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Research article

Joint Distribution of Overtaking and Being Overtaken in the M/M/c Queue

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Abstract: Multi-server $M/M/c$ queueing systems exhibit the overtaking phenomenon, in which the randomness of service times causes the departure order to differ from the arrival order. The distribution of the number of overtakings is well known, and it is also well established that the mean number of customers being overtaken equals the mean number of overtakings. However, for the number of customers being overtaken, except for the case $c = 2$, neither its distribution nor even its variance has been reported in the literature. In this work, we derive the joint probability generating function for the number of times a customer overtakes others and is overtaken in an $M/M/c$ queue with arbitrary $c \geq 2$. This joint probability generating function enables the derivation of formulas for higher-order joint moments of these two quantities and also allows numerical computation of their joint distribution via numerical inversion. In particular, we present recursive formulas for computing their variances and covariance and provide numerical examples illustrating both the joint moments and the joint distribution.

Keywords: M/M/c Queue; being overtaken; overtaking; discrimination; matrix-analytic method; probability generating function
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1. Introduction

Multi-server queueing systems serve as fundamental models in operations research and applied probability, with broad applications in telecommunications, healthcare operations, cloud computing, and service systems engineering. A critical structural feature of these systems is the distinction between the first-come, first-served (FCFS) service discipline and the first-in, first-out (FIFO) departure property. While the FCFS rule governs the order in which jobs begin service, the randomness of service durations implies that departure sequences do not necessarily preserve arrival order. That is, a later-arriving customer may complete service earlier than an earlier one, producing an *overtaking* event, while the earlier customer experiences being *overtaken*. These order-crossing events give rise to rich and intricate stochastic behavior.

This phenomenon was first examined rigorously by Fayolle et al. [1], who analyzed sojourn times in queueing networks with overtaking, and demonstrated that such systems can be reduced to challenging boundary value problems. Whitt [2] further characterized the amount of overtaking in general queueing networks and highlighted structural conditions under which FCFS ordering may or may not imply FIFO departures. Bertsimas and Mourtzinou [3] introduced a unified method to analyze overtake-free queueing systems, characterizing when departure order aligns with arrival order and establishing fundamental insights into system structure.

Recent research has refocused on overtaking phenomena within the classical M/M/c model. Kim and Lim [4] analyzed the number of events that an arbitrary customer overtakes others and the probability that an arbitrary customer is overtaken by the next customer. Baumann and Neumann [5] derived, for the two-server case ($c = 2$), the joint probability generating function (PGF) for the number of overtaking events and the number of times a customer is overtaken. Their analysis also included exact expressions for the first and second moments. They also observed that an arbitrary customer experiences the same expected number of overtaking others as the expected number of being overtaken by others. The overtaking phenomenon has also been investigated in other contexts. De Clercq et al. [6] studied overtaking with time-limited priority in discrete-time queues, while Erlichman and Hassin [7, 8] analyzed strategic overtaking in observable M/M/1 queues, showing how overtaking risk influences equilibrium behavior and decision-making. As a related contribution, Sandmann [9] introduced the concept of discrimination frequency in a queueing model as an unfairness measure for queueing systems operating under probabilistic arrival and service processes. The discrimination frequency counts two types of events a customer may suffer: (i) being overtaken by other customers, and (ii) waiting because of customers with larger service requirements. Neumann and Baumann [10] compared the discrimination frequency in an $M/M/2$ system and in two parallel $M/M/1$ systems, and showed that server pooling can reduce unfairness for two-server systems.

Despite these advances, a full probabilistic description of the overtaking phenomenon remains lacking. In particular, except for the case $c = 2$, in an $M/M/c$ system only the mean number of times an arbitrary customer is overtaken is known, while neither its distribution nor even its variance has been reported. In this paper, we investigate the $M/M/c$ system to derive not only the marginal distribution of the number of times an arbitrary customer is overtaken but also the joint distribution of the numbers of overtakings and times being overtaken. Knowledge of the joint distribution is essential for understanding the dependence structure between the two events, overtakings and being overtaken. For example, the covariance between these two quantities indicates whether customers who are frequently overtaken also tend to overtake others more often, or whether the two events exhibit little or no relationship.

For the joint distribution of the numbers of overtakings and times being overtaken, we derive the joint PGF for these quantities for a randomly selected customer in a stationary $M/M/c$ queue with arbitrary $c \geq 2$. This joint probability generating function enables the derivation of formulas for higher-order joint moments of these two quantities, and also allows numerical computation of their joint distribution via numerical inversion.In particular, we present recursive formulas for computing their variances and covariance and provide numerical examples illustrating both the joint moments and the joint distribution.

The paper is organized as follows. Section 2 introduces the model and derives the conditional joint PGF of the numbers of overtakings and times being overtaken, given the system state. In Section 3, we derive the unconditional joint PGF, which forms the central result of the paper. Section 4 is devoted to moment analysis. Numerical results, including inversion of the PGF to compute the joint distribution, are presented in Section 5. Section 6 concludes with a discussion and outlook. Appendices A-D present the proofs of the propositions and the lemma, and Appendix E provides a summary of the notation used throughout the paper.

2. Conditional joint distribution of overtaking and being overtaken

In this section, we begin by deriving the conditional joint probability generating function, given the system state upon the tagged customer's arrival, as a necessary step toward obtaining the unconditional joint distribution.

Consider an M/M/c queueing system with stability condition $\rho < 1$, where $\rho = \lambda/c\mu$. Let time 0 denote the present moment. Assume that there is at least one customer in the system at time 0. Among the customers currently in service, select one and designate this individual as the *tagged customer*. For each $t \geq 0$, let $N(t)$ denote the total number of customers in the system at time t, including the tagged customer. Let $J(t)$ denote the number of customers in service at time t who arrived *after* the tagged customer. Define $X(t)$ as the number of customers who *overtake* the tagged customer during the time interval $[0, t]$, and $Y(t)$ denotes the number of customers who are *overtaken* by the tagged customer over the same interval.

Let

$$\sigma = \inf\{t > 0 : N(t) = N(0) - 1\}$$

and τ denote the departure time of the tagged customer from the system.

For $n = 2, 3, \ldots$, $i = 0, 1, \ldots, \min\{n-1, c-1\}$, $j = 0, 1, \ldots, \min\{n-2, c-1\}$, and $|z| \leq 1$, define

$$G_{n,ij}(z) = \mathbb{E}\left[z^{X(\sigma)} \, \mathbb{1}_{\{\sigma<\tau,\ J(\sigma)=j\}} \,\middle|\, N(0) = n,\ J(0) = i \right].$$

Let $G_n(z)$ denote the $(\min\{n-1, c-1\}+1) \times (\min\{n-2, c-1\}+1)$ matrix whose (i, j)-entry is given by $G_{n,ij}(z)$, where $0 \leq i \leq \min\{n-1, c-1\}$ and $0 \leq j \leq \min\{n-2, c-1\}$. Note that $G_{n,ij}(z)$ remains the same for all $n \geq c+1$. Accordingly, the entire matrix $G_n(z)$ is also identical for all $n \geq c+1$. Hence, for notational simplicity, we write $G_{ij}(z)$ and $G(z)$ in place of $G_{n,ij}(z)$ and $G_n(z)$, respectively, for $n \geq c+1$. The matrix generating functions $G_n(z)$, $|z| \leq 1$, $n = 2, 3, \ldots$, can be obtained from Proposition 1 below. The proof of this proposition is given in Appendix A. In Proposition 1 and throughout the remainder of the paper, any entries not explicitly shown in a matrix are understood to be zeros.

