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Research article

Delayed control model for multi-stage closed-loop continuous MRP with rework and recycle

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Abstract: Inventory accumulation is a natural consequence of material flow in manufacturing systems, particularly under dynamic conditions. Manufacturing companies must manage a diverse range of inventory types, including raw materials, semi-finished goods, and finished products. This article proposes a novel multi-stage continuous material requirements planning (MRP) system formulated as a linear–quadratic optimal control model that incorporates production lead times, returns, rework, and recycling. The delayed-control framework provides the foundation for a sustainable production-planning technology roadmap through delay-dependent feedback, dynamic flow relationships, and a multi-stage recovery operation. The roadmap emphasizes the ability to synchronize production and inventory decisions across interrelated stages of production and inventory levels, producing operational, efficiency, and stability improvements. Applied to a fertilizer production case study, the model reduced inventory deviations, enhanced coordination of ordering and production activities, and improved responsiveness to demand changes. The inclusion of perishability and delay mechanisms produced more realistic production planning and waste minimization.

Keywords: optimal control; continuous material requirements planning; multi-stage production systems; reworking; recycling; lead time
Mathematics Subject Classification: 34H05, 90B05

1. Introduction

A substantial portion of a company's capital is tied up in inventory, and hence its effective management is necessary to maximize the utilization of the facility and optimal operations. Effective inventory control maintains the optimal stock level, preventing shortages and excesses, which reduces the impact of disruptions and minimizes unnecessary usage of capital. It enhances supply chain resilience, allowing for quicker recovery from disruptions [1].

In manufacturing, multi-stage manufacturing systems involve cumulative procedures that create work-in-progress (WIP) inventory, which is costly, and therefore, WIP control becomes significant. Lead time control is also essential for supply chain responsiveness, reducing inventory risk, improving customer service, and harmonizing production with demand fluctuation, ensuring consistent on-time delivery for customer satisfaction [2].

Material requirements planning (MRP) is a planned approach to material availability, ensuring quantities, locations, and times are adequate to fulfill production and customer orders [3]. It reduces costs by optimizing inventories and reorder points [4], and it aids operational priorities and capacity planning [5]. Traditional MRP models operate on discrete time intervals, which are unsuitable for continuous manufacturing environments (e.g., petrochemical industries), where inventory levels vary smoothly [6]. Continuous MRP bridges this gap by making real-time control possible. However, most models are yet to be single-stage and lack the multi-stage bill of materials (BOM) framework, or rule out the use of lead time as a control factor [6]. This highlights an important gap: there being no common MRP model that possesses continuous-time dynamics, multi-stage BOM, and lead-time handling.

To bridge this gap, we introduce a model of delayed control for continuous MRP with rework and recycling opportunities. It covers production lead time and return phases in an explicit multi-level framework. It also incorporates defect processing through rework and recycling to minimize waste and optimize resource utilization. We apply the model in an actual industrial fertilizer plant, where inadequate consideration of lead time resulted in overstocking, higher storage costs, and product obsolescence. By modeling lead time and incorporating a delayed control structure, our approach enhances inventory flow, reduces waste, and improves system responsiveness, even in a mature industrial setting.

Although the case study is based on a process for fertilizer production, the proposed framework is applicable to other multi-stage, continuous production processes. The model is particularly relevant in situations where product degradation, recycling in the plant, rework processes, and production lead times are operationally significant. The novelty of this work resides in the incorporation of reworking, recycling, and production lead-time constraints into a continuous-time optimal control structure following a linear-quadratic (LQ) structure. This approach enables the explicit consideration of production delays and the associated feedback from rework and recycling flows, resulting in an agile control structure for multi-stage manufacturing systems that becomes analytically tractable. These features together distinguish the proposed model from the literature and render it more relevant to real-world, complex manufacturing systems.

This work is organized as follows. Section 2 offers a review of literature related to this study. The proposed approach is described in Section 3, including two mathematical models for with lead time and without lead time modes, while Section 4 describes the proposed linear-quadratic optimal control model. Section 5 illustrates the proposed approach with the production system and data for fertilizer produced by the Yeganeh Yaran Shokoofeh Dasht Company in Iran. Section 6 will feature practical and theoretical discussions concerning the model and its simulation. Finally, concluding remarks and suggestions for future research are provided.

2. Related work

This section categorizes existing studies around four defining characteristics of the proposed model: (i) in-plant rework and recycling, (ii) deteriorating or perishable items, (iii) production lead times, and

(iv) other cost-driven and MRP-related models. This structure helps clarify the progression of prior work and emphasizes the novelty of the proposed model.

2.1. Rework and recycling in multi-stage systems

Numerous studies have focused on return stages, rework, or recycling in production systems. Pooya and Pakdaman [7] introduced an optimal control model using neural networks for single-stage systems. In a follow-up study, they extended the model to multi-stage production with lead times [8], but without incorporating core MRP features. In another work [6], they considered continuous MRP with returned items but excluded production lead times. Miri [9] proposed a partial differential equation and ordinary differential equation (PDE-ODE)-based optimization model using genetic algorithms, but ignored returns and lead time delays. Sowmica and Suvinthra [10] analyzed a deteriorating two-item system lacking rework, recycling, and lead time. Rachih et al. [11] addressed reverse logistics with rework, yet did not model production lead times. Öztürk [12] developed a rework and breakdown model of imperfect production without recycling. Eshaghnezhad et al. [13] applied bounded optimal control methods, again without lead times or return stages. Several contemporary studies have investigated sustainable or stochastic inventory systems that are combined with reworking and recycling aspects. For example, Aiello et al. [14] analyzed a closed-loop structure but noted that, in reality, full integration of multi-stage production and reworking is complicated. Recent contributions, such as Sharma et al. [15], introduced a green supply chain model incorporating inspection and rework using the adaptive neuro-fuzzy inference system (ANFIS) and metaheuristic optimization. Similarly, Mahata and Debnath [16] developed a multi-stage economic production rate (EPR) model integrating flexible production and carbon emission considerations. Our study integrates these dimensions into a continuous multi-stage MRP framework with full return cycles.

2.2. Perishable products and deteriorating items

Deterioration in inventory systems has been widely examined, especially in models concerned with demand forecasting and uncertainty. Hatami-Marbini et al. [17] implemented hedging-point policies for deteriorating items in a network context. Foul et al. [18] built adaptive models for inventory in a situation of uncertainty of deterioration. Jiayu et al. [19] presented deteriorating products in terms of uncertainty theory. Megoze et al. [20] investigated remanufacturing with deteriorating production based on Hamilton-Jacobi-Bellman equations. Ignaciuk [21] considered reverse logistics but excluded deteriorated item disposal stages. However, these models typically treat single-stage systems and are not framed within MRP-based multi-stage environments. Recently, Covei [22] proposed a regime-switching model incorporating deterioration dynamics and probabilistic transitions in continuous production systems. However, these models typically treat single-stage systems and are not framed within MRP-based multi-stage environments. Our work embeds perishability directly into a dynamic, multi-stage, continuous MRP system, allowing more accurate control of production and storage across the bill of materials. Attia et al. [23] suggested a recycling-based optimization model and explicitly stated that future research needs to include multi-stage effects and lead times, both of which motivated this study.

2.3. Production lead times and delayed systems

Lead time has been studied as a significant operational factor in production and supply chain models. Louly et al. [4] scheduled components in assembly systems with random lead times. Grubbstrom and Tang [5] analyzed BOM structure effects on production timing. Milne et al. [24] determined optimal lead times using mixed-integer programming in MRP. Rigatos et al. [25] proposed a nonlinear optimal control model with lead times but no recycling. Hedjar et al. [26] optimized production rates using optimal control theory. Dizbin and Tan [27] incorporated lead time into demand-processing correlation models, and Al-Khazraji et al. [28] worked on dynamic production-inventory control with delay systems. Recent work by Jin et al. [29] presented a stochastic optimization approach for managing large-scale supply chains with lead time uncertainty. Utama and Putri [30] developed a stochastic production–inventory model with imperfect and scrap items; however, their method was also limited to a single-stage process and did not include delay effects. Additionally, Hake et al. [31] evaluated artificial intelligence methods to predict dynamic lead times in complex automotive production systems. Yet, most of these works omit integration with rework, perishability, and continuous-time MRP logic. Our approach incorporates lead times as explicit delays in a continuous optimal control setting, aligned with the real-time behavior of continuous production systems.

