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Research article

Service oriented dual channel supply chain system with stochastic lead time and variable demand

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Abstract: Market demand in today's global environment is highly dynamic, shaped by globalization, evolving consumer preferences, and shifting priorities. To address these challenges, firms increasingly adopt dual-channel supply chain systems that integrate both online and offline channels. The integration of sales data across these channels provides valuable insights into consumer behavior, enabling effective resource allocation during peak seasons and periods of demand uncertainty. In this study, an integrated dual-channel supply chain model is developed under a stochastic lead time and variable demand, incorporating investment in lead time reduction and setup costs reduction. Demand is considered as a function of the selling price, the advertisement cost, and the consumer service cost for both channels. The model allows shortages and treats the reorder point as a decision variable, jointly determining the optimal selling price, order quantity, reorder point, lead time, machine failure rate, and cost allocations. A global solution is obtained analytically, and an iterative algorithm provides optimal numerical results. The analytical results show that dual-channel facilities significantly boost profitability. Profit decreases by 62.90% and 43.18% when the offline and online facilities are absent, respectively. Consumer service facilities are also critical, as profit declines by 10.99% without them. Similarly, profit decreases by 7.65% in the absence of advertisement facilities and by 3.70% when setup costs reduction is not considered. The findings also indicate that the absence of free delivery reduces profit by 2.25%.

Keywords: supply chain management; dual-channel system; consumer service; selling price; advertisement; stochastic lead time
Mathematics Subject Classification: 90B06, 90B05

1. Introduction

In today's highly competitive and fast-paced retail landscape, the garment industry is undergoing a trans formative change driven by digitization, evolving consumer preferences, and increased market volatility. Organizations are progressively adopting a dual-channel supply chain system, which combines traditional brick-and-mortar stores with online platforms to enhance visibility, customer reach, and service flexibility. The inventory dynamics between manufacturers and retailers in such systems further complicate the scenario. Typically, the manufacturer produces goods in larger batches and delivers the goods in partial shipments, while the retailer sells products at a slower and more variable rate. As visualized in the manufacturer–retailer inventory model, the mismatch in production and consumption rates requires precise planning to synchronize supply with demand. Inadequate coordination can lead to high holding costs, delayed replenishment, or lost sales, all of which erode profitability. Nowadays, supply chain management is significantly dependent on consumers' feedback and experiences. If the consumer is kind enough to provide good feedback for a specific product, another consumer can benefit from this feedback while purchasing that product [1]. However, the feedback and the experiences from the consumer may be considered for any channel, such as a single channel, a dual channel, or a multi-channel system. When a retailer makes an effort to sell the products to more than one channel, the retailer may earn more profit than a single channel [2]. A dual-channel retailing system is an approach used by retailers to sell their products through two distinct distribution channels [3]. Using a combination of online and offline platforms, such as physical storefronts, websites, different social media etc, a retailer can conduct business using a dual-channel strategy [4]. With the diversity of dual channels, a large number of consumers can be satisfied with different options. Businesses can reach a larger consumer base and satisfy the needs of various market segments. One channel is the traditional physical stores, where consumers can visit and make purchases in person to see, touch, and try things before buying. Consumers can use mobile devices to purchase products. Online shopping provides a large product range and the facilities for making comparisons. Depending on the nature of the consumer's purchases and personal preferences, many consumers choose both online and offline shopping [5]. Using random demand during the lead time, the retailer decides the best strategy to improve consumer experiences for dual-channel management. Selection of the selling price, advertisement activities, and good management of consumer service centers surely boost the profit [6]. One of the most influential aspects in determining a product's demand is its selling price. Providing competitive prices and maintaining quality are crucial for retailers. Retail prices affect consumer demand a lot. A proper pricing method optimizes supply chain inventory levels [7]. Pricing fluctuations may impact supply chain lead times and production planning. The selling price must be established at a level that enables the company to recoup all expenses and still turn a profit on the excess [8]. Price reductions may increase sales but reduce profit per unit if the demand is highly elastic. In contrast, if the demand for a product is inelastic, then raising the price can lead to higher profit. Another critical component in this system is the machine failure rate. Machines with a high setup costs or frequent breakdowns contribute to imperfect output, increased operational cost, and disrupt timely delivery, thereby weakening the retailer's ability to compete [9]. By investing in reliable systems and cost-effective setup procedures, firms can improve production consistency, reduce waste, and better support just-in-time delivery models. A company that uses effective advertising can bring new consumers by keeping them informed about new items [10]. Good advertising efforts enhance brand image, which in turn fosters brand loyalty and encourages repeat business [11]. Effective marketing initiatives can result in a considerable rise in sales revenue. Consumer satisfaction plays a vital role in expanding any business, especially for retailers. The consumers' trust can be enhanced, loyalty can be boosted, and recurring purchases can be encouraged if retailers offer excellent consumer service. By using the consumer's name and customizing the solution to fit their unique needs and preferences, retail enterprises can offer personalized consumer care service [12]. This leads to an increased degree of satisfaction for consumers. As technological support, essential consumer services allow retailers to sell their products freely to consumers both online and offline, without setting prices that are too high [13]. Home delivery is a crucial component of dual-channel business that helps many companies run smoothly in this competitive business era [14]. Consumers can enjoy the ease of obtaining goods or services right at their door without having to go to a physical site. It also promotes impulsive purchasing. A dual-channel system promotes business growth by expanding consumer reach and strengthening market presence, while enabling traditional firms to integrate advanced technologies for greater efficiency and profitability. Cost efficiency can be achieved through the reduction setup costs without compromising product quality, and consumer satisfaction is further enhanced by delivering defect-free products. Incorporating the machine failure rate in the estimation of production cost provides a more practical approach than assuming a constant cost, thus improving inventory management accuracy and supporting just-in-time production to reduce expenses. Moreover, product demand in a dual-channel system is strongly influenced by a balanced mix of selling price, advertising, and consumer service, with excellent consumer care serving as a key strategy for attracting and retaining customers.

1.1. Research motivation and objectives

This research is motivated by the following purposes.

1. Sometimes, customers select a product online but make the actual purchase by visiting a physical showroom. Conversely, some customers explore products in the showroom and later buy them through online platforms. Since customer purchasing behavior is often unpredictable, this study introduces dual-channel supply chain strategies to effectively address such uncertainties.

2. Customer's purchasing decisions are frequently influenced by product price. High prices may drive customers toward competitors, while low prices can create doubts about the product's quality. In today's market, customers place great importance on efficient and reliable customer service. They prefer retailers who provide accessible support and quick resolution of product-related issues, even when similar products are available from multiple sellers. Furthermore, reliable after-sales service and effective advertising play a crucial role in enhancing customer trust and stimulating demand. Therefore, this study is motivated to define demand as a function of the selling price, customer service facilities, and advertising within a dual-channel supply chain framework.

3. In real-world manufacturing systems, machine failure rates play a vital role in determining the overall production efficiency and cost. Frequent machine failures lead to unplanned downtime, increased maintenance expenses, and loss of productive time all of which raises the total production cost. Ignoring these failures results in unrealistic cost estimations and poor

decision-making. Therefore, this study is motivated to incorporate the effect of the machine failure rate into the production cost formulation to reflect more practical and reliable behavior of the system.

4. In a production system, machine failure is a common phenomenon that disrupts the entire production process and increases the total operational cost. Improved maintenance practices, skilled labor, and advanced equipment can help to reduce the machine failure rate, which can be achieved through strategic investments in improvements to the setup. However, for profit maximization, it is equally important to minimize the setup costs. Hence, a balance must be established between optimizing the setup processes to enhance the machines' reliability and controlling the setup expenses to ensure greater profitability. Therefore, this study is aimed to develop an effective investment strategy aimed at reducing the setup costs, thereby improving production efficiency and maximizing profit.

Here are some research objectives of this research.

1. This study aims to demonstrate how dual-channel strategies affect the total profit of any organization in the supply chain system. The major objective of this research is to optimize profit in an uncertain dual-channel system when the cost fluctuations of the products cannot be accurately predicted.

2. To find how demand is influenced by the selling prices, advertisement efforts, and cost of consumer care service in the dual-channel supply chain system. To maintain healthy competition, manufacturing firms need to align consumer service efforts with cost management practices.

3. Another intention of this study is to achieve long-term success and sustainable growth by considering the machine's failure rate to calculate production cost in a dual-channel supply chain system.

4. Furthermore, we aimed to find an appropriate investment strategy to reduce the setup costs and improve production efficiency to maximize profit.

The rest of the paper is organized as shown in Figure1.

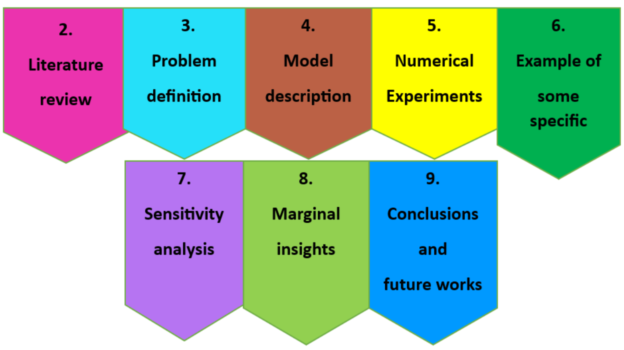


Figure 1. Paper orientation.

2. Literature review

Business management encompasses supply chain management, marketing strategies, consumer behavior, and store operations. Given its dynamic and evolving nature, research in supply chain systems spans a wide range of topics, including retailing trends, technological advancements, and pricing strategies.

2.1. Exploring the dynamics of dual-channel supply chain management

Kunomboua et al. [15] explored a dual-channel strategy for perishable products. Advanced technology was used to increase consumer satisfaction. Profit was increased by 29.12% compared with firms not offering these services. Profit decreased by 16.98% when the holding cost increased. Here, promotion strategies and investment for reducing setup costs were absent. Sarkar et al. [16] proposed a model where consumers received free home delivery under certain conditions. Classical optimization techniques and integer programming were used for numerical solutions. Implementing a free home delivery policy increased profit by 54.75%. The impact of the machine failure rate on production cost was not included in this study. Pal et al. [17] presented a sustainability model for dual-channel retailing. Financial and economic elements were combined with eco-friendly, social, and moral considerations. The core product was sold through traditional channels, and customized products were available online. Total cost was reduced by 14.0625% for having a dual-channel system rather than a single retailing system. This study did not consider any investment aimed at reducing the setup costs. Yao et al. [18] developed a model to forecast consumers' intentions in a dual-channel system. This study compared the "to-shop" and "to-home" channels in China, testing the hypotheses through structural equation modeling (SEM). Predictions were evaluated using the root mean square error (RMSE) statistic. The findings revealed that many consumers searched for products online but ultimately made purchases in offline stores. Das et al. [19] examined online–offline retailing systems and highlighted the role of re-manufacturing in sustainable supply chain management. This study showed that return rates exhibited stochastic behavior, and the re-manufacturing rate depended directly on the volume of returns. This study did not consider investments for reducing setup costs and advertisement strategies. Chen et al. [20] proposed a dual-channel system with pre-announced selling price techniques. Four types of models were discussed in this study. A backward induction approach was followed to find the solution of the model. How to increase the number of customers was not discussed in this study. Dey et al. [21] investigated the effectiveness of adopting a dual-channel in enhancing the operational efficiency of small-scale stores. The results indicated that integrating the online-to-offline (O2O) platform was an inventive and strategic method for small-scale online retailers that improved their efficiency compared with other methods. Yadav et al. [22] proposed an integrated supply chain model to identify the factors influencing online–offline service acquisition by small and medium enterprises. In this study, demand was linked to pricing systems. The real-life problem was formulated as a mixed-integer nonlinear optimization model. The solution was obtained using differential calculus. Strategies to minimize the setup costs were not investigated. Ullah et al. [23] proposed a dual-channel supply chain model for defective products without relying on dynamic pricing. This study found that demand uncertainties negatively affected logistics activities. Higher re-manufacturing rates reduced the setup costs and improved cost-effectiveness. It also recommended expressing the re-manufacturing cost as a percentage of the manufacturing cost rather than as a fixed value. Choi et al. [24] proposed an enhanced retailing approach in which both the retailer and the manufacturer jointly manage a single product. The retailer sold the product in both e-commerce and physical stores, while the manufacturer chose to sell only through physical stores. The company invested money to improve its services by reducing the lead time. This study suggested a partial payment system for profit maximization. Sett et al. [25] developed a supply chain model that enhanced customer service within an O2O retail framework. Here, the supplier deliberately reduced product quality to increase profit. A classical optimization strategy was used for solving the problem. To maintain production quality and address out-of-control states, proactive investment in process control was recommended. Moon et al. Here, the effect of machine failure rate on production quality was not addressed. [9] developed a production model where the machine failure rate had a great impact on the production system. A quasi-closed method was used to obtain an optimal solution. A reliable machine produced fewer defective items than an unreliable one. Dual-channel retailing was not addressed to increase the number of consumers.