Proposition 1. *For $|z| \leq 1$, $G(z)$ and $G_n(z)$, $n = 2, 3, \ldots, c$, are given as follows.*

- *For $0 \leq i, j \leq c-1$, the (i, j)-entries $G_{ij}(z)$ of $G(z)$, $|z| \leq 1$, can be obtained recursively as follows:*

For $j = 0, 1, \ldots, c-1$,

$$G_{ij}(z) = \begin{cases} \frac{\rho \sum_{k=i+1}^{j-1} G_{ik}(z) G_{kj}(z)}{1+\rho-\rho(G_{ii}(z)+G_{jj}(z))}, & i = 0, 1, \ldots, j-2, \\ \frac{1-\frac{j}{c}}{1+\rho-\rho(G_{ii}(z)+G_{jj}(z))}, & i = j-1, \\ \frac{1+\rho-\sqrt{(1+\rho)^2-4\frac{j}{c}\rho z}}{2\rho}, & i = j, \\ 0, & i = j+1, j+2, \ldots, c-1. \end{cases} \tag{2.1}$$

Furthermore, $G(z)$, $|z| \le 1$, *satisfies the following matrix quadratic equation*

$$G(z) = \frac{\rho}{1+\rho}(G(z))^2 + \frac{1}{1+\rho} B(z), \tag{2.2}$$

where $B(z)$ *is the* $c \times c$ *matrix given by*

$$B(z) = \frac{1}{c} \begin{bmatrix} 0 & c-1 & & & & \\ & z & c-2 & & & \\ & & 2z & c-3 & & \\ & & & \ddots & \ddots & \\ & & & & (c-2)z & 1 \\ & & & & & (c-1)z \end{bmatrix}.$$

- *For* $|z| \le 1$, $G_c(z)$ *is given by*

$$G_c(z) = \frac{1}{1+\rho} \left[I_c - \frac{\rho}{1+\rho} G(z) \right]^{-1} B_c(z), \tag{2.3}$$

where I_c *is the* $c \times c$ *identity matrix, and* $B_c(z)$ *is the* $c \times (c-1)$ *matrix given by*

$$B_c(z) = \frac{1}{c} \begin{bmatrix} c-1 & & & & \\ z & c-2 & & & \\ & 2z & c-3 & & \\ & & \ddots & \ddots & \\ & & & (c-2)z & 1 \\ & & & & (c-1)z \end{bmatrix}.$$

- *For* $n = 2, 3, \ldots, c-1$ *and* $|z| \le 1$, $G_n(z)$ *is obtained by the backward recursion by*

$$G_n(z) = \frac{1}{\frac{n}{c}+\rho} \left[I_n - \frac{\rho}{\frac{n}{c}+\rho} A_n G_{n+1}(z) \right]^{-1} B_n(z), \tag{2.4}$$

where I_n *is the* $n \times n$ *identity matrix,* A_n *is the* $n \times (n+1)$ *matrix given by*

$$A_n = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \end{bmatrix},$$

and $B_n(z)$ *is the* $n \times (n-1)$ *matrix given by*

$$B_n(z) = \frac{1}{c} \begin{bmatrix} n-1 & & & & \\ z & n-2 & & & \\ & 2z & n-3 & & \\ & & \ddots & \ddots & \\ & & & (n-2)z & 1 \\ & & & & (n-1)z \end{bmatrix}.$$

For $n = 1, 2, \ldots$, $i = 0, 1, \ldots, \min\{n-1, c-1\}$, and $|z| \le 1$, $|w| \le 1$, define

$$h_{n,i}(z,w) = \mathbb{E}\left[z^{X(\tau)} w^{Y(\tau)} \, \mathbb{1}_{\{\tau \le \sigma\}} \,\middle|\, N(0) = n,\ J(0) = i \right].$$

Let $\boldsymbol{h}_n(z,w)$ denote the $(\min\{n-1, c-1\}+1)$-dimensional column vector whose ith component is given by $h_{n,i}(z,w)$, where $0 \le i \le \min\{n-1, c-1\}$. Note that $h_{n,i}(z,w)$ remains the same for all $n \ge c$. Accordingly, the entire column vector $\boldsymbol{h}_n(z,w)$ is also identical for all $n \ge c$. Hence, for notational simplicity, we write $h_i(z,w)$ and $\boldsymbol{h}(z,w)$ in place of $h_{n,i}(z,w)$ and $\boldsymbol{h}_n(z,w)$, respectively, for $n \ge c$. The vector generating functions $\boldsymbol{h}_n(z,w)$, $|z| \le 1$, $|w| \le 1$, $n = 1, 2, \ldots$, can be obtained by Proposition 2 below. The proof of this proposition is given in Appendix B.

Proposition 2. *For $|z| \le 1$ and $|w| \le 1$, $\boldsymbol{h}(z)$ and $\boldsymbol{h}_n(z)$ $n = 1, 2, \ldots, c-1$, are given as follows.*

- *For $|z| \le 1$ and $|w| \le 1$,*

$$\boldsymbol{h}(z,w) = (I_c - \rho G(z))^{-1}\, \boldsymbol{b}(w), \tag{2.5}$$

where

$$\boldsymbol{b}(w) = \frac{1}{c}(w^{c-1},\ w^{c-2},\ \cdots, w,\ 1)^{\top}.$$

- *For $n = 1, 2, \ldots, c-1$, $|z| \le 1$ and $|w| \le 1$, $\boldsymbol{h}_n(z,w)$, are obtained by the backward recursion by*

$$\begin{aligned}
&\boldsymbol{h}_n(z,w) \\
&= \left[I_n - \frac{\rho}{\frac{n}{c} + \rho} A_n G_{n+1}(z) \right]^{-1} \left[\frac{\rho}{\frac{n}{c} + \rho} A_n \boldsymbol{h}_{n+1}(z,w) + \frac{1}{\frac{n}{c} + \rho} \boldsymbol{b}_n(w) \right],
\end{aligned} \tag{2.6}$$

where

$$\boldsymbol{b}_n(w) = \frac{1}{c}(w^{n-1},\ w^{n-2},\ \cdots, w,\ 1)^{\top}.$$

For $n = 1, 2, \ldots$, $i = 0, 1, \ldots, \min\{n-1, c-1\}$, and $|z| \le 1$, $|w| \le 1$, define

$$\phi_{n,i}(z,w) = \mathbb{E}\left[z^{X(\tau)} w^{Y(\tau)} \,\middle|\, N(0) = n,\ J(0) = i \right].$$

Let $\boldsymbol{\phi}_n(z,w)$ denote the $(\min\{n-1, c-1\}+1)$-dimensional column vector whose ith component is given by $\phi_{n,i}(z,w)$, where $0 \le i \le \min\{n-1, c-1\}$. Furthermore, let

$$\widetilde{\phi}_i(z,w) = \sum_{n=c}^{\infty} \rho^{n-c} \phi_{n,i}(z,w), \quad i = 0, 1, \ldots, c-1,$$

$$\widetilde{\boldsymbol{\phi}}(z,w) = \sum_{n=c}^{\infty} \rho^{n-c} \boldsymbol{\phi}_n(z,w).$$

The vector generating functions $\boldsymbol{\phi}_n(z,w)$, $n = 1, \ldots, c$, and $\widetilde{\boldsymbol{\phi}}(z,w)$, $|z| \le 1$, $|w| \le 1$, can be obtained by Proposition 3 below. The proof of this proposition is given in Appendix C.

Proposition 3. *For $|z| \le 1$ and $|w| \le 1$, $\boldsymbol{\phi}_n(z,w)$, $n = 1, \ldots, c$, and $\widetilde{\boldsymbol{\phi}}(z,w)$ are given as follows.*

$$\boldsymbol{\phi}_1(z,w) = \boldsymbol{h}_1(z,w), \tag{2.7}$$

$$\boldsymbol{\phi}_n(z,w) = \boldsymbol{h}_n(z,w) + G_n(z)\, \boldsymbol{\phi}_{n-1}(z,w), \quad n = 2, 3, \ldots, c, \tag{2.8}$$

$$\widetilde{\boldsymbol{\phi}}(z,w) = [I_c - \rho\, G(z)]^{-1} \left[\frac{1}{1-\rho} \boldsymbol{h}(z,w) + G_c(z)\, \boldsymbol{\phi}_{c-1}(z,w) \right]. \tag{2.9}$$

3. Joint distribution of overtaking and being overtaken

In this section, we derive the joint probability generating function of the number of overtakings and overtakens experienced by an arbitrary customer, based on the conditional probability generating function obtained in the previous section.

