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Research article

Multi-criteria group decision making method with dual normal clouds under an information environment of dual probabilistic linguistic variables

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Abstract: Human judgment is inherently subjective, and cognitive limitations often lead to uncertainty in decision-making. To address this, multi-criteria group decision making (MCGDM) widely uses linguistic and probabilistic linguistic variables to represent experts' preferences. This study focuses on decision problems where criteria values are expressed as probabilistic linguistic variables. First, the concept of dual probabilistic linguistic variables is introduced, extending traditional probabilistic linguistic representations. Second, the normal cloud model characterizes the interconnection between fuzziness and randomness, enabling the derivation of the dual normal cloud framework. A transformation method is developed to convert dual probabilistic linguistic variables into dual probabilistic linguistic normal clouds. Additionally, a dual probabilistic linguistic normal cloud (DPLNC)-weighted averaging operator is proposed to aggregate multiple DPLNCs, while a dual probabilistic linguistic normal cloud-gray relational degree (DPLNC-GRD) is established to enhance decision-making accuracy. A new MCGDM approach based on DPLNC-GRD is proposed and applied to a sustainable supplier selection case study. Sensitivity analysis and comparative evaluations with established methods confirm the robustness and reliability of the proposed framework, demonstrating its effectiveness in managing uncertainty and improving decision outcomes.

Keywords: MCGDM; dual probabilistic linguistic concept; dual probabilistic linguistic normal cloud model; $\mathfrak{R}_{\mathcal{D}_{agg}}$ operator; gray relational degree; DPLNC-GRD
Mathematics Subject Classification: 90B50, 03E72

1. Introduction

In an increasingly complex global landscape, organizations routinely face critical decisions with far-reaching economic, environmental, and social consequences. From selecting sustainable supply chains to evaluating large-scale infrastructure projects, these decisions are rarely made by a single individual. Instead, they rely on the collective wisdom of multiple experts through multi-criteria group decision-making (MCGDM) [1, 2]. The fundamental challenge of MCGDM lies in synthesizing the preferences, experiences, and judgments of diverse decision-makers who must evaluate multiple alternatives against often conflicting criteria. A significant complication arises from the inherent subjectivity and cognitive limitations of human experts. These experts frequently operate with incomplete information and naturally express judgments using qualitative, imprecise language rather than precise numerical values [3].

The reliance on crisp numbers forces experts into an artificial precision that fails to capture the genuine ambiguity of their thoughts. Recognizing this, decision sciences have witnessed a paradigm shift toward fuzzy logic and linguistic modeling. These approaches provide natural and realistic frameworks for experts to articulate assessments using terms such as "very good," "acceptable," or "high risk," which closely mirror human cognitive patterns [4]. To computationally operationalize these linguistic terms, a rich ecosystem of fuzzy set extensions has been developed, including interval-valued [5], triangular [6], and trapezoidal fuzzy sets [7], followed by sophisticated models like intuitionistic fuzzy sets [8], hesitant fuzzy sets [9], and their interval-valued counterparts [10].

A pivotal advancement in this evolution emerged with probabilistic linguistic term sets (PLTSs) introduced by Pang et al. [11]. PLTSs acknowledge that expert opinion is not always a single definitive linguistic term but often represents a distribution over several possible terms. For instance, when assessing a supplier's service level, an expert might express the assessment as "excellent with a 60% degree of belief" and "good with a 40% degree of belief," formally represented as excellent(0.6), good(0.4). The PLTS framework provides powerful tools to capture the relative importance or likelihood of different linguistic evaluations, offering a richer and more flexible representation of uncertain information than previous models [12]. It has become a cornerstone of modern MCGDM research, with numerous aggregation operators, distance measures, and integration with methods such as the technique for order preference by similarity to ideal solution (TOPSIS) [13] and the preference ranking organization method for enrichment evaluation (PROMETHEE) [14] having been developed.

Despite their expressiveness, traditional PLTSs and other fuzzy linguistic models capture only one dimension of the expert cognitive process: The evaluation of the object itself. However, in real-world high-stakes decision-making scenarios, a second crucial layer of uncertainty almost always exists the expert's confidence in their own evaluation. Consider a medical diagnosis, where a panel might describe symptoms as "highly indicative (70%) of Disease A," yet a junior doctor may attach low confidence to this assessment while a senior specialist exhibits high confidence in the same diagnosis. In financial risk assessment, a bank's committee might evaluate a startup as "risky (80%)," where a conservative analyst shows high certainty while an innovative analyst expresses uncertainty about traditional models' applicability. In sustainable supplier selection, an expert might evaluate environmental systems as "good (60%)" but assign only "likely (70%)" confidence if relying on second-hand reports.

In all cases, the primary evaluation (*good*, *risky*) and confidence in that evaluation (*certain*, *uncertain*) represent distinct but intertwined uncertain information. The primary evaluation manifests fuzziness (what does "good" truly mean?), while confidence exhibits randomness (how likely is the assessment to be accurate?). This dual layer of uncertainty in a fuzzy evaluation with a random confidence level constitutes a natural and pervasive component of expert judgment that existing probabilistic linguistic models fail to capture explicitly. Ignoring confidence levels can be misleading, as it treats hesitant guesses with the same weight as firm convictions. Research on incomplete probabilistic linguistic information by Xiao et al. [15] emphasized that partial ignorance in expert assessments must be properly modeled rather than forced into artificial distributions.

Recognizing the need for more complex linguistic structures, researchers have proposed several "dual" models, each with specific limitations that our work directly addresses. Xie et al. [16] developed dual probabilistic linguistic term sets (DPLTSs), structured as $\langle L(p), U(p)\rangle$, to model membership and non-membership degrees. While useful for certain types of ambiguity, DPLTSs fundamentally capture the balance between support and opposition for an alternative, not the confidence in a single evaluation. Furthermore, they introduce procedural complexity through required normalization steps. Wang et al. [17] proposed dual linguistic term sets (DLTSs) using ordered pairs like $(good, likely)$, where the second term linguistically describes confidence in the first. This model intuitively introduces a confidence layer; however, its critical limitation is that the confidence term itself (e.g., "likely") remains a crisp linguistic label without a probabilistic distribution. This prevents experts from expressing nuanced meta-uncertainty about their own confidence level (e.g., being "60% sure that confidence is 'likely' and 40% sure that it is 'very likely'").

In contrast to these models, the proposed dual probabilistic linguistic variable (DPLV) is designed to overcome these specific shortcomings. First, unlike DPLTSs, it explicitly models confidence in an evaluation rather than support/opposition balance. Second, unlike DLTSs, it represents confidence not as a crisp term but as a full probabilistic linguistic distribution $b_j(\mu'_j)$, allowing rich expression of uncertainty about uncertainty. This structure, $DPLV = \langle h_i(\mu_i), b_j(\mu'_j)\rangle$, provides a direct and natural representation for the two-tiered uncertainty inherent in expert judgment.

Contemporary research continues to explore this area. Liu et al. [18] introduced trust-based quantum probabilistic linguistic models incorporating interference effects to capture psychological factors, while recent studies have also developed consensus-reaching methods for dual probabilistic linguistic contexts. Recent advances in probabilistic linguistic decision-making continue to expand its applications and methodological depth. For instance, an artificial intelligence (AI)-powered decision-making framework using probabilistic linguistic Sugeno–Weber aggregation has been proposed for road safety optimization, demonstrating the utility of advanced aggregation operators in practical safety management. Similarly, Momena et al. [19] applied multi-criteria decision analysis to sustainable medicinal supply chains, addressing adaptability and challenge issues in logistics contexts. Chen et al. [20] integrated the probabilistic linguistic best-worst method (PL-BWM) and probabilistic linguistic three-way TOPSIS for multi-index decision-making, reflecting the ongoing evolution of hybrid methods in complex evaluations. Wu et al. [21] developed a mixed integer programming-driven consensus reaching process for probabilistic linguistic environments, while Rasool et al. [22] proposed an integrated criteria importance through intercriteria correlation (CRITIC)-evaluation based on distance from average solution (EDAS) model using linguistic T-spherical fuzzy aggregation operators. Other studies include that of Xu et al. [23], who used the vlsekriterijumska optimizacija I kompromisno resenje (VIKOR) method with PLTSs for foreign direct investment risk evaluation, while Li et al. [24] developed an improved tomada de decisão interativa e multicritério (TODIM) method based on a new distance measure for probabilistic linguistic MAGDM. Naz et al. [25] enhanced industrial robot selection through hybrid CRITIC-VIKOR methods with probabilistic uncertain linguistic information. An opinion dynamics model for group decision-making with probabilistic uncertain linguistic information has also been introduced [26], highlighting the growing interest in dynamic and interactive decision processes. Social network group decision-making methods based on stochastic multi-criteria acceptability analysis have been proposed by Xu et al. [27]. Additionally, Razak et al. [28] advanced linguistic variable frameworks for e-commerce decision-making, and Guo et al. [29] developed consensus-reaching processes using personalized modification rules in large-scale group decision-making. While these studies underscore the field's recognition of the problem, they often address different aspects (like social trust) or introduce significant computational overhead. The identified research gap therefore remains clear and significant: A unified, computationally tractable framework is lacking that can simultaneously and effectively model primary probabilistic linguistic evaluation (e.g., "good with 60% probability"), model probabilistic confidence in that evaluation (e.g., "likely with 70% probability that my 'good' assessment is accurate"), seamlessly integrate both fuzziness and randomness inherent in these layers, and provide a mathematically sound foundation for aggregating such information from multiple experts while ranking alternatives.

Compared with existing dual linguistic models, the novelty of the proposed DPLV lies in the nature of the second uncertainty layer and in how the two layers are jointly represented and processed. Existing DPLTSs mainly characterize paired membership and non-membership information, whereas DLTSs introduce a second linguistic component to describe confidence but do not allow this confidence itself to be represented probabilistically. Consequently, these models cannot fully describe situations in which both the primary evaluation and the decision-maker's confidence in that evaluation are uncertain. The proposed DPLV addresses this limitation by defining $\mathcal{D} = \left\langle h_i(\mu_i), b_j(\mu'_j)\right\rangle$, where both the primary linguistic evaluation and the associated confidence level are represented as probabilistic linguistic information. This structure therefore models not only uncertainty in the evaluation itself, but also uncertainty regarding the reliability of that evaluation.

In addition, the proposed framework does not stop at extending the information's representation. The DPLV is further integrated with the normal cloud model to form the dual probabilistic linguistic normal cloud (DPLNC), thereby providing a unified numerical representation of the fuzziness and randomness contained in the two uncertainty layers. Based on this representation, a dedicated DPLNC weighted aggregation operator and a DPLNC-based gray relational degree (GRD) are developed for group information fusion and alternative ranking. Therefore, the proposed framework differs from existing dual models at three levels: Information representation, uncertainty transformation, and decision aggregation and ranking.

To bridge this critical gap, this paper proposes a novel, unified, and computationally tractable MCGDM framework based on the normal cloud model, specifically designed for dual probabilistic linguistic uncertainty. The principal contributions of this work are as follows.

(1) We formally define the novel DPLV, $\mathcal{D} = \langle h_i(\mu_i), b_j(\mu'_j)\rangle$, to characterize two distinct layers of uncertainty in expert judgment: Uncertainty in the primary evaluation and uncertainty in the associated confidence. Unlike existing DPLTSs, which mainly represent paired support/opposition information, and DLTSs, which express confidence as a single linguistic term, the proposed DPLV represents both the evaluation and the associated confidence as probabilistic linguistic information, enabling a more comprehensive representation of two-level uncertainty.
(2) We integrate the DPLV with the normal cloud model [30] to create a DPLNC. This transformation leverages the cloud's three numerical parameters, expectation ($\mathbb{E}x$), entropy ($\mathbb{E}n$), and hyper-entropy ($\mathbb{H}e$) to inherently capture the interconnection between fuzziness (through $\mathbb{E}n$ and $\mathbb{H}e$) and randomness (through the random cloud drop distribution), thereby providing a robust and unified representation of dual-layer uncertainty.
(3) To operationalize the framework for group settings, we develop a new DPLNC-weighted averaging ($\mathfrak{R}_{\mathcal{D}_{agg}}$) operator. This operator synthesizes individual DPLNC evaluations from multiple

experts into a collective assessment while rigorously preserving the encapsulated uncertainty characteristics.

(4) We establish a new ranking mechanism by proposing the DPLNC-GRD. This measure enables effective comparison and ranking of alternatives under profound uncertainty by calculating their relational closeness to an ideal solution, extending gray system theory into the proposed cloud-based context.

(5) We encapsulate these components into a complete, step-by-step MCGDM methodology. Its practicality and effectiveness are rigorously validated through a detailed sustainable supplier selection case study, supplemented by comprehensive sensitivity and comparative analyses.

The structure of this paper is organized as follows. A review of the relevant literature is presented in Section 2. Section 3 introduces the novel concept of a DPLV, defines the dual normal cloud, and details the method for converting DPLVs into a DPLNC. The subsequent section, Section 4, introduces the $\mathfrak{R}_{\mathcal{D}_{agg}}$ operator for aggregating multiple DPLNCs. Building on this, Section 5 proposes a novel distance measure for DPLNCs and defines the DPLNC-GRD. Section 6 then synthesizes these components into a new MCGDM methodology based on the DPLNC-GRD. To demonstrate its practical application, Section 7 presents a case study on sustainable supplier selection, supplemented by sensitivity and comparative analyses to validate the approach. Finally, the conclusions and implications of the study are summarized in Section 8.

2. Literature review

The growing complexity of MCGDM problems has led scholars to explore increasingly sophisticated models for representing uncertainty and linguistic vagueness in expert assessments. Early research focused on hesitant and fuzzy linguistic models, which allowed decision-makers to express judgments with greater flexibility compared with traditional crisp numbers. For instance, Gou et al. [31] developed the double hierarchy hesitant fuzzy linguistic term set (DHHFLTS) in combination with the multi-objective optimization by ratio analysis plus the full multiplicative form (MULTIMOORA) method, providing a robust framework for evaluation environmental measures and demonstrating the ability of hierarchical hesitant structures to capture complex linguistic preferences. Similarly, Yang and Xu [32] introduced a matrix game-based approach using probabilistic triangular intuitionistic hesitant fuzzy information, while Singh et al. [33] extended the analytic hierarchy process (AHP) to hesitant probabilistic fuzzy linguistic sets. Both contributions emphasize more nuanced representations of uncertainty in group evaluations. In another domain, Tian et al. [34] integrated prospect theory with an extended picture fuzzy MULTIMOORA method to model decision-makers' behavioral biases in medical institution selection, highlighting the value of integrating cognitive elements into linguistic multicriteria decision-making (MCDM) methods.

A major breakthrough was achieved with the introduction of PLTSs, which represent expert opinions as distributions over multiple linguistic terms, thereby capturing uncertainty more naturally. The fundamental operations for PLTSs have been established, enabling their broader computational use. Building on this, Zhang et al. [35] proposed a deviation method to address disagreements among experts, while Merigó et al. [36] emphasized balancing subjective and objective components in linguistic group decision-making. Recent surveys, such as that of Ma et al. [37], systematically reviewed PLTS's foundations and future challenges, confirming their centrality to modern decision science. The importance of consensus-reaching mechanisms has also been recognized. Wan et al. [38] presented a dual-strategy consensus process for probabilistic linguistic MCGDM, whereas Saha et al. [39] proposed a dual probabilistic linguistic consensus-reaching method. Both studies underscore the necessity of reconciling divergent expert opinions in uncertain contexts.

Expanding beyond static frameworks, researchers have explored dynamic and socially informed extensions of probabilistic linguistic decision-making. Dong et al. [40] designed a dynamic MCGDM approach capable of handling evaluations across consecutive time points, reflecting the temporal dimension of real-world decision processes. Li et al. [41] incorporated social network structures into probabilistic linguistic decision-making, acknowledging that trust relationships and social influence shape collective outcomes. Furthermore, Wang et al. [42] proposed a method that explicitly considers the relative importance of both experts and attributes, while Xian et al. [43] introduced an interval-value Z-probabilistic double hierarchy linguistic model for evaluating city habitability, demonstrating the scalability of these approaches to large and hierarchical decision problems.

Despite these advances, many probabilistic linguistic models primarily emphasize either fuzziness or randomness, without fully integrating both dimensions of uncertainty. To overcome this limitation, Li et al. [44] introduced the cloud model, a cognitive representation that connects qualitative linguistic concepts with quantitative data using expectation, entropy, and hyper-entropy parameters. This foundation was extended by Wang et al. [45], who applied the cloud model to uncertain linguistic MCGDM, demonstrating its ability to reconcile qualitative judgments with numerical reasoning. More recently, Zhu et al. [46] applied cloud models to multi-stage multi-attribute decision-making under probabilistic interval-valued hesitant fuzzy environments, further showcasing the adaptability of this approach in complex decision contexts.

Contemporary research continues to explore this space with increasingly advanced frameworks. Liu et al. [18] introduced trust-based quantum probabilistic linguistic models incorporating interference effects to capture psychological and cognitive factors, while Saha et al. [39] proposed novel consensus-reaching methods tailored for dual probabilistic linguistic contexts. Although these contributions underscore the recognition of the problem, they often target specific aspects, such as social trust or collective consensus, and may introduce significant computational overhead. Consequently, the identified research gap remains both evident and significant: The current methods lack a unified, computationally tractable framework that can simultaneously and effectively model primary probabilistic linguistic evaluations (e.g., "good with 60% probability"), probabilistic confidence in those evaluations (e.g., "likely with 70% probability that my 'good' assessment is accurate"), and the seamless integration of fuzziness and randomness across these two layers. Moreover, the existing approaches provide limited mathematical grounding for aggregating such enriched information across multiple experts and ranking alternatives with reliability. To clarify the differences among the most relevant dual linguistic information models and to identify the specific limitations addressed by the proposed DPLV, a comparative summary is provided in Table 1.

Table 1. Comparison of existing dual linguistic models and the proposed DPLV.