2.2. *Investment for promotion in retailing system*

Numerous studies that examined the impact of sales efforts in the retail sector have been published in the literature. Taleizadeh et al. [26] studied a packaged food manufacturing model where products were sold through the conventional channel. Data were collected from 92 firms. This research highlighted the significant role of advertising in attracting consumers, showing that online promotion was more effective than offline promotion in the technological era. By applying advertising strategies, firms achieved 2.55% higher profit compared to those without promotional investments. The impact of the machine failure rate on production cost was not included in this study. Li et al. [27] examined the impact of advertising cost on product sales in a dual-channel system. This study considered the use of both online and offline coupons to attract consumers. The Black–Scholes equations were used to solve this model. Effective sales efforts increased company profits. Results indicated that advertising strategies generated a 0.02% increase in profit compared to cases without promotional investment. The impact of the machine failure rate on production cost was not included in this study. Zhang et al. [28] investigated omnichannel retailing systems where products were promoted through coupon campaigns. This study found that such campaigns increased consumer numbers, raised inventory levels, and positively influenced joint profits. To maximize profits, both manufacturers and retailers were required to adopt online–offline promotion strategies. The model was solved using game theory. This study did not include the role of consumer service facilities. Zhou et al. [29] compared sales efforts between live-streaming platforms and traditional platforms in supply chain systems. This study noted that consumers on traditional platforms often lacked awareness of live-streaming platforms and could not purchase from them. Live streaming platforms increased repeat consumers, and optimal advertising cost positively contributed to profit maximization. The impact of the machine failure rate on production cost was not included in this study. Ye et al. [30] examined the impact of live-streaming promotions on dual-channel retailing, focusing on the role of popular and regular influencers. This study compared differential and uniform pricing strategies and found that collaborating with leading influencers was not always the most profitable approach. Results indicated that adopting a differential pricing strategy increased profits for all supply chain participants. Fan et al. [31] investigated the impact of digital content marketing on product promotion and its alignment with product categories in supply chain management. This study used data from a

Chinese public university, where 65.85% of the sample was female. Researchers proposed a framework combining dual coding theory and the heuristic–systematic model. This study applied a two-sample t-test for hypothesis testing. Findings suggested that marketing effectiveness for women could be improved by integrating socially driven text with conceptual images. The study did not examine how investment could be used to reduce costs. Motlagh and Nasiri [32] studied a production model in a dual-channel system where product promotion was conducted exclusively online. This study applied a classical optimization strategy to determine the optimal profit levels. Findings revealed that the absence of promotional investment reduced the manufacturer's profit. Strategies to minimize setup costs were not investigated. Sarkar et al. [33] analyzed brand advertising approaches of manufacturers within a competitive model under supply uncertainty. This study evaluated different brand strategies and their effects on investors. The results showed that a low promotional cost was essential for profit maximization. When advertising expenditure was balanced with market expansion efforts, the equilibrium condition resulted in only one manufacturer undertaking promotional activities. The impact of the machine failure rate on the production cost was not included in this study.

2.3. Retail price management

Datta et al. [34] explored the dynamic pricing strategy for a dual-channel system. Black-Scholes equations were used to solve the proposed model. Advertisement strategies enhanced the number of new customers. This study did not explore approaches for reducing the setup costs. Sarkar et al. [35] examined how a retailer's pricing strategy was affected when the demand for a product depended not only on its own price but also on the prices of other related products. The production rate changed over time and was influenced by the frequency of system failures. The Euler-Lagrange method was used to solve the problem. Defective products were repaired or reworked by spending an additional cost so they became suitable for sale. This study did not include the role of consumer service facilities. Parvasi et al. [36] introduced a bi-level model for retail management between domestic and international retailers. This study applied the Stackelberg game strategy, considering the domestic firm as the leader and the retailer as the follower. The explicit enumeration method was used to determine the optimal selling price for domestic goods. The Lambert-W function was applied to maximize profit. This study did not consider any investment aimed at reducing the setup costs. Martin et al. [37] examined the significance of reference pricing on purchasing decisions. A two-period model was developed to analyze how participants in a dual-channel system determined pricing and advertising strategies. Stackelberg game theory was used to optimize profit. The researchers suggested discounted prices in the second period for profit maximization. Advertisement strategies were not accounted for in this analysis. Qiu et al. [38] customized retail pricing schemes for different types of consumers using data from smart electricity meter users. This study proposed a bi-level optimization model, which was transformed into a single-level mathematical program with equilibrium constraints using the Kuhn–Tucker approach. The impact of the machine failure rate on the production cost was not included in this study. Yan et al. [39] observed that higher selling prices reduced product demand, while lower prices increased it. This study explored a two-tier green supply chain and applied classical optimization to determine the optimal values for variable quantities. The advertisement policy was overlooked by the authors. Sun et al. [40] described a model showing that demand gradually increased when the selling price dropped below the benchmark price. This study extended the reference price updating mechanism to allow sudden jumps in the benchmark price value. A Stackelberg game methodology was utilized in the study to derive the optimal profit solution. Advertisement strategies and customer service facilities were not accounted for in this analysis. Shu et al. [41] developed a supply chain model that incorporated a two-period selling policy along with a return policy. This study applied the Stackelberg game approach to determine optimal profit. It also highlighted the challenges and critical decisions faced by the retail industry in managing newly launched products. This study did not consider investments for reducing setup costs or advertisement strategies.

2.4. *Stochastic lead time*

Lead time refers to the order-to-delivery cycle, from when an order is placed to when it is received. Managing both the lead time and its variability allows firms to maintain a faster, more reliable supply chain, improving inventory efficiency and customer satisfaction. Reducing the lead time strengthens the relationship between customers and retailers. In recent years, lead time reduction has emerged as a key research focus in supply chain management. Several researchers, including Dey et al. [42] and Choi et al. [24] have proposed various strategies for reducing the lead time. The authors introduced a crashing cost function to represent the trade-off between cost and reduced lead time. The authors assumed that lead time duration followed a normal distribution. Strategies to minimize setup costs and the influence of the machine failure rate on the production cost were not investigated in their study. Moon et al. [43] developed an inventory model based on maintaining a certain service level, assuming that customer demand was met without a stockout. They considered the additional cost to reduce the lead time to enhance service performance. The role of consumer service facilities was not examined in this study. Albana et al. [44] addressed lead time quotation in a make-to-order system with variable operating costs. They analyzed three types of models. The findings revealed that customer sensitivity to price and lead time significantly influences the quoted lead times, service levels, and pricing strategies. The results demonstrated that representing the operating costs as a lead-time-dependent function yielded substantially better outcomes compared with constant-cost benchmark models. Machine reliability and its impact on production cost were beyond the scope of this study. Dominguez et al. [45] designed an inventory model where the buyers paid later for imperfect production. This study emphasized lead time reduction as a strategy to enhance the overall supply chain performance. Shorter lead times improved inventory turnover and reduced total system cost under credit and inspection conditions. Hemapriya and Uthayakumar et al. [46] developed models to support manufacturers in determining the optimal order release timing and quantities across multiple periods within a make-to-order production environment. Here, two models were observed: A constant lead time and a fixed lead time. With a variable lead time, the model maintained lower inventory levels. This study did not address the contribution of consumer service facilities. Li [47] examined how cutting down the lead time and its variability improved the performance of inventory systems. Instead of using rough or estimated methods to calculate the cost, this study used an exact total cost equation that directly considers the relationship between actual inventory and backorders. This study described a more accurate analysis of how changes in lead time and its variance affected total cost. This study did not account for the influence of the machine failure rate on production costs. Similarly, Sett et al. [25] proposed an inventory model in which both demand and lead time duration were treated as stochastic variables. Faster replenishment reduced the need for excess inventory, resulting in lower storage and carrying expenses. Strategies to minimize the setup costs were not investigated. Sarada and Shenbagam [48] proposed a repair-based maintenance framework for a degradable system, taking variable lead time into account. Quasi-renewal processes were used to analyze the system with variable lead times for repairs. This study did not explore approaches for reducing the setup costs. A short summary of some selected literature is discussed in Table 1.

Table 1. Literature comparison and novelty.

	Demand influenced by selling price	Demand influenced by advertisement cost	Lead time reduction	Home delivery facilities	Consumer service facilities	Investment in advertisement	Investment reducing setup costs
	Shu et al. [41]	Dey et al. [42]	Choi et al. [24]	Li et al. [47]	Sarkar et al. [33]	Shu et al. [41]	Datta et al. [34]
	Datta et al. [34]		Moon et al. [9]	Sarkar et al. [33]	Datta et al. [34]	Datta et al. [34]	
	Amankou et al. [15]		Li et al. [47]		Shu et al. [41]	Amankou et al. [15]	
	Sarkar et al. [33]		Dey et al. [42]		Choi et al. [24]	Dey et al. [42]	
	Dey et al. [21]		Sarada et al. [48]		Amankou et al. [15]		
	Sett et al. [25]						
	Li et al. [47]						
	Qiu et al. [38]						
	Taleizadeh et al. [26]						
	Sarada et al. [48]						
Proposed model	√	√	√	√	√	√	√

According to Table 1, there are numerous supply chain models presented in the literature to address the problem of optimizing retailers' profit, and some of them address the issue of dynamic pricing conventionally. However, no research has addressed how to set online–offline prices while controlling the lead time under uncertain demand. Many researchers used different approaches for increasing the profit earlier by reducing different costs. Most studies did not include machine failure rates in the production cost or reliability analyses. Only a few studies have considered this in traditional retailing with imperfect production systems. Very few studies focused on strategies for reducing setup costs to optimize the total cost. Moreover, existing supply chain models rarely integrate the selling price, the customer service cost, the advertisement cost, reduction in the setup costs, and the lead time control within a single framework. The proposed dual-channel retailing model fills these gaps by incorporating the machine failure rate into production cost calculations, providing a more realistic assessment of operational reliability and cost. The proposed model directly incorporates two key operational factors, such as reducing the setup costs and controlling the lead time, into its calculations. Demand for this proposed model depends on the selling price, the advertisement cost, and the cost of consumer service facilities, which makes it more realistic. By addressing these gaps, the proposed model provides practical managerial insights for dual-channel retailers, enhancing decision-making for cost reduction, efficiency, and customer satisfaction.

3. Problem definition, notation, and assumptions

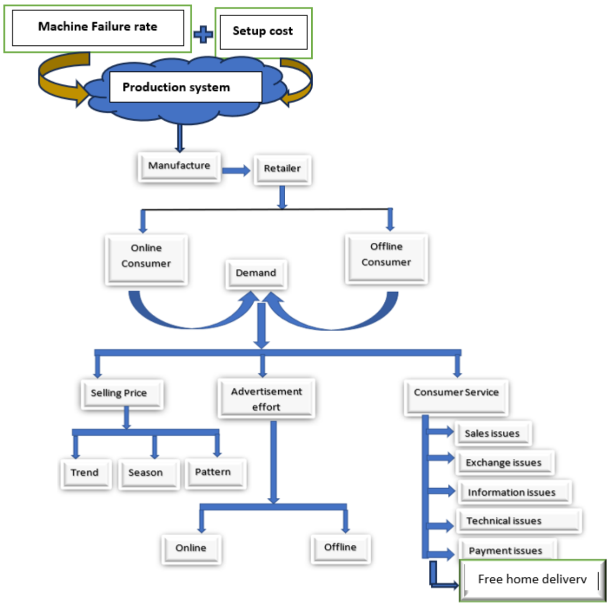


Figure 2. Proposed dual-channel supply chain system system.