Let $\mathcal{X}$ and $\mathcal{Y}$ denote the number of times an arbitrary customer is overtaken and overtakes others, respectively. Let $\Phi(z, w)$ be the joint probability generating function of $\mathcal{X}$ and $\mathcal{Y}$, i.e.,

$$\Phi(z, w) = \mathbb{E}[z^{\mathcal{X}} w^{\mathcal{Y}}].$$

Then, $\Phi(z, w)$ can be expressed as

$$\Phi(z, w) = \sum_{n=1}^{\infty} q_n \, \mathbb{E}[z^{\mathcal{X}} w^{\mathcal{Y}} \mid \mathcal{N} = n], \tag{3.1}$$

where $\mathcal{N}$ denotes the total number of customers in the system, including the arbitrary customer, immediately after that customer begins service. The probability q_n is defined by

$$q_n = \mathbb{P}(\mathcal{N} = n), \quad n = 1, 2, \ldots.$$

To derive an explicit expression for q_n, we introduce the following notation. For $\rho_0 > 0$, $0 < \rho < 1$, and $c \geq 1$, define

$$D_c(\rho_0, \rho) = \sum_{k=0}^{c-2} \frac{(\rho_0)^k}{k!} + \frac{(\rho_0)^{c-1}}{(c-1)!} \frac{1}{1-\rho},$$

with the convention that $D_0(\rho_0, \rho) = 0$. The following lemma presents a closed-form expression for q_n. The proof is given in Appendix D.

Lemma 1. *For* $n = 1, 2, \ldots,$

$$q_n = \begin{cases} \dfrac{1}{D_c(c\rho, \rho)} \dfrac{(c\rho)^{n-1}}{(n-1)!}, & \text{if } 1 \leq n \leq c-1, \\ \dfrac{1}{D_c(c\rho, \rho)} \dfrac{(c\rho)^{c-1}}{(c-1)!}(1+\rho), & \text{if } n = c, \\ \dfrac{1}{D_c(c\rho, \rho)} \dfrac{(c\rho)^{c-1}}{(c-1)!} \rho^{n-c+1}, & \text{if } n \geq c+1. \end{cases}$$

Recall that τ is the departure time of a tagged customer who is in service at time 0, and that $X(\tau)$ and $Y(\tau)$ are the numbers of overtakings and times being overtaken, respectively, during $[0, \tau]$ experienced by the tagged customer. Also, $N(0)$ is the total number of customers in the system at time 0, including the tagged customer, and $J(0)$ is the number of customers in service at time 0 who arrived after the tagged customer. Therefore, we have

$$\mathbb{E}[z^{\mathcal{X}} w^{\mathcal{Y}} \mid \mathcal{N} = n] = \mathbb{E}[z^{X(\tau)} w^{Y(\tau)} \mid N(0) = n, \ J(0) = 0], \quad n = 1, 2, \ldots,$$

and hence

$$\mathbb{E}[z^{\mathcal{X}} w^{\mathcal{Y}} \mid \mathcal{N} = n] = \phi_{n,0}(z, w), \quad n = 1, 2, \ldots.$$

Therefore, by (3.1) and Lemma 1, we have the following theorem.

Theorem 1. *The joint probability generating function $\Phi(z, w)$ of the number of times an arbitrary customer is overtaken, $\mathcal{X}$, and the number of times the arbitrary customer overtakes others, $\mathcal{Y}$, is given by*

$$\Phi(z,w) = \frac{1}{D_c(c\rho,\rho)}\left(\sum_{n=1}^{c}\frac{(c\rho)^{n-1}}{(n-1)!}\,\phi_{n,0}(z,w) + \frac{(c\rho)^c}{c!}\,\widetilde{\phi}_0(z,w)\right).$$

4. Moments for the overtaking and being overtaken

As noted by Neumann and Baumann [10], the number $\mathcal{X}$ of times an arbitrary customer is overtaken and the number $\mathcal{Y}$ of times the customer overtakes others have the same expected value. The common expectation is given by

$$\mathbb{E}[\mathcal{X}] = \mathbb{E}[\mathcal{Y}] = \frac{c\rho}{2}\cdot\frac{D_{c-1}(c\rho,\rho)}{D_c(c\rho,\rho)}. \tag{4.1}$$

The probability generating function of $\mathcal{Y}$ can be easily obtained as

$$\mathbb{E}\left[w^{\mathcal{Y}}\right] = \frac{1}{D_c(c\rho,\rho)}\left[\sum_{k=0}^{c-1}\frac{(c\rho)^k}{k!}\frac{1-w^{k+1}}{(k+1)(1-w)} + \frac{(c\rho)^c}{c!}\frac{1}{1-\rho}\frac{1-w^c}{c(1-w)}\right].$$

Moreover, based on the distribution of $\mathcal{Y}$, its second moment is given by

$$\begin{aligned}
\mathbb{E}[\mathcal{Y}^2] &= \frac{(c\rho)^2}{3}\frac{D_{c-2}(c\rho,\rho)}{D_c(c\rho,\rho)} + \frac{c\rho}{2}\frac{D_{c-1}(c\rho,\rho)}{D_c(c\rho,\rho)},\\
\mathrm{Var}(\mathcal{Y}) &= \frac{(c\rho)^2}{3}\frac{D_{c-2}(c\rho,\rho)}{D_c(c\rho,\rho)} + \frac{c\rho}{2}\frac{D_{c-1}(c\rho,\rho)}{D_c(c\rho,\rho)} - \left(\frac{c\rho}{2}\frac{D_{c-1}(c\rho,\rho)}{D_c(c\rho,\rho)}\right)^2.
\end{aligned} \tag{4.2}$$

In this section, we focus on the moments of $\mathcal{X}$ and $\mathcal{Y}$. The higher-order moments of $\mathcal{X}$ and $\mathcal{Y}$ can be derived from the joint probability generating function of $(\mathcal{X}, \mathcal{Y})$ presented in Theorem 1. We describe the method for computing the moments of $\mathcal{X}$ and $\mathcal{Y}$ and, in particular, provide a detailed derivation of $\mathrm{Var}(\mathcal{X})$ and $\mathrm{Cov}(\mathcal{X}, \mathcal{Y})$. While higher-order moments can also be obtained by taking successive partial derivatives of the joint probability generating function, their derivations become increasingly complex. Therefore, it is more practical to compute such moments using a dedicated computational algorithm rather than performing the calculations by hand. Accordingly, we restrict our attention to the manual derivation of $\mathrm{Var}(\mathcal{X})$ and $\mathrm{Cov}(\mathcal{X}, \mathcal{Y})$. Note that we already obtained an explicit formula for $\mathrm{Var}(\mathcal{Y})$ in (4.2).

For the derivation of $\mathrm{Var}(\mathcal{X})$ and $\mathrm{Cov}(\mathcal{X}, \mathcal{Y})$, we introduce the following notation:

$$\begin{aligned}
G_n^{(k)} &= \left.\frac{d^k}{dz^k}G_n(z)\right|_{z=1},\\
\boldsymbol{h}_n^{(k,l)} &= \left.\frac{\partial^{k+l}}{\partial z^k\,\partial w^l}\boldsymbol{h}_n(z,w)\right|_{z=1,\,w=1},\\
\boldsymbol{\phi}_n^{(k,l)} &= \left.\frac{\partial^{k+l}}{\partial z^k\,\partial w^l}\boldsymbol{\phi}_n(z,w)\right|_{z=1,\,w=1}.
\end{aligned}$$

Now, we describe how to compute $G_n^{(k)}$, $\boldsymbol{h}_n^{(k,l)}$, and $\boldsymbol{\phi}_n^{(k,l)}$, and subsequently derive $\mathrm{Var}(\mathcal{X})$ and $\mathrm{Cov}(\mathcal{X}, \mathcal{Y})$.