2.4. Other related MRP and cost-driven models

Sadeghian [3] introduced the continuous MRP (CMRP) framework and compared it to traditional discrete MRP (DMRP). Rossi et al. [32] developed an MRP model under capacity constraints. Le Thi and Tran [33] minimized costs in multi-stage systems via nonlinear integer programming. Mezghiche et al. [34] integrated production forecasting based on past demand and inventory. Singer and Khmelnitsky [35], ElHafsi et al. [36], and Gao [37] extended optimal control to stochastic systems, yet generally overlooked the full combination of rework, lead time, and BOM interactions. Dhaiban [38] and Nakhaeinejad et al. [39] also contributed optimal control-based models but did not include lead times or return flows. To address multi-stage uncertainty and cost-driven decisions, Schlenkrich et al. [40] developed a progressive hedging approach for stochastic lot sizing, while Tobares et al. [41] proposed a novel lot sizing framework grounded in physical system analogies.

While most research addresses individual features of continuous-time dynamics, rework, multi-stage systems, production lead times, finite capacity, recycling, optimization methods, and BOM integration in isolation, it is clear that there is a gap in terms of integrated models that can capture the full complexity of real-case production systems. To address this complexity, we propose a multi-stage MRP-based optimal control model that integrates production lead time, returns, rework, recycling, and finite capacity into a continuous-time framework.

3. Proposed model

This section presents the formulation of the proposed continuous-time multi-stage MRP model and its control structure.

The system involves combining raw materials (b and c) to form intermediate product A_1, which is then mixed with raw material d to produce the final product A. To produce one unit of A_1, α units of b and β units of c are required; one unit of A requires δ units of A_1 and γ units of d (see Figure 1).

Although we have modeled the example in a two-stage structure (e.g., two raw materials and one intermediate product produced in combination with another raw input), the model is scalable to follow any number of raw materials or intermediate stages. Since the equations and delay structures remain valid in such an extension, as long as the index sets for materials and production stages are simply extended, we can generalize the formulation to an n-level BOM system.

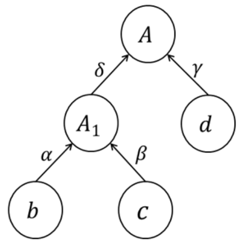


Figure 1. Three-level BOM structure of the product.

Pooya and Pakdaman [6, 8] proposed two related MRP models: one with finite capacity without lead time and another that incorporates production lead time. The model proposed in this paper has components of both of the models by Pooya and Pakdaman [6, 8]. The multi-stage continuous MRP production-inventory system in Figure 2 includes stages for recycling and reworking returned items. The proposed model addresses the disposal of deteriorating items as well.

In the proposed model illustrated in Figure 2, the first step orders the raw materials b and c, with order quantities $s_{mb}(t)$ and $s_{mc}(t)$, respectively. The inventory levels of these raw materials at time t are $I_{mb}(t)$ and $I_{mc}(t)$, respectively. The amounts of shipment of these items for production are $P_{mb}(t)$ and $P_{mc}(t)$.

In the first production stage, the two raw materials b and c are combined to produce the intermediate product A_1. In the second stage, this intermediate item A_1 is mixed with raw material d to generate the final product A. This description clarifies how the intermediate and final stages are connected within the production chain.

The first-stage inventory, $I_{mA_1}(t)$, is divided into two parts: the portion $P_{mA_1}(t)$ sent to the next stage for the production of A, and the remainder returned at rate ω_{m1} due to defects in production. Returned goods are either reworked at rate ω_{RA_1}, recycled at rate ω_{CA_1}, or disposed of at rate ω_{dA_1}. From the recycled inventory, the portions related to raw materials b and c, represented by $C_{rb}(t)$ and $C_{rc}(t)$, are added back to their respective inventories. Items that are not eligible for rework or recycling are treated as unusable intermediate products and are disposed of at a rate ω_{dA_1}. This ensures that the model properly accounts for material losses due to quality issues or degradation, and prevents unused intermediates from re-entering the production flow.

In the second production stage, the inventory of manufactured goods $I_{mA}(t)$ is divided into two parts. Part of the inventory is shipped to the customer according to the demand at time t, $D(t)$ (i.e., $P_{mA}(t) = D(t)$), and the remainder forms returned goods at rate ω_{m1}. Returned goods follow the same process as in the first stage: they may be reworked at rate ω_{RA}, recycled at rate ω_{CA}, or disposed of at rate ω_{dA}. Additionally, some shipped items, $R(t)$, may return from the customer and be added to the second-stage returned goods inventory.

For clarity, raw materials are indexed by $i \in \{b, c, d\}$, while intermediate and final products are indexed by $j \in \{A_1, A\}$. This distinction reflects their different roles in the production and return flows.

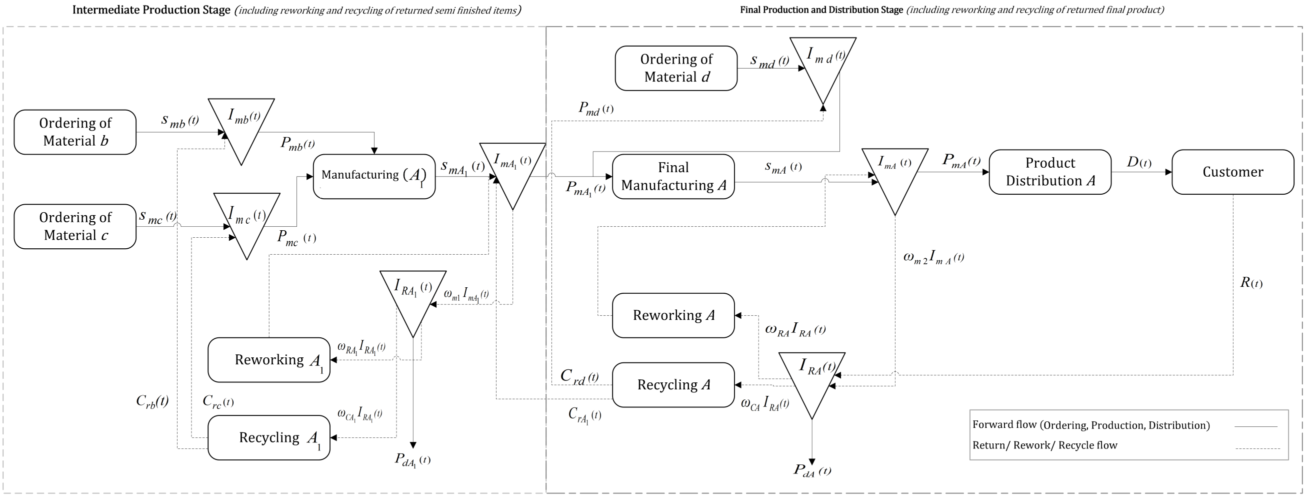


Figure 2. Schematic representation of the proposed multi-stage delayed MRP system.

The variables and parameters used are listed in Tables 1 to 3.

Table 1. State variables in the proposed model.

Notation	Description
$I_{mi}(t)$	Inventory of raw materials $i \in \{b,c,d\}$ at time t
$I_{mj}(t)$	Inventory of intermediate or final products $j \in \{A_1, A\}$ at time t
$I_{Rj}(t)$	Inventory of returned products $j \in \{A_1, A\}$ at time t

The numbers of shipped items b and c should be consistent with their usage in the BOM to prevent excessive shipment of raw materials to produce A_1. Specifically, the shipment of each raw material should not exceed the available inventory and should remain within practical release limits, as expressed in Eq. (3.1).

$$0 \le P_{mb}(t) \le \kappa_b I_{mb}(t), \qquad 0 \le P_{mc}(t) \le \kappa_c I_{mc}(t), \qquad 0 < \kappa_b, \kappa_c \le 1. \tag{3.1}$$

Here, κ_b and κ_c denote the maximum proportions of the available stock that can be released for production. This maintains a realistic link between inventory and shipment levels while preventing unrealistic dispatching of raw materials.