Model	Basic structure	Primary evaluation representation	Second-layer representation	Probabilistic representation of confidence	Main limitation	Improvement provided by DPLV
DPLTS	$\langle L(p), U(p)\rangle$	Probabilistic linguistic information	Membership/non-membership or support/opposition information	No	The second component does not explicitly represent confidence in the primary evaluation	DPLV explicitly interprets the second layer as confidence associated with the evaluation
DLTS	(h_i, b_j)	Linguistic evaluation	A linguistic confidence term	No	Confidence is represented by a single linguistic label and cannot express uncertainty in the confidence itself	DPLV represents confidence as probabilistic linguistic information
Proposed DPLV	$\langle h_i(\mu_i), b_j(\mu'_j)\rangle$	Probabilistic linguistic evaluation	Probabilistic linguistic confidence	Yes	—	Simultaneously represents uncertainty in the evaluation and uncertainty in the associated confidence

As shown in Table 1, the existing dual models capture different aspects of uncertainty but do not simultaneously provide probabilistic representations of both the primary evaluation and the associated confidence. DPLTSs mainly characterize paired support/opposition information, whereas DLTSs introduce a confidence-related linguistic component but represent it as a single linguistic term. The proposed DPLV addresses these limitations by allowing both the evaluation and confidence layers to be expressed probabilistically, thereby providing a more complete representation of two-level uncertainty in expert judgment.

Overall, the limitations of existing probabilistic linguistic decision-making models can be systematically summarized into three aspects. First, there is a lack of two-layer confidence modeling: The existing models generally focus on uncertainty in the primary evaluation and have limited ability to simultaneously represent uncertainty in both the evaluation and its associated confidence. Second, fuzziness and randomness are often modeled separately, resulting in insufficient integration of these two forms of uncertainty within a unified linguistic decision-making framework. Third, existing aggregation operators are insufficient for directly processing enriched dual probabilistic linguistic information, particularly when both the primary evaluation and the associated confidence are expressed probabilistically. These three limitations motivate the development of a unified framework capable of representing two-layer uncertainty, integrating fuzziness and randomness, and effectively aggregating enriched expert information in MCGDM problems.

Addressing this gap and the challenges of complex and uncertain linguistic information, this paper proposes a novel framework by fusing several advanced concepts. We first define DPLV, extending the work of Pang et al. on probabilistic linguistic variables [11] and that of Wang et al. on DLTSs [17]. The operational complexity of this new variable type presents a significant aggregation challenge for MCGDM. To overcome this, we leverage the cloud model [44], an effective tool for quantifying uncertainty through three parameters: Expectation, entropy, and hyper-entropy. This model facilitates an objective transformation between qualitative concepts and quantitative values [45]. Our specific contribution is the definition of a dual normal cloud, which serves as the equivalent representation for DPLVs. To aggregate the evaluations of multiple experts in an MCGDM setting, we introduce the DPLNC-weighted averaging ($\mathfrak{R}_{\mathcal{D}_{agg}}$) operator. This operator efficiently combines multiple cloud models into a unified aggregated DPLNC. For the final ranking of alternatives, we develop a new methodology that merges gray relational analysis with our cloud model. The proposed DPLNC-GRD is designed to handle the information incompleteness and fuzziness typical in MCDM. This hybrid approach not only enhances traditional gray relational theory but also provides a practical and novel ranking method for decision environments characterized by DPLV.

3. Dual probabilistic linguistic variable set

This section begins by establishing the fundamental definitions for the linguistic variable set. For convenience, the symbols used throughout the paper and their meanings are given in Table 2.

Table 2. Summary of main notations.

Symbol	Meaning	Symbol	Meaning
1. Core linguistic and probabilistic sets			
$\mathcal{H}$	Linguistic term set (evaluation)	h_i	A linguistic term in $\mathcal{H}$
ρ	Granularity of $\mathcal{H}$ $(2\rho + 1)$	$\mathcal{P}$	Linguistic probability term set
b_j	Linguistic probability term	ϖ	Granularity of $\mathcal{P}$ $(2\varpi + 1)$
$\mathcal{H}(\mu)$	Probabilistic linguistic variable set	$h_i(\mu_i)$	Probabilistic linguistic variable
$\mathcal{P}(\mu')$	Probabilistic linguistic probability set	$b_j(\mu'_j)$	Probabilistic linguistic probability term
2. Dual probabilistic linguistic variable (DPLV)			
$\mathcal{D}$	DPLV	$\langle h_i, b_j \rangle$	DPLV structure
$\mathcal{D}_{rql}$	DPLV for alternative r, criterion q, and expert l		
3. Cloud model parameters			
$\mathfrak{R}$	Normal cloud	$\mathfrak{R}_{\mathcal{D}}$	DPLNC
$\mathbb{E}x$	Expectation (central value)	$\mathbb{E}n$	Entropy (fuzziness)
$\mathbb{H}e$	Hyper-entropy	$\widetilde{\mathbb{E}x}$	DPLNC expectation
$\widetilde{\mathbb{E}n}$	DPLNC entropy	$\widetilde{\mathbb{H}e}$	DPLNC hyper-entropy
ξ	Cloud drop	$f(\xi)$	Certainty degree
Ω	Universe of discourse	$[\mho_{\min}, \mho_{\max}]$	Evaluation bounds

Continued on next page

Symbol	Meaning	Symbol	Meaning
4. Aggregation and operation symbols			
$\aleph$	Number of DPLNCs	$\mathfrak{R}_{\mathcal{D}_{agg}}$	DPLNC weighted average
ω_ℓ	Weight for the ℓ-th DPLNC	CWA_ω	Cloud weighted average
5. Decision-making framework			
m	Number of alternatives	n	Number of criteria
K	Number of experts	X_r	r-th alternative
$\mathcal{A}_q$	q-th criterion	E_l	l-th expert
W	Criteria weight vector	V	Expert weight vector
$\mathcal{M}^{(l)}$	Decision matrix of expert l	$\mathfrak{R}^{(l)}$	DPLNC matrix of expert l
$\mathfrak{R}$	Aggregated DPLNC matrix	$\mathcal{H}_r$	DPLNC sequence for alternative r
$\mathcal{H}^+$	Positive ideal sequence	$\mathcal{H}^-$	Negative ideal sequence
$\gamma(\mathcal{H}_r, \mathcal{H}^+)$	Gray relational degree to the positive ideal	$\gamma(\mathcal{H}_r, \mathcal{H}^-)$	Gray relational degree to the negative ideal
Γ_r	Closeness degree	ζ	Distinguishing coefficient
6. Distance and metric symbols			
$d(\mathfrak{R}_{\mathcal{D}_1}, \mathfrak{R}_{\mathcal{D}_2})$		Distance between two DPLNCs	

Definition 1. *[47, 48]. Let $\mathcal{H}$ be a set of linguistic terms $h_0, h_1, ..., h_{2\rho}$ formed by $2\rho + 1$. For any two linguistic variables h_i and h_j in $\mathcal{H}$, if the following basic operations are satisfied:*

(1) $h_i \oplus h_j = \min\{h_{i+j}, h_{2\rho}\}$;
(2) $h_i \ominus h_j = \max\{h_{i-j}, h_0\}$;
(3) $h_i \otimes h_j = \min\{h_{ij}, h_{2\rho}\}$;
(4) $\mathrm{neg}(h_i) = h_{2\rho-i}$;
(5) $\lambda h_i = h_{\lambda i}, \lambda \geq 0$.

Then, $\mathcal{H} = \{h_i \mid i \in [0, 2\rho], \rho \in \mathbb{N}\}$ is a linguistic variable set (LVS).

Based on the linguistic variable h_i previously described, we now present the formal definition of a probabilistic linguistic variable.

Definition 2. *[11]. Let $\mathcal{H} = \{h_i \mid i \in [0, 2\rho], \rho \in \mathbb{N}\}$ be a linguistic variable set. The probabilistic linguistic variable set is defined as*

$$\mathcal{H}(\mu) = \{h_i(\mu_i) \mid h_i \in \mathcal{H}, i \in [0, 2\rho]\} \tag{1}$$

where each linguistic concept h_i is assigned a corresponding probability value $\mu_i \in [0, 1]$, and the probability distribution $\mu = (\mu_0, \mu_1, \ldots, \mu_{2\rho})$ satisfies the condition $\sum_{i=0}^{2\rho} \mu_i \leq 1$.

Pang et al. [11] proposed a normalized probabilistic linguistic term set (NPLTS) to address computational challenges. To adjust the numbers of elements in two different PLVSs $\mathcal{H}_1(\mu) = \{h_i^1(\mu_i^1) \mid$

$i \in [0, 2\rho_1]\}$ and $\mathcal{H}_2(\mu) = \{h_i^2(\mu_i^2) \mid i \in [0, 2\rho_2]\}$, the process involves two steps: Quantity and weight adjustment.

The shorter PLTS is extended by adding the smallest linguistic terms, while the added terms' probabilities are set to zero to preserve the original information. Through these adjustments, NPLTSs can be directly obtained.

In practical decision-making processes, the assignment of linguistic variables $h_i \in \mathcal{H}$ and probabilistic linguistic variables $h_i(\mu_i)$ is typically based on the subjective judgment of decision experts. While these assessments are valuable, their reliability is inherently tied to the confidence level of the experts providing them. To ensure the rationality and objectivity of the decision-making process, it is essential to quantify the confidence of the experts through a measure known as the probabilistic information degree. The probabilistic information degree can be represented using crisp numbers "30% impossible, 10%very unlikely, 50%unlikely, 60%maybe", "70%very unsure, 40%unsure, 80%neutral" [11]. These terms allow for the incorporation of both the fuzziness degree and the probabilistic information degree into the original decision-making framework. By embedding these degrees into the decision-making information, a more comprehensive and robust representation can be achieved. Wang et al. [17] proposed the DLTS to address limitations in existing linguistic models, such as the reliance on crisp probabilities in PLTSs, which can lead to information loss and unrealistic assessments. They aim to provide a more natural and flexible way for decision-makers to express uncertainty using linguistic probabilities (e.g., "likely," "unlikely") rather than exact numbers, as defined in Definition 3.

Definition 3. *[17] . Let $\mathcal{H} = \{h_i \mid i \in [0, 2\rho], \rho \in \mathbb{N}\}$ be a linguistic variable set and let $\mathcal{P} = \{b_j \mid j \in [0, 2\varpi], \varpi \in \mathbb{N}\}$ be a linguistic probability set formed by $2\varpi + 1$ (e.g, $\mathcal{H} = \{h_0 = \textit{very poor},\ h_1 = \textit{poor},\ h_2 = \textit{medium},\ h_3 = \textit{good},\ h_4 = \textit{very good}\}$, $\mathcal{P} = \{b_0 = \textit{impossible},\ b_1 = \textit{very unlikely},\ b_2 = \textit{unlikely},\ b_3 = \textit{maybe},\ b_4 = \textit{likely},\ b_5 = \textit{very likely},\ b_6 = \textit{certain}\}$). The DLVS can be defined as*

$$\mathcal{L} = \{(A, B) \mid A \in \mathcal{H}, B \in \mathcal{P}\} \tag{2}$$

where A is the linguistic variable associated with the linguistic probability B.

Based on the DLTS given in Definition 3 and the concept of PLTSs from Definition 2, this work introduces the novel concept of DPLV, which is presented in Definition 4.

Definition 4. *Let $\mathcal{H}(\mu) = \{h_i(\mu_i) \mid h_i \in \mathcal{H}, i \in [0, 2\rho], \mu_i \geq 0, \sum_{i=0}^{2\rho} \mu_i \leq 1\}$ be a probabilistic linguistic variable set, and $\mathcal{P}(\mu') = \{b_j(\mu'_j) \mid b_j \in \mathcal{P}, j \in [0, 2\varpi], \mu'_j \geq 0, \sum_{j=0}^{2\varpi} \mu'_j \leq 1\}$ be a probabilistic linguistic possible set. The DPLV set is defined as*

$$\mathcal{D} = \{\langle h_i(\mu_i), b_j(\mu'_j)\rangle \mid h_i \in \mathcal{H}, b_j \in \mathcal{P}\} \tag{3}$$

where $h_i(\mu_i)$ is the probabilistic evaluation associated with the probabilistic linguistic confidence term $b_j(\mu'_j)$.

In cases where $\sum \mu_i < 1$ for $\mathcal{H}(\mu)$ or $\sum \mu'_j < 1$ for $\mathcal{P}(\mu')$, the remaining probability mass $1 - \sum \mu_i$ (or $1 - \sum \mu'_j$) represents the expert's degree of hesitation, partial ignorance, or belief that cannot be assigned to any predefined linguistic term. This models realistic scenarios where an expert's judgment is incomplete or uncertain, and does not force an artificial distribution of probability. The same interpretation applies to both the evaluation ($h_i(\mu_i)$) and confidence ($b_j(\mu'_j)$) components.

For example, given $\mathcal{H} = \{h_0 = \text{very poor},\ h_1 = \text{poor},\ h_2 = \text{medium},\ h_3 = \text{good},\ h_4 = \text{very good}\}$, $\mathcal{P} = \{b_0 = \text{very unlikely},\ b_1 = \text{unlikely},\ b_2 = \text{maybe},\ b_3 = \text{likely},\ b_4 = \text{very likely}\}$, then (30% it maybe 60% good) can be expressed as $\langle h_3(0.6), b_2(0.3)\rangle$, and (20% very unlikely to be 70% poor) can be expressed as $\langle h_1(0.7), b_0(0.2)\rangle$.

The DPLV structure is particularly valuable in complex, real-world decision scenarios where an expert's confidence in their own judgment is as crucial as the judgment itself. For instance, in medical diagnosis, a panel might assess a symptom as "highly indicative (70%) of Disease A," but a junior doctor may attach only "low confidence (40%)" to this assessment, while a senior specialist expresses "high confidence (90%)." This would be modeled as $\langle$highly indicative(0.7), low confidence(0.4)$\rangle$ and $\langle$highly indicative(0.7), high confidence(0.9)$\rangle$, respectively, a distinction that is lost in traditional models. Similarly, in financial risk assessment, an analyst might evaluate a startup as "risky (80%)" but with only "likely (70%)" confidence if relying on novel business models, expressed as $\langle$risky(0.8), likely(0.7)$\rangle$. These examples underscore how DPLVs directly capture the two-tiered uncertainty inherent in expert judgment, enabling more nuanced and reliable decision-making models.

3.1. DPLNCs

The normal cloud model, introduced by Li et al. [44], serves as a framework for addressing uncertainty by establishing a connection between qualitative conceptualizations and quantitative data. This model is particularly effective in representing the dual aspects of fuzziness and randomness that characterize human cognitive processes. Accordingly, it has found extensive applications across diverse fields, including both social and natural sciences.

Definition 5. *[44]. Consider a universe of discourse Ω and a qualitative concept T associated with Ω. An element $\xi \in \Omega$ is a random realization of T, with ξ following a normal distribution $\xi \sim N(\mathbb{E}x, \mathbb{E}n'^2)$, where $\mathbb{E}n'$ itself follows a normal distribution $\mathbb{E}n' \sim N(\mathbb{E}n, \mathbb{H}e^2)$. The certainty degree $f(\xi) \in [0, 1]$ of ξ belonging to T is a random number with a stable tendency, defined by the equation*

$$f(\xi) = exp\left\{-\frac{(\xi - \mathbb{E}x)^2}{2\mathbb{E}n'^2}\right\},$$

the distribution of ξ over Ω is referred to as a cloud, with each pair $(\xi, f(\xi))$ representing a cloud drop.

This model enables the qualitative-quantitative transformation via three parameters: Expectation $\mathbb{E}x$ (the central, most representative value), entropy $\mathbb{E}n$ (indicating concept granularity, with larger values implying broader scope), and hyper-entropy $\mathbb{H}e$ (the uncertainty of $\mathbb{E}n$, which reflects the cloud's thickness). A normal cloud is therefore formalized as $\mathfrak{R} = (\mathbb{E}x, \mathbb{E}n, \mathbb{H}e)$.

This formulation allows for a nuanced representation of concepts that inherently possess both fuzziness and randomness, making the cloud model a valuable tool in various scientific domains.

Definition 6. *[30, 49]. Let $\mathfrak{R}_i = (\mathbb{E}x_i, \mathbb{E}n_i, \mathbb{H}e_i),\ i = 1, 2$ represent two normal clouds. The primary operators are defined as follows:*

(1) $\mathfrak{R}_1 + \mathfrak{R}_2 = \left(\mathbb{E}x_1 + \mathbb{E}x_2,\ \sqrt{\mathbb{E}n_1{}^2 + \mathbb{E}n_2{}^2},\ \sqrt{\mathbb{H}e_1{}^2 + \mathbb{H}e_2{}^2}\right)$.

(2) $\mathfrak{R}_1 \times \mathfrak{R}_2 = \left(\mathbb{E}x_1\mathbb{E}x_2,\ \sqrt{(\mathbb{E}n_1\mathbb{E}x_2)^2 + (\mathbb{E}n_2\mathbb{E}x_1)^2},\ \sqrt{(\mathbb{H}e_1\mathbb{E}x_2)^2 + (\mathbb{H}e_2\mathbb{E}x_1)^2}\right)$.

(3) $\lambda\mathfrak{R}_1 = \left(\lambda\mathbb{E}x_1,\ \sqrt{\lambda}\,\mathbb{E}n_1,\ \sqrt{\lambda}\,\mathbb{H}e_1\right)$.

(4) $\mathfrak{R}_1{}^{\lambda} = \left(\mathbb{E}x_1{}^{\lambda},\ \sqrt{\lambda}\,\mathbb{E}x_1{}^{\lambda-1}\mathbb{E}n_1,\ \sqrt{\lambda}\,\mathbb{E}x_1{}^{\lambda-1}\mathbb{H}e_1\right)$.