Computation of $G_n^{(0)}$, $G_n^{(1)}$, and $G_n^{(2)}$ for $n \geq 2$

Note that, for each k, the matrices $G_n^{(k)}$ are identical for all $n \geq c+1$, and we denote this common matrix by $G^{(k)}$. The (i, j)-entry of $G^{(k)}$ is denoted by $G_{ij}^{(k)}$, so that $G_{ij}^{(k)} = G_{n,ij}^{(k)}$ for all $n \geq c+1$. By (2.1), the (i, j)-entries $G_{ij}^{(0)}$ of $G^{(0)}$, $0 \leq i, j \leq c-1$, can be obtained recursively as follows: For $j = 0, 1, \ldots, c-1$,

$$G_{ij}^{(0)} = \begin{cases} \frac{\rho \sum_{k=i+1}^{j-1} G_{ik}^{(0)} G_{kj}^{(0)}}{1+\rho-\rho\left(G_{ii}^{(0)}+G_{jj}^{(0)}\right)}, & i = 0, 1, \ldots, j-2, \\ \frac{1-\frac{j}{c}}{1+\rho-\rho\left(G_{j-1,j-1}^{(0)}+G_{jj}^{(0)}\right)}, & i = j-1, \\ \frac{1+\rho-\sqrt{(1+\rho)^2-4\frac{j}{c}\rho}}{2\rho}, & i = j, \\ 0, & i = j+1, j+2, \ldots, c-1. \end{cases}$$

Note that, by (2.2), $G^{(0)}$ satisfies

$$G^{(0)} = \frac{\rho}{1+\rho}(G^{(0)})^2 + \frac{1}{1+\rho} B^{(0)},$$

where $B^{(0)}$ is the $c \times c$ matrix given by

$$B^{(0)} = \frac{1}{c}\begin{bmatrix} 0 & c-1 & & & & \\ & 1 & c-2 & & & \\ & & 2 & c-3 & & \\ & & & \ddots & \ddots & \\ & & & & c-2 & 1 \\ & & & & & c-1 \end{bmatrix}.$$

By (2.2),

$$G^{(1)} = \frac{\rho}{1+\rho}\left(G^{(0)}G^{(1)} + G^{(1)}G^{(0)}\right) + \frac{1}{1+\rho} B^{(1)}, \tag{4.3}$$

$$G^{(2)} = \frac{\rho}{1+\rho}\left(G^{(0)}G^{(2)} + G^{(2)}G^{(0)}\right) + \frac{2\rho}{1+\rho}(G^{(1)})^2, \tag{4.4}$$

where $B^{(1)} = \frac{1}{c}\,\mathrm{diag}(0, 1, \cdots, c-1)$. Here $\mathrm{diag}(0, 1, \cdots, c-1)$ denotes the $c \times c$ diagonal matrix whose diagonal entries are $0, 1, \cdots, c-1$. It can be shown that the spectral radius of $G^{(0)}$ is strictly less than $\frac{1+\rho}{2\rho}$. Therefore $G^{(1)}$ and $G^{(2)}$ can be obtained as unique solutions of Sylvester equations (4.3) and (4.4), respectively. By (2.3),

$$G_c^{(0)} = \frac{1}{1+\rho}\left[I_c - \frac{\rho}{1+\rho} G^{(0)}\right]^{-1} B_c^{(0)},$$

$$G_c^{(1)} = \frac{1}{1+\rho}\left[I_c - \frac{\rho}{1+\rho} G^{(0)}\right]^{-1} [\rho G^{(1)} G_c^{(0)} + B_c^{(1)}],$$

$$G_c^{(2)} = \frac{\rho}{1+\rho}\left[I_c - \frac{\rho}{1+\rho} G^{(0)}\right]^{-1} [2G^{(1)} G_c^{(1)} + G^{(2)} G_c^{(0)}],$$

where $B_c^{(0)}$ and $B_c^{(1)}$ are the $c \times (c-1)$ matrices given by

$$B_c^{(0)} = \frac{1}{c}\begin{bmatrix} c-1 & & & & \\ 1 & c-2 & & & \\ & 2 & c-3 & & \\ & & \ddots & \ddots & \\ & & & c-2 & 1 \\ & & & & c-1 \end{bmatrix}$$

and

$$B_c^{(1)} = \frac{1}{c}\begin{bmatrix} 0 & & & & & \\ 1 & 0 & & & & \\ & 2 & 0 & & & \\ & & \ddots & \ddots & & \\ & & & c-2 & 0 & \\ & & & & c-1 & \end{bmatrix}.$$

By (2.4), for $n = 2, 3, \dots, c-2, c-1$,

$$\begin{aligned} G_n^{(0)} &= \frac{1}{\frac{n}{c}+\rho}\Big[I_n - \frac{\rho}{\frac{n}{c}+\rho}A_n G_{n+1}^{(0)}\Big]^{-1} B_n^{(0)}, \\ G_n^{(1)} &= \frac{1}{\frac{n}{c}+\rho}\Big[I_n - \frac{\rho}{\frac{n}{c}+\rho}A_n G_{n+1}^{(0)}\Big]^{-1}\Big[\rho A_n G_{n+1}^{(1)} G_n^{(0)} + B_n^{(1)}\Big], \\ G_n^{(2)} &= \frac{\rho}{\frac{n}{c}+\rho}\Big[I_n - \frac{\rho}{\frac{n}{c}+\rho}A_n G_{n+1}^{(0)}\Big]^{-1}\Big[2A_n G_{n+1}^{(1)} G_n^{(1)} + A_n G_{n+1}^{(2)} G_n^{(0)}\Big], \end{aligned}$$

where $B_n^{(0)}$ and $B_n^{(1)}$ are the $n \times (n-1)$ matrices given by

$$B_n^{(0)} = \frac{1}{c}\begin{bmatrix} n-1 & & & & & \\ 1 & n-2 & & & & \\ & 2 & n-3 & & & \\ & & \ddots & \ddots & & \\ & & & n-2 & 1 & \\ & & & & n-1 & \end{bmatrix}$$

and

$$B_n^{(1)} = \frac{1}{c}\begin{bmatrix} 0 & & & & & \\ 1 & 0 & & & & \\ & 2 & 0 & & & \\ & & \ddots & \ddots & & \\ & & & n-2 & 0 & \\ & & & & n-1 & \end{bmatrix}.$$

Computation of $h_n^{(k,l)}$ for $0 \le k+l \le 2$ and $n \ge 1$

Note that, for each (k,l), the matrices $\boldsymbol{h}_n^{(k,l)}$ are identical for all $n \ge c$, and we denote this common column vector by $\boldsymbol{h}^{(i,j)}$. By (2.5),

$$\begin{aligned} \boldsymbol{h}^{(0,0)} &= \Big[I_c - \rho G^{(0)}\Big]^{-1} \boldsymbol{b}^{(0)}, \\ \boldsymbol{h}^{(1,0)} &= \rho\Big[I_c - \rho G^{(0)}\Big]^{-1} G^{(1)} \boldsymbol{h}^{(0,0)}, \\ \boldsymbol{h}^{(2,0)} &= \rho\Big[I_c - \rho G^{(0)}\Big]^{-1} (2G^{(1)} \boldsymbol{h}^{(1,0)} + G^{(2)} \boldsymbol{h}^{(0,0)}), \\ \boldsymbol{h}^{(0,1)} &= \Big[I_c - \rho G^{(0)}\Big]^{-1} \boldsymbol{b}^{(1)}, \\ \boldsymbol{h}^{(1,1)} &= \rho\Big[I_c - \rho G^{(0)}\Big]^{-1} G^{(1)} \boldsymbol{h}^{(0,1)}, \end{aligned}$$

where $\boldsymbol{b}^{(0)}$ and $\boldsymbol{b}^{(1)}$ are c-dimensional column vectors given by $\boldsymbol{b}^{(0)} = \frac{1}{c}(1, \cdots, 1)^\top$ and $\boldsymbol{b}^{(1)} = \frac{1}{c}(c-1, c-2, \cdots, 1, 0)^\top$.

