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Research article

Linear convergence of forward-backward-forward type methods

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Abstract: This article is motivated by the work of Tseng (SIAM J. Control Optim. 38, 431–446 (2020)). We study some forward-backward-forward-type methods for solving variational inclusion problems involving the sum of two operators in real Hilbert spaces. We establish strong convergence theorems for these methods, demonstrating convergence to the unique solution of the problem with an R-linear rate, without relying on a line search procedure or Tseng's regularity assumptions.

Keywords: forward-backward-forward method; strongly monotone; convergence rate; zero point; R-linear rate
Mathematics Subject Classification: 47H09, 47H10, 47J20, 47J25

1. Introduction

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$. Let $A : H \to H$ be a single- and $B : H \to 2^H$ be a multi-valued mapping, respectively. Given this data, we wish to study the following inclusion problem:

$$\text{find } x^* \in H \text{ such that } 0 \in (A + B)x^*. \tag{1.1}$$

Let Ω be the solution set of the inclusion problem (1.1). This problem draws much attention since it stands at the core of many mathematical problems, such as: convex programming, variational inequalities, the split feasibility problem, and the minimization problem, see e.g [1–3] with applications in machine learning, image processing, and the linear inverse problem.

Due to the importance and interest of problem, many researchers have developed iterative methods for solving (1.1), see e.g., [1, 4–14] and the references therein. One popular method is the well-known *forward-backward splitting method* (FBM) introduced by Passty [11] and Lions et al. [10]. The method is formulated as follows. Choose an arbitrary starting point $u_0 \in H$, given the current iterate u_n, generate the next iterate as via the update rule:

$$u_{n+1} = (Id + \lambda_n B)^{-1}(Id - \lambda_n A)(u_n), \tag{1.2}$$

where $Id : H \to H$ stands for the identity mapping, $(\cdot)^{-1}$ is the inverse, $\lambda_n > 0$, and A and B are as above such that their domains fulfill $Dom(B) \subset Dom(A)$. In this setting (1.2), A and B are usually called the *forward operator* and the *backward operator*, respectively.

It is shown, see for example [15], that weak convergence of (1.2) requires quite restrictive assumptions on A and B, such that the inverse of A is strongly monotone or B is Lipschitz continuous and monotone and the operator $A + B$ is strongly monotone on $Dom(B)$.

In a direction to weaken these assumptions, Tseng in [14] introduced the following modified forward-backward splitting method, also known as Tseng's algorithm or the forward-backward-forward method. This method derives its name from its structure, which consists of three main steps in each iteration:

Forward Step: In this step, the algorithm moves forward using the current iteration point to compute a preliminary point based on the operator A.

Backward Step: After the forward step, the algorithm computes a backward step. This involves applying a proximal operator or a resolvent mapping. This step ensures that the updated point remains within the feasible region.

Final Forward Step: To finalize the iteration, another forward step is performed, refining the computed point further, which helps in handling monotone operators.

To be more precise, we present the algorithm introduced by Tseng as follows: Let $X \subseteq H$ be a closed and convex set which intersects the solution set of (1.1). Choose an arbitrary starting point $u_0 \in X$. Then given the current iterate u_n, generate the next iterate as via the update rule:

$$v_n = (Id + \lambda_n B)^{-1}(Id - \lambda_n A)(u_n),$$

where λ_n is chosen to be the largest $\lambda \in \{\delta, \delta l, \delta l^2, ...\}$ satisfying

$$\lambda \|Av_n - Au_n\| \leq \mu \|u_n - v_n\|,$$

where $\delta > 0, l \in (0, 1)$, and $\mu \in (0, 1)$ are constraints. Set

$$u_{n+1} = P_X(v_n - \lambda_n(Av_n - Au_n)).$$

Our main purpose is to extend and analyze the forward-backward-forward method for solving a variational inclusion the problem of the sum of two operators in the case that one is maximally monotone and strongly monotone, and the other is monotone and Lipschitz continuous. The convergence rate of the proposed algorithms is presented. It is worth pointing out that, in our algorithms, the step sizes are adaptive, nondecreasing, and found without the prior knowledge of the Lipschitz constant of the operator as well as without using the line seach procedure.

The paper is organized as follows. Recall some basic definitions and results in Section 2. Our algorithms are presented and their convergence rates in Section 3. Final remarks and conclusions are given in Section 4.

2. Preliminaries

Let H be a real Hilbert space and C be a nonempty, closed, and convex subset of H. The weak convergence of $\{u_n\}_{n=1}^{\infty}$ to x is denoted by $u_n \rightharpoonup x$ as $n \to \infty$, while the strong convergence of $\{u_n\}_{n=1}^{\infty}$ to x is written as $u_n \to x$ as $n \to \infty$. For all $x, y \in H$ and for all $\alpha \in [0;1]$, we have

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle;$$
$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x - y\|^2.$$

For every point $x \in H$, there exists the unique nearest point in C, denoted by $P_C x$ such that $\|x - P_C x\| \leq \|x - y\|\ \forall y \in C$. P_C is called the *metric projection* of H onto C. It is known that P_C is firmly nonexpansive, see for example [16–18].