Definition 7. *[47]. Let $\mathfrak{R}_\ell, \ell = 1, 2, ..., \aleph$ be a cloud set, and $CWA_\omega : \mathfrak{R}^{\aleph} \to \mathfrak{R}$. Then the cloud weighted averaging operator is*

$$CWA_w = \sum_{\ell=1}^{\aleph} \omega_\ell \mathfrak{R}_\ell = \left(\sum_{\ell=1}^{\aleph} \omega_\ell \mathbb{E}x_\ell,\ \sqrt{\sum_{\ell=1}^{\aleph} \omega_\ell \mathbb{E}n_\ell{}^2},\ \sqrt{\sum_{\ell=1}^{\aleph} \omega_\ell \mathbb{H}e_\ell{}^2} \right) \tag{4}$$

where $\omega_\ell \in [0, 1]$, $\ell = 1, 2, ..., \aleph$, $\sum_{\ell=1}^{\aleph} \omega_\ell = 1$.

To derive the numerical characteristics of $h_i \in \mathcal{H}$, three key parameters of the cloud model $\mathfrak{R}$ are used: The expectation $\mathbb{E}x$, the entropy $\mathbb{E}n$, and the hyper-entropy $\mathbb{H}e$. These parameters collectively define the properties of the cloud and provide a quantitative representation of the linguistic variable h_i. A method for generating cloud models based on the golden section ratio [44, 47] is used to characterize h_i as a normal cloud $\mathfrak{R} = (\mathbb{E}x, \mathbb{E}n, \mathbb{H}e)$. This approach ensures a systematic and mathematically rigorous representation of linguistic variables, as defined in Definition 8.

Definition 8. *[44, 47]. Let $[\mho_{\min}, \mho_{\max}]$ be the operational domain of a linguistic variable. A five-level clouds $(\mathbb{E}x_\ell, \mathbb{E}n_\ell, \mathbb{H}e_\ell)$ for $\ell = 0, 1, 2, 3, 4$, can be created using the golden section technique.*

The digital features for $\mathbb{E}x$ are computed as follows:

$$\begin{aligned} \mathbb{E}x_2 &= \tfrac{\mho_{min}+\mho_{max}}{2}, \mathbb{E}x_0 = \mho_{\min}, \mathbb{E}x_4 = \mho_{\max} \\ \mathbb{E}x_1 &= \mathbb{E}x_2 - 0.382\tfrac{\mho_{min}+\mho_{max}}{2} = \mathbb{E}x_2 - 0.191(\mho_{min} + \mho_{max}) \\ \mathbb{E}x_3 &= \mathbb{E}x_2 + 0.382\tfrac{\mho_{min}+\mho_{max}}{2} = \mathbb{E}x_2 + 0.191(\mho_{min} + \mho_{max}). \end{aligned}$$

For the digital features $\mathbb{E}n$ are derived as follows:

$$\begin{aligned} \mathbb{E}n_1 &= \mathbb{E}n_3 = 0.382\tfrac{\mho_{max}-\mho_{min}}{2} = \tfrac{191}{3000}(\mho_{max} - \mho_{min}) \\ \mathbb{E}n_2 &= 0.618\mathbb{E}n_3, \mathbb{E}n_0 = \mathbb{E}n_4 = \tfrac{\mathbb{E}n_3}{0.618} = \tfrac{500}{309}\mathbb{E}n_3. \end{aligned}$$

Lastly, the values of $\mathbb{H}e$ are computed as follows:

$$\mathbb{H}e_1 = \mathbb{H}e_3 = \tfrac{\mathbb{H}e_2}{0.618} = \tfrac{500}{309}\mathbb{H}e_2, \mathbb{H}e_0 = \mathbb{H}e_4 = \tfrac{\mathbb{H}e_3}{0.618} = \tfrac{500}{309}\mathbb{H}e_3.$$

As outlined in Definition 4, a DPLV, denoted $\mathcal{D} = \langle h_i(\mu_i), b_j(\mu'_j)\rangle$, is constructed from two constituent probabilistic linguistic variables: $h_i(\mu_i)$ and $b_j(\mu'_j)$. Using the transformation methodology provided in Definition 8, both linguistic variables h_i and b_j can be converted into their corresponding normal cloud representations.

Note 1. *The golden section method described here generates a standard five-level cloud, corresponding to a typical five-point linguistic scale. The procedure can be generalized to construct n-level clouds for linguistic sets of any odd cardinality. The core principle remains the same: Define the universe bounds $[\mho_{\min}, \mho_{\max}]$, set the central cloud $(\mathbb{E}x_{(n-1)/2}, \mathbb{E}n_{(n-1)/2}, \mathbb{H}e_{(n-1)/2})$, and then recursively apply the* *golden section ratio* (0.618 : 0.382) *to partition the intervals on both sides of the center to determine the parameters of the remaining* $(n-1)$ *clouds.*

Mathematically, for a scale with n terms (n odd), the expectation values $\mathbb{E}x_k$ *can be generated recursively from the center outward. Let the central index be* $c = (n-1)/2$ *with* $\mathbb{E}x_c = (\mho_{\min} + \mho_{\max})/2$. *Then, for* $k = 1, \ldots, c$*, the expectations of symmetric pairs are obtained as follows:*

$$\mathbb{E}x_{c\pm k} = \mathbb{E}x_c \pm 0.618^k \cdot \frac{\mho_{\max} - \mho_{\min}}{2}.$$

Correspondingly, entropy and hyper-entropy values are derived to ensure a monotonic and symmetric increase in uncertainty from the central (most certain) to the extreme (most vague) terms. This ensures a consistent, proportional distribution of cloud characteristics across the linguistic spectrum, preserving both cognitive consistency and the theoretical uncertainty gradient inherent in linguistic variables.

To capture both qualitative meaning and quantitative uncertainty in decision-making, the concept of probabilistic linguistic clouds, defined as follows.

Definition 9. *Let* Ω *represent a finite numerical universe. The probabilistic linguistic cloud* $\mathcal{H}$ *is a qualitative term defined on* Ω*, parameterized by three numerical characteristics, expectation* $(\mathbb{E}x(\mu))$*, entropy* $(\mathbb{E}n)$*, and hyper-entropy* $(\mathbb{H}e)$*, which is expressed as* $\mathscr{A}(\mathbb{E}x(\mu), \mathbb{E}n, \mathbb{H}e)$.

The DPLNC integrates both the evaluation $(h_i(\mu_i))$ and its confidence $(b_j(\mu'_j))$ into a unified cloud model, capturing the fuzziness and randomness inherent in human judgment. The expectation $\mathbb{E}x$ represents the weighted central tendency, while $\mathbb{E}n$ and $\mathbb{H}e$ quantify the scope of uncertainty and its variability, respectively. The DPLNC, represented as $\mathfrak{R}_{\mathcal{D}} = (\widetilde{\mathbb{E}x}, \widetilde{\mathbb{E}n}, \widetilde{\mathbb{H}e})$, is formally defined in Definition 10.

Definition 10. *Let* $\mathcal{D} = \langle h_i(\mu_i), b_j(\mu'_j)\rangle$ *be a DPLV, where* $h_i(\mu_i)$ *is a probabilistic linguistic variable from the set*

$$\mathcal{H}(\mu) = \{h_i(\mu_i) \mid h_i \in \mathcal{H}, i \in [0, 2\rho], \mu_i \geq 0, \sum_{i=0}^{2\rho} \mu_i \leq 1\},$$

representing an evaluation (e.g., "good" with probability 0.6),

and $b_j(\mu_j)$ *is a probabilistic linguistic possible from the set*

$$\mathcal{P}(\mu') = \{b_j(\mu'_j) \mid b_j \in \mathcal{P}, j \in [0, 2\varpi], \mu'_j \geq 0, \sum_{j=0}^{2\varpi} \mu'_j \leq 1\},$$

representing the confidence in the evaluation (e.g., "likely" with probability 0.7).

The DPLNC is a tuple $\mathfrak{R}_{\mathcal{D}} = (\widetilde{\mathbb{E}x}, \widetilde{\mathbb{E}n}, \widetilde{\mathbb{H}e})$*, where expectation* $\widetilde{\mathbb{E}x} = (\mathbb{E}x_{h_i} \cdot \mu_i) + \left(\mathbb{E}x_{b_j} \cdot \mu'_j\right)$*, entropy* $\widetilde{\mathbb{E}n} = \sqrt{(\mathbb{E}n_{h_i})^2 + (\mathbb{E}n_{b_j})^2}$*, and hyper-entropy* $\widetilde{\mathbb{H}e} = \sqrt{(\mathbb{H}e_{h_i})^2 + (\mathbb{H}e_{b_j})^2}$.

The $\mathbb{E}x_{h_i}$*,* $\mathbb{E}x_{b_j}$*,* $\mathbb{E}n_{h_i}$*,* $\mathbb{E}n_{b_j}$ $\mathbb{H}e_{h_i}$*, and* $\mathbb{H}e_{b_j}$ *are the expectations, entropies, and the hyper-entropies of the normal clouds for* h_i *and* b_j*, derived from their respective cloud models using the golden section method in Definition 8.*

The functional forms for synthesizing the cloud parameters from the constituent linguistic and confidence terms are derived from the properties of normal distributions and the standard operations for cloud models (Definitions 6 and 7).

First, the expectation ($\widetilde{\mathbb{E}}x$) represents the central tendency of the aggregated concept. A weighted arithmetic mean (i.e., a linear combination) is the natural and unbiased estimator for the mean of a combined distribution, preserving the centroid of the aggregated information. Second, the entropy ($\widetilde{\mathbb{E}}n$) and hyper-entropy ($\widetilde{\mathbb{H}}e$) characterize the dispersion and the uncertainty of dispersion, respectively. These are second-order moments. When combining independent normal clouds, their variances (which are proportional to $\mathbb{E}n^2$ and $\mathbb{H}e^2$) add. Therefore, the combined entropy and hyper-entropy are correctly computed as the square root of the weighted sum of squares, i.e., $\sqrt{\sum \mathbb{E}{n_i}^2}$ and $\sqrt{\sum \mathbb{H}{e_i}^2}$. This form ensures that the total uncertainty scope of the aggregated cloud reflects the combined variability of its components, following the rule of variance addition for independent random variables.

4. Aggregation operator of DPLNCs

Prior to defining the aggregation operator for DPLNCs, it is necessary to establish the operational rules governing these structures and examine their associated properties. Wang et al. [30] introduced a set of operational principles for normal clouds. Building on these principles, Definition 11 formulates the corresponding operational rules for DPLNCs.

Definition 11. *Let* $\mathfrak{R}_{\mathcal{D}_1} = (\widetilde{\mathbb{E}}x_1, \widetilde{\mathbb{E}}n_1, \widetilde{\mathbb{H}}e_1)$ *and* $\mathfrak{R}_{\mathcal{D}_2} = (\widetilde{\mathbb{E}}x_2, \widetilde{\mathbb{E}}n_2, \widetilde{\mathbb{H}}e_2)$ *be two DPLNCs. The following arithmetic operations are defined as follows:*

(1) $\mathfrak{R}_{\mathcal{D}_1} \oplus \mathfrak{R}_{\mathcal{D}_2} = \left(\widetilde{\mathbb{E}}x_1 + \widetilde{\mathbb{E}}x_2,\ \sqrt{\widetilde{\mathbb{E}}n_1^2 + \widetilde{\mathbb{E}}n_2^2},\ \sqrt{\widetilde{\mathbb{H}}e_1^2 + \widetilde{\mathbb{H}}e_2^2}\right).$

(2) $\lambda \otimes \mathfrak{R}_{\mathcal{D}_1} = \left(\lambda\widetilde{\mathbb{E}}x_1,\ \sqrt{\lambda}\,\widetilde{\mathbb{E}}n_1,\ \sqrt{\lambda}\,\widetilde{\mathbb{H}}e_1\right).$

(3) $\mathfrak{R}_{\mathcal{D}_1} \otimes \mathfrak{R}_{\mathcal{D}_2} = \left(\widetilde{\mathbb{E}}x_1\widetilde{\mathbb{E}}x_2,\ \sqrt{\left(\widetilde{\mathbb{E}}n_1\widetilde{\mathbb{E}}x_2\right)^2 + \left(\widetilde{\mathbb{E}}n_2\widetilde{\mathbb{E}}x_1\right)^2},\ \sqrt{\left(\widetilde{\mathbb{H}}e_1\widetilde{\mathbb{E}}x_2\right)^2 + \left(\widetilde{\mathbb{H}}e_2\widetilde{\mathbb{E}}x_1\right)^2}\right).$

(4) $\mathfrak{R}_{\mathcal{D}_1}^{\lambda} = \left(\mu_1\widetilde{\mathbb{E}}x_1^{\lambda},\ \sqrt{\lambda}\,\mu_1\widetilde{\mathbb{E}}x_1^{\lambda-1}\widetilde{\mathbb{E}}n_1,\ \sqrt{\lambda}\,\mu_1\widetilde{\mathbb{E}}x_1^{\lambda-1}\widetilde{\mathbb{H}}e_1\right).$

Theorem 1. *Let* $\mathfrak{R}_{\mathcal{D}_1}$*,* $\mathfrak{R}_{\mathcal{D}_2}$*, and* $\mathfrak{R}_{\mathcal{D}_3}$ *be three probabilistic linguistic clouds, and let* $\lambda > 0$*. Then the following algebraic properties hold:*

(1) $\mathfrak{R}_{\mathcal{D}_1} \oplus \mathfrak{R}_{\mathcal{D}_2} = \mathfrak{R}_{\mathcal{D}_2} \oplus \mathfrak{R}_{\mathcal{D}_1}.$
(2) $(\mathfrak{R}_{\mathcal{D}_1} \oplus \mathfrak{R}_{\mathcal{D}_2}) \oplus \mathfrak{R}_{\mathcal{D}_3} = \mathfrak{R}_{\mathcal{D}_1} \oplus (\mathfrak{R}_{\mathcal{D}_2} \oplus \mathfrak{R}_{\mathcal{D}_3}).$
(3) $\lambda(\mathfrak{R}_{\mathcal{D}_1} \oplus \mathfrak{R}_{\mathcal{D}_2}) = \lambda\mathfrak{R}_{\mathcal{D}_1} \oplus \lambda\mathfrak{R}_{\mathcal{D}_2}.$
(4) $\mathfrak{R}_{\mathcal{D}_1} \otimes \mathfrak{R}_{\mathcal{D}_2} = \mathfrak{R}_{\mathcal{D}_2} \otimes \mathfrak{R}_{\mathcal{D}_1}.$
(5) $(\mathfrak{R}_{\mathcal{D}_1} \otimes \mathfrak{R}_{\mathcal{D}_2}) \otimes \mathfrak{R}_{\mathcal{D}_3} = \mathfrak{R}_{\mathcal{D}_1} \otimes (\mathfrak{R}_{\mathcal{D}_2} \otimes \mathfrak{R}_{\mathcal{D}_3}).$

Proof. From Operation (1) in Definition 11, the following property is obtained.

$$\begin{aligned}\mathfrak{R}_{\mathcal{D}_1} \oplus \mathfrak{R}_{\mathcal{D}_2} &= \left(\widetilde{\mathbb{E}}x_1 + \widetilde{\mathbb{E}}x_2,\ \sqrt{\widetilde{\mathbb{E}}n_1^2 + \widetilde{\mathbb{E}}n_2^2},\ \sqrt{\widetilde{\mathbb{H}}e_1^2 + \widetilde{\mathbb{H}}e_2^2}\right)\\ &= \left(\widetilde{\mathbb{E}}x_2 + \widetilde{\mathbb{E}}x_1,\ \sqrt{\widetilde{\mathbb{E}}n_2^2 + \widetilde{\mathbb{E}}n_1^2},\ \sqrt{\widetilde{\mathbb{H}}e_2^2 + \widetilde{\mathbb{H}}e_1^2}\right) = \mathfrak{R}_{\mathcal{D}_2} \oplus \mathfrak{R}_{\mathcal{D}_1}.\end{aligned}$$

$$\begin{aligned}(\mathfrak{R}_{\mathcal{D}_1} \oplus \mathfrak{R}_{\mathcal{D}_2}) \oplus \mathfrak{R}_{\mathcal{D}_3} &= \left(\widetilde{\mathbb{E}}x_1 + \widetilde{\mathbb{E}}x_2,\ \sqrt{\widetilde{\mathbb{E}}n_1^2 + \widetilde{\mathbb{E}}n_2^2},\ \sqrt{\widetilde{\mathbb{H}}e_1^2 + \widetilde{\mathbb{H}}e_2^2}\right) \oplus \mathfrak{R}_{\mathcal{D}_3}\\ &= (\widetilde{\mathbb{E}}x_1 + \widetilde{\mathbb{E}}x_2 + \widetilde{\mathbb{E}}x_3,\ \sqrt{\widetilde{\mathbb{E}}n_1^2 + \widetilde{\mathbb{E}}n_2^2 + \widetilde{\mathbb{E}}n_3^2},\ \sqrt{\widetilde{\mathbb{H}}e_1^2 + \widetilde{\mathbb{H}}e_2^2 + \widetilde{\mathbb{H}}e_3^2}) = \mathfrak{R}_{\mathcal{D}_1} \oplus (\mathfrak{R}_{\mathcal{D}_2} \oplus \mathfrak{R}_{\mathcal{D}_3})\end{aligned}$$