By (2.6), for $n = c-1, c-2, \dots, 2, 1$,

$$\boldsymbol{h}_n^{(0,0)} = \Big[I_n - \frac{\rho}{\frac{n}{c}+\rho}A_n G_{n+1}^{(0)}\Big]^{-1}\Big(\frac{\rho}{\frac{n}{c}+\rho}A_n \boldsymbol{h}_{n+1}^{(0,0)} + \frac{1}{\frac{n}{c}+\rho}\boldsymbol{b}_n^{(0)}\Big),$$
$$\boldsymbol{h}_n^{(1,0)} = \frac{\rho}{\frac{n}{c}+\rho}\Big[I_n - \frac{\rho}{\frac{n}{c}+\rho}A_n G_{n+1}^{(0)}\Big]^{-1}A_n\Big(\boldsymbol{h}_{n+1}^{(1,0)} + G_{n+1}^{(1)}\boldsymbol{h}_n^{(0,0)}\Big),$$
$$\boldsymbol{h}_n^{(2,0)} = \frac{\rho}{\frac{n}{c}+\rho}\Big[I_n - \frac{\rho}{\frac{n}{c}+\rho}A_n G_{n+1}^{(0)}\Big]^{-1}A_n\Big(\boldsymbol{h}_{n+1}^{(2,0)} + 2G_{n+1}^{(1)}\boldsymbol{h}_n^{(1,0)} + G_{n+1}^{(2)}\boldsymbol{h}_n^{(0,0)}\Big),$$
$$\boldsymbol{h}_n^{(0,1)} = \Big[I_n - \frac{\rho}{\frac{n}{c}+\rho}A_n G_{n+1}^{(0)}\Big]^{-1}\Big(\frac{\rho}{\frac{n}{c}+\rho}A_n \boldsymbol{h}_{n+1}^{(0,1)} + \frac{1}{\frac{n}{c}+\rho}\boldsymbol{b}_n^{(1)}\Big),$$
$$\boldsymbol{h}_n^{(1,1)} = \frac{\rho}{\frac{n}{c}+\rho}\Big[I_n - \frac{\rho}{\frac{n}{c}+\rho}A_n G_{n+1}^{(0)}\Big]^{-1}A_n\Big(\boldsymbol{h}_{n+1}^{(1,1)} + G_{n+1}^{(1)}\boldsymbol{h}_n^{(0,1)}\Big),$$

where $\boldsymbol{b}_n^{(0)}$ and $\boldsymbol{b}_n^{(1)}$ are n-dimensional column vectors given by $\boldsymbol{b}_n^{(0)} = \frac{1}{c}(1, \cdots, 1)^\top$ and $\boldsymbol{b}_n^{(1)} = \frac{1}{c}(n-1, n-2, \cdots, 1, 0)^\top$.

Computation of $\phi_n^{(k,l)}$, $1 \le n \le c$, *and* $\widetilde{\phi}^{(k,l)}$ *for* $0 \le k+l \le 2$

By (2.7), for all (i, j),

$$\boldsymbol{\phi}_1^{(k,l)} = \boldsymbol{h}_1^{(k,l)}.$$

By (2.8), for $n = 2, \dots, c$,

$$\boldsymbol{\phi}_n^{(0,0)} = \boldsymbol{h}_n^{(0,0)} + G_n^{(0)}\boldsymbol{\phi}_{n-1}^{(0,0)},$$
$$\boldsymbol{\phi}_n^{(1,0)} = \boldsymbol{h}_n^{(1,0)} + G_n^{(1)}\boldsymbol{\phi}_{n-1}^{(0,0)} + G_n^{(0)}\boldsymbol{\phi}_{n-1}^{(1,0)},$$
$$\boldsymbol{\phi}_n^{(2,0)} = \boldsymbol{h}_n^{(2,0)} + G_n^{(2)}\boldsymbol{\phi}_{n-1}^{(0,0)} + 2G_n^{(1)}\boldsymbol{\phi}_{n-1}^{(1,0)} + G_n^{(0)}\boldsymbol{\phi}_{n-1}^{(2,0)},$$
$$\boldsymbol{\phi}_n^{(0,1)} = \boldsymbol{h}_n^{(0,1)} + G_n^{(0)}\boldsymbol{\phi}_{n-1}^{(0,1)},$$
$$\boldsymbol{\phi}_n^{(1,1)} = \boldsymbol{h}_n^{(1,1)} + G_n^{(1)}\boldsymbol{\phi}_{n-1}^{(0,1)} + G_n^{(0)}\boldsymbol{\phi}_{n-1}^{(1,1)}.$$

By (2.9),

$$\widetilde{\boldsymbol{\phi}}^{(0,0)} = \Big[I_c - \rho G^{(0)}\Big]^{-1}\Big(\frac{1}{1-\rho}\boldsymbol{h}^{(0,0)} + G_c^{(0)}\boldsymbol{\phi}_{c-1}^{(0,0)}\Big),$$
$$\widetilde{\boldsymbol{\phi}}^{(1,0)} = \Big[I_c - \rho G^{(0)}\Big]^{-1}\Big(\frac{1}{1-\rho}\boldsymbol{h}^{(1,0)} + G_c^{(1)}\boldsymbol{\phi}_{c-1}^{(0,0)} + G_c^{(0)}\boldsymbol{\phi}_{c-1}^{(1,0)} + \rho G^{(1)}\widetilde{\boldsymbol{\phi}}^{(0,0)}\Big),$$
$$\widetilde{\boldsymbol{\phi}}^{(2,0)} = \Big[I_c - \rho G^{(0)}\Big]^{-1}\Big(\frac{1}{1-\rho}\boldsymbol{h}^{(2,0)} + G_c^{(2)}\boldsymbol{\phi}_{c-1}^{(0,0)} + 2G_c^{(1)}\boldsymbol{\phi}_{c-1}^{(1,0)} + G_c^{(0)}\boldsymbol{\phi}_{c-1}^{(2,0)} + \rho G^{(2)}\widetilde{\boldsymbol{\phi}}^{(0,0)} + 2\rho G^{(1)}\widetilde{\boldsymbol{\phi}}^{(1,0)}\Big),$$
$$\widetilde{\boldsymbol{\phi}}^{(0,1)} = \Big[I_c - \rho G^{(0)}\Big]^{-1}\Big(\frac{1}{1-\rho}\boldsymbol{h}^{(0,1)} + G_c^{(0)}\boldsymbol{\phi}_{c-1}^{(0,1)}\Big),$$
$$\widetilde{\boldsymbol{\phi}}^{(1,1)} = \Big[I_c - \rho G^{(0)}\Big]^{-1}\Big(\frac{1}{1-\rho}\boldsymbol{h}^{(1,1)} + G_c^{(1)}\boldsymbol{\phi}_{c-1}^{(0,1)} + G_c^{(0)}\boldsymbol{\phi}_{c-1}^{(1,1)} + \rho G^{(1)}\widetilde{\boldsymbol{\phi}}^{(0,1)}\Big).$$

Computation of Var($\mathcal{X}$) and Cov($\mathcal{X},\mathcal{Y}$)

By Theorem 1, we have

$$\mathbb{E}[\mathcal{X}(\mathcal{X}-1)] = \frac{1}{D_c(c\rho,\rho)}\left(\sum_{n=1}^{c}\frac{(c\rho)^{n-1}}{(n-1)!}\,\phi_{n,0}^{(2,0)} + \frac{(c\rho)^c}{c!}\,\widetilde{\phi}_0^{(2,0)}\right), \tag{4.5}$$

$$\mathbb{E}[\mathcal{X}\mathcal{Y}] = \frac{1}{D_c(c\rho,\rho)}\left(\sum_{n=1}^{c}\frac{(c\rho)^{n-1}}{(n-1)!}\,\phi_{n,0}^{(1,1)} + \frac{(c\rho)^c}{c!}\,\widetilde{\phi}_0^{(1,1)}\right), \tag{4.6}$$

where $\phi_{n,0}^{(k,l)}$ denotes the first component of the column vector $\phi_n^{(k,l)}$, and $\widetilde{\phi}_0^{(k,l)}$ denotes the first component of the column vector $\widetilde{\phi}^{(k,l)}$.