The garment sector, operating across traditional and online channels, faces persistent challenges such as unstable pricing from fluctuating costs, rapidly changing fashion trends, and seasonal demand shifts that strain inventory management. Rising customer expectations for quality, delivery, and service intensify these pressures, while machine unreliability and imperfect production processes cause delays, defects, and wastage. High setup costs further burden financial resources, increasing the risk in competitive markets. Ineffective advertising strategies reduce brand visibility, weaken demand, and limit sales, while poor consumer service lowers satisfaction and loyalty. Together, these issues (pricing instability, inventory imbalance, demand surges, high cost, weak advertising, and poor service) create uncertainty and make it difficult for garment companies to maintain profitability in a fast-changing and highly competitive environment. As a result, this study proposes an advanced retail strategy based on dual-channel retailing, where product demand depends on the retail price, the advertising mechanism, and well-organized consumer care service that is illustrated graphically in Figure 2. Here, to reduce setup costs, this model adopts a capital investment approach. To provide better service to the consumers, free delivery of products is considered in this study. This model incorporates online-offline consumer service facilities to address various challenges faced in the garment sector. It handles sales-related issues to ensure smooth transactions and customer satisfaction. It manages exchange-related concerns by facilitating product returns and replacements efficiently. It also resolves information, technical, and payment issues to provide a seamless shopping experience for customers.

3.1. Notations

For easy access and better understanding, different types of symbols are used for the formulation of the supply chain system model. Throughout the paper, the following terminology is used.

Decision variables	
Ω_a	Online selling price ($/unit)
Ω_2	Offline selling price ($/unit)
Ω_3	Online consumer service price ($/unit)
Ω_4	Offline consumer service price ($/unit)
Ω_5	Price of investment for online advertisement ($/unit)
Ω_6	Price of investment for offline advertisement ($/unit)
L_b	Lead time by the retailer to complete the purchase (days)
m_l	Lot number of manufacturer-to-retailer delivery products in a single production cycle
f	Safety stock
Q_b	Retailer's order amount (unit)
S_e	Manufacturer's setup costs ($/setup)
θ	Failure rate of the machine
Parameters	
JNP	Total profit ($/unit time)
D_{dc}	Total amount of demand rate for dual channel (unit/year)
R_b	Retailer's reordering amount (unit)
S_{pv}	Selling price of manufacturer ($/unit/unit time)
n	The frequency of advertising
C_{ob}	Retailer's ordering cost ($/unit/unit time)
P_r	Rate of production per unit of time (unit/time)
S_{eo}	Manufacturer's initial setup costs ($/unit)
C_{pv}	Manufacturer paid unit production cost ($/unit/unit time)
C_{pbon}	Online wholesale price($/unit/unit time)
C_{pboff}	Offline wholesale price($/unit/unit time)
H_v	The holding cost rate for the manufacturer ($/unit/unit time)
H_b	The holding cost rate for the retailer ($/unit/unit time)
C_{tf}	Fixed cost of transportation ($)
C_{tv}	Variable transportation cost ($)
C_s	Capital investment to reduce setup costs ($)
t_u	The percentage decrease in S_e per dollar increase in I_{vs}
r_h	The yearly total amount of investment's fractional cost ($)
W	Demand for lead time (LTD) (units)
$E(.)$	Mathematical expectation
m_1	Product's percentage in online platform for delivery products at home without delivery charge (%)
m_2	Product's percentage in offline platform for delivery products at home without delivery charge (%)
l	Price-related scaling parameter for online demand
l_2	Price-related scaling parameter for offline demand
l_3	Consumer service related scaling parameter for online demand
l_4	Consumer service related scaling parameter for offline demand
l_5	Advertisement-investment-related scaling parameter for online demand
l_6	Advertisement-investment-related scaling parameter for offline demand
C_m	Fixed material cost ($)
λ	Tools cost(constant) ($)
A	Some fixed cost for production(not dependent on θ) ($)
B_t	Technology cost ($)
C_{ond}	The cost of home delivery for online purchases($/unit)
C_{offd}	The cost of home delivery for offline purchases ($/unit)
C_{lab}	Labor cost ($)
I_{vs}	Investment to reduce setup costs($)

3.2. Assumptions

To conduct a smooth investigation, the following assumptions were made.

- This study proposes a supply chain system strategy, where the selling price, advertisement cost, and consumer service cost are responsible for creating demand [49]. A complete cycle occurs when a specific product is delivered from the manufacturer to the retailer through a dual-channel sales system. Free delivery is also taken into consideration for shipping to the consumers.

- Demand is a function of the online-offline selling price, the online-offline advertisement cost, and the online-offline consumer care service cost. A shape parameter determines the way a function looks when it is plotted. Here, $\vartheta, g_2, g_3, g_4, g_5$, and g_6 are considered as the shape parameters for the online selling price, the offline selling price, the online consumer service cost, the offline consumer service cost, the online advertisement investment, the offline advertisement investment. The online demand is defined as D_{on}, which is calculated as

$$\begin{aligned} D_{on} &= \text{online selling price+ online advertisement cost+online consumer care cost} \\ &= \vartheta\Omega_a^{-l} + g_3\Omega_3^{l_3} + g_5\Omega_5^{l_5} \end{aligned} \tag{3.1}$$

The offline demand is defined as D_{off}, which is calculated as

$$\begin{aligned} D_{off} &= \text{offline selling price+ offline advertisement cost+offline consumer care cost} \\ &= g_2\Omega_2^{-l_2} + g_4\Omega_4^{l_4} + g_6\Omega_6^{l_6} \end{aligned} \tag{3.2}$$

By adding Equation 3.1 and 3.2, the total demand function for the dual-channel supply chain system is calculated as

$$D_{dc} = D_{on} + D_{off} = \vartheta\Omega_a^{-l} + g_2\Omega_2^{-l_2} + g_3\Omega_3^{l_3} + g_4\Omega_4^{l_4} + g_5\Omega_5^{l_5} + g_6\Omega_6^{l_6} \tag{3.3}$$

- To avoid the stock-out phase, the production rate is always higher than the rate of demand ($P_r > D_{dc}$). It increases the consumer satisfaction label as the consumer does not want to wait for a long period to receive the product. It helps to enhance brand loyalty. However, some technical issues or sudden increases in demand create a shortfall in the cycle. This is why the retailer needs to consider the shortage cost while calculating total cost from the retailer's perspective [50].

- The retailer continuously monitors the inventory and places a new order as soon as the stock level reaches the predetermined reorder point. In practice, it is not feasible for an industry manager to precisely determine the distribution function of lead time demand and subsequently resolve the lead time issue. However, when historical data are available, the mean and the standard deviation can be computed. Demands during lead time W having $D_{dc}L_b$ as the mean and $s_g\sqrt{L_b}$ as the standard deviation. Thus, the reordering point define as $R_b = D_{dc}L_b + fs_g\sqrt{L_b}$ [51]. Here, f is the safety stock that is an extra margin added to handle uncertainties like sudden changes in demand, supply delays, or system failures.

- The retailer places a new order at the reordering point R_b. $R_b - D_{dc}L_b$ is the average stock level before receiving an order. $Q_b + R_b - D_{dc}L_b$ is the average stock level instantly after delivering the products of Q_b. $R_b - D_{dc}L_b$ is the average stock level before receiving an order. $Q_b + R_b - D_{dc}L_b$ is the average stock level immediately after delivering the products of Q_b. The expected stock level for a cycle is defined as $\frac{Q_b}{2} + R_b - D_{dc}L_b$ [52].

- L_b(lead time) is composed of n components that are mutually independent. Here, l_i= least duration, r_i= regular length of duration, CR_i= crashing cost (per unit time) where, $CR_1 \leq CR2.... \leq CRn$. It is assumed that, $Lb_0 = \sum_{i=1}^{n}, Lb_i = Lb_0 - \sum_{i=1}^{n}(r_i - l_i)$ for i = 1, 2 ,3...,n. The cost for the lead time crashing cost for each cycle is $C_{Lb} = CR_i(Lb_{i-1} - Lb) + \sum_{i=1}^{j} -1(r_i - l_i)$[24]. If the lead time (delivery time) is reduced by paying extra (lead time crashing), all of that extra cost is paid by the retailer and not shared with the manufacturer.

- This model permits situations where customer demand temporarily exceeds the available stock. Even though items are out of stock, all customer orders are still recorded and promised for future delivery once the stock is replenished. Customers wait rather than cancelling their orders [42].

- When the cycle concludes, if some amount of products remain, they are sold at a discount price, which is very negligible. This is why this cost is not included as revenue.

- Consumers receive only good products. Defective products are scrutinized before delivery. They also sold at a discount price, which is very negligible. It does not affect revenue.

- This model assumes an ongoing, never-ending planning period with no set termination point.

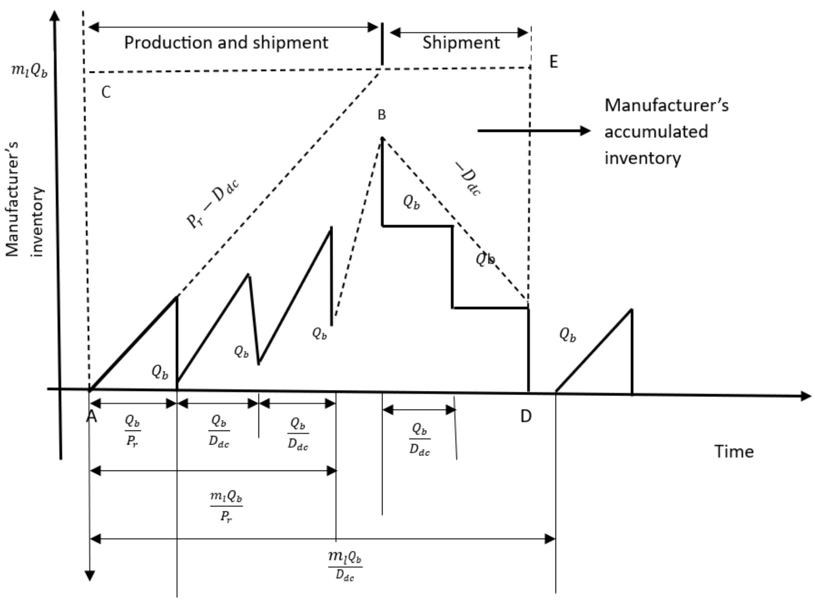


Figure 3. Manufacturer's inventory model.

4. Model description

This section presents mathematical formulations that describe and analyze the operations of both the manufacturer and the retailer. A smart dual-channel supply chain model is developed in this study for a single manufacturer and a single retailer, with an emphasis on continuous long-term planning. The supply chain follows a single-setup–multi-delivery (SSMD) policy, where the manufacturer produces in one setup and delivers to the retailer in multiple shipments within the same cycle.

4.1. Mathematical model for the manufacturer's perspective

The manufacturer's model is given in Figure 3. In this model, the manufacturer's demand depends on the retailer's demand. The manufacturer's production rate is P_r and the retailer's demand rate is D_{dc} and $P_r > D_{dc}$. The manufacturer makes a total of $m_l Q_b$ units in one production round. Instead of sending all $m_l Q_b$ units at once, the manufacturer delivers them to the retailer in m_l separate shipments, each of size Q_b. The manufacturer makes $m_l Q_b$ units to meet a total demand rate of D_{dc} units per cycle. Manufacturer's cycle length is $\frac{m_l Q_b}{D_{dc}}$. Each shipment takes a time of $\frac{Q_b}{P_r}$ to produce and a time of $\frac{Q_b}{D_{dc}}$ to be consumed by the retailer. The total time to consume all shipments except the first is $\frac{m_l Q_b}{D_{dc}}$. The manufacturer starts production at the beginning of a cycle. The stepwise line in the Figure 3 shows the retailer's inventory level. Each upward step happens when a shipment arrives. Each downward slope shows the inventory being used or sold over time.