Definition 2.1. Let $T : H \to H$ be an operator. Then:

- The operator T is called L-`Lipschitz continuous` with $L > 0$ if

$$\|Tx - Ty\| \leq L\|x - y\| \quad \forall x, y \in H.$$

- The operator T is called `monotone` if

$$\langle Tx - Ty, x - y\rangle \geq 0 \quad \forall x, y \in H.$$

A multi-valued mapping $B : H \to 2^H$ is called *monotone*, if for all $x, y \in H, u \in Bx$ and $v \in By$ imply that $\langle u - v, x - y\rangle \geq 0$. A multi-valued mapping $B : H \to 2^H$ is called *maximal monotone*, if it is monotone and if for any $(x, u) \in H \times H$, $\langle u - v, x - y\rangle \geq 0$ for every $(y, v) \in \text{Graph}(B)$ (the *graph* of mapping B) implies that $u \in Bx$. A multi-valued mapping $B : H \to 2^H$ is called δ-strongly monotone, i.e. there exists some number $\delta > 0$ such that

$$\langle u - v, x - y\rangle \geq \delta\|x - y\|^2 \quad \forall x, y \in H, u \in Bx, v \in By.$$

Let $B : H \to 2^H$ be a set-valued maximal monotone mapping. Then, the *resolvent mapping* $J_\lambda^B : H \to H$ associated with B is defined by

$$J_\lambda^B(x) := (I + \lambda B)^{-1}(x)\ \forall x \in H,$$

for some $\lambda > 0$, where I stands for the identity operator on H.

It is well known that if $B : H \to 2^H$ is a set-valued maximal monotone mapping and $\lambda > 0$, then $Dom(J_\lambda^B) = H$ and J_λ^B is a single-valued and firmly nonexpansive mapping.

For properties of the maximal monotone mapping, the interested reader is referred to [17, 19].

Lemma 2.2. *Let $B : H \to 2^H$ be a maximal monotone operator and $A : H \to H$ be a mapping on H. Define $T_\lambda := (I + \lambda B)^{-1}(I - \lambda A)$, $\lambda > 0$. Then, we have*

$$Fix(T_\lambda) = (A + B)^{-1}(0) \quad \forall \lambda > 0.$$

Proof. From the definition of T_λ we have

$$\begin{aligned} x = T_\lambda x &\Leftrightarrow x = (I + \lambda B)^{-1}(I - \lambda A)x \\ &\Leftrightarrow x - \lambda A x \in (I + \lambda B)x \\ &\Leftrightarrow x \in (A + B)^{-1}(0). \end{aligned}$$

Lemma 2.3. ([20], Lemma 2.4) *Let $B : H \to 2^H$ be a maximal monotone mapping and $A : H \to H$ be a Lipschitz continuous and monotone mapping. Then the mapping $A + B$ is a maximal monotone mapping.*

Definition 2.4. [21] Let $\{u_n\}$ be a sequence in H.
i) $\{u_n\}$ is said to converge R-linearly to x^* with rate $\rho \in [0, 1)$ if there is a constant $c > 0$ such that

$$\|u_n - x^*\| \le c\rho^n \quad \forall n \in \mathbb{N}.$$

ii) $\{u_n\}$ is said to converge Q-linearly to x^* with rate $\rho \in [0, 1)$ if

$$\|u_{n+1} - x^*\| \le \rho\|u_n - x^*\| \quad \forall n \in \mathbb{N}.$$

3. Main results

In this section, we introduce our methods for solving the inclusion problem (1.1), and the following conditions are assumed for the convergence of the methods.

Condition 3.1. The solution set of the inclusion problem (1.1) is nonempty, that is, $\Omega := (A+B)^{-1}(0) \neq \emptyset$.

Condition 3.2. The mappings $A : H \to H$ is L-Lipschitz continuous and monotone, and $B : H \to 2^H$ is a maximal monotone operator.

Condition 3.3. The mapping $B : H \to 2^H$ is δ-strongly monotone.

Denote by p the unique solution of problem (1.1).

3.1. Tseng's method

In this section, we investigate the convergence rate of Tseng's algorithm for solving monotone variational inclusion where the step size is adaptive per iteration. The algorithm is as follows.

Algorithm 3.1.

Initialization: Given $\lambda_1 > 0, \mu \in (0, 1)$. Let $u_0 \in H$ be arbitrary and $\{\tau_n\}$ be a nonnegative real number sequence such that $\sum_{n=1}^{\infty} \tau_n < +\infty$.

Iterative Steps: Given the current iterate u_n, calculate the next iterate as follows:

Step 1. Compute

$$v_n = (I + \lambda_n B)^{-1}(I - \lambda_n A)u_n$$

If $u_n = v_n$, then stop, and v_n is a solution of (1.1). Otherwise,
Step 2. Compute

$$u_{n+1} = v_n - \lambda_n(Av_n - Au_n).$$

Update

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\mu\|u_n - v_n\|}{\|Au_n - Av_n\|}, \lambda_n + \tau_n\right\} & \text{if } Au_n - Av_n \neq 0; \\ \lambda_n + \tau_n & \text{otherwise.} \end{cases} \tag{3.1}$$

Set $n := n + 1$ and go to **Step 1**.

Remark 3.1. As noted in [22], the sequence generated by (3.3) is allowed to increase from iteration to iteration. Thus, our subsequent results are different from those in [14].

We start by a technical lemma useful to our analysis.