The variance of $\mathcal{X}$ is then given by

$$\text{Var}(\mathcal{X}) = \mathbb{E}[\mathcal{X}(\mathcal{X}-1)] + \mathbb{E}[\mathcal{X}] - (\mathbb{E}[\mathcal{X}])^2,$$

where $\mathbb{E}[\mathcal{X}(\mathcal{X}-1)]$ is given in (4.5), and $\mathbb{E}[\mathcal{X}]$ is given in (4.1). Similarly, the covariance of $\mathcal{X}$ and $\mathcal{Y}$ is given by

$$\text{Cov}(\mathcal{X},\mathcal{Y}) = \mathbb{E}[\mathcal{X}\mathcal{Y}] - (\mathbb{E}[\mathcal{X}])^2,$$

where $\mathbb{E}[\mathcal{X}\mathcal{Y}]$ is given in (4.6).

5. Numerical examples

In this section, we provide numerical examples for the marginal and joint moments, as well as the marginal and joint distributions, of the number of customers by whom an arbitrary customer is overtaken, $\mathcal{X}$, and the number of customers the arbitrary customer overtakes, $\mathcal{Y}$. We first present the expected number of customers by whom the arbitrary customer is overtaken, $E[\mathcal{X}]$.Recall that this expected value is identical to the expected number of customers the arbitrary customer overtakes,$E[\mathcal{Y}]$. Figure 1 illustrates $\mathbb{E}[\mathcal{X}]$ (which equals $\mathbb{E}[\mathcal{Y}]$) as a function of ρ for different numbers of servers c.

Figure 1 shows that $\mathbb{E}[\mathcal{X}]$ (=$\mathbb{E}[\mathcal{Y}]$) increases with both ρ and c. The growth with respect to ρ is nearly linear and concave, in contrast to the convex growth typically observed in the mean number of customers in a queue. Moreover, the effect of increasing c is relatively modest, with each additional server raising the expected value by less than 0.5 even at high load. This behavior reflects the structural nature of overtaking. When the traffic intensity is high or the number of servers increases, there are indeed more opportunities for order reversals. Nevertheless, the average number of overtaking events per customer grows only slowly, which is explained by the fact that $\mathcal{Y}$ is bounded above by $(c-1)/2$. Furthermore, it can be shown analytically that $\mathbb{E}[\mathcal{X}] = \mathbb{E}[\mathcal{Y}]$ tends to 0 as $\rho \to 0$ and to $(c-1)/2$ as $\rho \to 1$, which is consistent with the numerical patterns observed in the figure.

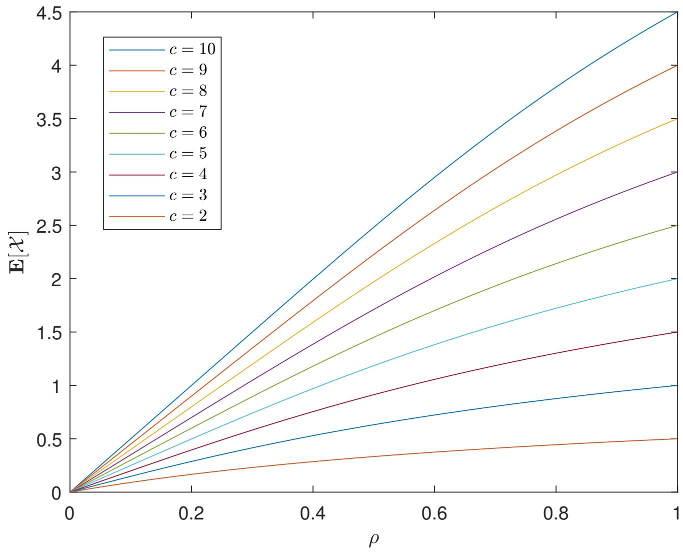


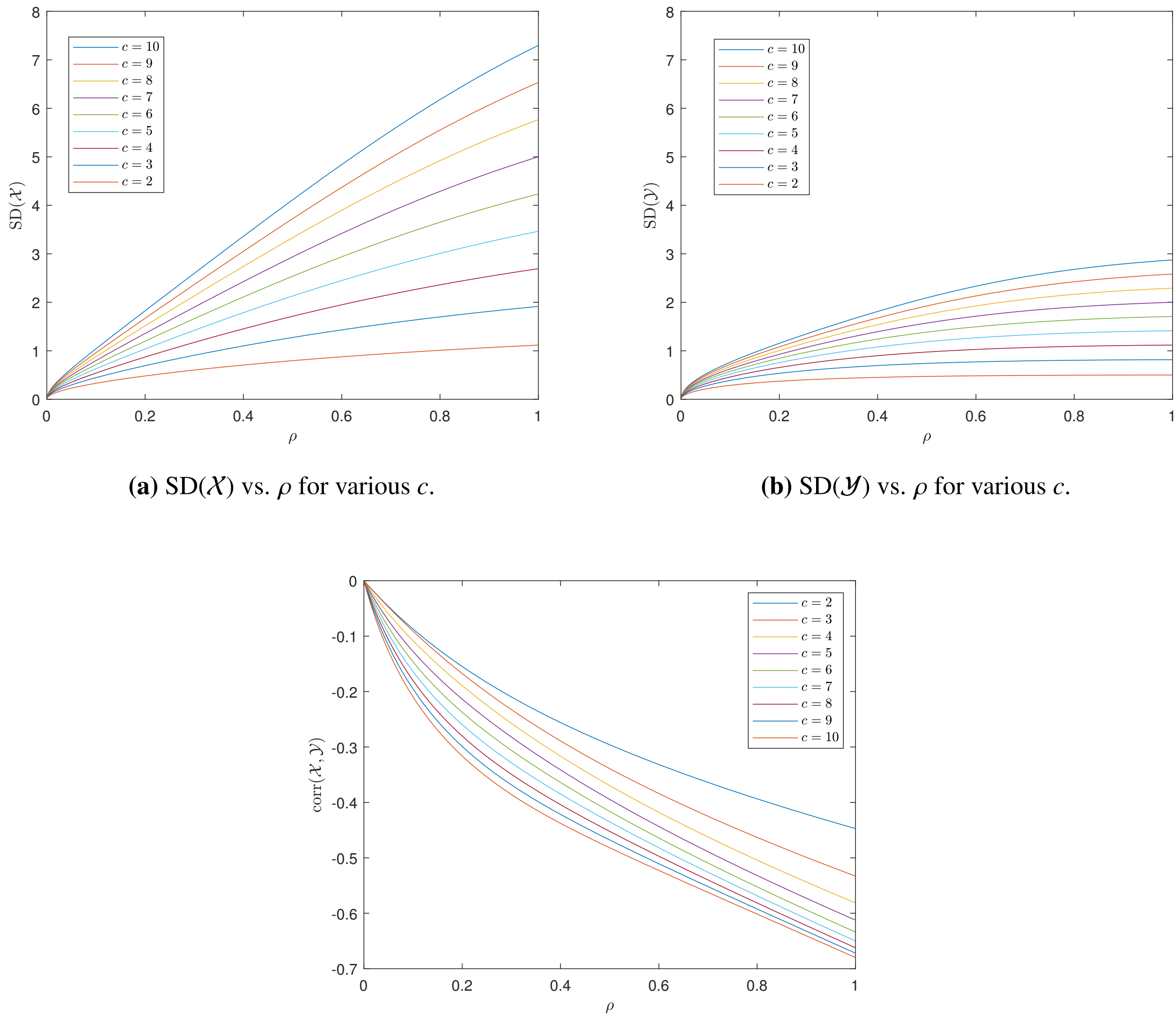
Figure 1. The expected number of overtaking events, $\mathbb{E}[\mathcal{X}]$ (equal to $\mathbb{E}[\mathcal{Y}]$), as a function of ρ for different numbers of servers c.