4.2. Cost calculation for the manufacturer's perspective

From the manufacturer's perspective, cost calculation generally includes all expenses directly or indirectly incurred in producing and delivering goods. This model calculates the manufacturer's total cost based on the setup costs, the production costs, and the holding costs.

4.2.1. Investment to reduce the setup costs

The setup costs play an important role in profit maximization. According to the literature review, the setup costs is always considered to be constant. In reality, the setup costs often changes with production volume, batch size, machine efficiency, labor cost, or the material's complexity. Here, it is considered a variable cost for more realistic issues [12],[13],[53]. A capital investment reduces the manufacturer's setup costs. It can be written in the following way:
$I_{vs} = (\frac{r_h}{t_u}) \ln \frac{S_{eo}}{S_e} = (\frac{r_h}{t_u})(\ln S_{eo} - \ln S_e)$ for $S_e > 0$ and $S_e \leq S_{eo}$
We know that the setup costs $= \left[\frac{\text{Total demand}}{\text{Batch size*Optimum quantity}}\right] = \frac{S_e D_{dc}}{m_l Q_b}$

Therefore, the setup costs with investment is given as

$$Cost_{S_e} = \frac{r_h}{t_u}\Big(\ln S_{eo} - \ln S_e\Big) + \frac{S_e D_{dc}}{m_l Q_b} \tag{4.1}$$

4.2.2. Production cost

A constant production cost is unrealistic because it depends on various factors. According to Sarkar et al. [35], machine failure rate is considered an important variable for calculation of the production cost. The failure rate of the machine system, represented by the parameter θ is

$\theta = \frac{\text{Total number of breakdowns}}{\text{Sum of machine operating hours}}.$

Lower values of θ signify higher reliability of the machine, which requires greater investment in advanced technology, whereas higher values of θ indicate lower reliability.

Here, the fixed material cost is C_m. The failure rate-dependent technology cost is $B_t e^{\left(\frac{R_d(\theta_{max}-\theta)}{(\theta-\theta_{min})}\right)}$.

The total development cost of the production system is $A + B_t e^{\left(\frac{R_d(\theta_{max}-\theta)}{(\theta-\theta_{min})}\right)}$. The per-unit development cost is $\frac{A+B_t e^{\left(\frac{R_d(\theta_{max}-\theta)}{(\theta-\theta_{min})}\right)}}{P_r}$. If the machine failure rate decreases, then its development cost increases. In the case of $\theta = \theta_{min}$, the development cost becomes the smallest, and for $\theta = \theta_{max}$, the development cost becomes the largest. The tooling cost per unit time is λP_r. The labor cost, depending on machine failure rate is $e^{\theta} B_e C_{lab}$, where B_e is a scaling parameter. Hence, the production cost is defined as follows:

Production cost = $Cost_{pr}(\theta)$ = fixed material cost+development cost+tooling cost+labor cost

$$= C_m + \frac{A + B_t e^{\left(\frac{R_d(\theta_{max}-\theta)}{(\theta-\theta_{min})}\right)}}{P_r} + \lambda P_r + e^{\theta} B_e C_{lab} \tag{4.2}$$

4.2.3. Holding cost

Several factors can affect the holding cost, including the size and location. The holding cost is calculated by dividing the total inventory in a cycle by the cycle length and multiplying the result by the holding cost per unit. From Figure 3, the cycle length is $\frac{m_l Q_b}{D_{dc}}$ and the manufacturer's average stock is as follows:

$$\frac{\text{Area ADEC-Area ACB-Area of the ladder}}{\text{Cycle length}} = \frac{m_l Q_b\left\{\frac{Q_b}{P_r}+(m_l-1)\frac{Q_b}{D_{dc}}\right\} - \frac{m_l^2 Q_b^2}{2P_r} - \frac{Q_b^2}{D_{dc}}\left[1+2+---+(m_l-1)\right]}{\frac{m_l Q_b}{D_{dc}}}$$

$$= \frac{m_l Q_b\{\frac{Q_b}{P_r} + (m_l-1)\frac{Q_b}{D_{dc}}\} - \frac{m_l^2 Q_b^2}{2P_r} - \frac{Q_b^2}{D_{dc}}[1+2+---+(m_l-1)]}{\frac{m_l Q_b}{D_{dc}}}$$

$$= \left[\left\{m_l Q_b\left(\frac{Q_b}{P_r} + (m_l-1)\frac{Q_b}{D_{dc}}\right) - \frac{m_l^2 Q_b^2}{2P_r}\right\} - \left\{\frac{Q_b^2}{D_{dc}}(1+2+---+(m_l-1))\right\}\right]\frac{D_{dc}}{m_l Q_b}$$

$$= \frac{Q_b}{2}\left[m_l\left(1-\frac{D_{dc}}{P_r}\right) - 1 + \frac{2D_{dc}}{P_r}\right]$$

The expected holding cost for manufacturer,

$Cost_{hm}$ = Unit holding cost $*$ manufacturer's average stock

$$= H_v \frac{Q_b}{2}\left[m_l\left(1-\frac{D_{dc}}{P_r}\right) - 1 + \frac{2D_{dc}}{P_r}\right] \tag{4.3}$$

4.2.4. Manufacturer's revenue

Revenue for the manufacturer is calculated as follows:

online wholesale price $*$ online demand + offline wholesale price $*$ offline demand

Thus, $REV_{man} = C_{pbon} D_{on} + C_{pboff} D_{off}$. (4.4)

4.2.5. Manufacturer's total cost

The total cost for the manufacturer is calculated as follows:
Thus, $TCost_{Mr} = Cost_{Se} + Cost_{pr}(\theta) + Cost_{hm}$.

4.2.6. Manufacturer's profit

Profit for the manufacturer is calculated as follows:
Profit= revenue - total cost= $Profit_{Manu}$=$REV_{man} - TCost_{Mr}$

$$= C_{pbon}D_{on} + C_{pboff}D_{off} - \left[C_m + \lambda P_r + \frac{A + B_t e^{\left(\frac{R_d(\theta max-\theta)}{(\theta-\theta min)}\right)}}{P_r} + \frac{r_h}{t_u}\left(\ln S_{eo} - \ln S_e\right).\right.$$
$$\left. + \frac{S_c D_b}{m_l Q_b} + e^{\theta} B_e C_d + H_v \frac{Q_b}{2}\left[m_l\left(1 - \frac{D_{dc}}{P_r}\right) - 1 + \frac{2D_{dc}}{P_r}\right]\right] \tag{4.5}$$

4.3. Mathematical model from the retailer's perspective

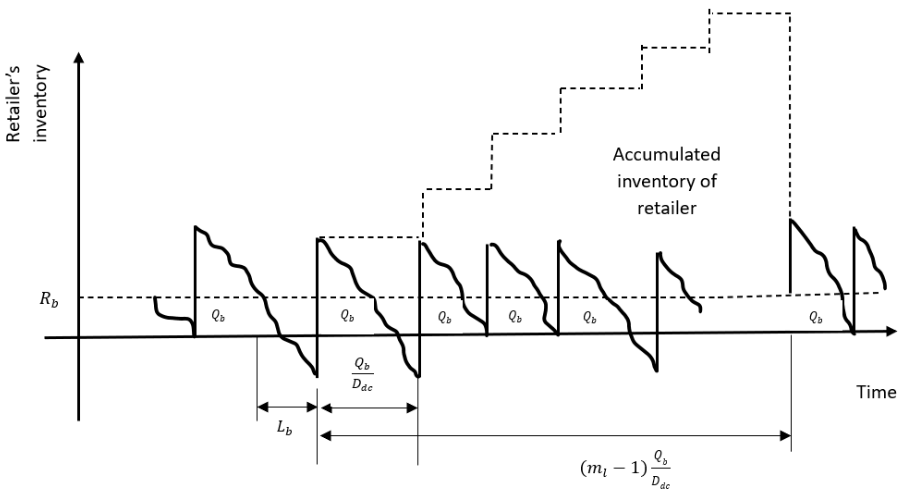


Figure 4. Retailer's inventory model.

The retailer's inventory model, where product demand is a function of the selling price, advertising expenditure, and customer service investment, is illustrated in Figure 4. The retailer's demand rate is D_{dc}. The retailer is always keeping track of their inventory level (continuous review). Here, the retailer receives and uses Q_b units at a time, so the retailer's cycle length is $\frac{Q_b}{D_{dc}}$. Each shipment takes a time of $\frac{Q_b}{D_{dc}}$ to be consumed by the retailer. The total time to consume all shipments except the first is $(m_l-1)\frac{Q_b}{D_{dc}}$. The retailer sets a specific reorder point R_b, when the minimum stock level before replenishment is needed. When the inventory falls to R_b, the retailer immediately orders a fixed quantity Q_b. This continuous review policy is designed so that the retailer does not keep excess inventory for too long, which reduces the holding cost. Just before placing the order, the available inventory is $R_b - D_{dc}L_b$, where $D_{dc}L_b$ is the average demand during the lead time. Once the Q_b units arrive, the inventory increases to $Q_b + R_b - D_{dc}L_b$. The average inventory over the cycle is therefore $\frac{Q_b}{2} + R_b - D_{dc}L_b$. This model considers an unknown *distribution function (c.d.f.) W* for the lead time demand in the class *U* of *c.d.f.'s* . Often, it is hard to know the exact probability distribution of lead time demand. Gathering this information can be expensive for managers. In such cases, the max-min distribution-free approach is helpful. This method works when the company only knows the mean and standard deviation of lead time demand, but not its exact distribution. This study considers the demands during the lead time W having $D_{dc}L_b$ as the mean and $s_g\sqrt{L_b}$ as the standard deviation. The retailer places new order at the reordering point R_b. Thus, the reordering point is defined as follows:

$$R_b = D_{dc}L_b + f s_g\sqrt{L_b}. \quad (4.6)$$

4.4. Cost calculation from the retailer's perspective

For retailer, the best retailing strategy is to consider the investment for controlling the lead time. Because the retailer can reduce the lead time but cannot reduce it to zero, the system must contain some non-zero lead time. Therefore, to calculate the revenue of the retailer, this study considers the cost to purchase the product from the manufacturer, the cost to order products, the cost to carry those products, and the cost to purchase from the manufacturer. The shortage cost, investment to control the lead time, investment for advertisement, home delivery cost, consumer care cost, and transportation cost are described below.

4.4.1. Purchasing cost

Purchasing cost is the total amount a business spends to buy goods or materials from a supplier. The manufacturer's selling price serves as the retailer's purchasing price, as outlined below:

$$\begin{aligned} Cost_{pc} &= \text{unit online wholesale price} * \text{online quantity purchased} \\ &+ \text{unit offline wholesale price} * \text{ offline quantity purchased} \\ &= C_{pbon}D_{on} + C_{pboff}D_{off} \end{aligned} \quad (4.7)$$

where, C_{pbon} *and* C_{pboff} are the prices of purchasing online and offline, respectively and D_{on} *and* D_{off} are the online and offline demand, respectively.