Therefore, both (1) and (2) hold.
On the basis of Operations (1) and (2) in Definition 11, we obtain

$$\begin{aligned}\lambda\left(\mathfrak{R}_{\mathcal{D}_1}\oplus\mathfrak{R}_{\mathcal{D}_2}\right)&=\left(\lambda(\widetilde{\mathbb{E}}x_1+\widetilde{\mathbb{E}}x_2),\ \sqrt{\lambda}\sqrt{\widetilde{\mathbb{E}}n_1^2+\widetilde{\mathbb{E}}n_2^2},\ \sqrt{\lambda}\sqrt{\widetilde{\mathbb{H}}e_1^2+\widetilde{\mathbb{H}}e_2^2}\right)\\&=\left(\lambda\widetilde{\mathbb{E}}x_1+\lambda\widetilde{\mathbb{E}}x_2,\ \sqrt{(\sqrt{\lambda}\widetilde{\mathbb{E}}n_1)^2+(\sqrt{\lambda}\widetilde{\mathbb{E}}n_2)^2},\ \sqrt{(\sqrt{\lambda}\widetilde{\mathbb{H}}e_1)^2+(\sqrt{\lambda}\widetilde{\mathbb{H}}e_2)^2}\right)\\&=\left(\lambda\widetilde{\mathbb{E}}x_1,\ \sqrt{\lambda}\widetilde{\mathbb{E}}n_1,\ \sqrt{\lambda}\widetilde{\mathbb{H}}e_1\right)\oplus\left(\lambda\widetilde{\mathbb{E}}x_2,\ \sqrt{\lambda}\widetilde{\mathbb{E}}n_2,\ \sqrt{\lambda}\widetilde{\mathbb{H}}e_2\right)=\lambda\mathfrak{R}_{\mathcal{D}_1}\oplus\lambda\mathfrak{R}_{\mathcal{D}_2}\end{aligned}$$

that is, the equality $\lambda\left(\mathfrak{R}_{\mathcal{D}_1}\oplus\mathfrak{R}_{\mathcal{D}_2}\right)=\lambda\mathfrak{R}_{\mathcal{D}_1}\oplus\lambda\mathfrak{R}_{\mathcal{D}_2}$ holds.
Property (4) follows from Operation (3) in Definition 11.

$$\begin{aligned}\mathfrak{R}_{\mathcal{D}_1}\otimes\mathfrak{R}_{\mathcal{D}_2}&=\left(\widetilde{\mathbb{E}}x_1\widetilde{\mathbb{E}}x_2,\ \sqrt{(\widetilde{\mathbb{E}}n_1\widetilde{\mathbb{E}}x_2)^2+(\widetilde{\mathbb{E}}n_2\widetilde{\mathbb{E}}x_1)^2},\ \sqrt{(\widetilde{\mathbb{H}}e_1\widetilde{\mathbb{E}}x_2)^2+(\widetilde{\mathbb{H}}e_2\widetilde{\mathbb{E}}x_1)^2}\right)\\&=\left(\widetilde{\mathbb{E}}x_2\widetilde{\mathbb{E}}x_1,\ \sqrt{(\widetilde{\mathbb{E}}n_2\widetilde{\mathbb{E}}x_1)^2+(\widetilde{\mathbb{E}}n_1\widetilde{\mathbb{E}}x_2)^2},\ \sqrt{(\widetilde{\mathbb{H}}e_2\widetilde{\mathbb{E}}x_1)^2+(\widetilde{\mathbb{H}}e_1\widetilde{\mathbb{E}}x_2)^2}\right)=\mathfrak{R}_{\mathcal{D}_2}\otimes\mathfrak{R}_{\mathcal{D}_1}\end{aligned}$$

For Property (5), from Operation (3) in Definition 11, we have

$$\begin{aligned}\left(\mathfrak{R}_{\mathcal{D}_1}\otimes\mathfrak{R}_{\mathcal{D}_2}\right)\otimes\mathfrak{R}_{\mathcal{D}_3}&=\left(\widetilde{\mathbb{E}}x_1\widetilde{\mathbb{E}}x_2\widetilde{\mathbb{E}}x_3,\quad\sqrt{\left(\sqrt{(\widetilde{\mathbb{E}}n_1\widetilde{\mathbb{E}}x_2)^2+(\widetilde{\mathbb{E}}n_2\widetilde{\mathbb{E}}x_1)^2}\cdot\widetilde{\mathbb{E}}x_3\right)^2+(\widetilde{\mathbb{E}}n_3\widetilde{\mathbb{E}}x_1\widetilde{\mathbb{E}}x_2)^2},\right.\\&\qquad\left.\sqrt{\left(\sqrt{(\widetilde{\mathbb{H}}e_1\widetilde{\mathbb{E}}x_2)^2+(\widetilde{\mathbb{H}}e_2\widetilde{\mathbb{E}}x_1)^2}\cdot\widetilde{\mathbb{E}}x_3\right)^2+(\widetilde{\mathbb{H}}e_3\widetilde{\mathbb{E}}x_1\widetilde{\mathbb{E}}x_2)^2}\right)\\&=\left(\widetilde{\mathbb{E}}x_1\widetilde{\mathbb{E}}x_2\widetilde{\mathbb{E}}x_3,\quad\sqrt{(\widetilde{\mathbb{E}}n_1\widetilde{\mathbb{E}}x_2\widetilde{\mathbb{E}}x_3)^2+(\widetilde{\mathbb{E}}n_2\widetilde{\mathbb{E}}x_1\widetilde{\mathbb{E}}x_3)^2+(\widetilde{\mathbb{E}}n_3\widetilde{\mathbb{E}}x_1\widetilde{\mathbb{E}}x_2)^2},\right.\\&\qquad\left.\sqrt{(\widetilde{\mathbb{H}}e_1\widetilde{\mathbb{E}}x_2\widetilde{\mathbb{E}}x_3)^2+(\widetilde{\mathbb{H}}e_2\widetilde{\mathbb{E}}x_1\widetilde{\mathbb{E}}x_3)^2+(\widetilde{\mathbb{H}}e_3\widetilde{\mathbb{E}}x_1\widetilde{\mathbb{E}}x_2)^2}\right)\end{aligned}$$

$$\begin{aligned}\mathfrak{R}_{\mathcal{D}_1}\otimes\left(\mathfrak{R}_{\mathcal{D}_2}\otimes\mathfrak{R}_{\mathcal{D}_3}\right)&=\left(\widetilde{\mathbb{E}}x_1\widetilde{\mathbb{E}}x_2\widetilde{\mathbb{E}}x_3,\quad\sqrt{(\widetilde{\mathbb{E}}n_1\widetilde{\mathbb{E}}x_2\widetilde{\mathbb{E}}x_3)^2+\left(\sqrt{(\widetilde{\mathbb{E}}n_2\widetilde{\mathbb{E}}x_3)^2+(\widetilde{\mathbb{E}}n_3\widetilde{\mathbb{E}}x_2)^2}\cdot\widetilde{\mathbb{E}}x_1\right)^2},\right.\\&\qquad\left.\sqrt{(\widetilde{\mathbb{H}}e_1\widetilde{\mathbb{E}}x_2\widetilde{\mathbb{E}}x_3)^2+\left(\sqrt{(\widetilde{\mathbb{H}}e_2\widetilde{\mathbb{E}}x_3)^2+(\widetilde{\mathbb{H}}e_3\widetilde{\mathbb{E}}x_2)^2}\cdot\widetilde{\mathbb{E}}x_1\right)^2}\right)\\&=\left(\widetilde{\mathbb{E}}x_1\widetilde{\mathbb{E}}x_2\widetilde{\mathbb{E}}x_3,\quad\sqrt{(\widetilde{\mathbb{E}}n_1\widetilde{\mathbb{E}}x_2\widetilde{\mathbb{E}}x_3)^2+(\widetilde{\mathbb{E}}n_2\widetilde{\mathbb{E}}x_1\widetilde{\mathbb{E}}x_3)^2+(\widetilde{\mathbb{E}}n_3\widetilde{\mathbb{E}}x_1\widetilde{\mathbb{E}}x_2)^2},\right.\\&\qquad\left.\sqrt{(\widetilde{\mathbb{H}}e_1\widetilde{\mathbb{E}}x_2\widetilde{\mathbb{E}}x_3)^2+(\widetilde{\mathbb{H}}e_2\widetilde{\mathbb{E}}x_1\widetilde{\mathbb{E}}x_3)^2+(\widetilde{\mathbb{H}}e_3\widetilde{\mathbb{E}}x_1\widetilde{\mathbb{E}}x_2)^2}\right).\end{aligned}$$

Thus, Property (5) holds. □.

Given the operational rules for DPLNCs (Definition 11) and the properties established in Theorem 1, Definition 12 introduces a novel weighted averaging operator $\mathfrak{R}_{\mathcal{D}_{agg}}$ for aggregating multiple DPLNCs.

Definition 12. *Let* $\Re_{\mathcal{D}_1}, \Re_{\mathcal{D}_2}, ..., \Re_{\mathcal{D}_\aleph}$ *be* $\aleph$ *DPLNCs, where* $\Re_{\mathcal{D}_\ell} = (\widetilde{\mathbb{E}x}_\ell, \widetilde{\mathbb{E}n}_\ell, \widetilde{\mathbb{H}e}_\ell)$, $\ell = 1, 2, ..., \aleph$, $\omega_\ell \in [0, 1]$, $\sum_{\ell=1}^{\aleph} \omega_\ell = 1$. *Then the weighted averaging operator* $\Re_{\mathcal{D}_{agg}}$ *is defined as follows:*

$$\Re_{\mathcal{D}_{agg}} = \sum_{\ell=1}^{\aleph} \omega_\ell \Re_{\mathcal{D}_\ell}. \tag{5}$$

To aggregate multiple DPLNCs, we integrate Definition 11, Definition 12, and Theorem 1, as shown in Theorem 2.

Theorem 2. *Let* $\Re_{\mathcal{D}_1}, \Re_{\mathcal{D}_2}, ..., \Re_{\mathcal{D}_\aleph}$ *be* $\aleph$ *DPLNCs, where* $\Re_{\mathcal{D}_\ell} = (\widetilde{\mathbb{E}x}_\ell, \widetilde{\mathbb{E}n}_\ell, \widetilde{\mathbb{H}e}_\ell)$, $\ell = 1, 2, ..., \aleph$, $\omega_\ell \in [0, 1]$, $\ell = 1, 2, ..., \aleph$, *and* $\sum_{\ell=1}^{\aleph} \omega_\ell = 1$. *Then the aggregated value of* $\Re_{\mathcal{D}_1}, \Re_{\mathcal{D}_2}, ..., \Re_{\mathcal{D}_\aleph}$ *using the* $\Re_{\mathcal{D}_{agg}}$ *operator is a DPLNC, which can be expressed as follows:*

$$\Re_{\mathcal{D}_{agg}} = \left(\sum_{\ell=1}^{\aleph} \omega_\ell \widetilde{\mathbb{E}x}_\ell,\ \sqrt{\sum_{\ell=1}^{\aleph} \omega_\ell \widetilde{\mathbb{E}n}_\ell^2},\ \sqrt{\sum_{\ell=1}^{\aleph} \omega_\ell \widetilde{\mathbb{H}e}_\ell^2} \right). \tag{6}$$

Proof. From Eq (5), along with Operational Principles (1) and (2) in Definition 11, it follows that

$$\begin{aligned}
\Re_{\mathcal{D}_{agg}} = \sum_{\ell=1}^{\aleph} \omega_\ell \Re_{\mathcal{D}_\ell} &= \sum_{\ell} \omega_\ell \left(\widetilde{\mathbb{E}x}_\ell, \widetilde{\mathbb{E}n}_\ell, \widetilde{\mathbb{H}e}_\ell \right) \\
&= \left(\omega_1 \widetilde{\mathbb{E}x}_1,\ \sqrt{\omega_1} \widetilde{\mathbb{E}n}_1,\ \sqrt{\omega_1} \widetilde{\mathbb{H}e}_1 \right) + \left(\omega_2 \widetilde{\mathbb{E}x}_2,\ \sqrt{\omega_2} \widetilde{\mathbb{E}n}_2,\ \sqrt{\omega_2} \widetilde{\mathbb{H}e}_2 \right) \\
&+ ... + \left(\omega_\aleph \widetilde{\mathbb{E}x}_\aleph,\ \sqrt{\omega_\aleph} \widetilde{\mathbb{E}n}_\aleph,\ \sqrt{\omega_\aleph} \widetilde{\mathbb{H}e}_\aleph \right) \\
&= \left(\begin{array}{l} \omega_1 \widetilde{\mathbb{E}x}_1 + \omega_2 \widetilde{\mathbb{E}x}_2 + ... + \omega_\aleph \widetilde{\mathbb{E}x}_\aleph,\ \sqrt{\omega_1 \widetilde{\mathbb{E}n}_1^2 + \omega_2 \widetilde{\mathbb{E}n}_2^2 + ... + \omega_\aleph \widetilde{\mathbb{E}n}_\aleph^2}, \\ \sqrt{\omega_1 \widetilde{\mathbb{H}e}_1^2 + \omega_2 \widetilde{\mathbb{H}e}_2^2 + ... + \omega_\aleph \widetilde{\mathbb{H}e}_\aleph^2} \end{array} \right) \\
&= \left(\sum_{\ell=1}^{\aleph} \omega_\ell \widetilde{\mathbb{E}x}_\ell,\ \sqrt{\sum_{\ell=1}^{\aleph} \omega_\ell \widetilde{\mathbb{E}n}_\ell^2},\ \sqrt{\sum_{\ell=1}^{\aleph} \omega_\ell \widetilde{\mathbb{H}e}_\ell^2} \right).
\end{aligned}$$

This concludes the proof. □.

5. Dual probabilistic linguistic normal cloud-gray relational degree

Based on the concept of DPLNCs introduced in Definition 10, a novel approach to measure the distance between any two DPLNCs is proposed, leading to the formulation of a DPLNC-GRD. Our objective is to develop a MCDM method using DPLNC-GRD to assess and rank the alternatives under investigation. Specifically, leveraging the fuzzy distance formula for normal clouds proposed by Wang et al. [45], a novel definition for calculating the distance between any two DPLNCs is introduced in Definition 13.

Definition 13. *Let* $\Re_{\mathcal{D}_1} = (\widetilde{\mathbb{E}x}_1, \widetilde{\mathbb{E}n}_1, \widetilde{\mathbb{H}e}_1)$ *and* $\Re_{\mathcal{D}_2} = (\widetilde{\mathbb{E}x}_2, \widetilde{\mathbb{E}n}_2, \widetilde{\mathbb{H}e}_2)$ *be two DPLNCs. The distance between* $\Re_{\mathcal{D}_1}$ *and* $\Re_{\mathcal{D}_2}$ *is defined as follows:*

$$d(\Re_{\mathcal{D}_1}, \Re_{\mathcal{D}_2}) = \left| \left(1 - \frac{\widetilde{\mathbb{E}n}_1 + \widetilde{\mathbb{H}e}_1}{\widetilde{\mathbb{E}x}_1} \right) \widetilde{\mathbb{E}x}_1 - \left(1 - \frac{\widetilde{\mathbb{E}n}_2 + \widetilde{\mathbb{H}e}_2}{\widetilde{\mathbb{E}x}_2} \right) \widetilde{\mathbb{E}x}_2 \right|. \tag{7}$$

The expression $\left(1-\frac{\widetilde{\mathbb{E}}n+\widetilde{\mathbb{H}}e}{\widetilde{\mathbb{E}}x}\right)\widetilde{\mathbb{E}}x$ conceptually represents the "certainty-adjusted" expectation of a DPLNC, where the total uncertainty $\widetilde{\mathbb{E}}n+\widetilde{\mathbb{H}}e$ (fuzziness and randomness) is discounted from the central value $\widetilde{\mathbb{E}}x$. Thus, the distance between two DPLNCs is the absolute difference of their certainty-adjusted expectations.

Subsequently, the properties of the distance measure introduced in Definition 13 are formally established in Theorem 3.

Theorem 3. *For any three DPLNCs $\Re_{\mathcal{D}_1}$, $\Re_{\mathcal{D}_2}$, and $\Re_{\mathcal{D}_3}$, the distance measure established in Definition 13 satisfies the following fundamental metric axioms:*

(1) $d\left(\Re_{\mathcal{D}_1},\Re_{\mathcal{D}_2}\right)\geq 0$ *(non-negativity)*;
(2) $d\left(\Re_{\mathcal{D}_1},\Re_{\mathcal{D}_2}\right)=d\left(\Re_{\mathcal{D}_2},\Re_{\mathcal{D}_1}\right)$ *(symmetry)*;
(3) $d\left(\Re_{\mathcal{D}_1},\Re_{\mathcal{D}_3}\right)\leq d\left(\Re_{\mathcal{D}_1},\Re_{\mathcal{D}_2}\right)+d\left(\Re_{\mathcal{D}_2},\Re_{\mathcal{D}_3}\right)$ *(triangle inequality).*

Proof. Axioms (1) and (2) are obviously valid. Therefore, it is sufficient to prove Axiom (3). The proof proceeds as follows.