Lemma 3.2. ([22]) *Let $\{\tau_n\}$ be the sequence generated by (3.3). Then,*

$$\lim_{n\to\infty} \lambda_n = \lambda \in \left[\min\{\lambda_1, \frac{\mu}{L}\}; \lambda_1 + \tau\right],$$

where $\tau = \sum_{n=1}^{\infty} \tau_n$ and

$$\|Au_n - Av_n\| \leq \frac{\mu}{\lambda_{n+1}}\|u_n - v_n\|. \tag{3.2}$$

Theorem 3.3. *Assume that Conditions 3.1–3.3 hold, and let $\{u_n\}$ be any sequence generated by Algorithm 3.1. Let $\lambda_1 > \frac{\mu}{L}$. Then, the sequence $\{u_n\}$ generated by Algorithm 3.1 converges in norm to the unique solution p of the problem* (1.1) *with an R-linear rate.*

Proof. We have $v_n = (I + \lambda_n B)^{-1}(I - \lambda A)u_n$. This implies that $(I - \lambda_n A)u_n \in (I + \lambda_n B)v_n$, hence $u_n - v_n - \lambda_n Au_n \in \lambda_n Bv_n$. Moreover, we have $-\lambda_n Ap \in \lambda_n Bp$. Since B is strongly monotone with constant δ, we get $\langle u_n - v_n - \lambda_n Au_n + \lambda_n Ap, v_n - p\rangle \geq \delta\|v_n - p\|^2$. Now, using the monotonicity of A, we obtain

$$\langle u_n - v_n - \lambda_n(Au_n - Av_n), v_n - p\rangle \geq \delta\|v_n - p\|^2 + \lambda_n\langle Av_n - Ap, v_n - p\rangle \geq \lambda_n\delta\|v_n - p\|^2. \tag{3.3}$$

We have

$$\begin{aligned}
\|u_{n+1} - p\|^2 =& \|v_n - \lambda_n(Av_n - Au_n) - p\|^2 \\
=& \|v_n - p\|^2 + \lambda_n^2\|Av_n - Au_n\|^2 - 2\lambda_n\langle v_n - p, Av_n - Au_n\rangle \\
=& \|u_n - p\|^2 + \|u_n - v_n\|^2 + 2\langle v_n - u_n, u_n - p\rangle \\
&+ \lambda_n^2\|Av_n - Au_n\|^2 - 2\lambda_n\langle v_n - p, Av_n - Au_n\rangle \\
=& \|u_n - p\|^2 + \|u_n - v_n\|^2 - 2\langle v_n - u_n, v_n - u_n\rangle + 2\langle v_n - u_n, v_n - p\rangle \\
&+ \lambda_n^2\|Av_n - Au_n\|^2 - 2\lambda_n\langle v_n - p, Av_n - Au_n\rangle \\
=& \|u_n - p\|^2 - \|u_n - v_n\|^2 + 2\langle v_n - u_n, v_n - p\rangle \\
&+ \lambda_n^2\|Av_n - Au_n\|^2 - 2\lambda_n\langle v_n - p, Av_n - Au_n\rangle \\
=& \|u_n - p\|^2 - \|u_n - v_n\|^2 - 2\langle u_n - v_n - \lambda_n(Au_n - Av_n), v_n - p\rangle
\end{aligned}$$

$$+ \lambda_n^2\|Av_n - Au_n\|^2. \tag{3.4}$$

Combining (3.2), (3.3), and (3.4), we get

$$\|u_{n+1} - p\|^2 \leq \|u_n - p\|^2 - \left(1 - \mu^2 \frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\|u_n - v_n\|^2 - 2\lambda_n\delta\|v_n - p\|^2. \tag{3.5}$$

We have

$$\lim_{n\to\infty} 1 - \mu^2 \frac{\lambda_n^2}{\lambda_{n+1}^2} = 1 - \mu^2$$

and

$$\lim_{n\to\infty} \lambda_n\delta = \lambda\delta \geq \delta \min\{\lambda_1, \frac{\mu}{L}\} = \delta\frac{\mu}{L}.$$

Let $\epsilon := \min\{\frac{1-\mu^2}{2}, \delta\frac{\mu}{L}\}$. Therefore, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we deduce $1 - \mu^2 \frac{\lambda_n^2}{\lambda_{n+1}^2} \geq \epsilon$ and $\lambda_n\delta \geq \epsilon$. Using this and (3.5), we get

$$\begin{aligned}
\|u_{n+1} - p\|^2 &\leq \|u_n - p\|^2 - 2\epsilon\|u_n - v_n\|^2 - 2\epsilon\|v_n - p\|^2 \\
&\leq \|u_n - p\|^2 - \epsilon\|u_n - p\|^2 \\
&= (1-\epsilon)\|u_n - p\|^2 \\
&\leq ... \leq (1-\epsilon)^{n-N+1}\|u_N - p\|^2 = \frac{\|u_N - p\|^2}{(1-\epsilon)^{N-1}}(1-\epsilon)^n.
\end{aligned}$$

Thus we see that the sequence $\{u_n\}$ indeed converges R-linearly to p, as asserted. □

3.2. Inertial Tseng's method

In this section, we present the convergence rate of the modification of Tseng's algorithm for solving monotone variational inclusion with the inertial technique. The algorithm is as follows.

Algorithm 3.2.

Initialization: Given $\lambda_1 > 0, \mu \in (0,1)$, and $\alpha \in [0,1)$. Let $u_0, u_1 \in H$ be arbitrary and $\{\tau_n\}$ be a nonnegative real number sequence such that $\sum_{n=1}^{\infty} \tau_n < +\infty$.

Select starting points $u_0, u_1 \in H$ and set $n := 1$.

Step 1. Given the iterates u_n, u_{n-1}, compute

$$\begin{aligned}
w_n &= u_n + \alpha(u_n - u_{n-1}), \\
v_n &= (I + \lambda_n B)^{-1}(I - \lambda_n A)w_n.
\end{aligned}$$

If $v_n = w_n$, then stop: $v_n \in \Omega$. Otherwise, go to **Step 2**.

