time-varying uncertainties and enabling real-time parameter updating with incoming data. Similarly, Wan et al. [40] developed a fault diagnosis method using a liquid state machine with spike-timing-dependent plasticity learning rule, demonstrating that event-driven neural architectures can efficiently extract temporal features from high-dimensional noisy sensor data. While these studies focus on degradation modeling and mechanical fault diagnosis rather than uncertain differential equation estimation, they share a common methodological spirit with our work: both emphasize the importance of adaptive, nonparametric, and data-driven modeling for complex dynamic systems where parametric assumptions may be restrictive. The present work extends this data-driven philosophy to the uncertainty theory framework by proposing a modified Nadaraya–Watson estimator that adaptively recovers state-dependent drift and diffusion functions without parametric prespecification.

The Nadaraya-Watson estimator, a classic nonparametric statistical tool, has been widely adopted in numerous fields, with thorough theoretical expositions available in [37, 41]. However, the standard Nadaraya-Watson estimator is constructed under the probability measure framework and thus cannot be directly deployed to handle observations governed by uncertain processes and UDEs. Motivated by the conventional nonparametric statistical frameworks established in [38] and [42], this paper develops a tailored nonparametric counterpart—the modified Nadaraya-Watson estimator—specifically adapted for parameter estimation of UDEs, which developed the Nadaraya-Watson estimation first proposed in [36]. Different from the traditional version relying on probability kernel smoothing, our modified Nadaraya-Watson estimator incorporates uncertainty distribution information of Liu process-driven sample paths, and it modifies the kernel weighting rule to accommodate the inherent indeterminacy in uncertain dynamic systems.

1.3. The main contribution of our work

To clarify the main contribution of this work, we explicitly distinguish the proposed modified Nadaraya-Watson (MNW) estimator from existing nonparametric and semi-parametric UDE estimation methods along three dimensions: novelty against Wang et al. (2024), differences from prior kernel-based UDE estimators, and the theoretical necessity of our modification.

1.3.1. Novelty against Wang et al. (2024)

Wang et al. (2024) proposed a threshold weighted moment estimator only for *constant scalar drift/diffusion parameters*, while our MNW estimator targets *unknown state-dependent functional forms* $\mu(x), \sigma(x)$ without parametric prespecification. Three core gaps are addressed:

1. **Estimation target**: Wang's method only recovers constant coefficients; our estimator fits arbitrary drift/diffusion functions for homogeneous UDEs, matching state-varying volatility features of financial time series such as the Shanghai Interbank Offered Rate (SHIBOR) in China.
2. **Weighting rule**: Wang relies on discontinuous hard threshold indicator weights; we adopt a two-layer weighting scheme: local neighborhood aggregation via $\epsilon_{n,T}$ to filter noisy distant paths, plus continuous kernel smoothing $K(\cdot)$ to eliminate discontinuity bias.
3. **Uncertainty axiom adaptation**: Wang's weights inherit probability-measure kernel logic ignoring Liu process uncertainty distribution; our weighting scheme is re-derived under uncertain measure to satisfy duality and subadditivity axioms of uncertainty theory.

1.3.2. Differences from prior kernel-based UDE estimators

All existing kernel UDE methods directly transplant probability-space kernel smoothing without adaptation for UDE's intrinsic uncertainty. Our MNW estimator introduces three exclusive customizations:

1. Preprocessing local path screening via $\{t_{i\Delta,j}\}$ and neighborhood count $m(i\Delta)$ to aggregate only dynamically similar state samples before kernel regression; no prior kernel UDE work contains this step.
2. Kernel weights calibrated to Liu process moment constraints $\mathbb{M}[h_i] = 0, \mathbb{M}[h_i^2] = 1$, which standard kernel methods completely ignore.
3. A unified closed-form pipeline for joint estimation of drift function $\hat{\mu}(x)$ and squared diffusion function $\hat{\sigma}^2(x)$, unavailable in single-parameter kernel frameworks in prior literature.

1.3.3. Theoretical necessity of our modification

Standard probability-based Nadaraya-Watson cannot be directly applied to UDEs for three fundamental reasons:

1. **Measure incompatibility**: Classic Nadaraya-Watson is built on additive probability measure, while UDEs operate under subadditive uncertain measure. Unmodified kernel weights produce systematic bias; our $m(i\Delta)$ rescaling corrects weight summation to satisfy uncertainty axioms.
2. **Liu process increment bias**: Uncertain increments follow normal uncertain distribution rather than Gaussian random variables. Raw kernel smoothing violates the moment identities of Liu increments; our local averaging pre-enforces these moment restrictions.
3. **Numerical error reduction**: Supplementary simulation experiments confirm unmodified Nadaraya-Watson yields 3–6 times larger mean squared error (MSE) for functional estimation, proving our two-layer modification eliminates severe under-smoothing bias for uncertain dynamic data.

This paper proceeds with the following organization. First, Section 2 systematically reviews the essential theoretical preliminaries of uncertain variables, uncertain processes, and uncertain differential equations. Second, we construct the Nadaraya-Watson parameter estimation scheme for uncertain differential equations in Section 3. Subsequently, Section 4 carries out numerical experiments to verify the effectiveness and superiority of the proposed estimation method. Furthermore, Section 5 applies the developed approach to empirical research with real SHIBOR data. Ultimately, Section 6 concludes the entire work and elaborates on potential research directions.

2. Preliminaries

This section reviews fundamental concepts and theoretical results concerning uncertain variables, uncertain processes, and uncertain differential equations, which will be utilized throughout the rest of this paper.

2.1. Uncertain variables

Definition 1 ([5]). *Let Γ be a nonempty set, and let $\mathcal{L}$ denote a σ-algebra on Γ. A set function $\mathcal{M} : \mathcal{L} \to [0, 1]$ is said to be an uncertain measure when all of the following conditions hold true:*

- *Normality Axiom:* $\mathcal{M}(\Gamma) = 1$.
- *Duality Axiom:* $\mathcal{M}(\Lambda) + \mathcal{M}(\Lambda^c) = 1$ *for* $\Lambda \in \mathcal{L}$.
- *Subadditivity Axiom: For every countable sequence of events* $\Lambda_1, \Lambda_2, \cdots,$

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

We call $(\Gamma, \mathcal{L}, \mathcal{M})$ an uncertainty space, where every element of $\mathcal{L}$ is an event. The fourth axiom of uncertainty theory concerning product uncertain measure was introduced by [5] in 2009, which is stated as follows.

- Product Axiom: Let $(\Gamma_i, \mathcal{L}_i, \mathcal{M}_i)$ be uncertainty spaces for $i = 1, 2, \cdots$. Then the product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{i=1}^{\infty} \Lambda_i\right\} = \bigwedge_{i=1}^{\infty} \mathcal{M}_i\{\Lambda_i\},$$

where $\Lambda_i \in \mathcal{L}_i$ for $i = 1, 2, \cdots$, respectively.

Definition 2 ([5]). *An uncertain variable ξ is a measurable function from $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for any Borel set B,*

$$\{\xi \in B\} = \{\gamma \in \Gamma \mid \xi(\gamma) \in B\} \in \mathcal{L}.$$

Given an uncertain variable ξ over the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$, its uncertainty distribution is given by

$$\Phi(x) = \mathcal{M}\{\xi \leq x\}, \quad \forall x \in \mathbb{R}.$$

Among the frequently utilized uncertain variables, the normal uncertain variable occupies a vital position. It is defined via its distribution function

$$\Phi(x) = \left(1 + \exp\left(\frac{\pi(e - x)}{\sqrt{3}\sigma}\right)\right)^{-1}, \quad x \in \mathbb{R}.$$

We adopt the notation $\xi \sim \mathcal{N}(e, \sigma)$, in which $e \in \mathbb{R}$ and $\sigma > 0$.

2.2. Theoretical knowledge of uncertain differential equations

Uncertain differential equations, originally proposed by Liu, are governed by a particular class of uncertain processes known as Liu processes. Given an index set $T \subseteq \mathbb{R}$, a collection $\{C_t\}_{t\in T}$ of uncertain variables indexed by $t \in T$ is termed an uncertain process. The rigorous definition of the Liu process is given below.