By using the triangle inequality for absolute values, we obtain

$$
\begin{aligned}
d(\Re_{\mathcal{D}_1},\Re_{\mathcal{D}_2})&=\left|\left(1-\frac{\widetilde{\mathbb{E}}n_1+\widetilde{\mathbb{H}}e_1}{\widetilde{\mathbb{E}}x_1}\right)\widetilde{\mathbb{E}}x_1-\left(1-\frac{\widetilde{\mathbb{E}}n_2+\widetilde{\mathbb{H}}e_2}{\widetilde{\mathbb{E}}x_2}\right)\widetilde{\mathbb{E}}x_2\right|,\\
d(\Re_{\mathcal{D}_2},\Re_{\mathcal{D}_3})&=\left|\left(1-\frac{\widetilde{\mathbb{E}}n_2+\widetilde{\mathbb{H}}e_2}{\widetilde{\mathbb{E}}x_2}\right)\widetilde{\mathbb{E}}x_2-\left(1-\frac{\widetilde{\mathbb{E}}n_3+\widetilde{\mathbb{H}}e_3}{\widetilde{\mathbb{E}}x_3}\right)\widetilde{\mathbb{E}}x_3\right|,\\
d(\Re_{\mathcal{D}_1},\Re_{\mathcal{D}_3})&=\left|\left(1-\frac{\widetilde{\mathbb{E}}n_1+\widetilde{\mathbb{H}}e_1}{\widetilde{\mathbb{E}}x_1}\right)\widetilde{\mathbb{E}}x_1-\left(1-\frac{\widetilde{\mathbb{E}}n_3+\widetilde{\mathbb{H}}e_3}{\widetilde{\mathbb{E}}x_3}\right)\widetilde{\mathbb{E}}x_3\right|.
\end{aligned}
$$

Then

$$
\begin{aligned}
d(\Re_{\mathcal{D}_1},\Re_{\mathcal{D}_3})&=\left|\left(1-\frac{\widetilde{\mathbb{E}}n_1+\widetilde{\mathbb{H}}e_1}{\widetilde{\mathbb{E}}x_1}\right)\widetilde{\mathbb{E}}x_1-\left(1-\frac{\widetilde{\mathbb{E}}n_3+\widetilde{\mathbb{H}}e_3}{\widetilde{\mathbb{E}}x_3}\right)\widetilde{\mathbb{E}}x_3\right|\\
&=\left|\left(\left(1-\frac{\widetilde{\mathbb{E}}n_1+\widetilde{\mathbb{H}}e_1}{\widetilde{\mathbb{E}}x_1}\right)\widetilde{\mathbb{E}}x_1-\left(1-\frac{\widetilde{\mathbb{E}}n_2+\widetilde{\mathbb{H}}e_2}{\widetilde{\mathbb{E}}x_2}\right)\widetilde{\mathbb{E}}x_2\right)\right.\\
&\quad\left.+\left(\left(1-\frac{\widetilde{\mathbb{E}}n_2+\widetilde{\mathbb{H}}e_2}{\widetilde{\mathbb{E}}x_2}\right)\widetilde{\mathbb{E}}x_2-\left(1-\frac{\widetilde{\mathbb{E}}n_3+\widetilde{\mathbb{H}}e_3}{\widetilde{\mathbb{E}}x_3}\right)\widetilde{\mathbb{E}}x_3\right)\right|\\
&\leq\left|\left(1-\frac{\widetilde{\mathbb{E}}n_1+\widetilde{\mathbb{H}}e_1}{\widetilde{\mathbb{E}}x_1}\right)\widetilde{\mathbb{E}}x_1-\left(1-\frac{\widetilde{\mathbb{E}}n_2+\widetilde{\mathbb{H}}e_2}{\widetilde{\mathbb{E}}x_2}\right)\widetilde{\mathbb{E}}x_2\right|\\
&\quad+\left|\left(1-\frac{\widetilde{\mathbb{E}}n_2+\widetilde{\mathbb{H}}e_2}{\widetilde{\mathbb{E}}x_2}\right)\widetilde{\mathbb{E}}x_2-\left(1-\frac{\widetilde{\mathbb{E}}n_3+\widetilde{\mathbb{H}}e_3}{\widetilde{\mathbb{E}}x_3}\right)\widetilde{\mathbb{E}}x_3\right|\\
&=d(\Re_{\mathcal{D}_1},\Re_{\mathcal{D}_2})+d(\Re_{\mathcal{D}_2},\Re_{\mathcal{D}_3}).
\end{aligned}
$$

Thus, the triangle inequality holds for the distance measure given in Definition 13. □.

Thereafter, given the distance measure for DPLNCs introduced in Eq (7), this study extends the conventional GRD [3, 50–52] to formulate the DPLNC-GRD. Prior to this extension, a foundational definition is presented.

Definition 14. *Let* $\mathcal{H} = \left(\Re_{\mathcal{D}_q}\right)_{q=1}^{n}$ *be the DPLNC sequence, and* $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_m$ *be m dual probabilistic linguistic cloud sequences, where each sequence* $\mathcal{H}_r = (\Re_{\mathcal{D}_{r1}}, \Re_{\mathcal{D}_{r2}}, \ldots, \Re_{\mathcal{D}_{rn}})$ *for* $r = 1, 2, \ldots, m$.
I. The positive ideal solution *is defined as*

$$\mathcal{H}^+ = \left(\Re_{\mathcal{D}_1^+}, \Re_{\mathcal{D}_2^+}, \ldots, \Re_{\mathcal{D}_n^+}\right) \tag{8}$$

where each component cloud $\Re_{\mathcal{D}_q^+} = \left(\widetilde{\mathbb{E}x}_q^+, \widetilde{\mathbb{E}n}_q^+, \widetilde{\mathbb{H}e}_q^+\right)$ *is determined by*

$$\text{For benefit criteria:} \quad \widetilde{\mathbb{E}x}_q^+ = \max_r \widetilde{\mathbb{E}x}_{rq}, \quad \widetilde{\mathbb{E}n}_q^+ = \min_r \widetilde{\mathbb{E}n}_{rq}, \quad \widetilde{\mathbb{H}e}_q^+ = \min_r \widetilde{\mathbb{H}e}_{rq}.$$

$$\text{For cost criteria:} \quad \widetilde{\mathbb{E}x}_q^+ = \min_r \widetilde{\mathbb{E}x}_{rq}, \quad \widetilde{\mathbb{E}n}_q^+ = \max_r \widetilde{\mathbb{E}n}_{rq}, \quad \widetilde{\mathbb{H}e}_q^+ = \max_r \widetilde{\mathbb{H}e}_{rq}.$$

II. The negative ideal solution *is defined as*

$$\mathcal{H}^- = \left(\Re_{\mathcal{D}_1^-}, \Re_{\mathcal{D}_2^-}, \ldots, \Re_{\mathcal{D}_n^-}\right) \tag{9}$$

where each component cloud $\Re_{\mathcal{D}_q^-} = \left(\widetilde{\mathbb{E}x}_q^-, \widetilde{\mathbb{E}n}_q^-, \widetilde{\mathbb{H}e}_q^-\right)$ *is determined by the following.*

$$\text{For benefit criteria:} \quad \widetilde{\mathbb{E}x}_q^- = \min_r \widetilde{\mathbb{E}x}_{rq}, \quad \widetilde{\mathbb{E}n}_q^- = \max_r \widetilde{\mathbb{E}n}_{rq}, \quad \widetilde{\mathbb{H}e}_q^- = \max_r \widetilde{\mathbb{H}e}_{rq}.$$

$$\text{For cost criteria:} \quad \widetilde{\mathbb{E}x}_q^- = \max_r \widetilde{\mathbb{E}x}_{rq}, \quad \widetilde{\mathbb{E}n}_q^- = \min_r \widetilde{\mathbb{E}n}_{rq}, \quad \widetilde{\mathbb{H}e}_q^- = \min_r \widetilde{\mathbb{H}e}_{rq}.$$

Consequently, $\mathcal{H}^+$ and $\mathcal{H}^-$ represent the positive and negative ideal normal clouds, respectively. In the proposed ranking method based on the DPLNC-GRD, $\mathcal{H}^+$ and $\mathcal{H}^-$ (defined in Definition 14) serve as the reference sequences, while $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_m$ constitute the m compared sequences. Unlike the classical GRD, where the reference and compared sequences are composed of real numbers, the sequences in this scope are constructed using DPLNCs. Consequently, the traditional GRD must be generalized to fashion DPLNC-based sequences. To address this, a novel extended GRD termed DPLNC-GRD is introduced in Definition 15.

Definition 15. *Let* $\mathcal{H}^+ = (\Re_{\mathcal{D}_1^+}, \Re_{\mathcal{D}_2^+}, \ldots, \Re_{\mathcal{D}_n^+})$ *be the positive ideal DPLNC sequence,* $\mathcal{H}^- = (\Re_{\mathcal{D}_1^-}, \Re_{\mathcal{D}_2^-}, \ldots, \Re_{\mathcal{D}_n^-})$ *be the negative ideal sequence, and* $\mathcal{H}_r = (\Re_{\mathcal{D}_{r1}}, \Re_{\mathcal{D}_{r2}}, \ldots, \Re_{\mathcal{D}_{rn}}), r = 1, 2, \ldots, m$ *be the compared sequences. The* ***DPLNC-GRD*** *is defined as follows:*

$$\gamma(\Re_{\mathcal{D}_{rq}}, \Re_{\mathcal{D}_q^+}) = \frac{\zeta \max_r \max_q d(\Re_{\mathcal{D}_{rq}}, \Re_{\mathcal{D}_q^+})}{d(\Re_{\mathcal{D}_{rq}}, \Re_{\mathcal{D}_q^+}) + \zeta \max_r \max_q d(\Re_{\mathcal{D}_{rq}}, \Re_{\mathcal{D}_q^+})}, \tag{10}$$

$$\gamma(\Re_{\mathcal{D}_{rq}}, \Re_{\mathcal{D}_q^-}) = \frac{\zeta \max_r \max_q d(\Re_{\mathcal{D}_{rq}}, \Re_{\mathcal{D}_q^-})}{d(\Re_{\mathcal{D}_{rq}}, \Re_{\mathcal{D}_q^-}) + \zeta \max_r \max_q d(\Re_{\mathcal{D}_{rq}}, \Re_{\mathcal{D}_q^-})}, \tag{11}$$

$$\gamma(\mathcal{H}_r, \mathcal{H}^+) = \sum_{q=1}^{n} \omega_q \gamma(\Re_{\mathcal{D}_{rq}}, \Re_{\mathcal{D}_q^+}), \quad \textit{and} \quad \gamma(\mathcal{H}_r, \mathcal{H}^-) = \sum_{q=1}^{n} \omega_q \gamma(\Re_{\mathcal{D}_{rq}}, \Re_{\mathcal{D}_q^-}), \tag{12}$$

where,

$$d(\mathfrak{R}_{\mathcal{D}_{rq}}, \mathfrak{R}_{\mathcal{D}_q^+}) = \left|\left(1 - \frac{\widetilde{\mathbb{E}}n_{rq} + \widetilde{\mathbb{H}}e_{rq}}{\widetilde{\mathbb{E}}x_{rq}}\right)\widetilde{\mathbb{E}}x_{rq} - \left(1 - \frac{\widetilde{\mathbb{E}}n_q^+ + \widetilde{\mathbb{H}}e_q^+}{\widetilde{\mathbb{E}}x_q^+}\right)\widetilde{\mathbb{E}}x_q^+\right| \tag{13}$$

$$d(\mathfrak{R}_{\mathcal{D}_{rq}}, \mathfrak{R}_{\mathcal{D}_q^-}) = \left|\left(1 - \frac{\widetilde{\mathbb{E}}n_{rq} + \widetilde{\mathbb{H}}e_{rq}}{\widetilde{\mathbb{E}}x_{rq}}\right)\widetilde{\mathbb{E}}x_{rq} - \left(1 - \frac{\widetilde{\mathbb{E}}n_q^- + \widetilde{\mathbb{H}}e_q^-}{\widetilde{\mathbb{E}}x_q^-}\right)\widetilde{\mathbb{E}}x_q^-\right| \tag{14}$$

$d(\cdot,\cdot)$ *is the distance measure from Definition 13,* $\zeta \in [0,1]$ *is the distinguishing coefficient (usually* $\zeta = 0.5$*), and* $\omega_q \in [0,1], q = 1,2,...,n, \sum_{q=1}^{n} \omega_q = 1$*, represents the criteria weights.*

Note 2. *The parameter* $\zeta \in [0,1]$ *controls the influence of the maximum distance in the GRD calculation, with* $\zeta = 0.5$ *representing a conventional, balanced setting. While a formal sensitivity analysis of* ζ *is not the central focus of this study, the DPLNC-GRD measure exhibits a well-known property from gray system theory: The final ranking of alternatives is typically insensitive to moderate variations of* ζ *(e.g., within the range* 0.3 *to* 0.7*) because the coefficient scales all relational coefficients uniformly. The ordinal ranking is determined by the relative distances between sequences, not the absolute magnitude of* ζ*.*

Definition 16. *Let* $\gamma(\mathcal{H}_r, \mathcal{H}^+)$ *and* $\gamma(\mathcal{H}_r, \mathcal{H}^-)$ *be the DPLNC-GRDs between the r-th alternative sequence* $\mathcal{H}_r$ *and the positive/negative ideal DPLNC sequences* $\mathcal{H}^+$*,* $\mathcal{H}^-$*, respectively, as defined in Definition 15. The comprehensive DPLNC-GRD for ranking the alternatives is given by*

$$\Gamma_r = \frac{\gamma(\mathcal{H}_r, \mathcal{H}^+)}{\gamma(\mathcal{H}_r, \mathcal{H}^+) + \gamma(\mathcal{H}_r, \mathcal{H}^-)}, \quad r = 1,2,\ldots,m, \tag{15}$$

where $\Gamma_r \in [0,1]$ *quantifies the relative closeness of* $\mathcal{H}_r$ *to the positive ideal* $\mathcal{H}^+$*. A larger value of* Γ_r *indicates better performance of the r-th alternative.*

6. MCGDM method based on DPLNC-GRD

On the basis of the foundations in Sections 3–5, this study introduces a new MCGDM approach using DPLNC-GRD. Our method adapts TOPSIS's ideal solution approach but uses a different evaluation process. TOPSIS ranks alternatives based on their geometric proximity to both positive and negative ideal solutions. TOPSIS selects the optimal alternative as the one nearest to the positive ideal solution and farthest from the negative ideal solution. Our proposed approach is fundamentally based on gray relational degree. For ranking alternatives with dual probabilistic linguistic data, we introduce a novel DPLNC-GRD approach based on gray relational degree. An alternative is optimal when it maximizes the DPLNC-GRD with the positive ideal solution and minimizes the DPLNC-GRD with the negative ideal solution. In summary, the DPLNC-GRD index quantifies similarity relationships between alternatives and the positive/negative ideal solution through relational analysis.

Let $X_r \in \mathbb{X}$ be the r-th alternative ($r = 1,2,\ldots,m$), $\mathcal{A}_q \in C$ be the q-th criterion ($q = 1,2,\ldots,n$), and E_l be the l-th decision expert ($l = 1,2,\ldots,K$). The weight vector of the n evaluation criteria is denoted $W = \omega_1, \omega_2, ..., \omega_n$, with $\omega_q \in [0,1], q = 1,2,...,n, \sum_{q=1}^{n} \omega_q = 1$, and the weight vector associated with the

K decision experts is represented as $V = (\upsilon_1, \upsilon_2, ..., \upsilon_K)$, with $\upsilon_l \in [0,1]$, $l = 1, 2, ..., K$, $\sum_{l=1}^{K} \upsilon_l = 1$. The evaluation value of X_r under $\mathscr{A}_q$ provided by E_l is represented as a DPLV

$$\mathcal{D}_{rql} = \langle h_{rql}(\mu_{rql}), b_{rql}(\mu'_{rql})\rangle,$$

where $h_{rql} \in \mathcal{H} = \{h_0, h_1, \ldots, h_{2\rho}\}$ (linguistic term set), $b_{rql} \in \mathcal{P} = \{b_0, b_1, \ldots, b_{2\varpi}\}$ (linguistic probability term set), $\mu_{rql} \in [0,1]$ and $\mu'_{rql} \in [0,1]$ are the associated probabilities, $\sum \mu_{rql} \leq 1$ and $\sum \mu'_{rql} \leq 1$. Then, the original decision-making information matrices can be denoted as

$$\mathcal{M}^{(l)} = \begin{pmatrix} \langle h_{11l}(\mu_{11l}), b_{11l}(\mu'_{11l})\rangle & \langle h_{12l}(\mu_{12l}), b_{12l}(\mu'_{12l})\rangle & \cdots & \langle h_{1nl}(\mu_{1nl}), b_{1nl}(\mu'_{1nl})\rangle \\ \langle h_{21l}(\mu_{21l}), b_{21l}(\mu'_{21l})\rangle & \langle h_{22l}(\mu_{22l}), b_{22l}(\mu'_{22l})\rangle & \cdots & \langle h_{2nl}(\mu_{2nl}), b_{2nl}(\mu'_{2nl})\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle h_{m1l}(\mu_{m1l}), b_{m1l}(\mu'_{m1l})\rangle & \langle h_{m2l}(\mu_{m2l}), b_{m2l}(\mu'_{m2l})\rangle & \cdots & \langle h_{mnl}(\mu_{mnl}), b_{mnl}(\mu'_{mnl})\rangle \end{pmatrix}, \quad l = 1, 2, \ldots, K$$

The aim is to assess and rank the m alternatives using K initial decision matrices $\mathcal{M}^{(l)} = \left(\mathcal{D}_{rql}\right)_{m\times n}$, containing DPLVs, to obtain the best alternative.

For MCGDM problems using dual probabilistic linguistic data, the evaluation process follows these steps.

Step 1: The expert committee defines the evaluation boundaries $[\mho_{\min}, \mho_{\max}]$ and initial hyper-entropy $\mathbb{H}e_0$, then generates five normal clouds to represent the linguistic term sets $h_0, h_1, ..., h_{2\rho}$ in $\mathcal{H}$, $b_0, b_1, ..., b_{2\varpi}$ in $\mathcal{P}$, using the method prescribed in Definition 8.

Step 2: Convert each DPLV $\mathcal{D}_{rql} = \langle h_{rql}(\mu_{rql}), b_{rql}(\mu'_{rql})\rangle$, in the original decision-making matrices $\mathcal{M}^{(l)} = \left(\mathcal{D}_{rql}\right)_{m\times n}$, into DPLNC matrices $\mathfrak{R}^{(l)} = \left(\mathfrak{R}_{\mathcal{D}_{rql}}\right)_{m\times n}$.

Step 3: Aggregate the K DPLNC matrices $\mathfrak{R}^{(1)}, \mathfrak{R}^{(2)}, ..., \mathfrak{R}^{(K)}$ into an aggregated DPLNC matrix $\mathfrak{R} = \left(\mathfrak{R}_{\mathcal{D}_{rq}}\right)_{m\times n}$ using the DPLNC-WA operator from Theorem 2 (Eq (6)).

Step 4: Using the aggregated decision matrix $\mathfrak{R} = \left(\mathfrak{R}_{\mathcal{D}_{rq}}\right)_{m\times n}$, generate the DPLNC sequences $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_m$ following the methodology in Definition 14.

Step 5: Identify the positive ideal cloud $\mathcal{H}^+$ and the negative ideal cloud $\mathcal{H}^-$ from the DPLNC sequences $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_m$, by applying Eqs (8) and (9) in Definition 14.