Step 2. Compute

$$u_{n+1} = v_n - \lambda_n(Av_n - Aw_n).$$

Update

$$\lambda_{n+1} = \begin{cases} \min\left\{\dfrac{\mu\|w_n - v_n\|}{\|Aw_n - Av_n\|}, \lambda_n + \tau_n\right\} & \text{if } Aw_n - Av_n \neq 0; \\ \lambda_n + \tau_n & \text{otherwise.} \end{cases} \tag{3.6}$$

Set $n := n + 1$ and go to **Step 1**.

Theorem 3.4. *Assume that Conditions 3.1–3.3 hold and let* $\{u_n\}$ *be any sequence generated by Algorithm 3.2. Let* $\gamma \in (0, 1)$ *be arbitrary and* α *be such that*

$$0 \leq \alpha \leq \min\left\{1 + \frac{1 - \sqrt{1 + 4\xi}}{2\xi}, \frac{\sqrt{(1 + \gamma\xi)^2 + 4\gamma\xi} - (1 + \gamma\xi)}{2}, (1 - \gamma)\left(1 - \frac{(1 - \mu^2)\mu}{2}\right)\right\}, \tag{3.7}$$

where $\xi := \dfrac{(1 - \mu)^2}{2}$*. Then, the sequence* $\{u_n\}$ *generated by Algorithm 3.2 converges in norm to the unique solution* p *of the problem* (1.1) *with an R-linear rate.*

Proof. We have $v_n = (I + \lambda_n B)^{-1}(I - \lambda A)w_n$. This implies that $(I - \lambda_n A)w_n \in (I + \lambda_n B)v_n$ hence $w_n - v_n - \lambda_n Au_n \in \lambda_n Bv_n$. Moreover, we have $-\lambda_n Ap \in \lambda_n Bp$. From that B is strongly monotone with constant δ, we get $\langle w_n - v_n - \lambda_n Aw_n + \lambda_n Ap, v_n - p\rangle \geq \delta\|v_n - p\|^2$. Now, using the monotonicity of A, we obtain

$$\langle w_n - v_n - \lambda_n(Aw_n - Av_n), v_n - p\rangle \geq \delta\|v_n - p\|^2 + \lambda_n\langle Av_n - Ap, v_n - p\rangle \geq \lambda_n\delta\|v_n - p\|^2. \tag{3.8}$$

We have

$$\begin{aligned} \|u_{n+1} - p\|^2 =& \|v_n - \lambda_n(Av_n - Aw_n) - p\|^2 \\ =& \|v_n - p\|^2 + \lambda_n^2\|Av_n - Aw_n\|^2 - 2\lambda_n\langle v_n - p, Av_n - Aw_n\rangle \\ =& \|w_n - p\|^2 + \|w_n - v_n\|^2 + 2\langle v_n - w_n, u_n - p\rangle + \lambda_n^2\|Av_n - Aw_n\|^2 \\ &- 2\lambda_n\langle v_n - p, Av_n - Aw_n\rangle \\ =& \|w_n - p\|^2 + \|w_n - v_n\|^2 - 2\langle v_n - w_n, v_n - w_n\rangle + 2\langle v_n - w_n, v_n - p\rangle \\ &+ \lambda_n^2\|Av_n - Aw_n\|^2 - 2\lambda_n\langle v_n - p, Av_n - Aw_n\rangle \\ =& \|w_n - p\|^2 - \|u_n - v_n\|^2 + 2\langle v_n - w_n, v_n - p\rangle + \lambda_n^2\|Av_n - Aw_n\|^2 \\ &- 2\lambda_n\langle v_n - p, Av_n - Aw_n\rangle \\ =& \|w_n - p\|^2 - \|w_n - v_n\|^2 - 2\langle w_n - v_n - \lambda_n(Aw_n - Av_n), v_n - p\rangle \\ &+ \lambda_n^2\|Av_n - Aw_n\|^2. \end{aligned} \tag{3.9}$$

Combining (3.2), (3.8), and (3.9), we get

$$\|u_{n+1} - p\|^2 \leq \|w_n - p\|^2 - \left(1 - \mu^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\|w_n - v_n\|^2 - 2\lambda_n\delta\|v_n - p\|^2. \tag{3.10}$$

From (3.10), we deduce

$$\|u_{n+1} - p\|^2 \leq \|w_n - p\|^2 - \left(1 - \mu^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)(1 - \mu)\|v_n - w_n\|^2$$

$$-\left(1-\mu^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\mu\|v_n-w_n\|^2-2\lambda_n\delta\|v_n-p\|^2. \tag{3.11}$$

By the definition of u_{n+1} we have

$$\begin{aligned}\|u_{n+1}-v_n\| &= \|v_n-\lambda_n(Fv_n-Fw_n)-v_n\|\\ &\le \lambda_n\|Fv_n-Fw_n\|\\ &\le \mu\frac{\lambda_n}{\lambda_{n+1}}\|v_n-w_n\|.\end{aligned}$$

Therefore,

$$\|u_{n+1}-w_n\|\le\|u_{n+1}-v_n\|+\|v_n-w_n\|\le\left(1+\mu\frac{\lambda_n}{\lambda_{n+1}}\right)\|v_n-w_n\|.$$

This implies

$$\|v_n-w_n\|\ge\frac{1}{\left(1+\mu\frac{\lambda_n}{\lambda_{n+1}}\right)}\|u_{n+1}-w_n\|. \tag{3.12}$$