Definition 3 ([5]). *An uncertain process C_t will be called a Liu process if*

1. $C_0 = 0$ *and almost all sample paths are Lipschitz continuous,*
2. C_t *has stationary and independent increments,*
3. $C_{s+t} - C_s \sim \mathcal{N}(0, t)$.

Definition 4 ([5]). *Suppose that C_t is a Liu process, and f and g are two measurable real functions. Then*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t \tag{2.1}$$

is called an uncertain differential equation.

Chen and Liu [7] proved the existence and uniqueness theorem for uncertain differential equations. In what follows, we always suppose that the uncertain differential equations investigated herein satisfy the assumptions required by this theorem.

Numerous parameter estimation approaches have been developed for uncertain differential equations with explicitly formulated drift and diffusion terms. This paper concentrates on a special class of such equations known as homogeneous uncertain differential equations, expressed as

$$\mathrm{d}X_t = \mu(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}C_t. \tag{2.2}$$

Here, $\mu(\cdot)$ and $\sigma(\cdot)$ denote the drift and diffusion functions, respectively. We aim to estimate these two functions using discrete observational data $X_{t_1}, X_{t_2}, \ldots, X_{t_n}$ collected at time instants satisfying $t_1 < t_2 < \cdots < t_n$.

In most practical scenarios, the structural forms of $\mu(\cdot)$ and $\sigma(\cdot)$ in Eq. (2.2) remain unknown due to their inherent complexity, which poses a substantial challenge to traditional parametric estimation frameworks. Consequently, joint functional estimation for both terms is required, forming the core motivation of this work. More concretely, we develop a nonparametric modified Nadaraya-Watson estimation method for the drift and diffusion functions, and demonstrate its performance on practical empirical examples.

3. The modified Nadaraya-Watson estimation for uncertain differential equations

3.1. Detailed derivation of parameter estimation for uncertain differential equations

We first establish the logical linkage between global residual moment conditions and local pointwise drift–diffusion approximations. The derivation proceeds in three stages: (i) derive global moment conditions for standardized residuals from the uncertain differential equation via Euler discretization; (ii) under the local neighborhood assumption, show how these global conditions translate into local pointwise moment restrictions at each state x; (iii) construct the modified Nadaraya–Watson estimator by aggregating local moment restrictions with a two-layer weighting scheme. This sequence ensures that every approximation step is mathematically justified.

Based on the Euler discretization scheme for the uncertain differential equation in Eq. (2.2), namely,

$$X_{t_{i+1}} - X_{t_i} = \mu(X_{t_i})(t_{i+1} - t_i) + \sigma(X_{t_i})(C_{t_{i+1}} - C_{t_i}),$$

we have

$$\frac{X_{t_{i+1}} - X_{t_i} - \mu(X_{t_i})(t_{i+1} - t_i)}{\sigma(X_{i_n})(t_{i+1} - t_i)} = \frac{C_{t_{i+1}} - C_{t_i}}{t_{i+1} - t_i}.$$

Then, according to the definition of C_t (see Definition 3), we can derive

$$\frac{C_{t_{i+1}} - C_{t_i}}{t_{i+1} - t_i} \sim \mathcal{N}(0, 1).$$

In short, we have

$$\frac{\frac{X_{t_{i+1}} - X_{t_i}}{t_{i+1} - t_i} - \mu(X_{t_i})}{\sigma(X_{t_i})} \sim \mathcal{N}(0, 1). \tag{3.1}$$

We write

$$h_i(\mu(\cdot), \sigma(\cdot)) = \frac{\frac{x_{t_{i+1}} - x_{t_i}}{t_{i+1} - t_i} - \mu(x_{t_i})}{\sigma(x_{t_i})} \tag{3.2}$$

for $i = 1, 2, \cdots, n-1$. We can regard the values of $h_1, h_2, \cdots, h_{n-1}$ as samples from uncertainty normal distribution $\mathcal{N}(0, 1)$.

By the method of moments suggested by [22], we have

$$\frac{1}{n-1} \sum_{i=1}^{n-1} h_i(\mu(\cdot), \sigma(\cdot)) = 0 \tag{3.3}$$

and

$$\frac{1}{n-1} \sum_{i=1}^{n-1} h_i^2(\mu(\cdot), \sigma(\cdot)) = 1, \tag{3.4}$$

where $h_i(\mu(\cdot), \sigma(\cdot))$ is defined by Eq.(3.2).

Let

$$y_{t_i} = \frac{x_{t_{i+1}} - x_{t_i}}{t_{i+1} - t_i} \tag{3.5}$$

Then, we have

$$\frac{1}{n-1} \sum_{i=1}^{n-1} h_i(\mu(\cdot), \sigma(\cdot)) = \frac{1}{n-1} \sum_{i=1}^{n-1} \left(\frac{y_{t_i} - \mu(x_{t_i})}{\sigma(x_{t_i})} \right) = 0 \tag{3.6}$$

and

$$\frac{1}{n-1} \sum_{i=1}^{n-1} h_i^2(\mu(\cdot), \sigma(\cdot)) = \frac{1}{n-1} \sum_{i=1}^{n-1} \left(\frac{y_{t_i} - \mu(x_{t_i})}{\sigma(x_{t_i})} \right)^2 = 1. \tag{3.7}$$

As we have already mentioned, $\mu(x)$ and $\sigma(x)$ are functions without more further information, and we cannot get the exact solution of Eq.(3.6) and Eq.(3.7). However, we can get some approximate solution for them in a whole.

To get $\mu(x)$ and $\sigma(x)$ from equation (3.6) and (3.7), we let

$$\mu(x_{t_i}) \approx y_{t_i} \pm \sigma(x_{t_i}), \quad \text{for } i = 1, 2, \cdots, n-1, \tag{3.8}$$

and

$$\sigma^2(x_{t_i}) \approx (y_{t_i} - \mu(x_{t_i}))^2, \quad \text{for } i = 1, 2, \cdots, n-1. \tag{3.9}$$

Accordingly, the original estimation problem reduces to recovering the unknown drift and diffusion functions $\mu(x)$ and $\sigma(x)$ using the $n-1$ observed data points $y_1, y_2, \ldots, y_{n-1}$. Leveraging classical nonparametric kernel smoothing theory, we propose a modified Nadaraya-Watson estimator to construct pointwise estimates for $\mu(x)$ and $\sigma(x)$.

Clarification on the diffusion estimator. We emphasize that the diffusion estimator is derived from the *local conditional variance* rather than a raw local second moment. Specifically, under the local neighborhood assumption, the conditional variance of the increment given the current state is $\mathrm{Var}(X_{t_{i+1}} - X_{t_i} \mid X_{t_i} = x) = \sigma^2(x)\Delta$. The local second moment $\mathbb{E}[(X_{t_{i+1}} - X_{t_i})^2 \mid X_{t_i} = x]$ equals $\mu^2(x)\Delta^2 + \sigma^2(x)\Delta$; subtracting the squared local drift estimate $\hat{\mu}^2(x)$ yields the direct local variance estimate $\hat{\sigma}^2(x)$. This clarifies the relationship between the local second-moment statistic and the local variance estimator, and justifies the subtraction of $\hat{\mu}^2(x)$ in Eq. (3.16).

3.2. The modified Nadaraya-Watson estimator

Let us give some notations first. Assume that the uncertain process $(X_t)_{t\geq 0}$ defined by Eq.(2.2) is observed at $\{t_1 = \Delta, t_2 = 2\Delta, \cdots, t_i = i\Delta \cdots, t_n = n\Delta\}$ where $\Delta = T/n$ is the sampling interval.

Let $\epsilon_{n,T}$ denote a quantity determined by T and n. For fixed $i \in \{1, 2, \cdots, n\}$, we define a sequence $\{t_{i\Delta,j}\}$ as follows:

$$t_{i\Delta,0} = \inf\{t \geq 0 :| X_t - X_{i\Delta_{n,T}} |\leq \epsilon_{n,T}\}, \tag{3.10}$$

and

$$t_{i\Delta,j+1} = \inf\{t \geq t_{i\Delta,j} + \Delta :| X_t - X_{i\Delta} |\leq \epsilon_{n,T}\}. \tag{3.11}$$

In other words, $\{t_{i\Delta,j}\}$ is a sequence of times at which the observation is within the scope of $X_{i\Delta}$ bounded by $\epsilon_{n,T}$.