The production rate of the intermediate product A_1 is governed by the most constrained input material, ensuring that the process obeys the mass balance principle and avoids overconsumption of any input, as shown in Eq. (3.2).

$$s_{mA_1}(t) = \min\left\{ \frac{P_{mb}(t)}{\alpha}, \frac{P_{mc}(t)}{\beta} \right\}. \tag{3.2}$$

For items b and c to be used in the correct proportions and maintain flow balance, the relationship in Eq. (3.3) should hold. This condition prevents either input from being over- or underutilized during the production of A_1.

$$\frac{P_{mb}(t)}{\alpha} = \frac{P_{mc}(t)}{\beta} \tag{3.3}$$

Table 2. Control variables in the proposed model.

Notation	Description
$s_{mi}(t)$	Ordering and releasing level of raw materials $i \in \{b, c, d\}$ at time t
$s_{mj}(t)$	Ordering and releasing level of products $j \in \{A_1, A\}$ at time t
$P_{mi}(t)$	Production level of raw materials $i \in \{b, c, d\}$ at time t
$P_{mj}(t)$	Production level of intermediate and final products $j \in \{A_1, A\}$ at time t
$C_{ri}(t)$	Recycling level of returned raw materials $i \in \{b, c, d\}$ at time t
$C_{rA_1}(t)$	Recycling level of returned intermediate product A_1 at time t
$P_{dj}(t)$	Deterioration level of returned products $j \in \{A_1, A\}$ at time t
$D(t)$	Released demand for final product A at time t
$R(t)$	Returned level of final product A from the customer at time t

Table 3. Parameters and other notation in the proposed model.

Notation	Description
ω_{mj}	Return rate of manufactured product $j \in \{A_1, A\}$
ω_{Rj}	Reworking rate of returned product $j \in \{A_1, A\}$
ω_{Cj}	Recycling rate of returned product $j \in \{A_1, A\}$
ω_{dj}	Deterioration rate of returned product $j \in \{A_1, A\}$
α	Units of b required to produce one unit of A_1
β	Units of c required to produce one unit of A_1
γ	Units of d required to produce one unit of A
δ	Units of A_1 required to produce one unit of A
κ_i	Maximum proportion of inventory of raw material $i \in \{b, c, d\}$ released to production
κ_{A_1}	Maximum proportion of inventory of intermediate product A_1 released to production
τ_j	Production lead time for product $j \in \{A_1, A\}$
T	Planning horizon
t	Continuous time variable, $t \in [0, T]$
$(\cdot)^\top$	Transpose of a vector or matrix

As a result, the shipment level corresponds directly to the balanced production rate of A_1, as indicated in Eq. (3.4).

$$s_{mA_1}(t) = \frac{P_{mb}(t)}{\alpha} = \frac{P_{mc}(t)}{\beta} \tag{3.4}$$

Together, Eqs. (3.1)–(3.4) ensure that the first-stage production flow respects both inventory availability and proportional material usage.

In the second production stage, the intermediate item A_1 and raw material d are combined to produce the final product A. The shipping levels of these materials are also limited by their available inventories, as expressed in Eq. (3.5).

$$0 \le P_{mA_1}(t) \le \kappa_{A_1} I_{mA_1}(t), \qquad 0 \le P_{md}(t) \le \kappa_d I_{md}(t), \qquad 0 < \kappa_{A_1}, \kappa_d \le 1. \tag{3.5}$$

Eq. (3.6) ensures that the production rate of A depends on the most limiting input among A_1 and d, aligning with the principle of constrained production.

$$s_{mA}(t) = \min\left\{\frac{P_{md}(t)}{\gamma}, \frac{P_{mA_1}(t)}{\delta}\right\} \tag{3.6}$$

To maintain coordination between the shipments of d and A_1, Eq. (3.7) enforces their proportional usage in the production of A, ensuring that both inputs are released in balanced quantities.

$$\frac{P_{md}(t)}{\gamma} = \frac{P_{mA_1}(t)}{\delta} \tag{3.7}$$

Eqs. (3.5)–(3.7) collectively maintain proportional material flow and prevent overproduction in the final manufacturing stage. By inserting Eqs. (3.3) and (3.7) into Eqs. (3.2) and (3.6), the overall production rates of the intermediate and final products can be expressed in an equivalent and compact form, as shown in Eq. (3.8). These relations capture the synchronized flow between input materials and production stages, reflecting both the limiting input rule (via the $\min\{\cdot\}$ operator) and the balanced proportionality conditions of Eqs. (3.3), (3.4), (3.6), and (3.7).

$$\begin{cases} s_{mA_1}(t) = \min\left\{\frac{P_{mb}(t)}{\alpha}, \frac{P_{mc}(t)}{\beta}\right\} = \frac{P_{mb}(t)}{\alpha} = \frac{P_{mc}(t)}{\beta} = \frac{1}{2}\left(\frac{P_{mb}(t)}{\alpha} + \frac{P_{mc}(t)}{\beta}\right), \\ s_{mA}(t) = \min\left\{\frac{P_{md}(t)}{\gamma}, \frac{P_{mA_1}(t)}{\delta}\right\} = \frac{P_{md}(t)}{\gamma} = \frac{P_{mA_1}(t)}{\delta} = \frac{1}{2}\left(\frac{P_{md}(t)}{\gamma} + \frac{P_{mA_1}(t)}{\delta}\right). \end{cases} \tag{3.8}$$

Eq. (3.8) therefore emerges as a direct result of combining the proportional shipment relations (Eqs. (3.3), (3.4), (3.6), and (3.7)) with the mass-balance-based production definitions (Eqs. (3.2) and (3.6)). This ensures that both intermediate and final production rates are physically consistent and synchronized with their respective material inputs.

Following Figure 2, recycling and deterioration rates are expressed as:

$$\begin{cases} C_{rb}(t) = \alpha\omega_{CA_1}I_{RA_1}(t) \\ C_{rc}(t) = \beta\omega_{CA_1}I_{RA_1}(t) \\ C_{rd}(t) = \gamma\omega_{CA}I_{RA}(t) \\ C_{rA_1}(t) = \delta\omega_{CA}I_{RA}(t) \\ P_{dA_1}(t) = \omega_{dA_1}I_{RA_1}(t) \\ P_{dA}(t) = \omega_{dA}I_{RA}(t) \end{cases} \tag{3.9}$$

The partition of returned items among reworking, recycling, and deterioration is expressed in Eqs. (3.10) and (3.11), indicating that all returned products in both intermediate and final stages are fully distributed among the reworking, recycling, and deterioration flows.

$$\omega_{RA} + \omega_{CA} + \omega_{dA} = 1 \tag{3.10}$$

$$\omega_{RA_1} + \omega_{CA_1} + \omega_{dA_1} = 1 \tag{3.11}$$

3.1. Model 1: Without consideration of lead time

The dynamic behavior of the proposed model, considering the balance relationships in Eqs. (3.10) and (3.11), can be expressed in terms of the relationships in Eq. (3.12).

$$\begin{cases} \dot{I}_{mb}(t) = s_{mb}(t) + \alpha\omega_{CA_1}I_{RA_1}(t) - P_{mb}(t) & I_{mb}(0) = I^0_{mb} \\ \dot{I}_{mc}(t) = s_{mc}(t) + \beta\omega_{CA_1}I_{RA_1}(t) - P_{mc}(t) & I_{mc}(0) = I^0_{mc} \\ \dot{I}_{mA_1}(t) = \frac{1}{2\alpha}P_{mb}(t) + \frac{1}{2\beta}P_{mc}(t) + \omega_{RA_1}I_{RA_1}(t) + \delta\omega_{CA_2}I_{RA_2}(t) - P_{mA_1}(t) - \omega_{m1}I_{mA_1}(t) & I_{mA_1}(0) = I^0_{mA_1} \\ \dot{I}_{md}(t) = s_{md}(t) + \gamma\omega_{CA}I_{RA}(t) - b\gamma I_{md}(t) & I_{md}(0) = I^0_{md} \\ \dot{I}_{mA}(t) = \frac{1}{2\gamma}P_{md}(t) + \frac{1}{2\delta}P_{mA_1}(t) + \omega_{RA}I_{RA}(t) - P_{mA} - \omega_{m2}I_{mA}(t) & I_{mA}(0) = I^0_{mA} \\ \dot{I}_{RA_1}(t) = \omega_{m1}(I_{mA_1}(t) - P_{mA_1}(t)) - I_{RA_1}(t) & I_{RA_1}(0) = I^0_{RA_1} \\ \dot{I}_{RA}(t) = \omega_{m2}(I_{mA}(t) - P_{mA}(t)) + R(t) - I_{RA}(t) & I_{RA}(0) = I^0_{RA} \end{cases} \tag{3.12}$$