4.4.2. Ordering cost

For conducting business smoothly, orders need to be placed easily and promptly. The ordering cost for retailers is calculated as follows:

$$Cost_{od} = \frac{\text{Cost associated with placing one order}}{\text{Cycle length}} = \frac{C_{ob}}{T} = \frac{C_{ob}D_{dc}}{Q_b} \quad (4.8)$$

Here, C_{ob} is the cost associated with placing one order, and T is the cycle length where $T = \frac{Q_b}{D_{dc}}$

4.4.3. Holding cost

The length of the cycle is $\frac{Q_b}{D_{dc}}$. When the stock reaches the reordering point R_b, the number of orders in Q_b is placed again. Thus, $R_b - D_{dc}L_b$is the average stock level before receiving an order. $Q_b + R_b - D_{dc}L_b$ is the average stock level instantly after delivering the products of Q_b. The expected stock level for a cycle is defined as $\frac{Q_b}{2} + R_b - D_{dc}L_b$. The calculation of the holding cost from the retailer's perspective is given as follows:

$$\begin{aligned} Cost_{hr} &= \text{unit holding cost} * \text{the expected stock level for a cycle} \\ &= H_b\Big[\frac{Q_b}{2} + R_b - D_{dc}L_b\Big] \end{aligned} \tag{4.9}$$

4.4.4. Shortage cost

Lead time demand follows a distribution with a known mean and standard deviation, e.g., the mean is $D_{dc}L_b$ and $s_g\sqrt{L_b}$ is the standard deviation. Thus, the reordering point is defined as $R_b = D_{dc}L_b + fs_g\sqrt{L_b}$. A shortage occurs if the actual demand during lead time W, exceeds R_b and the expected shortage quantity is given by $E(W-R_b)^+ = \int_{R_b}^{\infty}(w-R_b)\,\mathrm{d}F(w)$, which represents the average amount by which demand surpasses the reorder point. It is not possible to determine precisely because of the unknown probability distribution of W. $E(W - R_b)^+$ can be calculated with the help of the following lemma:

$E(W - R_b)^+ \leq \frac{1}{2} * \left[\sqrt{({s_g}^2 L_b + (R_b - D_{dc}L_b)^2)} - (R_b - D_{dc}L_b)\right] = \frac{1}{2}s_g\sqrt{L_b}[\sqrt{1+f^2} - f]$. The detailed calculations is given in Moon and Gallego [54]. Shortage costs are calculated as follows:

$$Cost_{SH} = \frac{\text{Unit shortage cost} * \text{Shortage quantity}}{\text{Cycle length}} = \frac{\pi D_{dc}s_g\sqrt{L_b}[\sqrt{1+f^2} - f]}{2Q_b}, \tag{4.10}$$

where π is the unit shortage cost.

4.4.5. Investment to control the lead time

Due to the dynamic character of business today, controlling the lead time assists a company in being competitive and meeting consumer expectations. Consumers can miss their deliveries due to unreasonable, excessive lead times. This can lead to consumer anger, cancelled orders, or missed opportunities in terms of business. The increased lead times can result in increased operational costs due to increased inventory holding costs, the cost of advanced shipping, overtime pay, and other costs necessary to satisfy the consumers. The retailer uses the cost C_{Lb} to cap the lead times, which guarantees that at all times, the lead times are controlled. The lead time expenditure is calculated as follows:

$$Cost_{Ld} = \frac{\text{Investment to reduce lead time}}{\text{Cycle length}} = \frac{C_{Lb}D_{dc}}{Q_b}. \tag{4.11}$$

4.4.6. Cost of advertisement

The retailer allocates resources for advertising since promotional activities are recognized as an effective driver of consumer demand. Optimizing these expenses allows businesses to boost consumer awareness and engagement, which are essential for building brand growth and loyalty. This study considers online and offline promotional investments. The cost calculation for the advertisement is as follows:

$$Cost_{Ad} = \frac{a_f\Omega_5^2}{2} + \frac{b_f\Omega_6^2}{2}. \tag{4.12}$$

4.4.7. Cost of home delivery service

The daily advancement of technology and changes in consumer preferences have increased the popularity of free home delivery day by day. This is applicable for both online and offline purchases. In this study, home delivery that is paid for and free delivery are both considered. The retailer sets a specified quantity that a customer has to purchase to qualify for free home delivery service. The cost calculation for the home-delivery service is given as follows:

$$\begin{aligned} Cost_{Hd} = &\text{ online home delivery cost} * \text{online demand} \\ &+ \text{offline home delivery cost} * \text{offline demand} \\ &= C_{ond}D_{on} + C_{offd}D_{off}. \end{aligned} \tag{4.13}$$

Here, C_{ond} is the online home delivery cost and C_{offd} is the offline home delivery cost.

4.4.8. Cost of consumer care service

The retailer establishes a consumer support center to improve the quality of service provided to customers. The associated cost of the consumer care service can be represented by the following equation:

$$Cost_{Cs} = \frac{e_f\Omega_3^2}{2} + \frac{k_f\Omega_4^2}{2}. \tag{4.14}$$

4.4.9. Transportation cost

The role of transportation is significant in the supply chain. Now, the retailer uses an SSMD transportation method to ship products to the retailer's warehouse, cutting down on the lead time and the retailer's holding cost. It should go without saying that transportation costs are not always constant. Therefore, to reflect the actual scenario, this analysis considers both constant and variable quantity costs associated with transportation. The equation of transportation cost is as follows:

$$\begin{aligned} Cost_{Tr} = &\text{ number of shipment} * \text{fixed transportation cost} \\ &+ \text{variable transportation cost} * \text{retailer's order amount} \\ &= m_l C_{tf} + m_l C_{tv} Q_b. \end{aligned} \tag{4.15}$$

4.4.10. Retailer's revenue

Revenue for the retailer is calculated as follows:

$$\begin{aligned} REV_{Ret} = &\text{ online selling price} * \text{online demand+offline selling price} * \text{offline demand} \\ &= \Omega_a D_{on} + \Omega_2 D_{off}. \end{aligned} \tag{4.16}$$

4.4.11. Retailer's total cost

The total cost for the retailer is calculated as follows:

$$TCost_{Re} = Cost_{pc} + Cost_{od} + Cost_{hr} + Cost_{SH} + Cost_{Ld} + Cost_{Ad} + Cost_{Hd} + Cost_{Cs} + Cost_{Tr}.$$

4.4.12. Retailer's profit

The profit for the retailer is calculated as follows:
Profit= revenue- total cost $=Profit_{Re}=REV_{Ret}-TCost_{Re}$

$$= \Omega_a D_{on} + \Omega_2 D_{off} - \Bigg[\frac{C_{ob}D_{dc}}{Q_b} + \frac{a_f\Omega_5^2}{2} + \frac{b_f\Omega_6^2}{2} + \frac{e_f\Omega_3^2}{2} + \frac{k_f\Omega_4^2}{2} + H_b\Big[\frac{Q_b}{2} + R_b - D_{dc}L_b\Big] + C_{pbon}D_{on} + C_{pboff}D_{off} + C_{ond}D_{on} + C_{offd}D_{off} + m_lC_{tf} + m_lC_{tv}Q_b + \frac{C_{Lb}D_{dc}}{Q_b} + \frac{\pi D_{dc}}{Q_b}s_g\sqrt{L_b}\zeta(f)\Bigg]. \tag{4.17}$$

4.4.13. Joint net profit

Joint net profit for the manufacturer and retailer is calculated as follows:
$JNP = Profit_{Manu} + Profit_{Re}$

$$= C_{pbon}D_{on} + C_{pboff}D_{off} + \Omega_a D_{on} + \Omega_2 D_{off} - \Bigg[C_m + \frac{A + B_t e^{\left(\frac{R_d(\theta max-\theta)}{(\theta-\theta min)}\right)}}{P_r} + e^{\theta}B_eC_d + \lambda P_r + \frac{r_h}{t_u}\Big(\ln S_{eo} - \ln S_e\Big) + \frac{S_eD_{dc}}{m_lQ_b} + C_{pbon}D_{on} + C_{pboff}D_{off} + C_{ond}D_{on} + C_{offd}D_{off} + H_v\frac{Q_b}{2}\Big[m_l\Big(1-\frac{D_{dc}}{P_r}\Big) + \frac{e_f\Omega_3^2}{2} + \frac{k_f\Omega_4^2}{2} + \frac{a_f\Omega_5^2}{2} + \frac{b_f\Omega_6^2}{2} + m_lC_{tf} + m_lC_{tv}Q_b - 1 + \frac{2D_{dc}}{P_r}\Big] + \frac{C_{ob}D_{dc}}{Q_b} + H_b\Big[\frac{Q_b}{2} + R_b - D_{dc}L_b\Big] + \frac{C_{Lb}D_{dc}}{Q_b} + \frac{\pi D_{dc}}{Q_b}s_g\sqrt{L_b}\zeta(f)\Bigg]. \tag{4.18}$$

4.5. Solution methodology

To achieve maximum profit, this model has been solved by applying the classical optimization method.
Here,

$$\frac{\partial JNP}{\partial Q_b} = \frac{1}{(Q_b)^2}\left(\frac{S_eD_{dc}}{m_l} + C_{ob}D_{dc} + C_{Lb}D_{dc} + \pi\sigma\sqrt{L_b}\zeta(f)D_{dc}\right) - m_lC_{tv} - \frac{1}{2}H_v\Big[m_l(1-\frac{D_{dc}}{P_r}) - 1 + \frac{2D_{dc}}{P_r}\Big] - \frac{1}{2}H_b = 0 \tag{4.19}$$

$$\frac{\partial JNP}{\partial \theta} = \frac{B(\theta_{max}-\theta_{min})}{P_r(\theta-\theta_{min})^2}e^{\frac{R_d(\theta max-\theta)}{(\theta-\theta_{min})}} + e^{\theta}C_d\beta_e = 0 \tag{4.20}$$

$$\frac{\partial JNP}{\partial S_e} = \frac{\alpha_s\rho}{\tau}\frac{1}{S_e} - \frac{D_{dc}}{m_lQ_b} = 0 \tag{4.21}$$

$$\frac{\partial JNP}{\partial m_l} = \frac{S_eD_{dc}}{Q_bm_l^2} - [\frac{H_vQ_b(1-\frac{D_{dc}}{P_r})}{2}] - C_{tv}Q_b - C_{tf} = 0 \tag{4.22}$$

$$\frac{\partial JNP}{\partial f} = -H_b\sigma\sqrt{L_b} - \frac{D_{dc}\pi\sigma\sqrt{L_b}}{2Q_b}\Big(\frac{f}{\sqrt{1+f^2}}\Big) + D_{dc}\frac{\pi\sigma\sqrt{L_b}}{2Q_b} = 0 \tag{4.23}$$

$$\frac{\partial JNP}{\partial L_b} = \frac{D_{dc}}{2Q_b\sqrt{L_b}}\pi\sigma\zeta(f) + \frac{D_{dc}}{Q_b} = 0 \tag{4.24}$$

$$\frac{\partial C_{Lb}}{\partial L_b} = \frac{H_b f\sigma}{2\sqrt{L_b}} = 0 \tag{4.25}$$

$$\frac{\partial JNP}{\partial \Omega_1} = g_1 l_1 \Omega_1^{-l_1-1}(Cond - X_1) + (1-l_1)D_{on} = 0 \tag{4.26}$$

$$\frac{\partial JNP}{\partial \Omega_2} = g_2 l_2 \Omega_2^{-l_2-1}(Coffd - X_1) + (1-l_2)D_{off} = 0 \tag{4.27}$$

$$\frac{\partial JNP}{\partial \Omega_3} = g_3 l_3 \Omega_3^{l_3-1}(\Omega_1 + X_1 - Cond) - \nu_3\Omega_3 = 0 \tag{4.28}$$

$$\frac{\partial JNP}{\partial \Omega_4} = g_4 l_4 \Omega_4^{l_4-1}(\Omega_2 + X_1 - Coffd) - \nu_4\Omega_4 = 0 \tag{4.29}$$

$$\frac{\partial JNP}{\partial \Omega_5} = g_5 l_5 \Omega_5^{l_5-1}(\Omega_1 + X_1 - Cond) - \nu_1\Omega_5 = 0 \tag{4.30}$$

$$\frac{\partial JNP}{\partial \Omega_6} = g_6 l_6 \Omega_6^{l_6-1}(\Omega_2 + X_1 - Coffd) - \nu_2\Omega_6 = 0 \tag{4.31}$$

For solving the Equation from 4.19 to 4.31, decision variables are calculated, and the mathematical expression is given below:

$$\theta^* = \theta_{min} + \sqrt{\frac{B_t(\theta_{min} - \theta_{max}) * e^{\frac{R_d(\theta_m ax - \theta_m in)}{\theta - \theta_{min}}}}{e^{\theta} C_d B_e Pr}} \tag{4.32}$$

$$S_e^* = \frac{r_h m_l Q_b}{(D_{dc} t_u} \tag{4.33}$$

$$Q_b^* = \sqrt{\frac{(D_{dc}(\frac{S_e}{m_l} + C_{ob} + C_{Lb} + \pi s_g\sqrt{L_b}\zeta(f))}{m_l C_{tv} + \frac{H_m}{2}[m_l(1-X_4) - 1 + 2X_4] + \frac{H_b}{2}}} \tag{4.34}$$

$$m_l^* = \sqrt{\frac{S_e(D_{dc}}{Q_b[\frac{H_v Q_b}{2}(1 - \frac{(D_{dc}}{P_r}) + C_{tf} + C_{tv}Q_b]}} \tag{4.35}$$

$$f^* = [-H_b s_g\sqrt{L_b} + \frac{(D_{dc}\pi s_g\sqrt{L_b}}{2Q_b}]\frac{2Q_b\sqrt{1+f^2}}{(D_{dc}\pi s_g\sqrt{L_b}} \tag{4.36}$$

$$\Omega_a^* = \left[\frac{\vartheta l(X_a - Cond)}{(1-l)D_{on}}\right]^{\frac{1}{l+1}} \tag{4.37}$$

$$\Omega_2^* = \left[\frac{g_2 l_2 (X_2 - Coffd)}{(1-l_2)D_{off}}\right]^{\frac{1}{l_2+1}} \tag{4.38}$$

$$\Omega_3^* = \left[\frac{g_3 l_3 \Omega_3^{l_3-1}(\Omega_a + X_a - Cond)}{e_f}\right] \tag{4.39}$$

$$\Omega_4^* = \left[\frac{g_4 l_4 \Omega_4^{l_4-1}(\Omega_2 + X_a - Coffd)}{k_f}\right] \tag{4.40}$$

$$\Omega_5^* = \left[\frac{g_5 l_5 \Omega_5^{l_5-1}(\Omega_a + X_a - Cond)}{a_f}\right] \tag{4.41}$$

$$\Omega_6^* = \left[\frac{g_6 l_6 \Omega_6^{l_6-1}(\Omega_2 + X_a - Coffd)}{b_f}\right] \tag{4.42}$$

Proposition 1.