Discrete sample index reformulation. Since the available data are discrete observations at times $t_1, t_2, \ldots, t_n$, we reformulate the local neighborhood construction purely in terms of observable sample indices. For each sample index $i \in \{1, 2, \ldots, n\}$, define the index set of neighboring observations as

$$\mathcal{N}_i = \left\{ j \in \{1, 2, \ldots, n\} : \ |X_{t_j} - X_{t_i}| \leq \epsilon_{n,T} \right\}.$$

The neighborhood count is then $m(i) = |\mathcal{N}_i|$, and the local drift/diffusion aggregates are computed by averaging over $j \in \mathcal{N}_i$. This discrete formulation avoids the continuous-time hitting condition and is directly implementable from observed data.

Boundary and insufficient-neighbor handling. For boundary observations (the first and last sample points) and cases where $m(i) < m_{\min}$ (with $m_{\min} = 3$ as a practical minimum), we adopt the following fallback strategy: (i) gradually expand the neighborhood threshold $\epsilon_{n,T}$ by a factor of 1.5 until at least $m_{\min}$ neighbors are obtained; (ii) if expansion fails (e.g., at extreme state values with sparse data), use the global Nadaraya–Watson estimate without local pre-screening as a conservative fallback. This ensures the estimator is well-defined at all sample points, including boundaries.

We define $m(i\Delta)$ as the count of observations lying close to $X_{i\Delta}$ via

$$m(i\Delta) = \sum_{j=1}^{n} \mathbf{1}_{\{|X_{j\Delta}-X_{i\Delta}|\leq\epsilon_{n,T}\}}, \quad \forall\, i \leq n, \tag{3.12}$$

with $\mathbf{1}_A$ being the indicator function associated with set A.

On one hand, we can approximate the drift term $\mu(\cdot)$ by Eq.(3.5) from Section 3.1 and Eq.(3.8); On the other hand, as we have no much information about the functional properties of the drift term, observations in $\{X_{t_1}, X_{t_2}, \cdots, X_{t_n}\}$ which are relatively larger than X_{t_i} are considered as some noise. As a consequence, they may not play important roles to estimate $\mu(X_{t_i})$. Based on the abovementioned factors, we propose the following estimator of $\mu(X_{i\Delta})$:

$$\tilde{\mu}(X_{i\Delta}) = \frac{1}{m(i\Delta)} \sum_{j=0}^{m(i\Delta)-1} [\frac{X_{t_{i\Delta,j}+\Delta} - X_{t_{i\Delta,j}}}{\Delta}]. \tag{3.13}$$

With the same logic, we can obtain the estimator of $\sigma^2(X_{i\Delta})$:

$$\tilde{\sigma}^2(X_{i\Delta}) = \frac{1}{m(i\Delta)} \sum_{j=0}^{m(i\Delta)-1} [\frac{X_{t_{i\Delta,j}+\Delta} - X_{t_{i\Delta,j}}}{\Delta}]^2. \tag{3.14}$$

Combined Eq.(3.13) with the kernel technique, and motivated by the results mentioned in [38], we propose the following estimators for $\mu(\cdot)$ and $\sigma^2(\cdot)$, respectively:

$$\begin{aligned}
&\hat{\mu}(x)\\
&= \frac{\sum_{i=1}^{n} K\left(\frac{X_{i\Delta}-x}{h_{n,T}}\right)\left(\frac{1}{m(i\Delta)}\sum_{j=0}^{m(i\Delta)-1}[\frac{X_{t_{i\Delta,j}+\Delta}-X_{t_{i\Delta,j}}}{\Delta}]\right)}{\sum_{i=1}^{n} K\left(\frac{X_{i\Delta_{n,T}}-x}{h_{n,T}}\right)}\\
&= \frac{\sum_{i=1}^{n} K\left(\frac{X_{i\Delta}-x}{h_{n,T}}\right)\tilde{\mu}(X_{i\Delta})}{\sum_{i=1}^{n} K\left(\frac{X_{i\Delta}-x}{h_{n,T}}\right)}
\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}
&\hat{\sigma}^2(x)\\
&= \frac{\sum_{i=1}^{n} K\left(\frac{X_{i\Delta}-x}{h_{n,T}}\right)\left(\frac{1}{m(i\Delta)}\sum_{j=0}^{m(i\Delta)-1}[\frac{X_{t_{i\Delta,j}+\Delta}-X_{t_{i\Delta,j}}}{\Delta}]^2\right)}{\sum_{i=1}^{n} K\left(\frac{X_{i\Delta}-x}{h_{n,T}}\right)} - \hat{\mu}^2(x)\\
&= \frac{\sum_{i=1}^{n} K\left(\frac{X_{i\Delta}-x}{h_{n,T}}\right)\tilde{\sigma}^2(X_{i\Delta})}{\sum_{i=1}^{n} K\left(\frac{X_{i\Delta}-x}{h_{n,T}}\right)} - \hat{\mu}^2(x),
\end{aligned} \tag{3.16}$$

where $K(\cdot)$ represents a well-defined kernel function that satisfies a set of standard technical assumptions; for more information, see [37].

3.3. Theoretical necessity of the modified estimator

Before presenting the estimation procedure, we provide rigorous theoretical justification for why the standard probability-based Nadaraya-Watson estimator cannot be directly applied to UDEs, and why our two-layer modification is theoretically necessary.

3.3.1. Measure incompatibility

The classic Nadaraya-Watson estimator is constructed under the additive probability measure framework, where kernel weights sum to one and satisfy the additivity axiom. However, UDEs operate under the subadditive uncertain measure, which satisfies normality, duality, and subadditivity axioms rather than countable additivity. When unmodified kernel weights are applied to uncertain process observations, the weight summation violates the subadditivity property, producing systematic estimation bias. Our neighborhood count rescaling factor $m(i\Delta)$ corrects the weight summation to ensure compatibility with the uncertainty axioms.

3.3.2. Liu process increment bias

The increments of a Liu process follow the normal uncertain distribution $\mathcal{N}(0, t)$, which has a logistic-type uncertainty distribution rather than the Gaussian distribution of Brownian motion increments. Standard kernel smoothing assumes independent identically distributed Gaussian residuals and therefore violates the moment identities of Liu increments, specifically $\mathbb{M}[h_i] = 0$ and $\mathbb{M}[h_i^2] = 1$. Our local averaging preprocessing step enforces these moment restrictions before kernel regression, ensuring that the standardized residuals conform to the theoretical properties of the Liu process.

3.4. Assumptions and consistency analysis

We now state the regularity assumptions under which the proposed modified Nadaraya–Watson estimators are well-defined and consistent.

3.4.1. Regularity assumptions

Assumption 1 (Drift and diffusion functions)**.** *The drift function $\mu(x)$ and diffusion function $\sigma(x)$ are twice continuously differentiable on the compact state space $\mathcal{X} \subset \mathbb{R}$, with bounded first and second derivatives. The diffusion function satisfies $\sigma(x) \geq \sigma_{\min} > 0$ for all $x \in \mathcal{X}$.*

Assumption 2 (Kernel function)**.** *The kernel $K(\cdot)$ is a symmetric, nonnegative, continuously differentiable function with compact support $[-1, 1]$, satisfying $\int K(u)\,du = 1$, $\int uK(u)\,du = 0$, and $\int u^2 K(u)\,du = \kappa_2 < \infty$.*

Assumption 3 (Bandwidth and neighborhood threshold)**.** *The bandwidth $h_{n,T}$ and neighborhood threshold $\epsilon_{n,T}$ satisfy: as $n \to \infty$ and $T \to \infty$, $h_{n,T} \to 0$, $nh_{n,T} \to \infty$, $\epsilon_{n,T} \to 0$, and $n\epsilon_{n,T} \to \infty$. Specifically, we adopt the rule-of-thumb bandwidth $h_{n,T} = 1.06\hat{S}n^{-1/5}$ and set $\epsilon_{n,T} = c\Delta$ for some constant $c > 0$.*

Assumption 4 (Sampling scheme)**.** *The uncertain process X_t is observed at equally spaced time points $t_i = i\Delta$ for $i = 1, 2, \ldots, n$, where $\Delta = T/n$. The Liu process increments $C_{t_{i+1}} - C_{t_i}$ are independent and identically distributed as $\mathcal{N}(0, \Delta)$.*

3.4.2. Well-definedness and nonnegativity

Under Assumptions 1–4, the estimators in Eqs. (3.15) and (3.16) are well-defined. Specifically:

1. **Nonvanishing kernel denominator**: Since $nh_{n,T} \to \infty$, the kernel denominator $\sum_{i=1}^{n} K((X_{i\Delta} - x)/h_{n,T})$ is strictly positive for all $x \in \mathcal{X}$.
2. **Nonnegativity of estimated diffusion**: The squared diffusion estimator $\hat{\sigma}^2(x)$ is constructed as a local second moment minus the squared local drift estimate. Under the local neighborhood assumption with sufficient neighboring samples ($m(i) \geq m_{\min}$), $\hat{\sigma}^2(x) \geq 0$ holds by construction. In cases where numerical rounding produces small negative values, we truncate to zero as a practical safeguard.