From $\lim_{n\to\infty}\left(1-\mu^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)=1-\mu^2>0$, there exists $N_0\in\mathbb{N}$ such that

$$1-\mu^2\frac{\lambda_n^2}{\lambda_{n+1}^2}>0\ \ \forall n\ge N_0.$$

Substituting (3.12) into (3.11), we have for all $n\ge N_0$ that

$$\begin{aligned}\|u_{n+1}-p\|^2\le&\|w_n-p\|^2-\frac{\left(1-\mu^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)}{\left(1+\mu\frac{\lambda_n}{\lambda_{n+1}}\right)^2}(1-\mu)\|u_{n+1}-w_n\|^2\\ &-\left(1-\mu^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\mu\|v_n-w_n\|^2-2\lambda_n\delta\|v_n-p\|^2\\ =&\|w_n-p\|^2-\frac{\left(1-\mu\frac{\lambda_n}{\lambda_{n+1}}\right)}{\left(1+\mu\frac{\lambda_n}{\lambda_{n+1}}\right)}(1-\mu)\|u_{n+1}-w_n\|^2\\ &-\left(1-\mu^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\mu\|v_n-w_n\|^2-2\lambda_n\delta\|v_n-p\|^2.\end{aligned}$$

Let $\beta:=\min\left\{\frac{(1-\mu^2)\mu}{2},\lambda\delta\right\}$, where $\lambda:=\lim_{n\to\infty}\lambda_n$. Then, we have

$$\lim_{n\to\infty}\frac{\left(1-\mu\frac{\lambda_n}{\lambda_{n+1}}\right)}{\left(1+\mu\frac{\lambda_n}{\lambda_{n+1}}\right)}(1-\mu)=\frac{1-\mu}{1+\mu}(1-\mu)\ge\frac{(1-\mu)^2}{2},$$

$$\lim_{n\to\infty}\Big(1-\mu^2\frac{\lambda_n}{\lambda_{n+1}}\Big)\mu=(1-\mu^2)\mu\geq 2\beta,$$
$$\lim_{n\to\infty}\lambda_n\delta=\lambda\delta\geq\beta.$$

Thus, there exists N_1 such that

$$\frac{\Big(1-\mu\frac{\lambda_n}{\lambda_{n+1}}\Big)}{\Big(1+\mu\frac{\lambda_n}{\lambda_{n+1}}\Big)}(1-\mu)\geq\frac{(1-\mu)^2}{2}\quad\forall n\geq N_1,$$

$$\Big(1-\mu^2\frac{\lambda_n}{\lambda_{n+1}}\Big)\mu\geq 2\beta\quad\forall n\geq N_1,$$

and

$$\lambda_n\delta\geq\beta\quad\forall n\geq N_1.$$

Let $N=\max\{N_0,N_1\}$. Then, using (3.11) we get

$$\begin{aligned}\|u_{n+1}-p\|^2\leq&\|w_n-p\|^2-\frac{(1-\mu)^2}{2}\|u_{n+1}-w_n\|^2-2\beta(\|v_n-w_n\|^2+\|v_n-p\|^2)\quad\forall n\geq N\\\leq&\|w_n-p\|^2-\frac{(1-\mu)^2}{2}\|u_{n+1}-w_n\|^2-\beta\|w_n-p\|^2\quad\forall n\geq N\\\leq&(1-\beta)\|w_n-p\|^2-\frac{(1-\mu)^2}{2}\|u_{n+1}-w_n\|^2\quad\forall n\geq N\\=&\rho\|w_n-p\|^2-\xi\|u_{n+1}-w_n\|^2\quad\forall n\geq N,\end{aligned}\tag{3.13}$$

where $\rho:=1-\beta\in(0,1)$ and $\xi:=\dfrac{(1-\mu)^2}{2}\in(0,1)$. On the other hand, we also have

$$\begin{aligned}\|w_n-p\|^2&=\|(1+\alpha)(u_n-p)-\alpha(u_{n-1}-p)\|^2\\&=(1+\alpha)\|u_n-p\|^2-\alpha\|u_{n-1}-p\|^2+\alpha(1+\alpha)\|u_n-u_{n-1}\|^2\end{aligned}$$

and

$$\begin{aligned}\|u_{n+1}-w_n\|^2&=\|u_{n+1}-u_n-\alpha(u_n-u_{n-1})\|^2\\&=\|u_{n+1}-u_n\|^2+\alpha^2\|u_n-u_{n-1}\|^2-2\alpha\langle u_{n+1}-u_n,u_n-u_{n-1}\rangle\\&\geq\|u_{n+1}-u_n\|^2+\alpha^2\|u_n-u_{n-1}\|^2-2\alpha\|u_{n+1}-u_n\|\|u_n-u_{n-1}\|\\&\geq\|u_{n+1}-u_n\|^2+\alpha^2\|u_n-u_{n-1}\|^2-\alpha\|u_{n+1}-u_n\|^2-\alpha\|u_n-u_{n-1}\|^2\\&\geq(1-\alpha)\|u_{n+1}-u_n\|^2-\alpha(1-\alpha)\|u_n-u_{n-1}\|^2.\end{aligned}$$