Step 6: Calculate the DPLNC-GRD values of $\gamma(\mathcal{H}_r, \mathcal{H}^+)$ and $\gamma(\mathcal{H}_r, \mathcal{H}^-)$ for each alternative using Eq (12) specified in Definition 15.

Step 7: Calculate the closeness degree Γ_r, for each alternative r ($r = 1, 2, \ldots, m$) relative to the ideal sequences using Eq (15) from Definition 16.

Step 8: Rank the m alternatives according to their corresponding values of Γ_r, $\quad r = 1, 2, \ldots, m$. A larger value of Γ_r indicates better performance for alternative r, allowing identification of the optimal choice.

The computational complexity of the proposed DPLNC-GRD method can be analyzed in terms of the numbers of alternatives (m), criteria (n), and decision experts (K). Transforming the original DPLVs into DPLNCs requires $O(Kmn)$ operations because each alternative-criterion evaluation provided by each expert must be transformed. Aggregating the K expert DPLNC matrices using the $\mathfrak{R}_{\mathcal{D}_{agg}}$ operator also requires $O(Kmn)$ operations. Constructing the positive and negative ideal DPLNC sequences, calculating the corresponding distances, and computing the DPLNC-GRD values each require $O(mn)$ operations. Finally, calculating the closeness degrees requires $O(m)$ operations, whereas ranking the m alternatives requires $O(m \log m)$ operations. Therefore, the overall computational complexity of the proposed method is $O(Kmn + m \log m)$, with $O(Kmn)$ representing the dominant computational component in typical MCGDM applications. This result indicates that the proposed method remains computationally tractable as the numbers of experts, alternatives, and criteria increase.

The complete step-by-step procedure of the proposed methodology is visually summarized in Figure 1.

7. Numerical illustration

This section presents a real-world supplier selection case study to demonstrate the proposed method's validity. Sensitivity analysis confirms its feasibility and effectiveness, while comparison with existing methods highlights its advantages.

7.1. Case description

Contemporary energy procurement strategies require the selection of reliable suppliers who can provide sustainable energy competitively and in compliance with environmental regulations. This paper illustrates this challenge through a case study in which a power generation company must select a supplier for a contract of 200 million tons of thermal coal from among four candidates. The selection is based on a framework of six criteria, derived from the core pillars of sustainable development as follows:

(i) Economic dimension: Production quality level ($\mathcal{A}_1$), Supplier service level ($\mathcal{A}_2$);

(ii) Environmental dimension: Particulate control level ($\mathcal{A}_3$), Environmental degradation level ($\mathcal{A}_4$);

(iii) Social dimension: Industrial relations track record ($\mathcal{A}_5$), Community engagement ($\mathcal{A}_6$).

The relative importance of these criteria is defined by the weight vector $W = (\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6) = (0.2, 0.2, 0.2, 0.1, 0.2, 0.1)$. These weights were determined subjectively by the expert panel through structured discussion, reflecting the specific strategic priorities of the power generation company in this high-stakes, long-term thermal coal procurement. The weighting rationale is justified by the context.

(i) Economic reliability (aggregated weight = 0.4): Criteria $\mathcal{A}_1$ (production quality) and $\mathcal{A}_2$ (supplier service) are each assigned a weight of 0.2. This reflects the fundamental business imperative to ensure a consistent, high-quality fuel supply and reliable logistical support, which are directly tied to the company's operational continuity and cost management.

(ii) Environmental compliance (aggregated weight = 0.3): Criteria $\mathcal{A}_3$ (particulate control) and $\mathcal{A}_4$ (environmental degradation) are assigned weights of 0.2 and 0.1, respectively. This prioritization reflects regulatory pressures and corporate sustainability commitments, where direct emission controls ($\mathcal{A}_3$) are often more stringently regulated and monitored than broader site rehabilitation metrics ($\mathcal{A}_4$).

(iii) Social license to operate (aggregated weight = 0.3): Criterion $\mathcal{A}_5$ (industrial relations track record) is assigned a weight of 0.2, acknowledging that labor disputes pose significant operational risks. Criterion $\mathcal{A}_6$ (community engagement) is assigned a weight of 0.1, reflecting the importance of mitigating reputational risk and aligning with the growing emphasis on social sustainability in extractive industries. This subjective weighting scheme is common in strategic supplier selection where objective data on criterion importance is scarce, and it aligns with established practices in the sustainable supply chain literature that balance economic, environmental, and social pillars with differing emphases depending on the purchasing context.

An expert panel of three senior managers, whose extensive industry experience informs their judgment, is assigned the decision weight vector $V = (v_1, v_2, v_3) = (0.4, 0.3, 0.3)$. These weights were determined by the organizing committee to calibrate their respective authority, primarily reflecting differences in their seniority, depth of direct experience in thermal coal procurement, and designated role within the selection committee. This subjective weighting approach is common in practice when objective metrics are unavailable and aligns with the acknowledged influence of experts' standing in MCGDM methodologies. The central decision problem involves identifying the optimal supplier amidst uncertainties related to (1) the present and future capabilities of the suppliers and (2) inherent variability in the assessments provided by the expert committee [47].

Suppose that the three experts provide evaluations for each supplier-criterion pair ($r = 1, 2 \ldots, m; q = 1, 2 \ldots, n$) using DPLVs $\mathcal{D}_{rql} = \langle h_{rql}(\mu_{rql}), b_{rql}(\mu'_{rql})\rangle$, where $l = 1, 2, 3$ denotes the expert. These assessments are organized into the decision matrices $\mathcal{M}^{(1)}$, $\mathcal{M}^{(2)}$, and $\mathcal{M}^{(3)}$ (see Table 3). The objective is to assess and rank the four candidate thermal coal suppliers using the assessment data in Table 3 to identify the optimal supplier.

Table 3. Dual probabilistic linguistic assessment values provided by the experts.

Expert (weight)	Criterion	X_1	X_2	X_3	X_4
E_1 (0.4)	$\mathcal{A}_1$	$\langle h_3(0.6), b_4(0.5)\rangle$	$\langle h_4(0.7), b_3(0.6)\rangle$	$\langle h_2(0.5), b_3(0.3)\rangle$	$\langle h_1(0.3), b_2(0.6)\rangle$
	$\mathcal{A}_2$	$\langle h_2(0.7), b_3(0.4)\rangle$	$\langle h_3(0.6), b_4(0.5)\rangle$	$\langle h_1(0.4), b_2(0.5)\rangle$	$\langle h_4(0.8), b_3(0.7)\rangle$
	$\mathcal{A}_3$	$\langle h_3(0.5), b_4(0.6)\rangle$	$\langle h_2(0.4), b_3(0.4)\rangle$	$\langle h_4(0.6), b_3(0.5)\rangle$	$\langle h_1(0.2), b_1(0.7)\rangle$
	$\mathcal{A}_4$	$\langle h_1(0.3), b_2(0.5)\rangle$	$\langle h_3(0.5), b_4(0.4)\rangle$	$\langle h_2(0.6), b_3(0.4)\rangle$	$\langle h_4(0.7), b_3(0.6)\rangle$
	$\mathcal{A}_5$	$\langle h_4(0.8), b_3(0.7)\rangle$	$\langle h_3(0.6), b_4(0.5)\rangle$	$\langle h_2(0.5), b_3(0.4)\rangle$	$\langle h_1(0.3), b_2(0.5)\rangle$
	$\mathcal{A}_6$	$\langle h_2(0.5), b_3(0.3)\rangle$	$\langle h_1(0.2), b_1(0.6)\rangle$	$\langle h_3(0.7), b_4(0.5)\rangle$	$\langle h_4(0.9), b_3(0.8)\rangle$
E_2 (0.3)	$\mathcal{A}_1$	$\langle h_2(0.5), b_3(0.4)\rangle$	$\langle h_3(0.6), b_4(0.5)\rangle$	$\langle h_4(0.8), b_4(0.7)\rangle$	$\langle h_1(0.2), b_1(0.7)\rangle$
	$\mathcal{A}_2$	$\langle h_3(0.6), b_4(0.5)\rangle$	$\langle h_2(0.5), b_3(0.4)\rangle$	$\langle h_1(0.3), b_2(0.5)\rangle$	$\langle h_4(0.9), b_4(0.8)\rangle$
	$\mathcal{A}_3$	$\langle h_4(0.7), b_4(0.6)\rangle$	$\langle h_3(0.5), b_4(0.5)\rangle$	$\langle h_2(0.5), b_3(0.4)\rangle$	$\langle h_1(0.3), b_2(0.5)\rangle$
	$\mathcal{A}_4$	$\langle h_2(0.4), b_3(0.3)\rangle$	$\langle h_1(0.2), b_1(0.6)\rangle$	$\langle h_3(0.6), b_4(0.5)\rangle$	$\langle h_4(0.8), b_3(0.7)\rangle$
	$\mathcal{A}_5$	$\langle h_3(0.5), b_4(0.4)\rangle$	$\langle h_4(0.8), b_3(0.7)\rangle$	$\langle h_1(0.2), b_1(0.6)\rangle$	$\langle h_2(0.6), b_3(0.5)\rangle$
	$\mathcal{A}_6$	$\langle h_1(0.3), b_2(0.5)\rangle$	$\langle h_2(0.6), b_3(0.4)\rangle$	$\langle h_3(0.5), b_4(0.4)\rangle$	$\langle h_4(0.7), b_3(0.6)\rangle$
E_3 (0.3)	$\mathcal{A}_1$	$\langle h_3(0.7), b_4(0.6)\rangle$	$\langle h_2(0.5), b_3(0.4)\rangle$	$\langle h_1(0.3), b_2(0.5)\rangle$	$\langle h_4(0.9), b_3(0.8)\rangle$
	$\mathcal{A}_2$	$\langle h_4(0.8), b_3(0.7)\rangle$	$\langle h_3(0.7), b_4(0.6)\rangle$	$\langle h_2(0.6), b_3(0.5)\rangle$	$\langle h_1(0.2), b_1(0.7)\rangle$
	$\mathcal{A}_3$	$\langle h_2(0.6), b_3(0.5)\rangle$	$\langle h_1(0.3), b_2(0.5)\rangle$	$\langle h_3(0.7), b_4(0.6)\rangle$	$\langle h_4(0.8), b_3(0.7)\rangle$
	$\mathcal{A}_4$	$\langle h_3(0.5), b_4(0.4)\rangle$	$\langle h_2(0.5), b_3(0.4)\rangle$	$\langle h_1(0.2), b_1(0.6)\rangle$	$\langle h_2(0.6), b_3(0.5)\rangle$
	$\mathcal{A}_5$	$\langle h_2(0.4), b_3(0.3)\rangle$	$\langle h_3(0.6), b_4(0.5)\rangle$	$\langle h_4(0.9), b_3(0.8)\rangle$	$\langle h_0(0.1), b_0(0.9)\rangle$
	$\mathcal{A}_6$	$\langle h_3(0.6), b_4(0.5)\rangle$	$\langle h_4(0.8), b_3(0.7)\rangle$	$\langle h_1(0.3), b_2(0.5)\rangle$	$\langle h_2(0.5), b_3(0.4)\rangle$

*Note: For the five-level linguistic evaluation scale, $\mathcal{H} = \{h_0, h_1, h_2, h_3, h_4\}$, the terms correspond to h_0 = very poor, h_1 = poor, h_2 = medium, h_3 = good, h_4 = very good. For the five-level linguistic confidence scale, $\mathcal{P} = \{b_0, b_1, b_2, b_3, b_4\}$, the terms correspond to b_0 = very unlikely, b_1 = unlikely, b_2 = maybe, b_3 = likely, b_4 = very likely.

Thus, $h_i(\mu_i)$ represents the primary linguistic evaluation with probability μ_i, while $b_j(\mu'_j)$ represents the associated linguistic confidence with probability μ'_j.

The complete decision-making framework is illustrated in Figure 1.

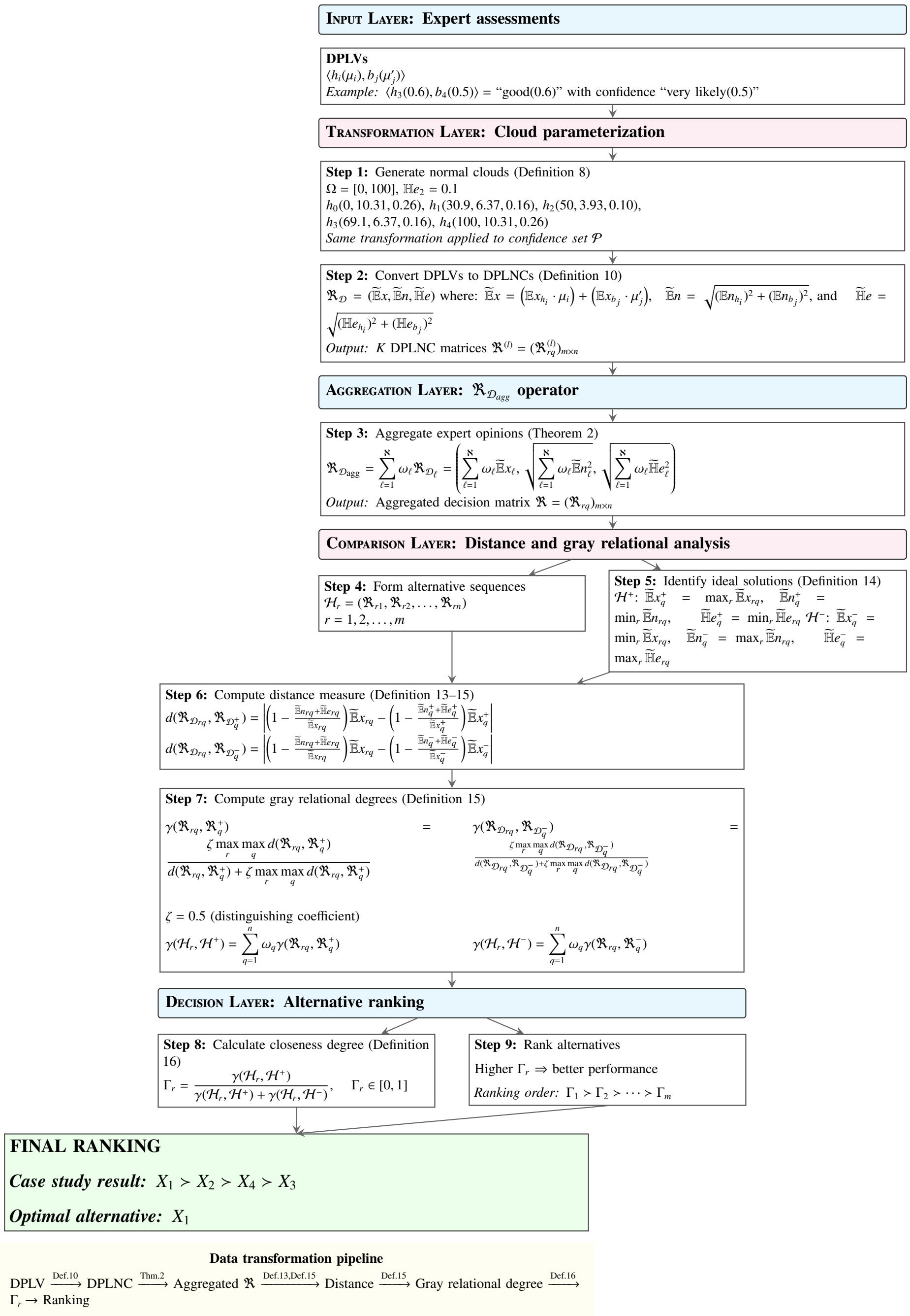

Figure 1. Detailed data transformation workflow of the proposed DPLNC-GRD methodology: From DPLVs through cloud parameterization, fusion, aggregation, and gray relational analysis, to final alternative ranking.

7.2. Decision-making process

The supplier selection methodology involves the following key steps.

(1) Normal cloud generation.

Consistent with established methods [47], we define the evaluation universe as $[\mho_{\min}, \mho_{\max}] =$

$[0, 100]$ with the hyper-entropy parameter $\mathbb{H}e_2 = 0.1$. Using the transformation method specified in Definition 8, we generate five normal clouds representing the linguistic term set h_0, h_1, h_2, h_3, h_4 in $\mathcal{H}$, with their numerical characteristics presented below:

$$\mathbb{E}x_2 = \frac{\mho_{\min} + \mho_{\max}}{2} = 50, \quad \mathbb{E}x_0 = 0, \quad \mathbb{E}x_4 = 100$$
$$\mathbb{E}x_1 = \mathbb{E}x_2 - 0.191(\mho_{\min} + \mho_{\max}) = 30.9, \quad \mathbb{E}x_3 = \mathbb{E}x_2 + 0.191(\mho_{\min} + \mho_{\max}) = 69.1$$
$$\mathbb{E}n_1 = \mathbb{E}n_3 = \frac{191}{3000}(\mho_{\max} - \mho_{\min}) = \frac{191}{30} = 6.37$$
$$\mathbb{E}n_2 = 0.618\mathbb{E}n_3 = 3.93, \quad \mathbb{E}n_0 = \mathbb{E}n_4 = \frac{\mathbb{E}n_3}{0.618} = \frac{500}{309}\mathbb{E}n_3 = 10.31$$
$$\mathbb{H}e_1 = \mathbb{H}e_3 = \frac{\mathbb{H}e_2}{0.618} = \frac{500}{309}\mathbb{H}e_2 = 0.16, \quad \mathbb{H}e_0 = \mathbb{H}e_4 = \frac{\mathbb{H}e_3}{0.618} = \frac{500}{309}\mathbb{H}e_3 = 0.26$$

A five-level clouds $(\mathbb{E}x_\ell, \mathbb{E}n_\ell, \mathbb{H}e_\ell)$ for $\ell = 0, 1, 2, 3, 4$ are $(0, 10.31, 0.26)$, $(30.9, 6.37, 0.16)$, $(50, 3.93, 0.1)$, $(69.1, 6.37, 0.16)$, and $(100, 10.31, 0.26)$.