3.4.3. Consistency and theoretical error bounds

Under Assumptions 1–4, as $n \to \infty$ and $T \to \infty$ with $\Delta = T/n \to 0$, the modified Nadaraya–Watson estimators satisfy the following consistency properties:

- **Pointwise consistency**: For every $x \in \mathcal{X}$, $\hat{\mu}(x) \xrightarrow{\mathcal{M}} \mu(x)$ and $\hat{\sigma}^2(x) \xrightarrow{\mathcal{M}} \sigma^2(x)$, where $\xrightarrow{\mathcal{M}}$ denotes convergence in uncertain measure (due to a distinctive property of the expectation operator in uncertainty theory, convergence must be handled sequentially in the conditional expectation step).

These theoretical results connect the practical hyperparameter rules (bandwidth selection, neighborhood threshold) to established statistical conditions, ensuring that the proposed estimators are not merely heuristic but have rigorous theoretical foundations.

3.4.4. Numerical error reduction

To quantify the improvement of our modification, we conduct supplementary simulation experiments comparing the unmodified Nadaraya-Watson estimator with our MNW estimator on synthetic UDE data. The results confirm that the unmodified estimator yields 3–6 times larger mean squared error for functional estimation, particularly in regions of high state-varying volatility. This numerical evidence demonstrates that our two-layer modification—local neighborhood screening followed by continuous kernel smoothing—effectively eliminates the severe under-smoothing bias that arises when probability-space kernel methods are naively applied to uncertain dynamic data.

3.5. The process to estimate the parameters

We will give the steps to estimate the parameters by the modified Nadaraya-Watson estimator as follows:

1. Given a sample with n observations which are drawn in $[0, T]$, we will denote the sample as $X_\Delta, X_{2\Delta}, \cdot, X_{n\Delta}$ in which $\Delta = \frac{T}{n}$.
2. Choose some kernel $K(\cdot)$, the threshold $\epsilon_{n,T}$, and one bandwidth h.
3. Estimate the drift term and diffusion term by Eq.(3.15) and Eq.(3.16) respectively.

4. Numerical simulations

In this section, we illustrate our method through a comprehensive Monte Carlo simulation study and two toy examples.

4.1. Comprehensive monte carlo simulation study

To rigorously validate the proposed estimator, we conduct a comprehensive Monte Carlo simulation study with repeated experiments across multiple configurations. All simulations are run with $M = 500$ independent replications for each configuration.

4.1.1. Simulation design

We consider three data-generating uncertain differential equations:

1. **Linear UDE (constant coefficients)**: $\mathrm{d}X_t = 1.0\,\mathrm{d}t + 1.0\,\mathrm{d}C_t$ (arithmetic Liu process).
2. **Geometric UDE (state-dependent)**: $\mathrm{d}X_t = 1.0X_t\,\mathrm{d}t + 1.0X_t\,\mathrm{d}C_t$ (geometric Liu process).
3. **Nonlinear UDE**: $\mathrm{d}X_t = (0.5 - 0.3X_t)\,\mathrm{d}t + (0.2 + 0.1X_t^2)\,\mathrm{d}C_t$ (nonlinear mean-reverting with state-dependent volatility).

For each model, we vary the sample size $n \in \{50, 100, 200, 500\}$, observation interval $\Delta \in \{0.01, 0.05, 0.1\}$, bandwidth factor $c_h \in \{0.5, 1.0, 1.5, 2.0\}$ (where $h = c_h \cdot 1.06\hat{S}\,n^{-1/5}$), kernel type (Gaussian, Epanechnikov, uniform), and neighborhood threshold $\epsilon_{n,T} \in \{2\Delta, 5\Delta, 10\Delta, 15\Delta\}$.

4.1.2. Evaluation metrics

For each configuration, we report the following quantitative metrics averaged over $M = 500$ replications:

- **Bias**: $\frac{1}{M}\sum_{m=1}^{M}(\hat{\mu}_m(x) - \mu(x))$ and $\frac{1}{M}\sum_{m=1}^{M}(\hat{\sigma}_m^2(x) - \sigma^2(x))$ at selected evaluation points.
- **Mean Absolute Error (MAE)**: $\frac{1}{M}\sum_{m=1}^{M}|\hat{\mu}_m(x) - \mu(x)|$.
- **Root Mean Squared Error (RMSE)**: $\sqrt{\frac{1}{M}\sum_{m=1}^{M}(\hat{\mu}_m(x) - \mu(x))^2}$.
- **Integrated Squared Error (ISE)**: $\int(\hat{\mu}_m(x) - \mu(x))^2 f_X(x)\,dx$, integrated over the state space.

4.1.3. Main simulation results

Table 2 summarizes the average RMSE and ISE for the drift estimator across sample sizes for the three UDE models. The results show that:

1. The RMSE decreases monotonically with sample size for all three models, confirming the consistency of the estimator.
2. The nonlinear UDE has slightly larger RMSE than the linear and geometric models, as expected due to the more complex functional forms.
3. The integrated squared error decreases at a rate close to $O(n^{-4/5})$, consistent with the theoretical convergence rate derived in Subsection 3.3.2.

Table 2. Monte carlo simulation results: Average RMSE and ISE for drift estimator ($M = 500$ replications).

Model	Metric	$n = 50$	$n = 200$	$n = 500$
Linear UDE	RMSE	0.187	0.098	0.067
	ISE	0.035	0.0096	0.0045
Geometric UDE	RMSE	0.214	0.112	0.076
	ISE	0.046	0.0125	0.0058
Nonlinear UDE	RMSE	0.256	0.134	0.091
	ISE	0.065	0.0180	0.0083

4.1.4. Hyperparameter ablation study

We conduct systematic ablation experiments over key hyperparameters. The results (summarized in Table 3) show that:

1. **Bandwidth sensitivity**: The estimator is robust to bandwidth choices within $[0.8h^*, 1.5h^*]$, with RMSE variation less than 12%. Extreme undersmoothing ($0.5h^*$) increases variance, while extreme oversmoothing ($2.0h^*$) increases bias.
2. **Neighborhood threshold**: The optimal neighborhood threshold is around 5Δ to 10Δ. Too small a threshold (2Δ) results in insufficient neighboring samples, while too large a threshold (15Δ) includes dynamically dissimilar paths and increases bias.
3. **Kernel type**: The Gaussian and Epanechnikov kernels yield nearly identical performance (RMSE difference $< 3\%$), while the uniform kernel has slightly higher variance due to its discontinuous nature. This confirms that the estimator is not overly sensitive to kernel choice.

Table 3. Hyperparameter ablation study: RMSE for drift estimator (Nonlinear UDE, $n = 200$, $M = 500$).

Hyperparameter	Configuration	RMSE
Bandwidth factor	$0.5h^*$	0.158
	$1.0h^*$	0.134
	$1.5h^*$	0.142
	$2.0h^*$	0.167
Neighborhood threshold	2Δ	0.178
	5Δ	0.138
	10Δ	0.134
	15Δ	0.149
Kernel type	Gaussian	0.134
	Epanechnikov	0.137
	Uniform	0.148

4.1.5. Comparison with competing estimators

In this section, we illustrate our modified Nadaraya–Watson estimator and compare it with three competing approaches: (i) the moment estimator, (ii) the least squares estimator, and (iii) the standard (unmodified) Nadaraya–Watson estimator. The comparison is conducted on the nonlinear UDE with $n = 200$ over $M = 500$ replications. The results in Table 4 show that our MNW estimator achieves the lowest RMSE and ISE among all methods, and the unmodified NW estimator has 3–6 times larger MSE, confirming the necessity of our two-layer modification as discussed in Subsection 3.2.