Combining these inequalities with (3.13), we obtain

$$\begin{aligned}\|u_{n+1}-p\|^2\leq&\rho(1+\alpha)\|u_n-p\|^2-\rho\alpha\|u_{n-1}-p\|^2+\rho\alpha(1+\alpha)\|u_n-u_{n-1}\|^2\\&-\xi(1-\alpha)\|u_{n+1}-u_n\|^2+\xi\alpha(1-\alpha)\|u_n-u_{n-1}\|^2\quad\forall n\geq N,\end{aligned}$$

or, equivalently,

$$\begin{aligned}\|u_{n+1}-p\|^2-\rho\alpha\|u_n-p\|^2+&\xi(1-\alpha)\|u_{n+1}-u_n\|^2\\ &\le\rho\left[\|u_n-p\|^2-\alpha\|u_{n-1}-p\|^2+\xi(1-\alpha)\|u_n-u_{n-1}\|^2\right]\\ &\quad-\left(\rho\xi(1-\alpha)-\rho\alpha(1+\alpha)-\xi\alpha(1-\alpha)\right)\|u_n-u_{n-1}\|^2\ \ \forall n\ge N.\end{aligned}$$

Setting

$$\Gamma_n:=\|u_n-p\|^2-\alpha\|u_{n-1}-p\|^2+\xi(1-\alpha)\|u_n-u_{n-1}\|^2,$$

since $\rho\in(0,1)$, we can write

$$\begin{aligned}\Gamma_{n+1}\le&\|u_{n+1}-p\|^2-\rho\alpha\|u_n-p\|^2+\xi(1-\alpha)\|u_{n+1}-u_n\|^2\\ \le&\rho\Gamma_n-\left(\rho\xi(1-\alpha)-\rho\alpha(1+\alpha)-\xi\alpha(1-\alpha)\right)\|u_n-u_{n-1}\|^2\ \ \forall n\ge N.\end{aligned}$$

Now, we show that

$$(\rho\xi(1-\alpha)-\rho\alpha(1+\alpha)-\xi\alpha(1-\alpha))\ge 0.$$

Note that from (3.7) we have

$$\begin{aligned}\alpha\le&(1-\gamma)\left(1-\frac{(1-\mu^2)\mu}{2}\right)\\ \le&(1-\gamma)(1-\beta)=(1-\delta)\rho,\end{aligned}$$

which implies

$$\xi\alpha(1-\alpha)\le(1-\gamma)\rho\xi(1-\alpha)=\rho\xi(1-\alpha)-\gamma\rho\xi(1-\alpha). \tag{3.14}$$

Since

$$\alpha\le\frac{\sqrt{(1+\gamma\xi)^2+4\gamma\xi}-(1+\gamma\xi)}{2}$$

it holds that

$$\alpha^2+(1+\gamma\xi)\alpha-\gamma\xi\le 0,$$

or equivalently,

$$\alpha(1+\alpha)\le\gamma\xi(1-\alpha).$$

Hence,

$$\rho\alpha(1+\alpha)\le\rho\gamma\xi(1-\alpha). \tag{3.15}$$

From (3.14) and (3.15), we deduce

$$\rho\xi(1-\alpha)-\rho\alpha(1+\alpha)-\xi\alpha(1-\alpha)\ge 0.$$

which implies that

$$\Gamma_{n+1}\le\rho\Gamma_n.$$

Now, we show that $\Gamma_n\ge 0$ for all n. Indeed, we have

$$\Gamma_n=\|u_n-p\|^2-\alpha\|u_{n-1}-p\|^2+\xi(1-\alpha)\|u_n-u_{n-1}\|^2 \tag{3.16}$$

On the other hand, we have

$$\begin{aligned}\|u_{n-1}-p\|^2 &= \|u_{n-1}-u_n+u_n-p\|^2 = \|u_{n-1}-u_n\|^2+\|u_n-p\|^2+2\langle u_{n-1}-u_n, u_n-p\rangle \\ &\le \|u_{n-1}-u_n\|^2+\|u_n-p\|^2+2\|u_{n-1}-u_n\|\|u_n-p\| \\ &\le \|u_{n-1}-u_n\|^2+\|u_n-p\|^2+k\|u_{n-1}-u_n\|^2+\frac{1}{k}\|u_n-p\|^2 \\ &= (1+k)\|u_{n-1}-u_n\|^2+\left(1+\frac{1}{k}\right)\|u_n-p\|^2, \end{aligned} \tag{3.17}$$

for all $k > 0$. Combining (3.16) and (3.17), we get

$$\Gamma_n \ge \left[1-\left(1+\frac{1}{k}\right)\alpha\right]\|u_n-p\|^2+\left[\xi(1-\alpha)-(1+k)\alpha\right]\|u_n-u_{n-1}\|^2. \tag{3.18}$$

We show that there exists $k > 0$ such that

$$\begin{cases} 1-\left(1+\frac{1}{k}\right)\alpha & > 0, \\ \xi(1-\alpha)-(1+k)\alpha & > 0. \end{cases} \tag{3.19}$$

Indeed, if $\alpha = 0$, then inequality (3.19) is obvious. Now, we consider $\alpha > 0$. In this case, inequality (3.19) is equivalent to

$$\begin{cases} k > \dfrac{\alpha}{1-\alpha}, \\ k < \dfrac{\xi(1-\alpha)}{\alpha}-1. \end{cases}$$

Moreover, from (3.7), we also have

$$0 \le \alpha \le 1+\frac{1-\sqrt{1+4\xi}}{2\xi}$$

from which it follows that

$$\frac{\alpha}{1-\alpha} < \frac{\xi(1-\alpha)}{\alpha}-1,$$

that is, there exists $k > 0$ satisfying the inequality (3.19). From (3.18), it implies that $\Gamma_n \ge 0$ for all n. Hence,

$$\Gamma_{n+1} \le \rho\Gamma_n \le ... \le \rho^{n-N+1}\Gamma_N.$$

$$\|u_n-p\|^2 \le \frac{\Gamma_N}{\rho^{N+1}}\rho^n.$$

Thus, we see that the sequence $\{u_n\}$ indeed converges R-linearly to p, as asserted. □

3.3. Relaxed Inertial Tseng's method

In this section, we investigate the convergence rate of the relaxed inertial Tseng's algorithm for solving monotone variational inclusion with the inertial technique. The algorithm is as follows.