Figure 2 illustrates the second-moment characteristics of $\mathcal{X}$ and $\mathcal{Y}$ as functions of the traffic intensity ρ for different numbers of servers c. Panel (a) shows that the standard deviation of $\mathcal{X}$, denoted by SD($\mathcal{X}$), increases monotonically with ρ, and its magnitude increases substantially with c. In particular, for larger c, the growth of SD($\mathcal{X}$) is more pronounced, indicating that the variability of $\mathcal{X}$ becomes more significant in systems with more servers. Panel (b) presents the standard deviation of $\mathcal{Y}$, SD($\mathcal{Y}$), which also increases with ρ but remains much smaller in scale compared to SD($\mathcal{X}$). Moreover, the effect of increasing c on SD($\mathcal{Y}$) is relatively modest. Panel (c) depicts the correlation between $\mathcal{X}$ and $\mathcal{Y}$, denoted by corr($\mathcal{X}, \mathcal{Y}$), which is negative for all ρ and c. The figure shows that the negative correlation between $\mathcal{X}$ and $\mathcal{Y}$ becomes stronger as both the traffic intensity ρ and the number of servers c increase. For a fixed c, the correlation decreases with ρ, that is, it becomes more negative under heavier load. Similarly, for a fixed ρ, the correlation becomes more negative as c increases. Hence, |corr($\mathcal{X}, \mathcal{Y}$)| increases with both ρ and c, indicating that the negative dependence between $\mathcal{X}$ and $\mathcal{Y}$ strengthens in multi-server systems with more servers and under heavier load. This phenomenon can be explained as follows. Customers with shorter service times tend to overtake many other customers while being overtaken by only a few, whereas those with longer service times overtake fewer others but are overtaken by many. Therefore, $\mathcal{X}$ and $\mathcal{Y}$ exhibit a negative correlation. This negative dependence becomes stronger when many customers have service times that overlap with that of a given customer. As ρ increases, more customers are likely to be in service simultaneously, enhancing this overlapping effect and thus increasing the absolute value of corr($\mathcal{X}, \mathcal{Y}$). Furthermore, since overtaking and being overtaken occur through customers served at other servers rather than the same one, a larger proportion of customers being served at different servers also intensifies this negative dependence. Consequently, the absolute value of corr($\mathcal{X}, \mathcal{Y}$) increases as c becomes larger.

For $\rho = 0.7$, Figure 3 shows the marginal probability mass functions(PMFs) of $\mathcal{X}$ and $\mathcal{Y}$ for $c = 5$ and $c = 10$. The PMFs were obtained by applying the numerical inversion method of Abate and Whitt [11] to the probability generating functions of $\mathcal{X}$ and $\mathcal{Y}$ derived in Section 3. Recall that $\mathcal{X}$ and

$\mathcal{Y}$ have the same mean, while the support of $\mathcal{X}$ is the set of all nonnegative integers and the support of $\mathcal{Y}$ is the finite set $\{0, 1, \ldots, c-1\}$. The figure also shows that the variance of $\mathcal{X}$ is larger than that of $\mathcal{Y}$.

Figures 4 and 5 illustrate the conditional and joint distributions of $\mathcal{X}$ and $\mathcal{Y}$ when $c = 5$ and $\rho = 0.7$. Figure 4 shows the conditional probability mass functions of $\mathcal{X}$ given $\mathcal{Y} = m$ for several values of m, while Figure 5 presents the joint probability mass function of $\mathcal{X}$ and $\mathcal{Y}$ in the form of a heat map. Both figures are obtained by applying the numerical inversion method of Abate and Whitt [11] to the joint probability generating function derived in Section 3. As expected, the conditional distribution of $\mathcal{X}$ given $\mathcal{Y} = m$ becomes stochastically smaller as m increases, which implies that $\mathcal{X}$ and $\mathcal{Y}$ are negatively correlated.



(a) SD($\mathcal{X}$) vs. ρ for various c.

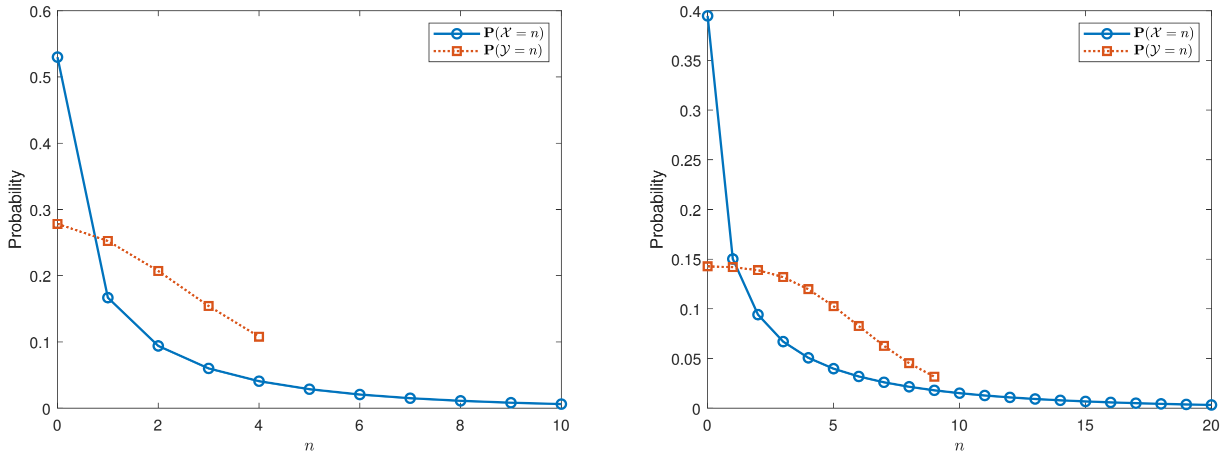
(b) SD($\mathcal{Y}$) vs. ρ for various c.

(c) corr($\mathcal{X}, \mathcal{Y}$) vs. ρ for various c.

Figure 2. The second moments of $\mathcal{X}$ and $\mathcal{Y}$, as a function of ρ for different numbers of servers c.

Let $\mathcal{D} = \mathcal{X} - \mathcal{Y}$. The random variable $\mathcal{D}$ represents the difference between the number of times the arbitrary customer is overtaken and the number of customers the arbitrary customer overtakes, and it can be regarded as a measure of discrimination. Since $\mathcal{X}$ and $\mathcal{Y}$ have the same mean, the expectation of $\mathcal{D}$ is zero. The standard deviation of $\mathcal{D}$ is given by

$$\mathrm{SD}(\mathcal{D}) = \sqrt{\mathrm{Var}(\mathcal{X}) + \mathrm{Var}(\mathcal{Y}) - 2\,\mathrm{Cov}(\mathcal{X}, \mathcal{Y})}.$$



(a) $c = 5$. **(b)** $c = 10$.

Figure 3. Marginal probability mass functions for $\mathcal{X}$ and $\mathcal{Y}$ when $\rho = 0.7$.

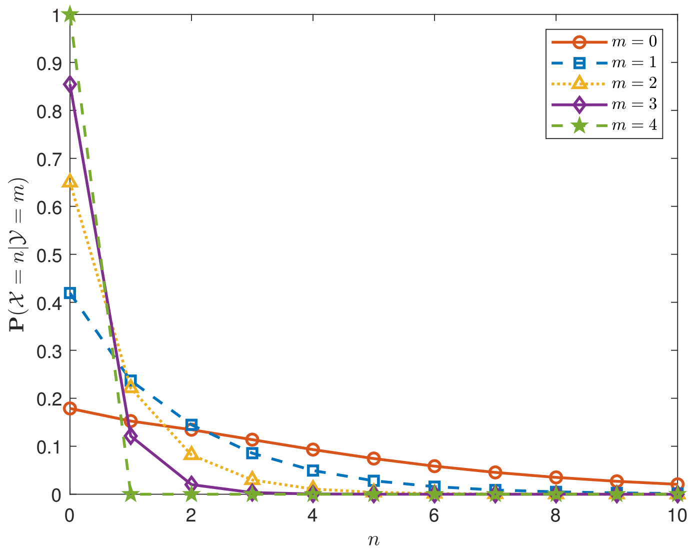


Figure 4. Conditional probability mass functions of $\mathcal{X}$ given $\mathcal{Y} = m$ for $m = 0, 1, 2, 3, 4$, when $c = 5$ and $\rho = 0.7$.