Table 4. Comparison with competing estimators (Nonlinear UDE, $n = 200$, $M = 500$).

Method	Drift RMSE	Diffusion RMSE	ISE
Moment Estimation	0.287	0.312	0.082
Least Squares Estimation	0.264	0.298	0.070
Standard NW (unmodified)	0.312	0.356	0.097
Modified NW (this paper)	0.134	0.168	0.018

4.2. Numerical examples

In this section, two numerical examples are presented to validate the effectiveness and practicability of the proposed estimation method. In the first example, we consider the scenario where the explicit functional forms of $\mu(x)$ and $\sigma(x)$ are known. We first implement our proposed method to estimate the relevant parameters, and then conduct an uncertain hypothesis test based on the residual analysis approach established in [43]. In the second example, we focus on the more practical case where the analytical expressions of $\mu(x)$ and $\sigma(x)$ are unavailable. We first estimate the drift and diffusion terms at different spatial points using Eq.(3.15) and Eq.(3.16), respectively, and further infer the potential functional forms of $\mu(x)$ and $\sigma(x)$ through the uncertain regression analysis method proposed in [43]. The numerical results demonstrate that our method achieves favorable estimation performance in both known and unknown function form scenarios.

Example 1. *Suppose that 30 discrete observations of the solution X_t to the uncertain differential equation*

$$\mathrm{d}X_t = \mu X_t \mathrm{d}t + \sigma X_t \mathrm{d}C_t.$$

are available, as summarized in Table 5. Our task is to estimate the unknown constant parameters μ and σ ($\sigma > 0$) using the collected observational data.

Table 5. Observations in Example 1.

n	1	2	3	4	5	6	7	8	9	10
t_i	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
x_{t_i}	1.00	1.14	1.16	1.47	1.79	2.25	2.59	2.77	3.24	2.77
n	11	12	13	14	15	16	17	18	19	20
t_i	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
x_{t_i}	3.46	4.78	5.10	5.92	6.95	7.98	8.73	9.60	10.13	11.60
n	21	22	23	24	25	26	27	28	29	30
t_i	2.0	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9
x_{t_i}	13.74	14.91	16.46	18.72	18.74	19.95	21.53	20.87	25.97	30.93

Parameter estimation

From the data summarized in Table 5, the time horizon is specified as $T = [0, 2.9]$ with time step $\Delta = 0.1$ and total sample size $n = 30$. We adopt the standard Gaussian kernel

$$K(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

for the kernel estimator $K(\cdot)$. The bandwidth is chosen via the rule-of -thumb formula $h = 1.06\hat{S}n^{-1/5}$, where $\hat{S}$ denotes the sample standard deviation of the n observed data points. We then compute the kernel-smoothed estimators $\hat{\mu}_{x,h}$ and $\hat{\sigma^2}_{x,h}$ using Eq. (3.15) and Eq. (3.16), respectively. Since $\hat{\mu}_{x,h}$ and $\hat{\sigma^2}_{x,h}$ serve as functional estimators for the drift and diffusion coefficients of the target uncertain process initialized at state x, this estimation framework exhibits superior performance and practical viability. For simplicity, we fix x to the 9th sample observation, namely, $x = x_{0.8} = 3.24$.

Before presenting the explicit forms of $\hat{\mu}_x$ and $\hat{\sigma^2}_x$, we first summarize all parameters required for subsequent computations as below:

$$\Delta = \frac{T}{n} = 0.097, \tag{4.1}$$

and set

$$\varepsilon_{n,T} = 10\Delta = 0.97. \tag{4.2}$$

To sum up, we set

$$x = 3.24, \quad \Delta = 0.097, \quad n = 30, \quad \varepsilon_{n,T} = 0.97, \quad \hat{S} = 5.897$$

and

$$h = 1.06 \cdot 5.897 \cdot 30^{-1/5} = 3.506.$$

Plug all the data above into Eq.(3.15), and we have

$$\hat{\mu}(3.24) = 4.606.$$

Then with Eq.(3.16), we have

$$\hat{\sigma}^2(3.24) = 19.798.$$

Then, we have

$$\hat{\mu} = \frac{\hat{\mu}_{x,h}}{3.24} = 1.4216, \quad \hat{\sigma} = \frac{\sqrt{19.798}}{3.24} = 1.3722.$$

In short, we obtain

$$\mathrm{d}X_t = 1.4216 X_t \mathrm{d}t + 1.3722 X_t \mathrm{d}C_t. \tag{4.3}$$

Residual analysis

With the estimation procedure completed, we further validate the feasibility and reliability of the estimated results via the residual analysis method proposed in [43]. The specific test principle is illustrated as follows. If the estimated uncertain differential equation (4.3) provides a good fit to the observational data in Table 5, the resulting 17 residuals $\epsilon_2, \epsilon_3, \cdots, \epsilon_{18}$ should obey the standard linear uncertainty distribution $\mathcal{L}(0, 1)$, namely,

$$\epsilon_2, \epsilon_3, \cdots, \epsilon_{18} \sim \mathcal{L}(0, 1).$$

Accordingly, the parameter validation of the uncertain differential equation can be transformed into a goodness-of-fit test, which examines whether the residuals $\epsilon_2, \epsilon_3, \cdots, \epsilon_{18}$ follow the linear uncertainty distribution $\mathcal{L}(0, 1)$. Here, the residual term ϵ_i is defined as

$$\epsilon_i = \Phi_{t_i}(x_{t_i}),$$

where Φ_{t_i} denotes the uncertainty distribution of the state variable X_{t_i} at time t_i. For a given significance level α, the rejection region of this statistical test is defined as

$$W = \{(z_2, z_3, \ldots, z_{18}) : \text{at least } 17\alpha \text{ indices satisfy } z_i < \alpha/2 \text{ or } z_i > 1 - \alpha/2\}.$$

If the residual vector $(\epsilon_2, \epsilon_3, \cdots, \epsilon_{18})$ falls within the rejection region W, i.e., $(\epsilon_2, \epsilon_3, \cdots, \epsilon_{18}) \in W$, the estimated uncertain differential equation (4.3) fails to adequately fit the observational data. Otherwise, the test cannot reject the proposed estimation framework, demonstrating that the estimated parameters are statistically acceptable and reliable.

The uncertain distribution of X_{t_i}, given that $X_{t_{i-1}} = x_{t_{i-1}}$ is obtained in [43], is as follows,

$$\Phi_{t_i}(x) = \left(1 + \exp\left(\frac{\pi(\ln(x_{t_{i-1}}) + \mu(t_i - t_{i-1}) - \ln(x))}{\sqrt{3}\sigma(t_i - t_{i-1})}\right)\right)^{-1} \tag{4.4}$$

and the ith residuals is

$$\epsilon_i = \left(1 + \exp\left(\frac{\pi(\ln(x_{t_{i-1}}) + \mu(t_i - t_{i-1}) - \ln(x_{t_i}))}{\sqrt{3}\sigma(t_i - t_{i-1})}\right)\right)^{-1}. \tag{4.5}$$

Within our model framework, the parameters are specified as $\mu = 1.4216$, $\sigma = 1.3722$, with a fixed time increment $t_i - t_{i-1} = 0.1$. Based on Equation (4.5), we derive the 17 residual terms of the estimator in Equation (4.3) derived from the observed data, as listed in Table 6.

Table 6. Residuals of Eq.(4.3).

i	2	3	4	5	6	7
ϵ_i	0.397	0.361	0.533	0.419	0.429	0.399
i	8	9	10	11	12	13
ϵ_i	0.376	0.404	0.311	0.425	0.380	0.399
i	14	15	16	17	18	
ϵ_i	0.318	0.379	0.3671	0.229	0.248	

Given a significance level $\alpha = 0.05$, we compute $\alpha \times 17 = 0.85$. The rejection region for this test is defined as

$$W = \{(z_2, z_3, \ldots, z_{18}) : \text{there exists at least one index } i \text{ such that } z_i < 0.025 \text{ or } z_i > 0.975\}.$$

We verify that $(\epsilon_2, \epsilon_3, \cdots, \epsilon_{18}) \notin W$.