Column matrices $\tilde{x}(t)$ and $\tilde{u}(t)$ comprise state and control variables, respectively, and are defined in Eqs. (3.13) and (3.14).

$$\tilde{x}(t) = \begin{bmatrix} I_{mb}(t) & I_{mc}(t) & I_{mA_1}(t) & I_{md}(t) & I_{mA}(t) & I_{RA_1}(t) & I_{RA}(t) \end{bmatrix}^\top \tag{3.13}$$

$$\tilde{u}(t) = \begin{bmatrix} s_{mb}(t) & s_{mc}(t) & s_{md}(t) & P_{mb}(t) & P_{mc}(t) & P_{mA_1}(t) & P_{md}(t) & P_{mA}(t) & R(t) \end{bmatrix}^\top \tag{3.14}$$

By embedding Eqs. (3.8) and (3.9) into Eq. (3.12), the model dynamics can be represented in matrix form. The matrices M and N, whose entries form the coefficients of state and control variables, respectively, in Eqs. (3.13) and (3.14), are defined in matrices (3.15) and (3.16).

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \alpha\omega_{CA_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta\omega_{CA_1} & 0 \\ 0 & 0 & -\omega_{m1} & 0 & 0 & \omega_{RA_1} & \delta\omega_{CA} \\ 0 & 0 & 0 & 0 & 0 & 0 & \gamma\omega_{CA} \\ 0 & 0 & 0 & 0 & -\omega_{m2} & 0 & \omega_{RA} \\ 0 & 0 & \omega_{m1} & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & \omega_{m2} & 0 & -1 \end{bmatrix} \tag{3.15}$$

$$N = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}\alpha & \frac{1}{2}\beta & \omega_{m1} - 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\delta & \frac{1}{2}\gamma & \omega_{m2} - 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\omega_{m1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\omega_{m2} & 1 \end{bmatrix} \tag{3.16}$$

Given matrices M and N, Eq. (3.17) encapsulates a unified state-space representation, simplifying system analysis and control.

$$\dot{\tilde{x}} = M\tilde{x}(t) + N\tilde{u}(t), \quad \tilde{x}(0) = \tilde{x}_0 \tag{3.17}$$

To obtain the target values of the state and control variables, the difference relationship is defined in Eq. (3.18), where f is the control or state variable and $\hat{f}$ is the target value of f.

$$\Delta f(t) = f(t) - \hat{f}(t) \tag{3.18}$$

As such, the vectors $x(t)$ and $u(t)$ are defined in Eqs. (3.19) and (3.20). These vectors represent the deviations of the state and control variables from their target values.

$$x(t) = \begin{bmatrix} \Delta I_{mb}(t) & \Delta I_{mc}(t) & \Delta I_{mA_1}(t) & \Delta I_{md}(t) & \Delta I_{mA}(t) & \Delta I_{RA_1}(t) & \Delta I_{RA}(t) \end{bmatrix}^\top \tag{3.19}$$

$$u(t) = \begin{bmatrix} \Delta s_{mb}(t) & \Delta s_{mc}(t) & \Delta s_{md}(t) & \Delta P_{mb}(t) & \Delta P_{mc}(t) & \Delta P_{mA_1}(t) & \Delta P_{md}(t) & \Delta P_{mA}(t) & \Delta R(t) \end{bmatrix}^\top \tag{3.20}$$

The goals for the state and control variables are the constraints that apply to these variables in the model. The purpose of producing, reworking, or recycling items is the finite capacity for which the workstation is intended, and the purpose of inventories is the finite capacity that should be considered for warehouses. Therefore, when the objective function minimizes Δf_i, the control and the state variables converge to their target values. As such, the problem of linear binomial optimal control with finite time is obtained.

3.2. Model 2: With consideration of lead time

This model incorporates production lead times into the delayed control framework using time-shifted variables in the differential equations to represent internal delays. It handles internal lead times with the ability to change reorder points and safety stock to manage the control of supplier lead times. For example, work-in-progress product A_1 production lead time is τ_{A_1}, while that for the finished product A is τ_A. The change in inventory of A_1 is expressed in (3.21).

$$\begin{aligned} \dot{I}_{mA_1}(t) = {} & \frac{1}{2\alpha} P_{mb}(t - \tau_{A_1}) + \frac{1}{2\beta} P_{mc}(t - \tau_{A_1}) + \omega_{RA_1} I_{RA_1}(t) + \delta \omega_{CA} I_{RA}(t) \\ & - P_{mA_1}(t) - \omega_{m1} I_{mA_1}(t), \qquad I_{mA_1}(0) = I^0_{mA_1}. \end{aligned} \tag{3.21}$$

Eq. (3.22) is obtained considering the lead time for final product A.

$$\begin{aligned} \dot{I}_{mA}(t) = {} & \frac{1}{2\gamma} P_{md}(t - \tau_A) + \frac{1}{2\delta} P_{mA_1}(t - \tau_A) + \omega_{RA} I_{RA}(t) \\ & - P_{mA}(t) - \omega_{m2} \left(I_{mA}(t) - P_{mA}(t) \right), \\ & I_{mA}(0) = I^0_{mA}. \end{aligned} \tag{3.22}$$

Eqs. (3.21) and (3.22) reflect the inventory dynamics of work-in-process item A_1 and final product A, respectively, with production lead times occurring in the form of time-delayed terms. These equations adhere to the principle of flow conservation where the rate of change of the inventory equals inflows minus outflows. For Eq. (3.21), the inflow consists of items b and c being processed into A_1. Since production is not instantaneous, the contribution of these raw materials is modeled using delayed inventory terms $I_{mb}(t-\tau_{A_1})$ and $I_{mc}(t-\tau_{A_1})$. The constants κ_b and κ_c reflect the combined effect of shipment policies and proportional BOM usage in the delayed production process, respectively. Additional inflows to the inventory of A_1 come from reworked and recycled items, $\omega_{RA_1}I_{RA_1}(t)$ and $\delta\omega_{CA}I_{RA}(t)$, respectively. Outflows include shipment to the next stage, $P_{mA_1}(t)$, and deterioration during storage, $\omega_{m1}I_{mA_1}(t)$.

Similarly, Eq. (3.22) captures the dynamics of the final product A, where the inflow is composed of time-delayed inventory levels of A_1 and raw material d. These appear as $I_{mA_1}(t-\tau_A)$ and $I_{md}(t-\tau_A)$, respectively, with equal weight (coefficients κ_{A_1} and κ_d) to reflect balanced contribution in the BOM. Reworked items, $\omega_{RA}I_{RA}(t)$, are also added to the inventory. The outflows consist of shipping to the customer, $P_{mA}(t)$, and inventory deterioration, $\omega_{m2}I_{mA}(t)$. This modeling of production delays through drifted arguments in state equations is similar to classical formulations of delayed differential equations (DDEs) in control theory, where the system inputs do not immediately affect the state but only later appear after lead times. The first-order Taylor expansion used in Eqs. (3.21) and (3.22) produces a good and computationally efficient approximation of the delay effects in the case of systems with moderately small production lead times. The first-order expansion maintains the analytical tractability and physical interpretability of the model. Higher-order expansions could be utilized to improve the quality of the time-delay representation in the case of systems with much larger delays, without modifying the overall model structure.