(2) Convert the original assessment information.

The DPLVs from Table 3 are converted into their corresponding DPLNCs using the transformation method defined in Definition 10, yielding three transformed matrices presented in Table 4.

Table 4. DPLNC matrix corresponding to experts.

Expert (weight)	Criterion	X_1	X_2	X_3	X_4
E_1 (0.4)	$\mathcal{A}_1$	(91.46, 12.12, 0.31)	(111.46, 12.12, 0.31)	(45.73, 7.48, 0.19)	(39.27, 7.48, 0.19)
	$\mathcal{A}_2$	(62.64, 7.48, 0.19)	(91.46, 12.12, 0.31)	(37.36, 7.48, 0.19)	(128.37, 12.12, 0.31)
	$\mathcal{A}_3$	(94.55, 12.12, 0.31)	(47.64, 7.48, 0.19)	(94.55, 12.12, 0.31)	(27.81, 9.01, 0.23)
	$\mathcal{A}_4$	(34.27, 7.48, 0.19)	(74.55, 12.12, 0.31)	(57.64, 7.48, 0.19)	(111.46, 12.12, 0.31)
	$\mathcal{A}_5$	(128.37, 12.12, 0.31)	(91.46, 12.12, 0.31)	(52.64, 7.48, 0.19)	(34.27, 7.48, 0.19)
	$\mathcal{A}_6$	(45.73, 7.48, 0.19)	(24.72, 9.01, 0.23)	(98.37, 12.12, 0.31)	(145.28, 12.12, 0.31)
E_2 (0.3)	$\mathcal{A}_1$	(52.64, 7.48, 0.19)	(91.46, 12.12, 0.31)	(150.00, 14.58, 0.37)	(27.81, 9.01, 0.23)
	$\mathcal{A}_2$	(91.46, 12.12, 0.31)	(52.64, 7.48, 0.19)	(34.27, 7.48, 0.19)	(170.00, 14.58, 0.37)
	$\mathcal{A}_3$	(130.00, 14.58, 0.37)	(84.55, 12.12, 0.31)	(52.64, 7.48, 0.19)	(34.27, 7.48, 0.19)
	$\mathcal{A}_4$	(40.73, 7.48, 0.19)	(24.72, 9.01, 0.23)	(91.46, 12.12, 0.31)	(128.37, 12.12, 0.31)
	$\mathcal{A}_5$	(74.55, 12.12, 0.31)	(128.37, 12.12, 0.31)	(24.72, 9.01, 0.23)	(64.55, 7.48, 0.19)
	$\mathcal{A}_6$	(34.27, 7.48, 0.19)	(57.64, 7.48, 0.19)	(74.55, 12.12, 0.31)	(111.46, 12.12, 0.31)
E_3 (0.3)	$\mathcal{A}_1$	(108.37, 12.12, 0.31)	(52.64, 7.48, 0.19)	(34.27, 7.48, 0.19)	(145.28, 12.12, 0.31)
	$\mathcal{A}_2$	(128.37, 12.12, 0.31)	(108.37, 12.12, 0.31)	(64.55, 7.48, 0.19)	(27.81, 9.01, 0.23)
	$\mathcal{A}_3$	(64.55, 7.48, 0.19)	(34.27, 7.48, 0.19)	(108.37, 12.12, 0.31)	(128.37, 12.12, 0.31)
	$\mathcal{A}_4$	(74.55, 12.12, 0.31)	(52.64, 7.48, 0.19)	(24.72, 9.01, 0.23)	(64.55, 7.48, 0.19)
	$\mathcal{A}_5$	(40.73, 7.48, 0.19)	(91.46, 12.12, 0.31)	(145.28, 12.12, 0.31)	(0.00, 14.58, 0.37)
	$\mathcal{A}_6$	(91.46, 12.12, 0.31)	(128.37, 12.12, 0.31)	(34.27, 7.48, 0.19)	(52.64, 7.48, 0.19)

(3) Aggregate DPLNC information.

Using the DPLNC-WA operator of Eq (6) in Theorem 2 to aggregate the three DPLNC matrices $\mathfrak{R}^{(1)}$, $\mathfrak{R}^{(2)}$, and $\mathfrak{R}^{(3)}$ into an aggregated DPLNC matrix as shown in Table 5.

Table 5. Aggregated DPLNC matrix.

Criteria	X_1	X_2	X_3	X_4
$\mathcal{A}_1$	$(84.89, 10.94, 0.28)$	$(87.81, 10.94, 0.28)$	$(73.57, 10.15, 0.26)$	$(67.63, 9.53, 0.24)$
$\mathcal{A}_2$	$(91.00, 10.51, 0.27)$	$(84.89, 10.94, 0.28)$	$(44.59, 7.48, 0.19)$	$(110.69, 12.12, 0.31)$
$\mathcal{A}_3$	$(96.18, 11.80, 0.30)$	$(54.70, 9.12, 0.23)$	$(86.12, 10.94, 0.28)$	$(59.92, 9.66, 0.25)$
$\mathcal{A}_4$	$(48.29, 9.12, 0.23)$	$(53.03, 9.99, 0.26)$	$(57.91, 9.53, 0.24)$	$(102.46, 10.94, 0.28)$
$\mathcal{A}_5$	$(85.93, 10.94, 0.28)$	$(102.53, 12.12, 0.31)$	$(72.06, 9.53, 0.24)$	$(33.07, 10.15, 0.26)$
$\mathcal{A}_6$	$(56.01, 9.12, 0.23)$	$(65.69, 9.66, 0.25)$	$(71.99, 10.94, 0.28)$	$(107.34, 10.94, 0.28)$

(4) Identify positive and negative ideal clouds.

Table 5 provides four DPLNC sequences ($\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4$), with each column forming a distinct sequence. Using Eqs (8) and (9) in Definition 14, we derive the positive and negative ideal reference clouds ($\mathcal{H}^+$ and $\mathcal{H}^-$) as shown below.

$$\begin{aligned}\mathcal{H}^+ =&((87.81, 9.53, 0.24), (110.69, 7.48, 0.19), (96.18, 9.12, 0.23),\\ &(102.46, 9.12, 0.23), (102.53, 9.53, 0.24), (107.34, 9.12, 0.23))\end{aligned}$$

$$\begin{aligned}\mathcal{H}^- =&((67.63, 10.94, 0.28), (44.59, 12.12, 0.31), (54.70, 11.80, 0.30),\\ &(48.29, 10.94, 0.28), (33.07, 12.12, 0.31), (56.01, 10.94, 0.28))\end{aligned}$$

(5) Calculate the DPLNC-GRD values.

Using the DPLNC sequence $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4$ as comparison sequences, and the positive and negative ideal clouds references ($\mathcal{H}^+$, $\mathcal{H}^-$), we compute the DPLNC-GRD values $\gamma(\mathcal{H}_r, \mathcal{H}^+)$ and $\gamma(\mathcal{H}_r, \mathcal{H}^-)$ by using Eq (12) in Definition 15. The calculated values are given in Table 6.

Table 6. Gray relational coefficients.

(a) $\gamma(\Re_{\mathcal{D}_{rq}}, \Re_{\mathcal{D}_q^+})$

	X_1	X_2	X_3	X_4
$\mathcal{A}_1$	0.88914	0.96027	0.70198	0.63462
$\mathcal{A}_2$	0.60588	0.54425	0.34652	0.88043
$\mathcal{A}_3$	0.92725	0.45799	0.74606	0.48769
$\mathcal{A}_4$	0.39285	0.41052	0.43802	0.94935
$\mathcal{A}_5$	0.66008	0.92946	0.53495	0.33333
$\mathcal{A}_6$	0.40577	0.45366	0.48499	0.94935

(b) $\gamma(\Re_{\mathcal{D}_{rq}}, \Re_{\mathcal{D}_q^-})$

	X_1	X_2	X_3	X_4
$\mathcal{A}_1$	0.66801	0.63249	0.83727	0.95992
$\mathcal{A}_2$	0.41950	0.45554	0.87946	0.34444
$\mathcal{A}_3$	0.45571	0.92663	0.51813	0.82416
$\mathcal{A}_4$	0.94891	0.85880	0.75830	0.39066
$\mathcal{A}_5$	0.39110	0.33333	0.45470	0.94503
$\mathcal{A}_6$	0.94891	0.75962	0.68487	0.40356

Accordingly, the DPLNC-GRD values $\gamma(\mathcal{H}_r, \mathcal{H}^+)$ and $\gamma(\mathcal{H}_r, \mathcal{H}^-)$ are obtained as follows.
$\gamma(\mathcal{H}_1, \mathcal{H}^+) = 0.6963, \gamma(\mathcal{H}_2, \mathcal{H}^+) = 0.6648, \gamma(\mathcal{H}_3, \mathcal{H}^+) = 0.5582, \gamma(\mathcal{H}_4, \mathcal{H}^+) = 0.6571;$
$\gamma(\mathcal{H}_1, \mathcal{H}^-) = 0.5766, \gamma(\mathcal{H}_2, \mathcal{H}^-) = 0.6314, \gamma(\mathcal{H}_3, \mathcal{H}^-) = 0.6822, \gamma(\mathcal{H}_4, \mathcal{H}^-) = 0.6941.$

(6) Calculate the degree of closeness.

Using Eq (15) in Definition 16, the degree of closeness $\Gamma_r, \quad r = 1, 2, 3, 4$, is computed for each alternative r with respect to the best supplier.

$$\Gamma_r = \frac{\gamma(\mathcal{H}_r, \mathcal{H}^+)}{\gamma(\mathcal{H}_r, \mathcal{H}^+) + \gamma(\mathcal{H}_r, \mathcal{H}^-)}$$
$$\Gamma_1 = 0.547, \Gamma_2 = 0.513, \Gamma_3 = 0.450, \Gamma_4 = 0.486$$

(7) Rank the candidate suppliers.
The computed closeness degrees $\Gamma_1 \succ \Gamma_2 \succ \Gamma_4 \succ \Gamma_3$ yield the following supplier ranking:

$$X_1 \succ X_2 \succ X_4 \succ X_3$$

Consequently, supplier X_1 emerges as the optimal thermal coal supplier.

7.3. Sensitivity analysis

Effective MCGDM methodologies should demonstrate stability, with the ranking outcomes remaining consistent despite variations in the criterion weights [5, 53]. To validate the robustness of the proposed approach (Section 6), we conduct a sensitivity analysis to examine how weight adjustments affect the supplier rankings.

The sensitivity analysis uses optimization techniques to determine, for each criterion weight, the minimum and maximum allowable values that preserve the original supplier ranking. This process yields the permissible weight range and its corresponding interval length, ensuring the robustness of the ranking results against weight variations.

Following the methodology in Section 6, we derive the closeness degree functions $\Gamma_r, r = 1, 2, 3, 4$, as mathematical expressions dependent on the criterion weights $W = \omega_1, \omega_2, ..., \omega_6$.

$$\begin{aligned}
\Gamma_1 &= \frac{(0.889\omega_1 + 0.606\omega_2 + 0.927\omega_3 + 0.393\omega_4 + 0.660\omega_5 + 0.406\omega_6)}{\begin{array}{l}(0.889\omega_1 + 0.606\omega_2 + 0.927\omega_3 + 0.393\omega_4 + 0.660\omega_5 + 0.406\omega_6)\\ \quad + (0.668\omega_1 + 0.419\omega_2 + 0.456\omega_3 + 0.949\omega_4 + 0.391\omega_5 + 0.949\omega_6)\end{array}},\\
\Gamma_2 &= \frac{(0.960\omega_1 + 0.544\omega_2 + 0.458\omega_3 + 0.411\omega_4 + 0.929\omega_5 + 0.454\omega_6)}{\begin{array}{l}(0.960\omega_1 + 0.544\omega_2 + 0.458\omega_3 + 0.411\omega_4 + 0.929\omega_5 + 0.454\omega_6)\\ \quad + (0.632\omega_1 + 0.456\omega_2 + 0.927\omega_3 + 0.859\omega_4 + 0.333\omega_5 + 0.759\omega_6)\end{array}},\\
\Gamma_3 &= \frac{(0.702\omega_1 + 0.347\omega_2 + 0.746\omega_3 + 0.438\omega_4 + 0.535\omega_5 + 0.485\omega_6)}{\begin{array}{l}(0.702\omega_1 + 0.347\omega_2 + 0.746\omega_3 + 0.438\omega_4 + 0.535\omega_5 + 0.485\omega_6)\\ \quad + (0.837\omega_1 + 0.879\omega_2 + 0.518\omega_3 + 0.758\omega_4 + 0.455\omega_5 + 0.68487\omega_6)\end{array}},\\
\Gamma_4 &= \frac{(0.635\omega_1 + 0.880\omega_2 + 0.488\omega_3 + 0.949\omega_4 + 0.333\omega_5 + 0.949\omega_6)}{\begin{array}{l}(0.635\omega_1 + 0.880\omega_2 + 0.488\omega_3 + 0.949\omega_4 + 0.333\omega_5 + 0.949\omega_6)\\ \quad + (0.960\omega_1 + 0.344\omega_2 + 0.824\omega_3 + 0.391\omega_4 + 0.945\omega_5 + 0.404\omega_6)\end{array}}.
\end{aligned}$$

The closeness degrees for the four potential suppliers are derived from the ranking results presented in Subsection 7.2.

$$\Gamma_1 > \Gamma_2 > \Gamma_4 > \Gamma_3 \tag{16}$$

Consider the allowable variation range of the weight $\omega_q, q = 1, 2, 3, 4, 5, 6$ associated with the criteria q as $\tilde{\omega} = \left[\omega_{qL}, \omega_{qR}\right]$, which preserves the ranking order in Eq (16). Therefore, the following optimization problems must be satisfied:

$$\begin{aligned} &\min \omega_q = \omega_{qL} \\ &subject\ to \\ &\begin{cases} \Gamma_1 > \Gamma_2 > \Gamma_4 > \Gamma_3 \\ 0 \le \omega_q \le 1,\ q = 1,2,3,4,5,6 \\ \sum\limits_{q=1}^{6} \omega_q = 1 \end{cases} \end{aligned} \tag{17}$$

and

$$\begin{aligned} &\max \omega_q = \omega_{qR} \\ &subject\ to \\ &\begin{cases} \Gamma_1 > \Gamma_2 > \Gamma_4 > \Gamma_3 \\ 0 \le \omega_q \le 1,\ q = 1,2,3,4,5,6 \\ \sum\limits_{q=1}^{6} \omega_q = 1 \end{cases} \end{aligned} \tag{18}$$

Lingo ver. 21.0 is used to solve Eqs (17) and (18), resulting in the following change intervals for the six criteria weights: $\tilde{\omega}_1 = [0, 0.768]$, $\tilde{\omega}_2 = [0, 0.750]$, $\tilde{\omega}_3 = [0, 0.457]$, $\tilde{\omega}_4 = [0, 0.433]$, $\tilde{\omega}_5 = [0, 0.512]$, and $\tilde{\omega}_6 = [0, 0.419]$.

Let the length of the change interval $\tilde{\omega} = \left[\omega_{qL}, \omega_{qR}\right]$ be defined as $L_q = \omega_{qR} - \omega_{qL}, q = 1,2,3,4,5,6$. Therefore, the following values are obtained:

$$L_1 = 0.768,\ L_2 = 0.750,\ L_3 = 0.457,\ L_4 = 0.433,\ L_5 = 0.512,\ L_6 = 0.419.$$

The interval length L_q provides a direct indication of the sensitivity of the final ranking to the weight of criterion $\mathcal{A}_q$. Specifically, a shorter L_q implies that only a relatively small variation in the corresponding criterion weight can be tolerated without changing the ranking of the alternatives, indicating higher sensitivity to that criterion. Conversely, a longer L_q indicates that the criterion weight can vary over a wider range while preserving the original ranking, reflecting lower sensitivity and greater ranking stability with respect to that criterion. Therefore, criteria associated with smaller L_q values can be regarded as more influential in determining the stability of the final ranking.

According to these outcomes, for the rankings in Eq (16) to remain unchanged, the lengths $L_q, q = 1,2,3,4,5,6$ are required to satisfy the following condition:

$$L_1 > L_2 > L_5 > L_3 > L_4 > L_6.$$

The L_q ranking results indicate that weight of the sixth criterion has the narrowest allowable variation range. This suggests that $\mathcal{A}_6$ (*community engagement*) shows the highest sensitivity, meaning it exerts the strongest influence on the final ranking outcome. The subsequent criteria in order of influence are criteria $\mathcal{A}_4$ and $\mathcal{A}_3$. The weight of the first criterion permits the widest variation range, indicating that $\mathcal{A}_1$ (*production quality level*) shows the lowest sensitivity and consequently the least influence on ranking outcomes. The analysis reveals a minimum allowable weight variation of $L_6 = 0.419$ across all six criteria. Since this threshold exceeds 0.4 and most criteria maintain stable rankings even with weight fluctuations below 0.45, the proposed methodology demonstrates strong robustness against parameter perturbations. These findings confirm the approach's high stability in generating consistent ranking outcomes.

To further validate the robustness of the proposed methodology, an extended sensitivity analysis was conducted by simultaneously varying three key model parameters: The universe bounds $[\mho_{\min}, \mho_{\max}]$, the initial hyper-entropy $\mathbb{H}e_2$, and expert weights υ_l to a combined scenario where $[0, 100]$ was reduced to $[0, 80]$, $\mathbb{H}e_2$ was lowered from 0.1 to 0.05, and the expert weight vector was changed from $V = (0.4, 0.3, 0.3)$ to $V = (0.5, 0.25, 0.25)$. After regenerating the five-level normal clouds (Definition 8), transforming all DPLVs to DPLNCs, and aggregating with the $\mathfrak{R}_{\mathcal{D}_{agg}}$ operator, the resulting DPLNC matrix (Table 7) exhibited the expected proportional scaling: The expectations and entropies reduced to approximately 80% of their original values, while the hyper-entropies decreased to about 50%. Recalculation of the DPLNC-GRD and closeness degrees Γ_r yielded an identical ranking order $X_1 \succ X_2 \succ X_4 \succ X_3$ (Table 8), confirming the framework's robustness to substantial parametric perturbations. This stability reinforces the reliability of the method's recommendations and justifies conventional parameter choices such as $[0, 100]$ as a standardized scale and $\mathbb{H}e_2 = 0.1$ as a moderate meta-uncertainty setting, enabling confident application across diverse calibration scenarios without compromising decisions' consistency.