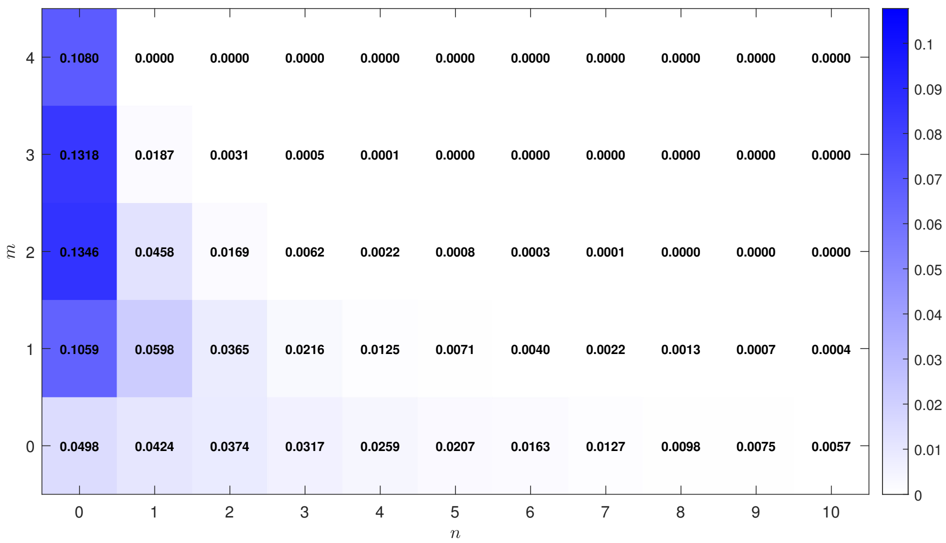


Figure 5. Joint probability mass function $\mathbb{P}(\mathcal{X} = n, \mathcal{Y} = m)$ represented as a heat map when $c = 5$ and $\rho = 0.7$.

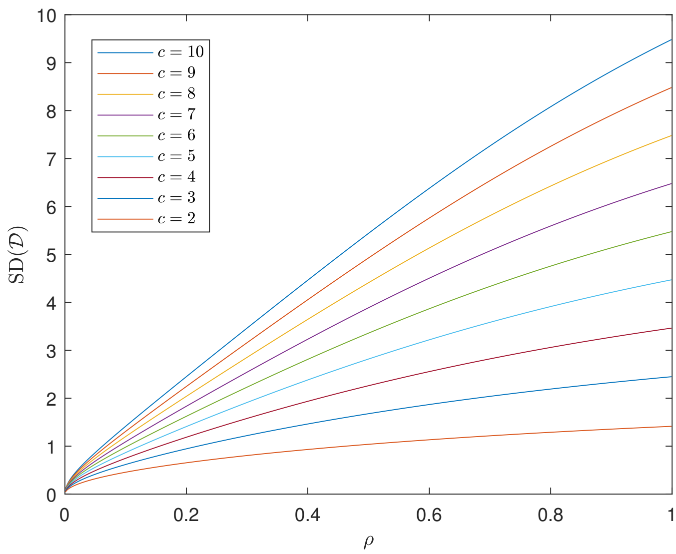


Figure 6. Standard deviation of $\mathcal{D} = \mathcal{X} - \mathcal{Y}$ as a function of ρ for different numbers of servers c.

Figure 6 presents the standard deviation of $\mathcal{D}$ as a function of ρ for different values of c. As ρ increases, the standard deviation grows for all c, reflecting greater variability in overtaking events under heavier traffic. The growth is concave, with a rapid increase at low values of ρ and a slower rate of growth as ρ approaches one. Moreover, the standard deviation becomes larger as the number of servers c increases, indicating that systems with more servers exhibit greater variability in the difference between $\mathcal{X}$ and $\mathcal{Y}$. Figure 7 shows the probability mass functions of $\mathcal{D} = \mathcal{X} - \mathcal{Y}$ for $c = 5$ and $c = 10$

when $\rho = 0.7$. In both cases the distribution is centered at zero, reflecting the fact that $\mathcal{X}$ and $\mathcal{Y}$ have the same mean, so that $\mathcal{D}$ has mean zero. However, the spread of the distribution is clearly larger for $c = 10$, indicating greater variability of $\mathcal{D}$ as the number of servers increases.

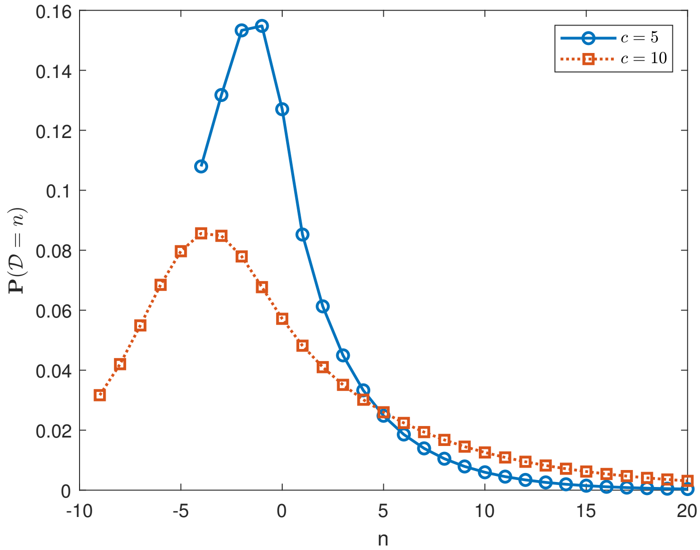


Figure 7. Probability mass function of $\mathcal{D} = \mathcal{X} - \mathcal{Y}$ for $c = 5$ and $c = 10$ when $\rho = 0.7$.

6. Conclusion

In this paper we studied the joint distribution of the numbers of times an arbitrary customer is overtaken and overtakes others in the $M/M/c$ queue. Using the matrix-analytic method, we derived the joint probability generating function of these two random variables. The computation based on the matrix-analytic method remains numerically stable for moderate values of c. By numerically inverting the generating function, we obtained the joint probability mass function, and we also showed that the joint moments can be computed by differentiating the joint generating function. Numerical experiments were provided to illustrate the marginal and joint distributions and to quantify the variability in overtaking behavior.

Although our analysis focused on homogeneous servers, the same framework could in principle be extended to heterogeneous-server systems, where the service speeds differ across servers. Such an extension would inevitably involve more technical complexity, but it would provide a natural and valuable direction for further research. Another limitation of our model is the assumption of Poisson arrivals and exponential service times. It would be worthwhile to investigate whether similar analytical approaches can be applied to more general interarrival or service-time distributions.

Beyond methodological extensions, our results also suggest several potential applications. In particular, the distributional properties of overtaking and being overtaken offer new quantitative measures of fairness in queueing systems. We expect this framework to serve as a foundation for further studies on fairness metrics, scheduling policies, and performance evaluation in multi-server systems. Furthermore, future work could explore alternative analytical frameworks for quantifying uncertainty, drawing from other complex domains like structural dynamics [12].

Author contributions

All authors listed in this paper have made substantial, direct, and intellectual contributions to the work and have equally dedicated themselves to every stage of this research. The successful completion of this manuscript is a result of the collective effort and devoted collaboration of all contributing authors.

Use of Generative-AI tools declaration

The authors declare that they did not utilize any artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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