Since Eqs. (3.21) and (3.22) form a linear system of differential equations with lead time, inserting them into Eq. (3.12) and using a Taylor expansion results in (3.23).

$$
\begin{cases}
\dot{I}_{mb}(t) = s_{mb}(t) + \alpha\omega_{CA_1}I_{RA_1}(t) - P_{mb}(t) & I_{mb}(0) = I^0_{mb} \\
\dot{I}_{mc}(t) = s_{mc}(t) + \beta\omega_{CA_1}I_{RA_1}(t) - P_{mc}(t) & I_{mc}(0) = I^0_{mc} \\
\dot{I}_{mA_1}(t) = \kappa_b\alpha^2\left(I_{mb}(t) - \tau_{A_1}\dot{I}_{mb}(t)\right) + \kappa_c\beta^2\left(I_{mc}(t) - \tau_{A_1}\dot{I}_{mc}(t)\right) + \omega_{RA_1}I_{RA_1}(t) + \delta\omega_{CA}I_{RA}(t) & \\
\qquad - P_{mA_1}(t) - \omega_{m1}(I_{mA_1}(t) - P_{mA_1}(t)) & I_{mA_1}(0) = I^0_{mA_1} \\
\dot{I}_{md}(t) = s_{md}(t) + \gamma\omega_{CA}I_{RA}(t) - b\gamma I_{md}(t) & I_{md}(0) = I^0_{md} \\
\dot{I}_{mA}(t) = \frac{\kappa_d}{2}\left(I_{md}(t) - \tau_A\dot{I}_{md}(t)\right) + \frac{\kappa_{A_1}}{2}\left(I_{mA_1}(t) - \tau_A\dot{I}_{mA_1}(t)\right) + \omega_{RA}I_{RA}(t) - P_{mA}(t) & \\
\qquad - \omega_{m2}(I_{mA}(t) - P_{mA}(t)) & I_{mA}(0) = I^0_{mA} \\
\dot{I}_{RA_1}(t) = \omega_{m1}(I_{mA_1}(t) - P_{mA_1}(t)) - I_{RA_1}(t) & I_{RA_1}(0) = I^0_{RA_1} \\
\dot{I}_{RA}(t) = \omega_{m2}(I_{mA}(t) - P_{mA}(t)) + R(t) - I_{RA}(t) & I_{RA}(0) = I^0_{RA}
\end{cases}
\tag{3.23}
$$

Matrices M' and N', whose entries form the coefficients of state variables and control variables, respectively, are defined in matrices (3.24) and (3.25).

$$
M' = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & \alpha\omega_{CA_1} & 0 \\
0 & 0 & 0 & 0 & 0 & \beta\omega_{CA_1} & 0 \\
(\kappa_b\alpha^2 + \tau_{A_1}(\kappa_b)^2\alpha^3) & (\kappa_c\beta^2 + \tau_{A_1}(\kappa_c)^2\beta^3) & -\omega_{m1} & 0 & 0 & -(\tau_{A_1}\kappa_b\alpha^3\omega_{CA_1} + \kappa_b\beta^3\tau_{A_1}\omega_{CA_1} - \omega_{RA_1}) & \delta\omega_{CA} \\
0 & 0 & 0 & 0 & 0 & 0 & \gamma\omega_{CA} \\
-\frac{\kappa_d 2}{\tau}{}_A(\kappa_c\alpha^2 + \tau_{A_1}a^2\alpha^3) & -\frac{\kappa_{A_1}}{2}\tau_A(\kappa_b\beta^2 + \tau_{A_1}a^2\beta^3) & \left(\frac{\kappa_d 2}{+}\frac{\kappa_{A_1}}{2}\tau_A\omega_{m1}\right) & \left(\frac{\kappa_{A_1}}{2} - \frac{(\kappa_d)^2}{2}\gamma\tau_A\right) & -\omega_{m2}\frac{\kappa_d}{2}\tau_A & (\tau_{A_1}\kappa_c\alpha^3\omega_{CA_1} + \kappa_b\beta^3\tau_{A_1}\omega_{CA_1} - \omega_{RA_1}) & \left(\frac{\kappa_{A_1}}{2}\gamma\omega_{CA}\tau_A - \frac{\kappa_{A_1}}{2}\tau_A\delta\omega_{CA} + \omega_{RA}\right) \\
0 & 0 & \omega_{m1} & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & \omega_{m2} & 0 & -1
\end{bmatrix}
\tag{3.24}
$$

$$N' = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -\kappa_b\alpha^2\tau_{A_1} & -\kappa_c\beta^2\tau_{A_1} & 0 & 0 & 0 & \omega_{m1}-1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ \frac{1}{2}\kappa_b\alpha^2\tau_A\tau_{A_1} & \frac{1}{2}\kappa_c\beta^2\tau_A\tau_{A_1} & -\frac{1}{2}\tau_A & 0 & 0 & \frac{1}{2}\tau_A & 0 & \omega_{m2}-1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\omega_{m1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\omega_{m2} & 1 \end{bmatrix} \tag{3.25}$$

4. Solution approach

The system of differential equations related to the proposed model will be obtained and solved in this section. Although the LQ formulation is an established control method [42], its application here addresses specific dynamics in a multi-stage production system: inventory management, rework, and recycling. The linear quadratic (LQ) control formulation was adopted because of its analytical tractability and well-established stability guarantees for continuous-time systems represented by coupled differential equations. The quadratic cost structure explicitly balances inventory deviations and control efforts while preserving convexity and ensuring an optimal analytical solution. Although other performance indices such as H_∞ or model predictive control (MPC) could be used, these approaches are typically intended for discrete-time or strongly disturbed systems. Given that the present model focuses on continuous deterministic dynamics with moderate time delays, the LQ structure provides the most appropriate and computationally efficient control mechanism. Future extensions may explore robust or predictive control methods for stochastic or large-delay systems. By adapting the LQ approach to these manufacturing challenges, we demonstrate its versatility in managing complex supply chain constraints and provide insights beyond traditional LQ applications.

Matrices Q, K, and R were determined by the standard linear quadratic regulator (LQR) formulation for stability and responsiveness to disturbances in the inventory control system. Specifically, Q penalizes the deviations from actual inventory levels and desired inventory levels, while R penalizes the control effort to keep production and shipment rates smooth and efficient. K is the terminal matrix that enforces stability at the end of the control horizon. By balancing the two weighting factors (Q and R), stable inventory dynamics are achieved with responsive disturbances to an underlying production process. This form of selection aligns with previous parametric LQR applications in inventory systems [43] that also used similar tuning strategies to achieve stable trajectories and returned producers to capacity utilization. If $Q = \text{diag}\{q_i\}$, $K = \text{diag}\{k_i\}$, and $R = \text{diag}\{r_j\}$ are defined as diagonal matrices where $i = 1,2,3,\ldots,7$ and $j = 1,2,3,\ldots,9$, these matrices represent penalty factors, and the optimal control model can be summarized in Eq. (4.1), where K and Q are both real diagonal matrices of dimension 7, and R is the real diagonal matrix of dimension 9, all with positive entries. Eq. (4.2) constrains the optimization problem by defining the system dynamics through a set of differential equations. It ensures that the state vector $x(t)$ evolves over time according to the linear system defined by matrices M and N, with initial conditions given by $x(0) = x^0$.

$$\min x(T)^t Kx(T) + \int_0^T [x(t)^t Qx(t) + u(t)^t Ru(t)]\, dt \tag{4.1}$$

$$\text{s.t.} \quad \dot{x}(t) = Mx(t) + Nu(t), \quad x(0) = x^0 \tag{4.2}$$

From the Hamiltonian framework, the following optimal state trajectories and control inputs are derived. Eqs. (4.3) and (4.4) represent the dynamics of the system under the optimal control law. Since the formulation follows a standard continuous linear quadratic regulator (LQR) framework, an optimal control approach that minimizes a quadratic cost function subject to linear system dynamics, the detailed derivatives are omitted for brevity.

$$\dot{\lambda}(t) = -(2Qx(t) + M^t\lambda(t)), \quad \lambda(T) = 2Kx(T) \tag{4.3}$$

$$\dot{x}(t) = Mx(t) - \frac{1}{2}NR^{-1}N^t\lambda(t), \quad x(0) = x^0 \tag{4.4}$$

5. Illustrative case study

We illustrate the proposed model with a case study driven by the production of sulfur bentonite phosphate (SBP), produced by the Yeganeh Yaran Shokoofeh Dasht Company in Iran. The SBP production system is a three-stage continuous production flow using sulfur, bentonite soil, and potassium, as shown in Figure 3. The system uses important operational aspects like mixing materials, reworking, recycling, and bunch returns. The case study is an example segmentation application to model evaluation by its two versions, with and without production lead time. In the first version of the model, lead times at each stage of the production are assumed to be zero. In the second version, lead times are included. The parameters and the initial inventory amounts are specified in Table 4 and Table 5, respectively, for both versions of the model.