Table 7. Aggregated DPLNC matrix for the combined sensitivity scenario.

Criterion	X_1	X_2	X_3	X_4
$\mathcal{A}_1$	$(67.91, 8.75, 0.22)$	$(70.25, 8.75, 0.22)$	$(58.86, 8.12, 0.21)$	$(54.10, 7.62, 0.19)$
$\mathcal{A}_2$	$(72.80, 8.41, 0.22)$	$(67.91, 8.75, 0.22)$	$(35.67, 5.98, 0.15)$	$(88.55, 9.70, 0.25)$
$\mathcal{A}_3$	$(76.94, 9.44, 0.24)$	$(43.76, 7.30, 0.18)$	$(68.90, 8.75, 0.22)$	$(47.94, 7.73, 0.20)$
$\mathcal{A}_4$	$(38.63, 7.30, 0.18)$	$(42.42, 7.99, 0.21)$	$(46.33, 7.62, 0.19)$	$(81.97, 8.75, 0.22)$
$\mathcal{A}_5$	$(68.74, 8.75, 0.22)$	$(82.02, 9.70, 0.25)$	$(57.65, 7.62, 0.19)$	$(26.46, 8.12, 0.21)$
$\mathcal{A}_6$	$(44.81, 7.30, 0.18)$	$(52.55, 7.73, 0.20)$	$(57.59, 8.75, 0.22)$	$(85.87, 8.75, 0.22)$

Table 8. Final closeness degrees and ranking for combined sensitivity scenario.

Alternative	$\gamma(\mathcal{H}_r, \mathcal{H}^+)$	$\gamma(\mathcal{H}_r, \mathcal{H}^-)$	Γ_r	**Rank**
X_1	0.6963	0.5766	0.5470	1
X_2	0.6648	0.6314	0.5129	2
X_3	0.5582	0.6822	0.4500	4
X_4	0.6571	0.6941	0.4863	3

7.4. Comparative analysis with other methods

To provide a comprehensive evaluation of the proposed method, this section conducts two complementary comparative analyses. First, an ablation analysis is performed within the proposed methodological framework to isolate and assess the contribution of the probabilistic confidence component. Second, the proposed DPLNC-GRD method is compared with several existing MCGDM approaches to examine its methodological characteristics and ranking performance.

7.4.1. Ablation analysis of the probabilistic confidence component

To specifically examine the contribution of probabilistic confidence information and avoid confounding its effect with differences among decision-making algorithms, a controlled ablation analysis was conducted within the proposed methodological framework. Two experimental settings were considered. In the full model, each assessment is represented by the DPLV $\mathcal{D}_{rql} = \left\langle h_{rql}(\mu_{rql}), b_{rql}(\mu'_{rql}) \right\rangle$, where $h_{rql}(\mu_{rql})$ denotes the primary probabilistic linguistic evaluation and $b_{rql}(\mu'_{rql})$ represents the associated probabilistic confidence information. In the ablated setting, the confidence component $b_{rql}(\mu'_{rql})$ is removed, while the primary evaluation $h_{rql}(\mu_{rql})$ is retained.

Importantly, all other components of the decision-making pipeline are kept unchanged in the two settings, including the expert weights, criterion weights, cloud-based representation, aggregation procedure, construction of the positive and negative ideal solutions, distance measure, gray relational degree calculation, and final closeness-degree calculation. Therefore, the comparison isolates the effect of the probabilistic confidence component rather than differences between distinct decision-making methodologies.

For the evaluation-only setting, $h_{rql}(\mu_{rql})$ is transformed into a normal cloud according to

$$\Re^{E}_{rql} = \left(\mathbb{E}x_{h_{rql}(\mu_{rql})}, \mathbb{E}n_{h_{rql}}, \mathbb{H}e_{h_{rql}}\right).$$

The resulting clouds are subsequently aggregated using the same expert weights and processed through the same DPLNC-GRD pipeline used in the full model. Thus, the only experimental difference between the two settings is whether probabilistic confidence information is incorporated.

The controlled ablation analysis results are summarized in Tables 9 and 10.

Table 9. Ranking results with confidence information.

Alternative	$\gamma(\mathcal{H}_r, \mathcal{H}^+)$	$\gamma(\mathcal{H}_r, \mathcal{H}^-)$	Γ_r	**Rank**
X_1	0.6963	0.5766	0.5470	1
X_2	0.6648	0.6314	0.5129	2
X_3	0.5582	0.6822	0.4500	4
X_4	0.6571	0.6941	0.4863	3

Table 10. Ranking results without confidence information.

Alternative	$\gamma(\mathcal{H}_r, \mathcal{H}^+)$	$\gamma(\mathcal{H}_r, \mathcal{H}^-)$	Γ_r	**Rank**
X_1	0.6725	0.5699	0.5413	1
X_2	0.6493	0.6049	0.5177	2
X_3	0.5767	0.6580	0.4671	4
X_4	0.6341	0.6794	0.4828	3

As shown in Tables 9 and 10, the full DPLV-based model produces the closeness degrees $\Gamma_1 = 0.5470$, $\Gamma_2 = 0.5129$, $\Gamma_3 = 0.4500$, and $\Gamma_4 = 0.4863$, whereas the evaluation-only setting produces $\Gamma_1 = 0.5413$, $\Gamma_2 = 0.5177$, $\Gamma_3 = 0.4671$, and $\Gamma_4 = 0.4828$.

Both settings yield the same ordinal ranking: $X_1 \succ X_2 \succ X_4 \succ X_3$.

The unchanged ranking indicates that the final supplier selection is robust to the removal of the confidence component in the present case study. However, the confidence information produces noticeable changes in the closeness degrees and, more importantly, in the degree of separation among competing alternatives.

For instance, without confidence information, the difference between the two highest-ranked alternatives Γ_1 and Γ_2 is 0.0236, whereas incorporating probabilistic confidence increases this difference to 0.0341. An even more pronounced effect is observed between X_4 and X_3. Their separation increases from 0.0157 in the evaluation-only setting to 0.0363 when probabilistic confidence information is incorporated.

These findings provide a more nuanced interpretation of the role of confidence information. In the present case, the confidence component does not artificially alter the final ranking; instead, it modifies the relative strength of the assessments and increases the discrimination among alternatives. In particular, alternatives that may appear relatively close when only primary evaluations are considered become more clearly differentiated once the experts' confidence in those evaluations is incorporated.

Therefore, the value of the proposed dual-layer representation lies not necessarily in producing a different ranking in every decision problem, but in preserving additional information regarding the reliability of expert judgments. This additional layer can refine the relative closeness of alternatives and provide decision-makers with a more informative basis for distinguishing between closely competing alternatives. At the same time, the unchanged ranking in this application provides additional evidence of the robustness of the final decision.

7.4.2. Comparative analysis with existing MCGDM methods

The proposed DPLNC-GRD method is further compared with several established MCGDM approaches to examine its methodological characteristics and ranking behavior. It should be emphasized that these benchmark methods differ from the proposed framework not only in their treatment of confidence information but also in their information representations, aggregation mechanisms, and ranking procedures. Therefore, the purpose of this comparative analysis is to contextualize the proposed method relative to representative existing approaches rather than to attribute ranking differences exclusively to the confidence component. The specific contribution of probabilistic confidence information has already been isolated and examined through the controlled ablation analysis in Subsection 7.4.1.

First, in the case study presented in Subsection 7.1, expert assessments are represented as DPLVs $\mathcal{D}_{rql} = \left\langle h_{rql}(\mu_{rql}), b_{rql}(\mu'_{rql}) \right\rangle$. When the confidence component is omitted, the assessment information reduces to conventional probabilistic linguistic information. Under this simplified representation, the case study data can be processed using the method proposed in Ref. [11], denoted as **Approach 1**. This approach normalizes the probabilistic linguistic information, aggregates it using probabilistic linguistic weighted averaging (PLWA) operators, and ranks the alternatives using an extended TOPSIS framework and a score-deviation-based PLWA method. The resulting ranking is reported in Table 11. The difference between this ranking and that obtained using the proposed method reflects the combined effects of different information representations, aggregation procedures, and ranking mechanisms. Accordingly, this comparison should be interpreted as a methodological benchmark rather than as an isolated test of the confidence component.

Second, the proposed method is compared with the probabilistic linguistic TOPSIS approach in Ref. [13], denoted as **Approach 2**. This method represents assessments using PLTSs and applies a probabilistic linguistic (PL)-TOPSIS procedure for ranking alternatives. For the present case study, the DPLV assessments are converted into the information format required by this benchmark. As shown in Table 11, the resulting ranking differs from that of the proposed DPLNC-GRD method. This difference reflects the distinct information-processing mechanisms of the two approaches. In particular, the benchmark operates on conventional probabilistic linguistic information, whereas the proposed framework jointly represents probabilistic evaluation and probabilistic confidence and subsequently transforms them into DPLNCs for aggregation and gray relational analysis.

Third, the proposed method is compared with the quantum probability-based heterogeneous MCGDM approach proposed by Yan et al. [54], denoted as **Approach 3**. This method incorporates quantum probability theory and interference effects to characterize psychological interactions under incomplete probabilistic linguistic preference information. Although it provides a sophisticated mechanism for modeling behavioral and interaction effects, it does not explicitly represent the confidence associated with an evaluation as a probabilistic linguistic distribution. After adapting the case study data to the information structure required by this approach, the resulting ranking is $X_2 \succ X_1 \succ X_4 \succ X_3$. The difference from the proposed ranking illustrates that alternative uncertainty representations and aggregation mechanisms can lead to different decision outcomes.

Fourth, the probabilistic linguistic decision-making method based on PL-BWM and the improved three-way TODIM proposed by Chen and Yin [55], denoted as **Approach 4**, is considered. This method combines the BWM for criterion weighting with an improved TODIM procedure for alternative ranking under probabilistic linguistic information. Its methodological focus is primarily on behavioral characteristics, such as loss aversion, rather than on the explicit representation of two-layer uncertainty in evaluation and confidence. When applied to the present case using the information structure required by the benchmark, the resulting ranking is $X_4 \succ X_1 \succ X_2 \succ X_3$. Again, the ranking difference should be interpreted in light of the distinct information structures, weighting mechanisms, and ranking principles used by the two methods.

Fifth, the proposed method is compared with the dual-strategy consensus-reaching process proposed by Wan et al. [38], denoted as **Approach 5**. This approach combines assessment adjustment and expert-weight modification strategies to reconcile differences among experts in probabilistic linguistic environments. When applied to the transformed case study information, Approach 5 yields $X_1 \succ X_2 \succ X_4 \succ X_3$, which is consistent with the ranking obtained using the proposed DPLNC-GRD method. This agreement indicates that the two approaches produce the same ordinal ranking in the present case despite relying on different information processing mechanisms. Whereas Approach 5 emphasizes iterative consensus adjustment, the proposed framework directly incorporates probabilistic confidence into the information representation and processes the resulting DPLNC information through aggregation and gray relational analysis.

Overall, the comparative results reported in Tables 11 and 12 demonstrate that different uncertainty representations, aggregation mechanisms, and ranking procedures can produce different ordinal outcomes. These differences should not be attributed solely to the treatment of confidence information because the benchmark methods also differ in their underlying methodological structures. The principal methodological distinction of the proposed DPLNC-GRD framework is its explicit representation of both the primary probabilistic linguistic evaluation and the associated probabilistic confidence within a unified DPLNC structure. Therefore, the benchmark comparisons provide complementary evidence regarding the behavior and applicability of the proposed method relative to existing MCGDM approaches, whereas the controlled ablation analysis in Section 7.4.1 provides a direct assessment of the contribution of probabilistic confidence information.

Table 11. Ranking comparisons of different decision-making methods.

Approach	**Ranking order**
Proposed (DPLNC-GRD)	$X_1 \succ X_2 \succ X_4 \succ X_3$
Approach 1 [11]	$X_4 \succ X_2 \succ X_1 \succ X_3$
Approach 2 [13]	$X_1 \succ X_4 \succ X_2 \succ X_3$
Approach 3 [54]	$X_2 \succ X_1 \succ X_4 \succ X_3$
Approach 4 [55]	$X_4 \succ X_1 \succ X_2 \succ X_3$
Approach 5 [38]	$X_1 \succ X_2 \succ X_4 \succ X_3$

Table 12. Summary of the compared MCGDM methods.

Method	**Models confidence?**	**Primary uncertainty mechanism**	**Core ranking technique**
Proposed (DPLNC-GRD)	**Yes** (as a probabilistic distribution)	DPLNCs	GRD
Approach 1 [11]	No	PLTSs	TOPSIS and Score-Deviation
Approach 2 [13]	No	PLTSs	TOPSIS
Approach 3 [54]	No (models psychological interference)	Quantum probabilistic linguistic sets	Heterogeneous preference aggregation
Approach 4 [55]	No	PLTSs	BWM and improved TODIM
Approach 5 [38]	No (focuses on consensus)	PLTSs	Dual-strategy consensus

8. Conclusions

This research has introduced and developed a framework for MCGDM using DPLVs. The core of this work lies in the extension of probabilistic linguistic variables into a dual form, enabling a more nuanced representation of uncertain linguistic information. To operationalize this concept within a decision-making context, several key methodological contributions were made.

First, the concept of a DPLNC was defined, along with a transformation method that converts DPLVs into their cloud-based equivalents. This allows for a more effective capture of uncertainty inherent in qualitative assessments.

Second, a new aggregation operator, the DPLNC-weighted averaging ($\mathfrak{R}_{\mathcal{D}_{agg}}$) operator, was designed to synthesize evaluation information from multiple experts into a unified DPLNC.

Third, the DPLNC-GRD was proposed as a novel measure for comparing and ranking alternatives, integrating the strengths of gray system theory with the proposed cloud model.

The practicality and effectiveness of the proposed method were demonstrated through a sustainable supplier selection case study. Sensitivity analysis confirmed the robustness of the approach, while comparative analysis highlighted its advantages over several existing methods in terms of justification and reasonableness of results.

Beyond the numerical results of the case study, the proposed framework is particularly suitable for MCGDM problems involving two distinct layers of uncertainty: Uncertainty in the primary evaluation and uncertainty in the confidence associated with that evaluation. Its advantages are more evident in decision environments where experts provide probabilistic linguistic assessments with heterogeneous or uncertain confidence levels. Compared with conventional PLTS-based approaches, which primarily characterize uncertainty in the evaluation itself, and dual linguistic models that represent confidence using a single linguistic term, the proposed DPLV provides a more flexible representation when both the evaluation and the associated confidence are uncertain. Furthermore, the integration of DPLV with the normal cloud model makes the framework particularly applicable to decision problems involving both fuzziness and randomness in linguistic information. The proposed DPLNC-GRD framework is therefore especially suitable for complex expert-based group decision-making problems characterized by uncertain confidence information and closely competing alternatives, such as supplier selection, risk assessment, technology evaluation, and project prioritization.

The specific contributions of this study are summarized as follows.

(i) Generalization of probabilistic linguistic variables: We propose the DPLV as a generalized form of the probabilistic linguistic variable [11], enhancing its expressiveness and applicability in complex decision environments.
(ii) Uncertainty modeling via cloud models: The DPLNC is introduced, enabling the transformation of decision data into a form that supports effective aggregation and processing under fuzzy and uncertain conditions.
(iii) A novel aggregation operator: The $\Re_{\mathcal{D}_{agg}}$ operator is presented to combine multiple DPLNCs. It is shown to possess desirable properties such as idempotency and commutativity that ensure reliable and consistent information fusion.
(iv) A new MCGDM ranking method: The DPLNC-GRD is defined and embedded within a new MCGDM method for alternative ranking. Empirical analyses indicate that the method is both robust and capable of producing more justifiable results compared with existing approaches.

The sensitivity analysis confirms the greater robustness of the proposed framework, while the comparative analysis indicates its ability to yield more defensible results than established benchmark methods.

Future research will extend the proposed DPLV/DPLNC-GRD framework along several technically feasible directions. First, for incomplete DPLV information, probability completion mechanisms can be developed to estimate or redistribute unassigned probability mass when $\sum \mu_i < 1$ or $\sum \mu'_j < 1$. Such mechanisms may incorporate proportional redistribution, optimization-based completion, or entropy-based estimation while preserving the original preference structure and avoiding artificial distortion of expert judgments. Second, although the revised framework distinguishes benefit and cost criteria in the construction of the ideal DPLNC sequences, further research can develop more systematic criterion orientation and ideal cloud transformation schemes. In particular, normalized or monotonic transformation functions can be investigated to unify heterogeneous criterion directions while preserving the semantic and uncertainty characteristics of the DPLNC parameters. This would further improve the generality of the ideal solution construction for complex MCGDM problems involving mixed criterion types. Third, the proposed DPLNC-GRD model can be integrated with other established MCGDM techniques to construct hybrid decision-making frameworks. For example, combining DPLNC-GRD with VIKOR may enable compromise-based ranking under dual probabilistic linguistic uncertainty, while integration with the decision-making trial and evaluation laboratory (DEMATEL) method may support the identification and quantification of causal relationships among interdependent criteria. Such hybrid models would extend the applicability of the proposed framework to decision environments that require not only alternative ranking but also compromise analysis and structural evaluation of criterion interactions.

Author contributions

Abdulrahman Almandeel: Methodology, investigation, software, resources, writing–original draft; Congjun Rao: Methodology, conceptualization, supervision, formal analysis; Xiaolong Zhang: Methodology, visualization, validation, writing–review and editing.

Use of Generative-AI tools declaration

The authors declare that they did not utilize any artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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