The sufficient condition for the global optimality claims that a function is called globally optimum at the singular point, if the principal minors of the Hessian matrix are alternate in the sign for the optimal values of the decision variables.

Let, Q_b^*,f^*, S_e^*, Ω_a^*, Ω_2^*, Ω_3^*, Ω_4^*, Ω_5^* and, Ω_6^* be the optimum values of Q_b,f, S_e, Ω_a, Ω_2, Ω_3, Ω_4, Ω_5, and, Ω_6 . Then, assuming a fixed value m_l and $L_b \in [L_{bi}, L_{bi-1}]$ the global maximum value of the joint profit JNP can be obtained at Q_b^*, f^*, S_e^*, Q_b^*,f^*, S_e^*, Ω_a^*, Ω_2^*, Ω_3^*, Ω_4^*, Ω_5^*, and, Ω_6^* if the following hold.

1. $H11 < 0$.
2. $H22 > 0$ when, $\gamma_1\gamma_2 > \gamma_{18}^2$
3. $H33 < 0$ when $\gamma_{11}^2\gamma_2 + \gamma_{18}^2\gamma_3 < \gamma_1\gamma_2\gamma_3$.
4. $H44 > 0$ when,$\gamma_1\gamma_2\gamma_3\gamma_4 > \gamma_{11}^2\gamma_2\gamma_4 + \gamma_{18}^2\gamma_3\gamma_4$.
5. $H55 < 0$ when, $2\gamma_{12}\gamma_{18}\gamma_{19}\gamma_3\gamma_4 + \gamma_1\gamma_{19}^2\gamma_3\gamma_4 + \gamma_{12}^2\gamma_2\gamma_3\gamma_4 + \gamma_{11}^2\gamma_2\gamma_5\gamma_4 + \gamma_{18}^2\gamma_3\gamma_5\gamma_4 < \gamma_4\gamma_{11}^2\gamma_{19}^2 - \gamma_1\gamma_2\gamma_3\gamma_5\gamma_4$.
6. $H66 > 0$ when, $\gamma_{11}\Pi_3 + \gamma_{12}\Pi_4 > \gamma_1\Pi_1 + \gamma_{18}\Pi_2 + \gamma_{13}\Pi_5$
7. $H77 < 0$ when,$\gamma_{11}\Pi_8 + \gamma_{12}\Pi_9 + \gamma_{14}\Pi_{11} < \gamma_1\Pi_6 + \gamma_{18}\Pi_7 + \gamma_{13}\Pi_{10}$.
8. $H11 < 0$ when, $\gamma_{11}\Pi_{14} + \gamma_{12}\Pi_{15} + \gamma_{14}\Pi_{17} > \gamma_1\Pi_{12} + \gamma_{18}\Pi_{13} + \gamma_{13}\Pi_{16}$.

Proof. See Appendix B.

4.6. Proposed algorithm

In this subsection, we have presented an algorithm to find the optimal solutions of our proposed model.

Step I. Assign $i = 1$ and define all parameters by entering their values.
Step II. For every L_{bi}, carryout steps IIa-IId for each $i = 1, 2, ---n$.
Step III. Assign $Q_{bi}, \theta, f_i, S_{ei}, \Omega_{mp}$ with their initial values , mp=1,2,.....,6.
Step IV. Then, set $i = i + 1$. Find the values of D_{dc}.
Step V. Using D_{dc}, find the value $Q_{bi}, \theta, f_i, S_{ei}$, and Ω_{mp}
Step VI. Continue repeating III-VI until no change occurs in the value of $Q_{bi}, \theta, f_i, S_{ei}$, and Ω_{mp} for which the values are denoted as $Q^*_{bi}, \theta^*, f^*_i, S^*_{ei}$, and Ω^*_{mp}.

Step VII. If $S^*_{ei} < S_{eo}$, then Step VII follows. As an alternative, set $S^*_{ei} = S_{eo}$ and use (21)-(31) to calculate new optimum values $Q^*_{bi}, \theta^*, f^*_i, S^*_{ei}, \Omega^*_{mp}$. Follow the same procedure outlined in Step II, then proceed to Step VII. Calculate JNP $(Q^*_{bi}, \theta^*, f^*_i, S^*_{ei}, \Omega^*_{mp})$ and $Max_{i=1,2,--n} JNP(Q^*_{bi}, \theta^*, f^*_i, S^*_{ei}, \Omega^*_{mp}.)$.When $Max_{i=1,2,--n} JNP = JNP$, $JNP\ (Q^*_{bi}, \theta^*, f^*_i, S^*_{ei}, \Omega^*_{mp}.)$ is considered as the optimum solution for a given fixed n.

5. Numerical experiment

Through practical applications of real-world concepts, we can foster our understanding and the usability of models, alongside validating them through various concepts. The use of diverse examples helps in validating the model's robustness, accuracy, and practicality. All simulations in this study have been performed using MATLAB R2021b for numerical analysis and generating results, and figures have been drawn using Mathematica 11.3. The run time for this proposed model was approximately 0.03 microseconds.

5.1. Example

The data given below is adapted from Sarkar et al. [12], with the input values adjusted to best fit this model. $C_{pboff} = 5$ \$/unit/unit time, $Cm = 25$ \$, $r_h = 45$, m_l =1, $L_b = 28$, $S_{pv} = 100$ \$/unit/unit time, $C_{ob} = 160$\$/unit/unit time, $P_r = 1590$ unit/time, $S_{eo} = 30$, $C_{pv} = 4$ \$/unit/unit time, $C_{ond} = 0.02$ \$/unit, $C_{offd} = 0.02$\$/unit, $H_v = 25$ \$/unit/unit time, $H_b = 25$ \$/unit/unit time, $C_{tf} = 20$ \$, $C_{tv} = 30$ \$, $C_s = 1$, $t_u = 0.01$, $\pi = 18$, $r_h = 7$, $\vartheta = 120$, $g_2 = 150$, $g_3 = 30$, $g_4 = 20$, $g_5 = 150$, $g_6 = 250$, $l = .001$, $l_2 = .005$, $l_3 = .002$, $l_4 = 0.004$, $l_5 = .00007$, $l_6 = .00008$, $C_{ml} = 5$,$Cd = 1.2$, $Be = 0.94$, $\lambda = 0.001$, $A = 25$ \$, $B = 15$\$, $R_d = 0.02$, $\theta_{max} = 0.25$, $\theta_{min} = 0.19$, $Clb = 2.4$, $s_g = 70$, $a_f = 0.01$,$b_f = 0.02$, $e_f = 0.01$, and $k_f = 0.04$.

Table 2. Decision variable values for determining the optimal profit.

Decision variable	Optimal value	Decision variable	Optimal value
Q_b (unit)	173.41	Ω_2(\$/unit)	256.66
S_e (\$/setup)	10.9	Ω_3(\$/unit)	55.89
f	3.4	Ω_4(\$/unit)	31.36
θ	0.19	Ω_5(\$/unit)	47.87
Ω_a (\$/unit)	179.24	Ω_6(\$/unit)	22.90

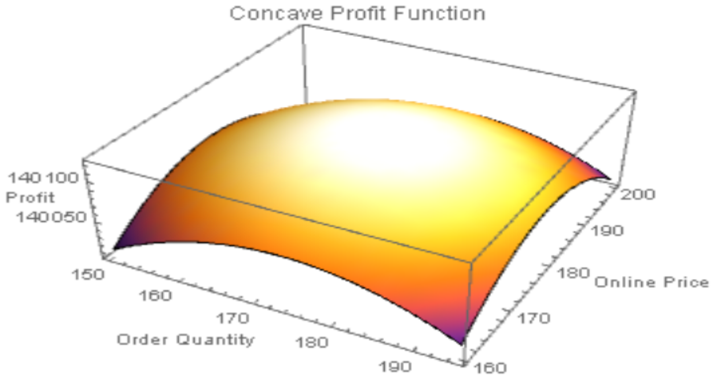


Figure 5. Global concavity of the joint profit in connection with OnSp and the size of optimal order.

Table 2 describes the optimal value of the order amount, setup costs, safety stock, machine failure rate, online sales price (OnSp), offline sales price (OffSp), online consumer service price, offline consumer service price, online advertisement price, and offline advertisement price for $m_l = 1$ and $L_b = 28$ for the dual-channel supply chain system. With the help of these decision variables, joint profit is maximized, which is graphically presented in Figure 5. This approach reduces the need for storage space, cutting down on the cost related to warehousing, insurance, and the risk of the inventory becoming outdated. Through the Hessian matrix, global concavity is proven. The principal minors of the Hessian matrix are as follows: $|H_{11}| = -0.18(< 0)$, $|H_{22}| = 0.10547(> 0)$, $|H_{33}| = -448051(< 0)$, $|H_{44}| = 2.31 \times 10^6(> 0)$, $|H_{55}| = -1511.37(< 0)$, $|H_{66}| = 3.82(> 0)$, $|H_{77}| = -0.0058(< 0)$, $|H_{88}| = 0.00015(> 0)$, $|H_{99}| = -0.96 \times 10^{-8}(< 0)$, $|H_{1010}| = 9.74 \times 10^{-10}(> 0)$.

5.2. *Special cases*

This section describes some specific cases to highlight the relevance of this analysis. Examining extreme cases helps verify the model's robustness, detect potential biases, and test its generalization. This analysis checks if the model works well in both general and specific situations, which builds confidence in its reliability.

5.2.1. Offline functionalities are temporarily unavailable

A physical outlet builds trust, brand recognition, and allows personalized services. It enables product trials, encourages impulse purchases, and reduces returns. In contrast, relying only on online channels weakens consumer confidence, increases competition risk, and raises return rates, reducing profitability. From the Table 3, when offline facilities are unavailable, the offline selling price is absent, the demand decreases by 56.25%, and the retailer's order quantity is increased by 34.71%. When demand is occasionally decreased, the retailer jumps on more orders because the retailer wants to make sure of having enough stock to satisfy consumer demand whenever they need. Also, the joint profit decreases by 62.90% when the offline facility is absent. The global concavity of the joint profit for unavailable offline facilities is presented in Figure 6(b).

Table 3. Decision variable values for different values of m_l.