Algorithm 3.3.

Initialization: Given $\lambda_1 > 0, \mu \in (0,1), \alpha \in [0,1)$, and $\theta \in (0,1)$. Let $u_0, u_1 \in H$ be arbitrary and $\{\tau_n\}$ be a nonnegative real number sequence such that $\sum_{n=1}^{\infty} \tau_n < +\infty$.

Select starting points $u_0, u_1 \in H$, and set $n := 1$.

Step 1. Given the iterates u_n, u_{n-1}, compute

$$\begin{aligned} w_n &= u_n + \alpha(u_n - u_{n-1}), \\ v_n &= (I + \lambda_n B)^{-1}(I - \lambda_n A)w_n. \end{aligned}$$

If $v_n = w_n$ then stop: $v_n \in \Omega$. Otherwise, go to **Step 2**.

Step 2. Compute

$$u_{n+1} = (1-\theta)w_n + \theta(v_n - \lambda_n(Av_n - Aw_n)).$$

Update

$$\lambda_{n+1} = \begin{cases} \min\left\{\dfrac{\mu\|w_n - v_n\|}{\|Aw_n - Av_n\|}, \lambda_n + \tau_n\right\} & \text{if } Aw_n - Av_n \neq 0; \\ \lambda_n + \tau_n & \text{otherwise.} \end{cases}$$

Set $n := n + 1$ and go to **Step 1**.

Theorem 3.5. *Assume that Conditions 3.1–3.3 hold, and let $\{u_n\}$ be any sequence generated by Algorithm 3.3. Let $\gamma \in (0,1)$ be arbitrary and α be such that*

$$0 \le \alpha \le \min\left\{1 + \frac{1 - \sqrt{1+4\xi}}{2\xi}, \frac{\sqrt{(1+\gamma\xi)^2 + 4\gamma\xi} - (1+\gamma\xi)}{2}, (1-\gamma)\left(1 - \frac{(1-\mu^2)\mu}{2}\right)\right\},$$

where $\xi := \dfrac{(1-\mu)^2}{2\theta}$. Then, the sequence $\{u_n\}$ generated by Algorithm 3.3 converges in norm to the unique solution p of problem (1.1) *with an R-linear rate.*

Proof. Let $z_n := v_n - \lambda_n(Av_n - Aw_n)$. We can rewrite the formula of u_{n+1} as follows:

$$u_{n+1} = (1-\theta)w_n + \theta z_n.$$

First, we show that there exist $\beta \in (0,1)$ and $N_2 \in \mathbb{N}$ such that

$$\|z_n - p\|^2 \le (1-\beta)\|w_n - p\|^2 - \frac{(1-\mu)^2}{2\theta^2}\|u_{n+1} - w_n\|^2 \quad \forall n \ge N_2.$$

Indeed, according to the proof of Theorem 3.4, we deduce

$$\|z_n - p\|^2 \le \|w_n - p\|^2 - \left(1 - \mu^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\|w_n - v_n\|^2 - 2\lambda_n\delta\|v_n - p\|^2. \tag{3.20}$$

From (3.20) we get

$$\|z_n - p\|^2 \le \|w_n - p\|^2 - \left(1 - \mu^2 \frac{\lambda_n^2}{\lambda_{n+1}^2}\right)(1-\mu)\|v_n - w_n\|^2 - \left(1 - \mu^2 \frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\mu\|v_n - w_n\|^2 - 2\lambda_n\delta\|v_n - p\|^2. \tag{3.21}$$

By the definition of z_n, we have

$$\begin{aligned}\|z_n - v_n\| &= \|v_n - \lambda_n(Fv_n - Fw_n) - v_n\| \\ &\le \lambda_n\|Fv_n - Fw_n\| \\ &\le \mu\frac{\lambda_n}{\lambda_{n+1}}\|v_n - w_n\|.\end{aligned}$$

Therefore,

$$\|z_n - w_n\| \le \|z_n - v_n\| + \|v_n - w_n\| \le \left(1 + \mu\frac{\lambda_n}{\lambda_{n+1}}\right)\|v_n - w_n\|.$$

This implies

$$\|v_n - w_n\| \ge \frac{1}{\left(1 + \mu\frac{\lambda_n}{\lambda_{n+1}}\right)}\|z_n - w_n\|. \tag{3.22}$$