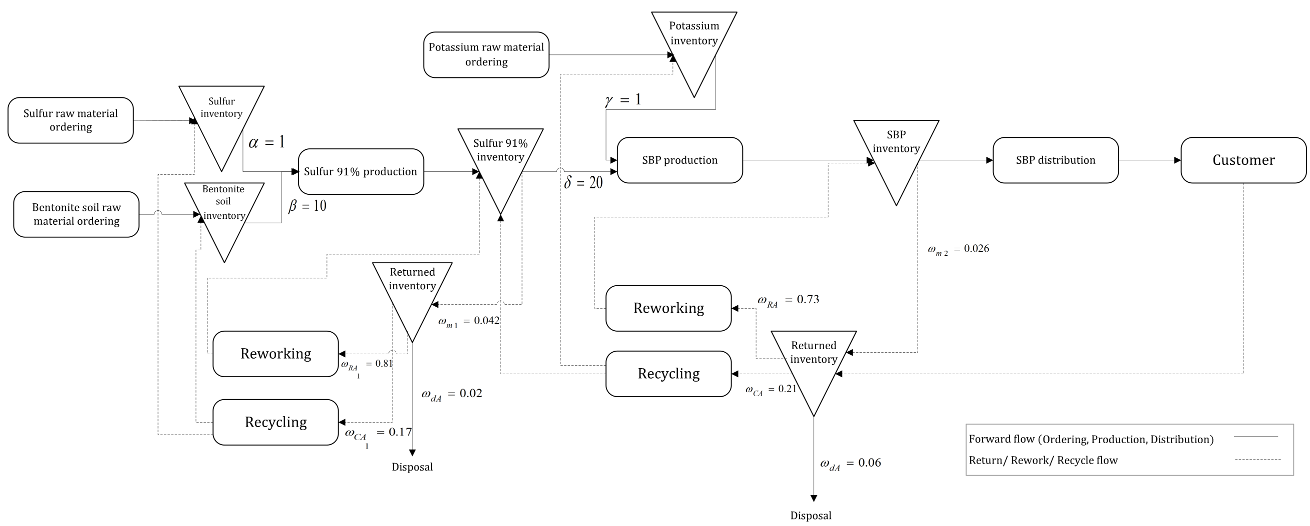


Figure 3. Production–inventory model of the sulfur bentonite phosphate (SBP) system with key rate parameters.

Table 4. Initial inventory values of the state variables.

I_{mb}^0	I_{mc}^0	$I_{mA_1}^0$	I_{mA}^0	I_{md}^0	$I_{RA_1}^0$	I_{RA}^0
104	98	22	5	19	3	4

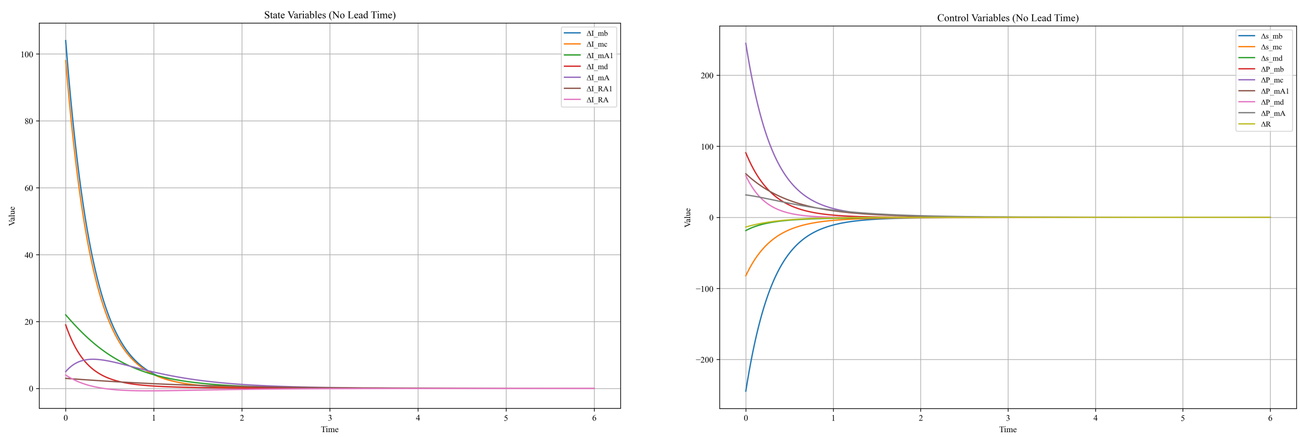
Table 5. Fixed parameter values used in the SBP system.

α	β	γ	δ	ω_{m1}	ω_{m2}	ω_{RA_1}	ω_{RA}	ω_{CA_1}	ω_{CA}	ω_{dA_1}	ω_{dA}	τ_{A_1}	τ_A	T	t	a	b
1	10	1	20	0.042	0.026	0.81	0.73	0.17	0.21	0.02	0.06	0.1	0.05	6	1	0.01	0.1

5.1. Model 1: Without consideration of lead time

By placing the specified parameters in Table 4 and Table 5 in Eq. (3.12), a dynamic system is obtained in which the coefficients of state and control variables form the entries of matrices M and N.

Now consider the linear binomial optimal control model in Eqs. (4.1) and (4.2). If the present penalty factors are defined as $Q = \text{diag}\{15, 16, 10, 14, 3, 8, 9\}$, $K = \text{diag}\{120, 190, 140, 110, 145, 130, 125\}$, and $R = \text{diag}\{2, 6, 4, 5, 2, 2, 1, 1, 2\}$, then the result is depicted in Figure 4. These matrices are selected so that they reflect common trade-offs between inventory control and production control. The weights in Q and K are chosen to prefer control action deviations from safety stock levels and end-period inventory targets to be minimized, respectively. The entries of R penalize control action swings that are large, such as ordering and shipping, to promote smoother behavior. Since company-specific cost coefficients were not obtainable, numerical scaling was employed to achieve stable convergence performance with consistency in managerial priorities for order responsiveness and inventory stability. The approach is generally employed in LQR-based production-inventory models where empirical cost data are sparse. Therefore, variables of state and control converge to their target values. Since the state and control functions are the deviation between the answers and their target functions, according to Figure 4, when $\Delta f(t) \to 0$ (where $f(t)$ can be a state or control variable), this suggests that $f(t) - \hat{f}(t)$ converges to zero or $f(t) \to \hat{f}(t)$, meaning that functions (state or control) are converging according to their targets. Convergence to target values means that inventory variables and the level of orders and shipments have reached the desired values for managers to control.