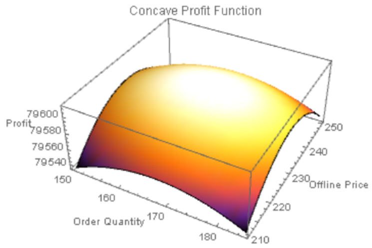
	Q_b	S_e	Ω_a	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6	Demand	Profit
Offline facilities are unavailable	233.61	20.37	192.60	—	10.31	—	1.84	—	313.08	51972.44
Online facilities are unavailable	167.75	18.13	—	230.65	—	75.43	—		416.14	79612.00
Consumer service unavailable	171.54	11.60	179.21	242.83	—	—	66.86	31.96	665.43	124712.15
Investment for advertisement unavailable	324.57	28	178.93	163.62	17.92	14.43	—	—	315.52	129393.97
Investment to reduce the setup costs unavailable	174.4	—	179.20	244.20	69.71	40.18	63.2	30.21	715.70	134893.25
Free delivery facilities unavailable	173.41	10.90	179.20	244.06	70.16	40.47	63.70	30.45	705.7	136970.65

5.2.2. Online functionalities are temporarily unavailable

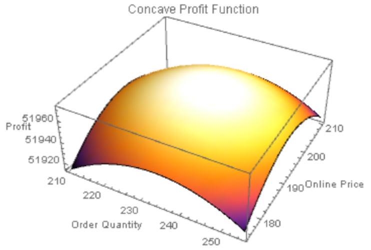
Eliminating online retailing confines sales to physical stores, leading to lost customers who avoid in-store visits. Consumers who depend on online stock checks or assistance may become dissatisfied, weakening their experience and brand loyalty. From Table 3, when online facilities are unavailable, customers have limited access to products, leading to a 41.85% drop in demand. This forces retailers to reduce their order quantity by 5.57% to avoid overstocking. Consequently, profit decreases by 43.18% when the online facility is absent. The global concavity of the joint profit for unavailable online facilities is presented in Figure 6(a). Uncertain demand leads retailers to cut orders, reducing the financial risk and operational cost for greater efficiency.

5.2.3. Consumer service functionalities are temporarily unavailable

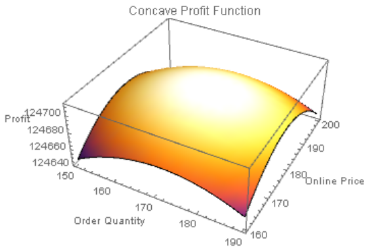
Consumer service resolves product or service issues, preventing returns and extra cost. Poor service leads to lost sales, damaged reputation, negative reviews, and loss of customer loyalty. From Table 3, the temporary unavailability of consumer service reduces customers' confidence in problem resolution, leading to a 7.02% decline in demand. Consequently, retailers reduce their order quantity by 1.07% and adjust offline selling prices downward by 5.38% to attract customers. As a result, the profit decreases by 10.99%. The lack of consumer service reduces trust and satisfaction, pushing customers toward alternatives and lowering demand. Retailers respond with price cuts and smaller orders to manage cost, but this further reduces their profit margins, creating a chain reaction that weakens demand, prices, orders, and profit. The global concavity of the joint profit for unavailable consumer services is presented in Figure 6(c).



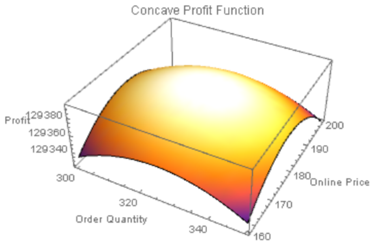
6(a) Global concavity of the joint profit when online facilities are unavailable.



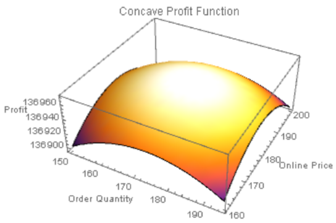
6(b) Global concavity of the joint profit when offline facilities are unavailable.



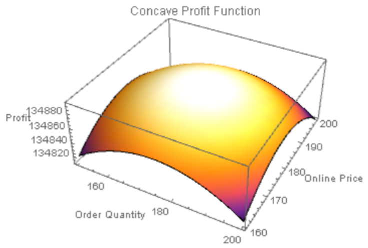
6(c) Global concavity of the joint profit when consumer services are unavailable.



6(d) Global concavity of the joint profit when advertisement facilities are unavailable.



6(e) Global concavity of the joint profit when when free delivery facilities are unavailable.



6(f) Global concavity of the joint profit when investment to reduce setup costs facility is unavailable.

Figure 6. Global concavity of joint profit for different special circumstances.

5.2.4. Investment for advertising is temporarily unavailable

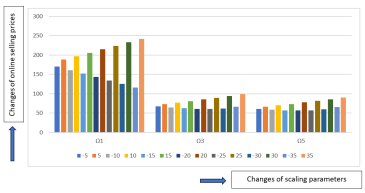
Investment in advertising is essential for informing consumers, attracting new buyers, and sustaining profit growth. Without it, firms rely only on existing customers, which weakens brand penetration and sales opportunities. From Table 3, the absence of advertising reduces demand by 55.91% and profit by 7.65%. The global concavity of the joint profit for the absence of advertisement investment is presented in Figure 6(d). The absence of advertisement not only reduces demand and profit but also pushes retailers to cut their selling prices. Specifically, online prices fall slightly by 0.17%, while offline prices drop sharply by 36.25%. Ultimately, the lack of advertising leads to a reliance on price competition and inventory adjustments, resulting in reduced profitability and market presence.

5.2.5. Free delivery facilities are temporarily unavailable

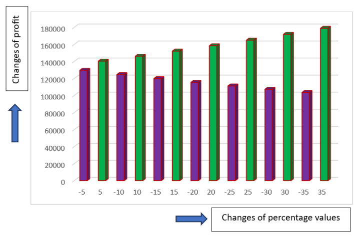
When free home delivery facilities are withdrawn, consumers face additional delivery charges that increase the overall purchase cost. This lowers the product's value in the eyes of consumers, causing them to buy less, choose competitors with free delivery, or shop offline instead. From Table 3, when free delivery is temporarily unavailable, consumers perceive higher total purchase cost, leading to a 1.36% drop in demand. The lower demand reduces overall turnover, resulting in a 2.25% decrease in profit. Retailers slightly increase prices online by 0.02% and offline by 4.9% to recover some of the lost profit. The global concavity of the joint profit for unavailable free delivery facilities is presented in Figure 6(e).

5.2.6. Investment for reducing setup costs is temporarily unavailable

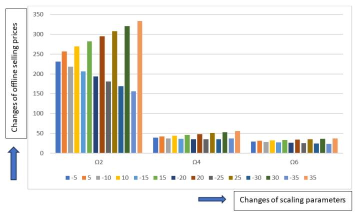
High setup costs reduce competitiveness for firms without cost control. Without investing in advanced technology or automation, operational costs stay high and profit declines. The absence of investment in reducing the setup costs leads to lower demand, driven by consumer dissatisfaction. From Table 3, when the setup costs investment is absent, total demand decreases by 2.37% because of receiving low-quality products. At the same time, profit falls by 3.7% as demand decreases. The retailer increases online and offline customer service costs by 24.72% and 28.12%, respectively, to manage additional customer complaints about products. The global concavity of the joint profit for the lack of investment to reduce the setup costs is presented in Figure 6(f).



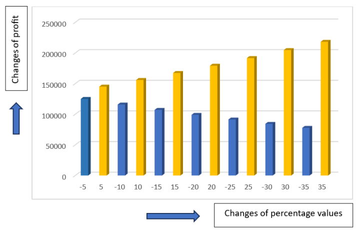
7(a) The significance of online scaling parameters for selecting online selling prices.



7(b) Effects of online scaling parameters in joint profit.



7(c) The significance of offline scaling parameters for selecting online selling prices.



7(d) Effects of offline scaling parameters in joint profit.

Figure 7. Significance of scaling parameters in online-offline selling price strategies.

5.2.7. Significance of scaling parameters in online-offline selling price strategies

When the scaling parameters of the selling price are low, then the selling price becomes low, and when the scaling parameters of the selling price are high, then the selling price becomes high. Lower scaling parameters tend to higher profit, and higher scaling parameters tend to lower profit. One strategy is to have more clients in its purview or remain competitive as the selling price increases when scaling reduces. Figure 7 represents how the online-offline selling price depends on the scaling

parameters. The variation of the scaling parameters for the online-offline selling price varies from -5% to -35%, and from 5% to 35%. When the scaling parameter for the online selling price decreases by 5% and 35%, the online selling price decreases by 4.99% and 5%, the online consumer service cost decreases by 4.36%, and 5.57%, and the online investment for advertisements decreases by 4.11%and 6.67%, respectively. As all online selling prices decrease, joint profit also decreases by 3.88% and 23.12%. Similarly, when the scaling parameter for the online selling price increases by 5% and 35%, the online selling price increases by 5% and 34.98%, the online consumer service cost increases by 4.84% and 41.36%, and the online investment for advertisement increases by 4.71% and 42.06%, respectively. As all online selling prices increase, joint profit also increases by 8.36% and 32.73%.
When the scaling parameter for the offline selling price decreases by 5% and 35%, the offline selling price decreases by 5.18% and 36.04%, the offline consumer service cost decreases by 4.10% and 7.68%, and the offline investment for advertisements decreases by 3.05% and 21.47%, respectively. As all offline selling prices decrease, joint profit also decreases by 7.24% and 42.30%. Similarly, when the scaling parameter for the offline selling price increases by 5% and 35%, the offline selling price increases by 5.17% and 36.38%, the offline consumer service cost increases by 4.49%, 37.60%, and offline investment for advertisements increases by 2.98%and 21.05%, respectively. As all offline selling prices increase, joint profit also increases by 7.64% and 61.88%.

5.3. A comparative analysis

Table 4. A detailed comparison with related works available in the literature.

	Choi et al. [24]	Pal et al. [17]	Sarkar et al. [16]	This model
Production rate (units)	Constant	Constant	Variable	Constant
setup costs ($)	Constant	Constant	Variable	Variable
Advertisement investment ($)	Not considered	Not considered	Considered	Considered
Consumer service ($)	Not considered	Not considered	Not considered	Considered
Total joint profit ($)	1455.97	54931.6	49217.3	140123.75

Table 4 presents a detailed comparison among the models proposed by Choi et al. [24], Pal et al. [17], Sarkar et al. [16], and the current study. The comparison considers key parameters such as the production rate, the setup costs, the advertisement investment, the cost of consumer service, and total joint profit. In this study by Choi et al. [24], the production rate was 2100 units with a setup costs of $30, where both were fixed, and the total joint profit was $1455.97. The demand in this model depended on the dual-channel service level. Here, both advertising investment and consumer service costs were excluded from consideration. In Pal et al. [17], the production rate was 1000 units, and the total joint profit obtained was $54931.6. The demand in this model was determined by the green investment level and the selling price. Here, investment for advertisement and cost for consumer service were not considered. In Sarkar et al. [16], the production rate and the setup costs were 253.96 units, $20.06 , where both were variable, and the total joint profit obtained was $49217.3. The demand in this model was determined by selling price, advertising cost, and automation policy. Here, the cost for consumer service was not considered. In contrast, the proposed model has a production rate of 1590 units, the setup costs of \$10.90 , and a total joint profit of \$140123.75 . The model also has included advertisement investment of \$47.84 for online and 22.88\$ for offline advertising. Here, the consumer service facilities cost \$55.85 for online and \$31.35 for offline customer care. Here, the demand is influenced by online–offline selling prices, advertisement investment, and consumer service facilities. The integration of dual-channel pricing, advertisement investment, and enhanced consumer service facilities, combined with lower setup costs, allows the proposed model to generate a significantly higher total joint profit compared with existing models.

6. Sensitivity analysis

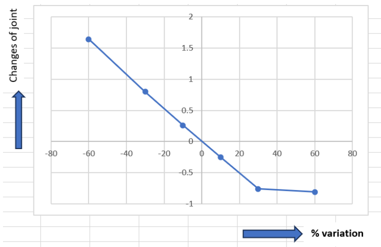
Sensitivity analysis provides a transparent view of possible outcomes under varying scenarios, enabling stronger decision-making. It validates models by testing their response to input changes and helps identify the most influential variables affecting the results. This section examines a sensitivity analysis of various parameter values and unit cost with respect to the overall profit. The parameter values change by $-10\%, -30\%, -60\%, -90\%, 10\%, 30\%, 60\%$, and 90%. Within this segment, a sensitivity investigation is described regarding various parameter values and unit prices concerning the profitability of the entire system. Figure 8 and Figure 9 represent the impact of different parameters related to this study on the joint profitability for the retailer as well as the manufacturer.