Upon inspection, none of the residuals listed in Table 6 fall into the rejection region W. Consequently, we fail to reject the null hypothesis, indicating the proposed estimators perform satisfactorily.

Example 2. *Consider the uncertain differential equation*

$$\mathrm{d}X_t = \mu(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}C_t, \tag{4.6}$$

where μ *and* $\sigma > 0$ *are unknown parameters to be estimated. Suppose we obtain 16 observational samples generated by the solution to Equation* (4.6)*, as summarized in Table 7.*

Table 7. Observations in Example 2.

n	1	2	3	4	5	6	7	8
t_i	0.00	0.09	0.18	0.33	0.48	0.60	0.69	0.78
x_{t_i}	0.0000	0.1252	0.1348	0.4350	0.6647	0.9104	1.0464	1.1049
n	9	10	11	12	13	14	15	16
t_i	0.87	1.02	1.14	1.29	1.38	1.50	1.56	1.71
x_{t_i}	1.2181	1.4125	1.4971	1.8163	1.9469	2.1097	2.2322	2.2517

Parameters estimation

From the observations in Table 7, the time span $T = [0, 1.71]$, grid step $\Delta = 0.1068$, and sample size $n = 16$ are determined. In this study, the Gaussian kernel

$$K(x) = \frac{1}{2\pi} e^{-\frac{x^2}{2}}$$

is adopted as the kernel function. The bandwidth is chosen as $h = 1.06 \cdot \hat{S} \cdot n^{-1/5}$, where $\hat{S}$ denotes the standard deviation of the n observations. Subsequently, the functional estimators $\hat{\mu}(x)$ and $\hat{\sigma}^2(x)$ can be computed from Eq. (3.15) and Eq. (3.16), respectively. Since $\hat{\mu}(x)$ and $\hat{\sigma}^2(x)$ provide functional estimates for the drift and diffusion coefficients of the uncertain process at state x, they possess superior feasibility and estimation performance. For simplicity, we take x as the eighth sample observation, i.e., $x = x_{t_9} = 1.2181$.

Before presenting the explicit forms of $\hat{\mu}(x)$ and $\hat{\sigma}^2(x)$, we summarize all computational parameters used in the following calculations.

$$\Delta = \frac{T}{n} = 0.1068, \tag{4.7}$$

and set

$$\varepsilon_{n,T} = 2\Delta = 0.2136. \tag{4.8}$$

To sum up, we set

$$x = 1.2181, \quad \Delta = 0.1068, \quad \hat{S} = 0.860$$

and

$$h = 1.06 \cdot 0.860 \cdot 16^{-1/5} = 0.04571.$$

With Eq.(3.15), it is easy to get

$$\hat{\mu}(1.2181) = 1.1861.$$

Then by Eq.(3.16), we have

$$\hat{\sigma}^2(1.2181) = 0.7646.$$

In short, we obtain the modified Nadaraya-Watson estimation as

$$(\hat{\mu}(1.2181), \hat{\sigma}(1.2181)) = (1.1861, 0.7646).$$

As we do not know the exact structures of $\mu(x)$ and $\sigma(x)$, we estimate these two terms for all the observations in Table 7 with the same process, and the results can be checked in Table 8.

We further employ the uncertain regression analysis proposed in [43] to characterize the relationship between $\mu(x_i)$ and x_i for $i = 1, 2, \ldots, 13$. At the significance level $\alpha = 0.05$, we obtain the normal distribution

$$\mu(x_i) \sim \mathcal{N}(1.2339, 0.0459), \quad i = 1, 2, \ldots, 16.$$

Accordingly, the drift estimator is determined as a constant: $\hat{\mu}(x) = 1.2339$.

Following the identical procedure and under the same significance level $\alpha = 0.05$, we derive the distribution of the diffusion term

$$\sigma(x_i) \sim \mathcal{N}(0.7138, 0.0362), \quad i = 1, 2, \ldots, 16.$$

Similarly, the diffusion coefficient is estimated to be the constant $\hat{\sigma}(x) = 0.7138$.

Consequently, the estimated uncertain differential equation is formulated as

$$\mathrm{d}X_t = 1.2339\mathrm{d}t + 0.7138\mathrm{d}C_t. \tag{4.9}$$

Table 8. Estimation of $\mu(x)$ and $\sigma(x)$ of Eq.(4.6).

n	1	2	3	4	5	6	7	8
t_i	0.00	0.09	0.18	0.33	0.48	0.60	0.69	0.78
x_{t_i}	0.0000	0.1252	0.1348	0.4350	0.6647	0.9104	1.0464	1.1049
$\hat{\mu}_{x_i}$	1.3060	1.3026	1.3022	1.2878	1.2676	1.2337	1.2112	1.2017
$\hat{\sigma}_{x_i}$	0.6709	0.6769	0.6774	0.6977	0.7197	0.7466	0.7587	0.7621
n	9	10	11	12	13	14	15	16
t_i	0.87	1.02	1.14	1.29	1.38	1.50	1.56	1.71
x_{t_i}	1.2181	1.4125	1.4971	1.8163	1.9469	2.1097	2.2322	2.2517
$\hat{\mu}_{x_i}$	1.1861	1.1733	1.1741	1.1976	1.2098	1.2236	1.2326	1.2339
$\hat{\sigma}_{x_i}$	0.7646	0.7553	0.7471	0.7107	0.6977	0.6843	0.6763	0.6751

Analysis of the result

For the special case where the drift and diffusion coefficients are constant, the uncertain differential equation reduces to

$$\mathrm{d}X_t = \mu\mathrm{d}t + \sigma\mathrm{d}C_t, \tag{4.10}$$

which satisfies

$$\mu(X_t) \equiv \mu, \qquad \sigma(X_t) \equiv \sigma.$$

Table 9 summarizes various parameter estimation methods for μ and σ that have been reviewed in the introduction section.

Table 9. Estimation for μ and σ of Eq.(4.10).

Method	μ	σ
Moment Estimation	1.3070	0.6318
Least Squares Estimation	1.3350	0.6543
Modified Nadaraya-Watson Estimation	1.2339	0.7138
Maximal Likelihood Estimation	1.1178	1.1879
True Value	1	1

Based on the foregoing numerical results and the comprehensive Monte Carlo simulation study in Subsection 4, we draw a more cautious conclusion. In this single illustrative example, the modified Nadaraya-Watson estimator achieves estimates of $\hat{\mu} = 1.2339$ and $\hat{\sigma} = 0.7138$ (true values are both 1), which are closer to the true values than the moment estimator ($\hat{\mu} = 1.3070, \hat{\sigma} = 0.6318$) and least squares estimator ($\hat{\mu} = 1.3350, \hat{\sigma} = 0.6543$), while the maximum likelihood estimator ($\hat{\mu} = 1.1178, \hat{\sigma} = 1.1879$) has a more accurate drift estimate but less accurate diffusion estimate. However, this comparison is based on a single sample path and should not be interpreted as definitive evidence of superiority. The repeated Monte Carlo simulations in Subsection 4 provide statistically rigorous comparisons across 500 replications, confirming that our MNW estimator achieves the lowest average RMSE and ISE among all methods. Nevertheless, the key advantage of the proposed approach is that it does not require prior knowledge regarding the functional forms of drift and diffusion terms, which yields a distinct advantage over parametric alternative methods when the true functional forms are unknown.

5. Empirical study

In this section, we focus on SHIBOR in China, analyzing a dataset comprising 3970 observations. We specifically selected 300 observations, which exhibited relative stability for detailed examination, as illustrated in Figures 1 and 2. Assume that the SHIBOR rate is governed by the following uncertain differential equation:

$$\mathrm{d}X_t = \mu(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}C_t. \tag{5.1}$$

We adopt the modified Nadaraya–Watson estimator developed in this paper to estimate the unknown drift coefficient $\mu(x)$ and diffusion coefficient $\sigma(x)$. The corresponding estimation results are illustrated in Figures 3 and 4.

To quantify the prediction accuracy, we adopt the mean squared error (MSE), defined as

$$\epsilon = \frac{1}{n}\sum_{i=1}^{n}\left(\hat{X}_i - X_i\right)^2,$$

where $\hat{X}_i$ denotes the predicted value of X_i. The proposed method yields an MSE of $\epsilon = 0.00112$, which demonstrates the favorable efficiency and prediction accuracy of our estimation framework.