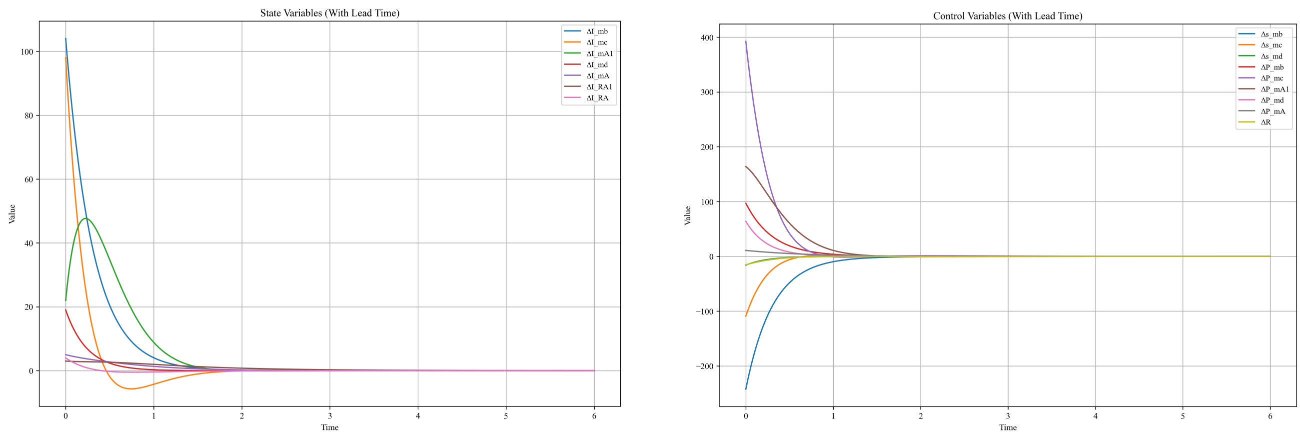


(a) **(b)**

Figure 4. Final solution for (a) state variables and (b) control variables, both without consideration of lead times.

5.2. *Model 2: With consideration of lead time*

Now, if the specified parameters in Table 4 and Table 5 are included in the optimal control model with lead time in Eq. (3.23), the matrix of coefficients of the state and control variables M' and N' are obtained. Penalty factors in the current linear binomial optimal control are considered the same as in the previous model, defined as matrices $Q = \text{diag}\{15,16,10,14,3,8,9\}$, $K = \text{diag}\{120,190,140,110,145,130,125\}$, and $R = \text{diag}\{2,6,4,5,2,2,1,1,2\}$. The results of solving this model are shown in Figure 5.



(a) **(b)**

Figure 5. Final solution for (a) state variables and (b) control variables, under lead-time conditions.

Comparing Figure 4(a) and Figure 5(a), the convergence velocity deviation of the inventories ΔI_{mA_1}, ΔI_{RA_1}, ΔI_{mA}, and ΔI_{RA} in the model without lead time is greater than the model with lead time, but the values of variables ΔI_{mb}, ΔI_{mc}, and ΔI_{md} in both with- and without-lead-time models are similar because there is no lead time for inventories related to raw materials.

Because no additional bentonite soil and sulfur raw materials should be present in the 91% sulfur production stage, the diagrams related to the deviation from the target level of the two variables of sulfur and bentonite soil $(\Delta I_{mb}, \Delta I_{mc})$ are equal in the case without lead times. When lead time is considered in a system, the rate at which the convergence deviates from the target level for bentonite soil is more significant than that for sulfur. This is because the consumption coefficient of bentonite soil is greater than that of sulfur. When comparing a 91% sulfur inventory, the deviation from the target level first increases and then converges to zero in the case of a system with lead time. However, the behavior of the diagram for the foam's raw material is the same in both models.

Comparing Figure 4(b) and Figure 5(b), which are related to the deviation from the target level of control variables, the convergence rate is higher in the model without lead time than in the model with lead time.

Table 6 compares the objective function values for the proposed model under scenarios with and without production lead time, which shows that including lead time results in a higher total cost.

Table 6. Comparison of the objective functions.

	With lead time	Without lead time
Value of objective J^*	351,897	216,881

5.3. *Comparison of convergence of both models*

We now consider the speed of convergence of three control variables P_{mb}, P_{mc}, and S_{mb} in the two models: with and without lead time. As depicted in Figure 6, the convergence speed is higher for all three variables when the lead time is zero. This suggests that a model without lead time may demonstrate faster convergence to target levels. However, considering lead time provides a more accurate and realistic representation of the production process, which is essential for effective decision-making and long-term efficiency.

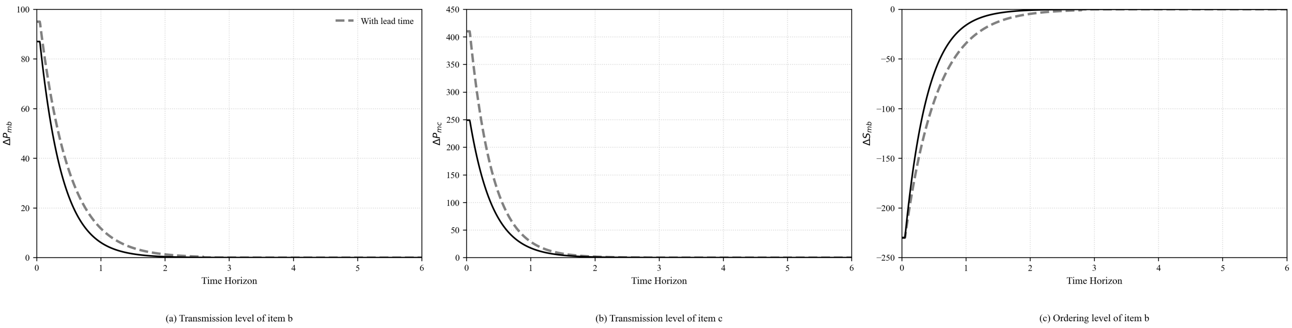


Figure 6. Comparison of the speed of convergence of the (a) transmission level of item b, (b) transmission level of item c, and (c) ordering level of item b for both models.

6. Discussion

In previous research, several optimal control models have been proposed for managing production and inventory of MRP systems. However, these models often fail to consider lead time or include the return and reworking stages. This study introduces a production-inventory optimal control model that addresses these shortcomings. The implications of the model are presented in two sections: theoretical implications, which mainly involve comparing the proposed model with previous models, and practical implications, which include scenarios related to interpreting the results of the proposed model.

6.1. *Theoretical implications*

Based on Pooya and Pakdaman's [6] approach, an optimal continuous MRP control model can be modified to use computational aspects of CMRP. For example, state variables for inventory, control variables for production, ordering, and demand programs, a BOM system, and demand dependence on BOM can all indicate CMRP computational requirements. The model proposed here is a continuous MRP model considering the delay due to production lead time [8] and a multi-stage production-inventory system [39]. Since time is considered a continuous parameter and the lead time is considered in each production stage, the proposed model is more realistic and feasible for a variety of industries. It is also a proposed model for determining production values at any given time in discrete production processes such as a workshop, handling the flow, and applied assembly lines.

One of the significant differences between CMRP and DMRP is lead time. In the CMRP system, the lead time can take on continuous values. Alternatively, in the DMRP system provided by Ignaciuk and Bartoszewicz [44], the lead time must be an integer number (i.e., if the lead time is a non-integer number, it is rounded to the nearest larger integer). However, the number of orders received is delayed. If the lead time is rounded to the first smaller number, orders will be received sooner, which will create a surplus inventory and thus increase maintenance costs, and MRP objectives are not met.

According to the model proposed by Foul and Tadj [45], returned items will only be returned for reworking from the market. In the model proposed here, the produced items will be examined before distribution in the market to check the quality of distributed items, resulting in a reduction in the percentage of defective items in the market.

Also, unlike the models proposed by Mishra [46], in the model proposed here, in addition to rework, the recycling of items is considered. The recycling stage of the returned items is a benefit of this model. Since recycled items could be used in the production process, for economic reasons, manufacturers understand the benefits of product recycling. Returned items, in addition to manufactured goods, could also be work-in-process goods. Returned items can be work-in-process and manufactured items, some of which cannot be recycled and will be disposed of as spoiled items. Returned items may be reworked or recycled. The reworking and recycling processes are assumed to be complete, and all inputs are converted to the desired output. Otherwise, if there is waste in the recycling and reworking processes, the model will automatically adapt to new coefficients by changing its structure.

Compared to other models in the literature, a significant advantage of our proposed model is that the amount of items sent to the next stage is exactly the same as what is used for production. This is a ratio of the inventory amount and its usage ratio in the BOM. This prevents the creation of excess inventory. An accurate answer could be obtained since the proposed optimal control model is linear quadratic. However, approximate approaches can be used for large-scale problems. The positive values of the state variables suggest surplus inventory, which is more than the safety stocks. Alternatively, negative values represent that inventories are less than the safety stocks. As shown in the figures, the inventories are close to the target values after a short time when the order programs are set.