- In Figure 8, when holding cost for manufacturer and retailer were decreased by 10%, joint profit was increased by 0.26% and 0.78%, respectively. When the holding cost for the manufacturer and retailer were decreased by 60%, joint profit was increased by 1.64% and 5.71%, respectively. When holding cost for manufacturer and retailer were increased by 10%, joint profit was decreased by 0.25% and 0.77%, respectively. When the holding cost for the manufacturer and retailer were increased by 60%, joint profit was decreased by 0.81% and 18.16% respectively. When the holding cost reduced, the total cost for inventory purposes reduced, which directly improved the overall profit margins. Effective inventory management reduced the risk of overstocking. Excess inventory held for long periods became obsolete or expired, leading to losses.

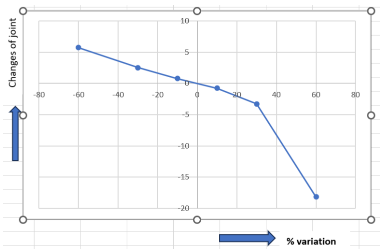
- In Figure 9, when the production rate increased by 10%, joint profit decreased by 0.23%. When the production rate decreased by 10%, joint profit increased by 0.20%. When the production rate increased by 60%, joint profit decreased 1.01%. When the production rate decreased by 60%, joint profit increased by 3.49%. Higher production rates required overtime pay, hiring additional workers, or adding more shifts, increasing the labor cost disproportionately. When the production rate increased, the labor cost, manufacturing cost, and holding cost increased simultaneously, which led to decreased profitability.

- In Figure 8, when the retailer's ordering cost increased by 10%, joint profit decreased by 0.034%. When the retailer's ordering cost increased by 60%, joint profit decreased by 0.20%. When the retailer's ordering cost decreased by 10%, joint profit increased by 0.03%. When the retailer's ordering cost decreased by 60%, joint profit increased by 0.21%. When ordering costs were reduced, shipping, handling, and administrative costs were also reduced simultaneously. Higher ordering costs resulted in less efficient inventory management, leading to stockouts or excess inventory that decreased joint profit.

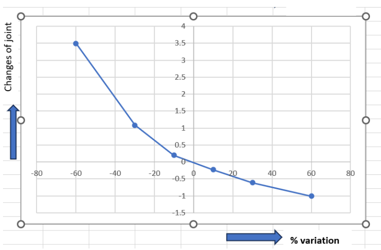
- The transportation cost was a very significant cost for determining the profitability. In Figure 8, it has been observed that when the variable transportation cost decreased by 10%, joint profit increased by 0.68%. When the transportation cost decreased by 60%, joint profit increased by 4.85%. When the transportation cost increased by 10%, joint profit decreased by 0.64%. When the transportation cost increased by 60%, joint profit decreased by 3.48%. When the transportation cost decreased, profit increased due to a reduction in overall operating expenses. Lower transportation cost directly reduced the cost of delivering goods or services, which, in turn, improved the contribution margin. This allowed the company to retain more revenue from each sale, thereby boosting overall profitability.



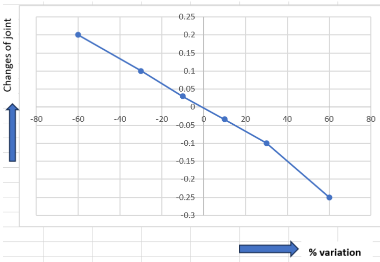
8(a) Profit variation with respect to the manufacturer's holding cost.



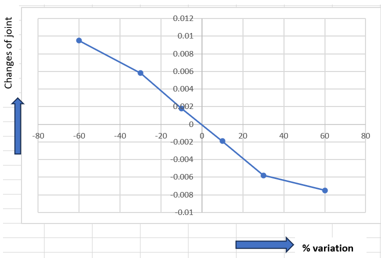
8(b) Profit variation with respect to the retailer's holding cost.



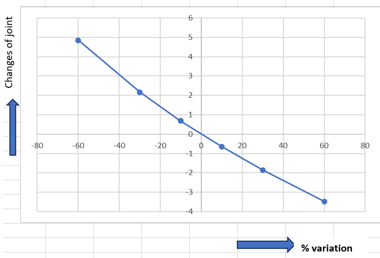
8(c) Profit variation with respect to the production rate.



8(d) Profit variation with respect to the ordering cost.



8(e) Profit variation with respect to total investment.

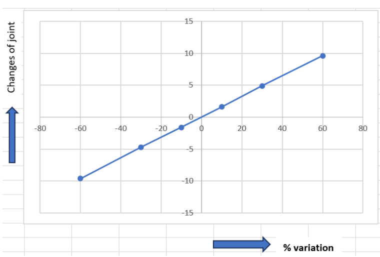


8(f) Profit variation with respect to the transportation cost.

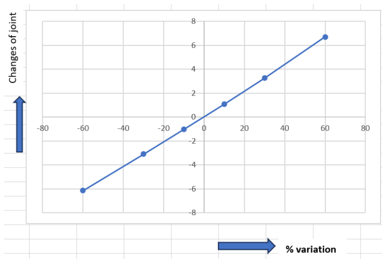
Figure 8. Sensitivity analysis of various parameter values with respect to the total joint profit.

- Figure 9 shows that when the scaling parameter for the online and offline selling price, online-offline consumer service cost, and online and offline advertisement cost decreased by 10%, joint profit decreased by 1.59%, 2%, 1.05%, 0.95%, 5.14%, and 11.45%, respectively.

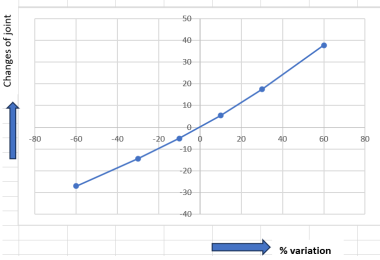
When the scaling parameter for the online and offline selling price, online and offline consumer service cost, and online and offline advertisement cost decreased by 60%, joint profit decreased by 9.5%, 12.36%, 6.15%, 5.64%, 27.10%, and 44.96%, respectively. When the scaling parameter for online & offline selling price, online-offline consumer service cost, and online and offline advertisement cost increased by 10%, joint profit decreased by 1.59%, 2%, 1.05%, 0.95%, 5.14%, and 11.45%, respectively. When the scaling parameter for online & offline selling price, online-offline consumer service cost, and online-offline advertisement cost increased by 60%, joint profit decreased by 9.5%, 11.87%, 1.05%, 6.63%, 5.85%, and 47.77%, respectively. The selling prices decreased when the scaling parameters increased. When the prices were reduced, the cost per unit was covered by the increase in volume and a reduction in variable costs, which impacted profitability directly.



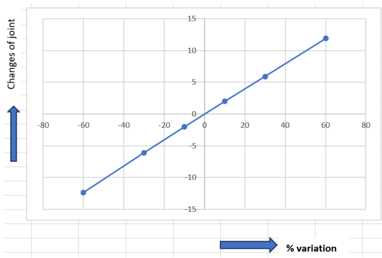
9(a) Global concavity of the joint profit to the shape parameter for OSP.



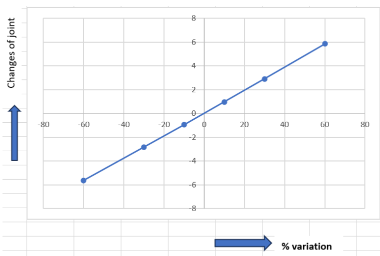
9(b) Global concavity of the joint profit to the shape parameter for OCS.



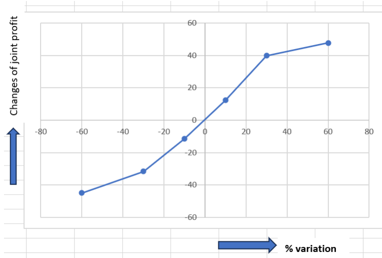
9(c) Profit variation with respect to the shape parameter for OAI.



9(d) Profit variation with respect to the shape parameter for OFSP.



9(e) Profit variation with respect to the shape parameter for OFCS.



9(f) Profit variation with respect to the shape parameter for OFAI.

OSP-online selling price; OCS-online consumer service; OAI-online advertisement investment; OFSP-offline selling price; OFCS-offline consumer service; OFAI-offline advertisement investment.

Figure 9. Sensitivity analysis of various parameter values with respect to the total joint profit.

7. Managerial insights

- Dual-channel retail helps manufacturers reach more customers. It improves the customer experience, provides useful data for better decisions, supports marketing, and reduces risks. Overall, it makes business smoother, increases brand value, and builds a stronger market position. This research guides manufacturers and retailers in making key decisions about the dual-channel supply chain system.
- Cost minimization is the foremost target of any business. The selection of appropriate selling prices is also a challenging task. Using this study, managers can make appropriate decisions about the best-selling prices of the different channels and the optimal order quantity. This study helps manufacturers make investment decisions, particularly by focusing on reducing the setup costs on the basis of its findings.
- Managers can take guidance from this study to reduce the setup costs, which allows firms to allocate resources more efficiently, lower overall production expenses, improve operational flexibility, respond faster to market demand, and achieve better profit margins.
- Managers can take guidance from this study to monitor machine failure rates for accurate the production cost, plan maintenance, reduce unexpected expenses, improve reliability, lower production costs, and achieve higher profitability through preventive maintenance.
- Consumer service facilities are important in the dual-channel supply chain system to improve the consumers' satisfaction. The members of the consumer service team help consumers with the initiation and smoothest possible procedure to receive a product refund or replacement of the goods in a hassle-free manner. From this research, manufacturers and retailers will be able to make retailing systems more consumer-friendly by considering effective consumer service facilities.
- Advertising is an important component of business as it performs the dual function of marketing goods and services. Hence, effective investment of advertising funds will improve profit. Retail sector managers can make appropriate investment decisions for advertising with the help of this study to maximize profit.

8. Conclusion

This study discussed how dual-channel strategies affect the total profit of a manufacturing firm in the supply chain system. Consumer demand was observed to be affected by the price of the sale, the efforts to advertise, and the cost of the consumer care service in the dual channel system for the supply chain system. Investment for reducing setup costs enhanced the accuracy of the inventory management, facilitating just-in-time production methods that further reduced cost and boosted profitability. An algorithm was suggested to find the optimal profit. A case study for certain scenarios was discussed here. It was found that, when offline facilities were absent, profit decreased by 62.90%. According to this study, online facilities also played a vital role in profit maximization, as profit decreased by 43.18% when online facilities were unavailable. Consumer service facilities had a pivotal role in profit maximization, as profit decreased by 10.99% when these facilities were absent. Without investment in advertising, the business didn't work as well as expected profit decreased by4.03%. When free delivery facilities stopped, though demand decreased by 1.36% and profit decreased by 2.25% because as demand dropped, the company lost sales, leading to decreased profitability. Investment to reduce setup costs was essential as profit decreased by 7.37% because of the lack of this facility. The scaling parameters for all types of selling prices had a great impact on the selection of the optimum selling prices. The major drawback of this study was that it considered a fixed production rate, whereas demand fluctuation is a very common scenario nowadays. Another limitation was considering only perfect production, which is not 100% possible nowadays. Imperfect production is a frequent occurrence in almost all sectors of the supply chain system. Further exploration of this study can be done by considering imperfect production as a key factor in satisfying consumers. Dealing with multiple products can be another extension of this study to increase profit in the supply chain system. In future work, the model can be extended to include eco-friendly supply chain practices, carbon emission regulations, and green investment decisions. It can be extended to examine how investment in research and development improves production efficiency, reduces the costs, and supports environmental sustainability.

Author contributions

Nandita Barman: Methodology, writing original draft, writing review, editing, software, and validation. Adrijit Goswami: Investigation, conceptualization, and supervision. Biswajit Sarkar: Conceptualization, resources, and supervision. Mitali Sarkar: Conceptualization, resources, and supervision.

Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

Use of Generative-AI tools declaration

The authors declare that they did not utilize any artificial intelligence (AI) tools in the creation of this article.

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