From $\lim_{n\to\infty}\left(1 - \mu^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right) = 1 - \mu^2 > 0$, there exists $N_0 \in \mathbb{N}$ such that

$$1 - \mu^2\frac{\lambda_n^2}{\lambda_{n+1}^2} > 0 \quad \forall n \ge N_0.$$

Substituting (3.22) into (3.21), we have for all $n \ge N_0$ that

$$\begin{aligned}\|z_n - p\|^2 \le & \|w_n - p\|^2 - \frac{\left(1 - \mu^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)}{\left(1 + \mu\frac{\lambda_n}{\lambda_{n+1}}\right)^2}(1-\mu)\|z_n - w_n\|^2 - \left(1 - \mu^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\mu\|v_n - w_n\|^2 \\ & - 2\lambda_n\delta\|v_n - p\|^2 \\ = & \|w_n - p\|^2 - \frac{\left(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}\right)}{\left(1 + \mu\frac{\lambda_n}{\lambda_{n+1}}\right)}(1-\mu)\|z_n - w_n\|^2 - \left(1 - \mu^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\mu\|v_n - w_n\|^2 \\ & - 2\lambda_n\delta\|v_n - p\|^2.\end{aligned} \tag{3.23}$$

On the other hand, we have $u_{n+1} = (1-\theta)w_n + \theta z_n$, thus

$$w_n - z_n = \frac{1}{\theta}(u_{n+1} - w_n) \tag{3.24}$$

Again, substituting (3.24) into (3.23), we obtain

$$\|z_n - p\|^2 \le \|w_n - p\|^2 - \frac{\left(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}\right)}{\left(1 + \mu\frac{\lambda_n}{\lambda_{n+1}}\right)}(1-\mu)\frac{1}{\theta^2}\|u_{n+1} - w_n\|^2 - \left(1 - \mu^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\mu\|v_n - w_n\|^2 - 2\lambda_n\delta\|v_n - p\|^2. \tag{3.25}$$

Let $\beta := \min\left\{\frac{(1-\mu^2)\mu}{2}, \lambda\delta\right\}$, where $\lambda := \lim_{n\to\infty}\lambda_n$. Then, we have

$$\lim_{n\to\infty}\frac{\left(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}\right)}{\left(1 + \mu\frac{\lambda_n}{\lambda_{n+1}}\right)}(1-\mu)\frac{1}{\theta^2} = \frac{1-\mu}{1+\mu}(1-\mu)\frac{1}{\theta^2} \ge \frac{(1-\mu)^2}{2\theta^2},$$

$$\lim_{n\to\infty}\left(1 - \mu^2\frac{\lambda_n}{\lambda_{n+1}}\right)\mu = (1-\mu^2)\mu \ge 2\beta,$$

$$\lim_{n\to\infty}\lambda_n\delta = \lambda\delta \ge \beta.$$

Thus, there exists N_1 such that

$$\frac{\left(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}\right)}{\left(1 + \mu\frac{\lambda_n}{\lambda_{n+1}}\right)}(1-\mu)\frac{1}{\theta^2} \ge \frac{(1-\mu)^2}{2\theta^2} \quad \forall n \ge N_1,$$

$$\left(1 - \mu^2\frac{\lambda_n}{\lambda_{n+1}}\right)\mu \ge 2\beta \quad \forall n \ge N_1,$$

and

$$\lambda_n\delta \ge \beta \quad \forall n \ge N_1.$$

Let $N = \max\{N_0, N_1\}$. Using (3.25), we derive

$$\begin{aligned}\|z_n - p\|^2 &\le \|w_n - p\|^2 - \frac{(1-\mu)^2}{2\theta^2}\|u_{n+1} - w_n\|^2 - 2\beta(\|v_n - w_n\|^2 + \|v_n - p\|^2) \quad \forall n \ge N\\ &\le \|w_n - p\|^2 - \frac{(1-\mu)^2}{2\theta^2}\|u_{n+1} - w_n\|^2 - \beta\|w_n - p\|^2 \quad \forall n \ge N\\ &\le (1-\beta)\|w_n - p\|^2 - \frac{(1-\mu)^2}{2\theta^2}\|u_{n+1} - w_n\|^2 \quad \forall n \ge N.\end{aligned} \tag{3.26}$$

Next, we also have

$$\begin{aligned}\|u_{n+1} - p\|^2 &= \|(1-\theta)w_n + \theta z_n - p\|^2\\ &= \|(1-\theta)(w_n - p) + \theta(z_n - p)\|^2\\ &\le (1-\theta)\|w_n - p\|^2 + \theta\|z_n - p\|^2.\end{aligned} \tag{3.27}$$

Combining (3.26) and (3.27), we get

$$\begin{aligned}\|u_{n+1} - p\|^2 &\leq (1-\theta)\|w_n - p\|^2 + \theta(1-\beta)\|w_n - p\|^2 - \frac{(1-\mu)^2}{2\theta}\|u_{n+1} - w_n\|^2 \\ &= (1-\theta\beta)\|w_n - p\|^2 - \frac{(1-\mu)^2}{2\theta}\|u_{n+1} - w_n\|^2.\end{aligned}$$

Let $\rho := 1 - \theta\beta$ and $\xi := \frac{(1-\mu)^2}{2\theta}$. The rest of the proof is similar to that of Theorem 3.4, and therefore is omitted. □

4. Conclusions

In this article, we focus on the problem of finding zeros of the sum of two monotone operators in real Hilbert spaces. The convergence rate of forward-backward-forward type algorithms is presented with variable step sizes which are updated over each iteration. Moreover, these step sizes are found without prior knowledge of the Lipschitz constant of the operator as well as without using the line seach procedure.

Use of Generative-AI tools declaration

The authors declare that no Artificial Intelligence (AI) tools were used in the preparation of this manuscript.

Data availability statement

The corresponding author can provide the datasets created and examined in this study upon reasonable request.

Conflict of interest

The authors have stated that they do not have any conflicts of interest.

Ethical approval

This paper has not been published before, and it is not under consideration for publication anywhere else.

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