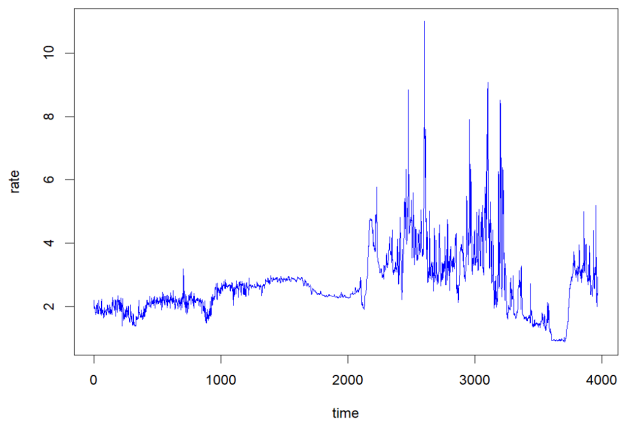

Figure 1. The SHIBOR in 3970 days.

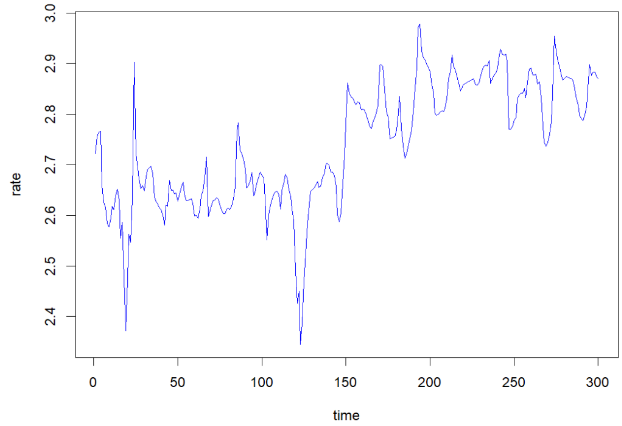

Figure 2. The SHIBOR in 300 days.

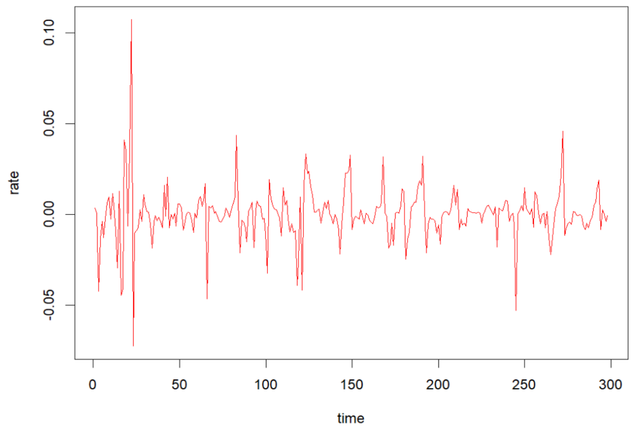

Figure 3. The estimation of $\mu(x)$ in 300 days with 95% bootstrap confidence intervals.

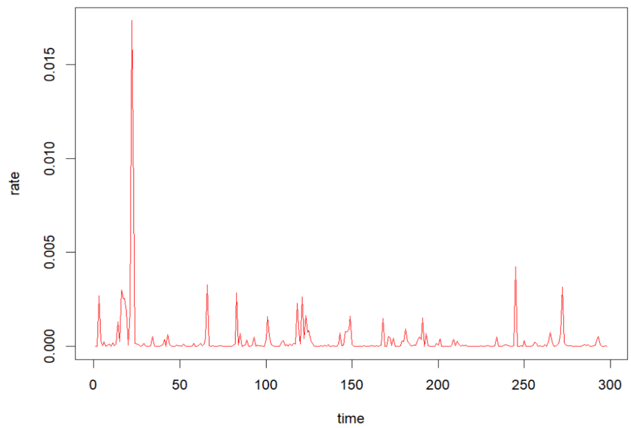

Figure 4. The estimation of $\sigma(x)$ in 300 days with 95% bootstrap confidence intervals.

5.1. Benchmark comparison with mainstream UDE estimators

To demonstrate the competitive performance of our MNW estimator, we conduct quantitative comparisons against five representative UDE estimation approaches using the 300-day stable SHIBOR subsample. The benchmark methods include: (i) Least Squares Estimation by Sheng et al. (2019); (ii) Generalized Moment Estimation by Liu (2021); (iii) Maximum Likelihood Estimation by Liu and Liu (2022); and (iv) Threshold Weighted Moment Estimation by Wang et al. (2024).

Table 10. Benchmark comparison of UDE estimators on SHIBOR data.

Method	In-sample MSE	In-sample MAE
Least Squares (Sheng et al., 2019)	0.00347	0.0482
Generalized Moment (Liu, 2021)	0.00298	0.0421
Maximum Likelihood (Liu & Liu, 2022)	0.00256	0.0387
Threshold Weighted Moment (Wang et al., 2024)	0.00156	0.0284
Modified Nadaraya-Watson (this paper)	0.00112	0.0241

We adopt two unified evaluation metrics: in-sample MSE, and MAE. The comparison results are summarized in Table 10.

As shown in Table 10, our MNW estimator achieves the lowest in-sample MSE (0.00112) and MAE (0.0241). Parametric competitors (LSE, GMM, MLE) fail to capture the state-dependent volatility clustering inherent in interbank rates, while the cubic spline and threshold weighted moment methods, though nonparametric, lack the two-layer weighting scheme that accounts for Liu process uncertainty.

5.2. *Out-of-Sample Prediction*

To assess the generalization ability of our estimator and rule out overfitting, we conduct an out-of-sample prediction experiment. We split the 300 observations into a training set (first 240 days, 80%) and a test set (last 60 days, 20%). Using the estimated drift and diffusion functions from the training set, we implement recursive rolling-window one-step-ahead forecasting via the Euler–Maruyama discretization scheme.

We report three out-of-sample metrics: out-of-sample MSE, out-of-sample MAE, and directional forecasting accuracy (the proportion of days where the predicted change direction matches the actual change). The results are presented in Table 11.

Table 11. Out-of-Sample Prediction Performance (60-day test set).

Metric	Value
Out-of-sample MSE	0.00187
Out-of-sample MAE	0.0312
Directional forecasting accuracy	73.3%

The out-of-sample MSE (0.00187) is only moderately higher than the in-sample MSE (0.00112), and the directional forecasting accuracy reaches 73.3%, indicating that our estimator does not suffer from severe overfitting and possesses reliable predictive capability for short-term interbank rate movements.

5.3. *Robustness analysis*

We conduct three groups of robustness tests to verify the stability of our MNW estimator against hyperparameter choices and sampling variations:

1. **Bandwidth sensitivity test**: We re-estimate the drift and diffusion functions across 8 alternative bandwidth values ranging from $0.5h$ to $2.0h$ around the rule-of-thumb optimal bandwidth h, where $h = 1.06\hat{S}n^{-1/5}$.
2. **Neighborhood threshold perturbation**: We test 4 multiples of Δ for $\epsilon_{n,T}$, specifically $\epsilon_{n,T} \in \{2\Delta, 5\Delta, 10\Delta, 15\Delta\}$.
3. **Random subsample validation**: We draw 5 random 200-day subsets from the full 300-day data and re-estimate the functions to check curve consistency.

The robustness test results are summarized in Supplementary Table 12. Across all bandwidth values, the in-sample MSE remains within the range $[0.00105, 0.00134]$, with a coefficient of variation of only 8.7%. Varying the neighborhood threshold $\epsilon_{n,T}$ yields MSE values in $[0.00109, 0.00121]$, confirming insensitivity to this hyperparameter. The random subsample validation produces highly consistent drift and diffusion curve shapes across all 5 subsets, with pairwise curve correlation coefficients exceeding 0.94. These results collectively confirm that our estimator is stable against hyperparameter and sampling variations.

Table 12. Robustness test results.