6.2. Practical implications

This study presents a novel approach to optimizing multi-stage continuous MRP systems with finite capacity, considering production lead times and the deterioration of inventory items. This model considers purchase and manufacturing orders, shipments, and quantities returned from the market as control variables, while inventories serve as system states. Production lead times encompass the time required for both production and purchase orders. Notably, our proposed system permits the transfer of returned items from one inventory stage to the preceding one, facilitating recycling and reworking processes.

As the conservation equations are well established in the literature, our proposed model offers a clear advantage by focusing on using these principles to maintain the inventory balance at different stages of production. In particular, our model emphasizes the integration of bill of materials (BOM) usage coefficients with dynamic inventory levels, ensuring that the amount of items sent to the next stage is neither excessive nor insufficient. This allows the model to seamlessly manage the production flow and prevent inventory build-up in the manufacturing process while maintaining continuous production. While prior models may individually encompass features such as lead times and rework stages, our comprehensive model integrates all these aspects to offer a holistic framework that captures the realities of production-inventory systems.

A three-level BOM model is presented in this study, which is designed with a flexible framework to support higher BOM levels. However, as BOM levels increase, it presents potential computational challenges. Fortunately, using the modular structure of the model allows each additional BOM level to be seamlessly integrated with the same equations and control rules. By using sparse matrix representations and parallel processing, the complexity of the model is manageable, confirming its scalability in the extended BOM hierarchy. Computational scalability and feasibility are important for the usability and practicality of the model. Although this work uses a three-level BOM structure, the modularity of the formulation ensures that the model scales very well to more complex BOM hierarchies. Extra levels can be added with the same control logic and equations underlying each extra level. From a computational perspective, the case study demonstrated both state and control variables converging rapidly to their set levels by typical numerical procedures. This also indicates that the model is computationally manageable within normal industry operating conditions. The computation time can be reduced even more, with larger system scalability enhanced, through the inclusion of the utilization of sparse matrix computations and parallel processing—thus defining the method for medium to large-scale manufacturing operation.

The finite capacity of activities at workstations leads to limitations such as production, reconstruction, and recycling goals. The inventory objective is to minimize the difference in inventory level from its safety stock level. The purpose of disposing of items indicates the target level of disposal of deteriorated items and the capacity to order, release, and transport raw parts, among other goals. Also, in the master production schedule (MPS), the target demand for the final product is usually defined as planned demand. Therefore, minimizing the difference between demand and the demand target at any given time creates consistency in the production process with the planned demand.

The proposed model has utilitarian value for designing production systems with complex BOM structures and perishable items. Through the incorporation of lead times and return flows in a continuous MRP setting, the model enables planning and decision-making with greater precision in capacity-constrained environments. The model releases material in appropriate quantities between stages, limiting excessive inventory and aligning production with actual demand. The addition of perishability constraints minimizes waste and obsolescence of inventory, which is especially valuable for time-sensitive industries. In all, the case study validates that the addition of lead time decelerates convergence but increases system realism and responsiveness.

According to Figure 4(a), the potassium inventory deviation rapidly converges to the target value. If Figure 3 is considered, the shipped level deviation of potassium is exited from this inventory, and the ordering level deviation of the raw material is entered. Figure 4(b) shows that the convergence velocity of sent items is greater than the order variable's convergence velocity. This means that the input rate has decreased relative to the output, so the potassium inventory convergence velocity has increased. The amount of the state variable of the 91% sulfur inventory level has increased from the target predetermined value over the time period $[0, 0.6]$ if Figure 5(a) is considered. Because in this period, the convergence velocity of the control variables and the input state were higher than the convergence velocity of the output variables. However, after this period, the convergence velocity in the output variables has increased, causing a decrease in the 91% sulfur inventory deviation level from its target value.

The findings have significant implications for managers who work in multi-level production systems that produce perishable goods. First, the inclusion of perishability considerations in the planning phase substantially reduces the risk of obsolescence and write-offs of products, particularly during the later phases of production. Second, the delayed control feature of the model allows managers to actively reduce the levels of production, thereby preventing the buildup of unwanted stocks of products with limited shelf lives. Third, perishability constraints induce ordering patterns towards shorter-order completion lead times and reduced inventory cushions with heightened stage coordination. Fourth, the results identify that in such conditions, omitting perishability from the model leads to higher inventory levels and attendant higher holding costs. Such results necessitate the incorporation of perishability in MRP systems to heighten responsiveness, minimize waste, and improve service reliability in time-critical manufacturing networks.

7. Concluding remarks

A quadratic linear optimal control model is developed in this paper for a multi-stage production-inventory MRP system, which considers production lead time. In the proposed model, the objective function's value indicates the convergence rate of the variables (state and control functions) to their target values. The inventories are state variables, while the order variables, the amount of shipped goods and the amount of demand for work-in-process goods, are control variables. This model focuses on internal lead times that can be considered as the time required to set up and process machines and the time required to transport items between parts of the system; however, it provides flexibility to incorporate supplier lead times when inventory strategies require it. When lead times were accounted for, the total cost increased from 216,881 to 351,897 (approximately 62% higher), while the convergence speed of control and state variables decreased by about **35%**. This indicates that production delays increase the realism of the model at the cost of higher total costs, which better represents actual production behavior in continuous production systems. One of the innovations in the proposed optimal control model compared to previous models is that by providing explicit functions, the amount of items sent to following stages is as much as needed for production, thus preventing surplus inventory. The optimal control problem with lead time is approximated to a problem without lead time using Taylor expansion. Finally, the exact answer to the problem without lead time and the approximate answer to the problem with lead time were calculated and compared. In the fertilizer production system analyzed, the lead times ($\tau_{A_1} = 0.1$, $\tau_A = 0.05$) are relatively small when compared to the overall production horizon ($T = 6$). Therefore, the first-order Taylor expansion applied in the delayed control model is sufficiently accurate in representing the essential dynamic behavior. The higher-order expansions may be hypothetically applied when delays become very significant; however, there is no rationale for doing so in this industrial context. Sensitivity tests demonstrated that moderate changes in lead times ($\pm 25\%$) do not have an impact on the qualitative behavior of the system, which supports the adequacy of the approximation applied. A comparison of these responses showed that the convergence of the response without lead time to its target value is faster than that with the lead time assumption. One of the advantages of the proposed model is that, unlike other models in the literature, the production lead time can be continuous and a fraction of the unit of time.

Although the research confirms the model with a real case study from the fertilizer industry, the structure is not industry-specific. The model can be extended to other process-oriented industries with analogous manufacturing complexities, such as pharmaceuticals, chemicals, and food processing, by adjusting BOM patterns to stocks and attributes to lead time. These industries have multi-stage manufacturing flows, perishable components, process rework, and capacity limitations; thus, the structure possesses high generality to continuous production systems.

This paper expresses form work (order, send, and demand) and state work (inventory) in terms of time to find an open-loop solution. Future work can advance to closed-loop models where they are expressed instead in terms of inventory to allow a dynamic response from the system. Although comprehensive application to higher levels within the BOM lies outside the domain of work here, future work can explore methods such as decomposition techniques, sparse matrix calculations, or parallel processing to tackle the increased complexity in an effective manner. Expanding the alternate ordering policy beyond the lot-for-lot system and adding limits to the control and inventory variables can improve the optimization process. The inclusion of uncertainty techniques (i.e., robust control) could more effectively address variability in the system. A rolling horizon or a model predictive control (MPC) strategy could improve the robustness of the proposed model in the case of real-time disturbances to the system. MPC permits frequent re-optimization from new system states and hence is well adapted to dynamic manufacturing systems with uncertainty and noise being primary drivers.

Author contributions

Mahdi Miri conducted the conceptualization, methodology development, data curation, and writing – original draft. Kash Barker contributed to the investigation, supervision, and writing – review and editing. Andres D. Gonzalez contributed to the investigation and writing – review and editing. Roberto Sacile contributed to the investigation and writing – review and editing.

Use of generative-AI tools declaration

Generative AI tools were used only to improve language clarity and grammar.

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Data availability

All data used in this study are included within the article. No additional datasets were generated or analyzed.

Conflict of interest

All authors declare no conflicts of interest in this paper.

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