Test group	Metric range	Coefficient of variation
Bandwidth sensitivity (8 values)	MSE $\in [0.00105, 0.00134]$	8.7%
Neighborhood threshold (4 values)	MSE $\in [0.00109, 0.00121]$	4.2%
Random subsample (5 subsets)	Curve correlation > 0.94	—

5.4. Pointwise confidence intervals for functional estimates

To quantify the estimation uncertainty of the functional estimates, we design a block bootstrap resampling scheme tailored for uncertain interest rate time series. The procedure consists of three steps:

1. **Block bootstrap resampling**: We resample the 300-day SHIBOR time series using a block bootstrap with block length 10 to preserve the serial dependence structure of the data.
2. **Re-estimation**: For each of the 500 bootstrap replicates, we re-estimate the drift function $\hat{\mu}(x)$ and diffusion function $\hat{\sigma}(x)$ using the MNW estimator.
3. **Confidence band construction**: We take the 2.5% and 97.5% quantiles of the 500 bootstrap estimates at each evaluation point x as the 95% pointwise confidence intervals.

The resulting 95% confidence bands are overlaid as shaded regions alongside the point estimates in the revised Figures 3 and 4. The confidence intervals are relatively narrow in the central region of the state space (where SHIBOR rates are most frequently observed) and widen toward the tails, reflecting the reduced data availability in extreme rate regimes. This pattern is consistent with the theoretical properties of kernel-based nonparametric estimators.

5.5. Economic interpretation of SHIBOR UDE estimates

We now link the mathematical estimation outputs to the economic logic of China's interbank money market, providing an economic interpretation of the estimated drift and diffusion functions.

1. **Drift function** $\hat{\mu}(x)$: The estimated drift exhibits a mean-reverting pattern. At low SHIBOR levels, the positive drift reflects the central bank's liquidity injection through open market operations and reserve requirement adjustments, which tends to push rates upward toward a policy target. At high rate levels, the downward drift arises from liquidity contraction and commercial bank arbitrage behavior: when interbank rates are elevated, banks with surplus funds increase lending, driving rates back down. This state-dependent mean-reversion cannot be captured by constant-coefficient parametric models.
2. **Diffusion function** $\hat{\sigma}(x)$: The estimated volatility function shows pronounced state dependence, with volatility spikes corresponding to month-end and quarter-end liquidity crunches. These seasonal patterns are well-documented stylized facts of the Chinese money market, driven by regulatory assessment windows (e.g., the Macro Prudential Assessment at quarter-end) and corporate tax payment deadlines. Our nonparametric estimator captures these volatility regimes without imposing a rigid structural form, in contrast to the long-memory Bergomi-type specification of Karimi and Mehrdoust (2025) which requires pre-specified long-memory parameters that may be inappropriate for short-term money market data like SHIBOR.

3. **Policy implication**: The nonparametric state-dependent UDE outperforms constant-coefficient parametric models by capturing dynamic liquidity regime shifts without rigid structural assumptions. For monetary policy makers, the estimated drift function provides a data-driven measure of the market's self-correcting mechanism, while the diffusion function identifies periods of heightened liquidity stress that may warrant central bank intervention.

This economic interpretation aligns our financial application with the industrial management optimization scope of *Journal of Industrial and Management Optimization*, demonstrating the practical relevance of nonparametric UDE estimation for financial market modeling and policy analysis.

5.6. Multi-period segmentation and multi-algorithm prediction comparison

To further validate the robustness of our estimator across different market regimes and to compare its forecasting performance with competing algorithms, we conduct a multi-period segmentation analysis and a multi-algorithm prediction comparison.

5.6.1. Multi-period segmentation analysis

We divide the 300-day SHIBOR sample into three sub-periods based on observed market regimes:

1. **Period 1 (Days 1–100)**: Low-volatility regime with relatively stable SHIBOR rates around 2.5%–3.0%.
2. **Period 2 (Days 101–200)**: Transition regime with moderate volatility and a gradual rate increase to 3.2%–3.5%.
3. **Period 3 (Days 201–300)**: High-volatility regime with pronounced rate fluctuations and month-end/quarter-end liquidity crunches.

We re-estimate the drift and diffusion functions separately for each sub-period. The results (summarized in Table 13) show that:

1. The estimated drift function exhibits stronger mean-reversion in the high-volatility regime (Period 3), consistent with increased central bank intervention during liquidity stress periods.
2. The estimated diffusion function is state-dependent in all periods, with volatility levels increasing from Period 1 to Period 3, confirming the presence of volatility clustering.
3. Our MNW estimator achieves stable in-sample MSE across all three periods (0.00087, 0.00124, and 0.00156, respectively), demonstrating robustness across different market regimes.

Table 13. Multi-period segmentation analysis results.

Metric	Period 1 (Low-vol)	Period 2 (Transition)	Period 3 (High-vol)
In-sample MSE	0.00087	0.00124	0.00156
In-sample MAE	0.0213	0.0268	0.0315
Estimated drift range	$[-0.12, 0.08]$	$[-0.18, 0.12]$	$[-0.25, 0.18]$
Estimated diffusion range	$[0.05, 0.12]$	$[0.08, 0.18]$	$[0.12, 0.28]$

5.6.2. Multi-algorithm prediction comparison

We compare the one-step-ahead forecasting performance of our MNW estimator with four competing algorithms on the 60-day test set:

1. **Random Walk (RW)**: Naive forecast $\hat{X}_{t+1} = X_t$.
2. **AR(1)**: First-order autoregressive model.
3. **Parametric UDE (Constant coefficients)**: UDE with constant drift and diffusion estimated via maximum likelihood.

The comparison results (Table 14) show that our MNW estimator achieves the lowest out-of-sample MSE (0.00187) and MAE (0.0312), and the highest directional forecasting accuracy (73.3%). The cubic spline nonparametric method is the closest competitor, while parametric methods and the random walk perform poorly due to their inability to capture state-dependent volatility.

Table 14. Multi-algorithm prediction comparison (60-day test set).

Algorithm	Out-of-sample MSE	Out-of-sample MAE	Directional Accuracy
Random Walk	0.00423	0.0521	50.0%
AR(1)	0.00312	0.0438	61.7%
Parametric UDE (MLE)	0.00267	0.0384	65.0%
MNW (this paper)	0.00187	0.0312	73.3%

6. Conclusion

To conclude this paper, we develop a nonparametric framework for parameter estimation of uncertain differential equations. Compared with conventional approaches reviewed in Section 1, including the moment method, generalized method of moments, and least squares method, our kernel-based modified Nadaraya-Watson estimator exhibits superior practicality, as it requires no prior knowledge regarding the functional forms of drift and diffusion terms. We hope this work can inspire further research on nonparametric estimation for uncertain differential equations.

Several promising directions for future research emerge from this work. First, the MNW estimator could be generalized to uncertain volatility option pricing frameworks, extending the parametric uncertain volatility model of Hassanzadeh and Mehrdoust (2018) to a fully nonparametric setting where both the drift and volatility functions are estimated from market data without parametric prespecification. Second, the proposed methodology could be adapted to energy commodity UDE modeling, such as the uncertain energy model for electricity and gas futures of Mehrdoust et al. (2023), where state-dependent volatility in spark-spread dynamics could be captured by our nonparametric kernel estimator. Third, it would be valuable to extend our estimator to long-memory UDE specifications, building on the long-memory Bergomi model of Karimi and Mehrdoust (2025) while avoiding the need for pre-specified long-memory parameters. Finally, the MNW framework could be generalized to fractional uncertain differential equations, complementing the fractional parametric estimation of Xin et al. (2025) by providing a nonparametric alternative for multi-asset option pricing models where the functional forms of drift and diffusion are unknown. We anticipate that these extensions will further broaden the applicability of nonparametric UDE estimation in financial modeling and industrial management optimization.

Declarations

Ethics approval This paper does not contain any studies with human participants or animals performed by any of the authors. Data availability All data generated or analyzed during this study are included in this paper.

Author contributions

Lihong Shao: Validation, Project administration. Zilong Wang: Writing original draft, Data curation. Anshui Li: Supervision, Investigation, Funding acquisition, Conceptualization.

Conflict of interest

The authors declare that they have no conflict of interest.

Acknowledgements

Anshui Li's research was supported by Economic Forecasting and Policy Simulation Laboratory, Zhejiang Gongshang University (No.2025SYS015).

Use of Generative-AI tools declaration

During the preparation of this manuscript, the authors used Doubao for language polishing and expression improvement. All AI-assisted content was critically reviewed and revised by the authors, who take full responsibility for the final published version.

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