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Research article

Parameter estimation in uncertain multiple-delay differential equations

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Abstract: Uncertain multiple-delay differential equations (UMDDEs) driven by Liu process are critical for modeling systems with multiple-delay interactions and environmental noise. This paper proposes the method of moment estimation to estimate the parameters for UMDDEs with known or unknown delays. When the time interval of the observed data is particularly large, the parameters estimated by the above moment estimation are not very good. In order to overcome this shortcoming, the concept of residuals is introduced, and then we use the method of residual estimation to estimate the parameters for UMDDEs. Moreover, some numerical validations are investigated to show the effectiveness of the above methods for UMDDEs. Besides, the paradox of the stochastic multiple delay Logistic model is proved. Therefore, an uncertain multiple delay Logistic model is defined and applied to describe the dynamics of U.S. population dynamics.

Keywords: Uncertain multiple-delay differential equations; parameter estimation; liu processes
Mathematics Subject Classification: 34D20, 93D15

1. Introduction

Delay differential equations have been applied to numerous systems, such as ecosystems [1], mechanical automatic control systems [2], and chemical kinetics [3]. In 1977, Litching [4] discovered that the life cycle of Australian flies exhibits multiple delay characteristics. Subsequently, Braddock and Driessche [5] proposed a differential equation with two delay times. Furthermore, differential equations with two or more delay times are referred to as multiple delay differential equations, which have been used to describe iterative learning control systems [6], antigen T-cell immune systems [7], and epidemiological models [8].

In reality, systems are always subject to “noise”. If the “noise” is modeled using the Wiener process within the framework of probability theory, delay differential equations and multiple delay differential equations with the Wiener process are called stochastic delay differential equations and stochastic multiple delay differential equations, respectively. The applications of stochastic delay differential equations are extensive. For example, Yang et al. [9] used stochastic delay differential equations to model stock prices in financial markets, Ali and Khan [10] employed them to describe the spread of infectious diseases, and Jin et al. [11] applied them to model chemical reaction processes. Similarly, stochastic multiple delay differential equations have broad applications. For instance, Frank [12] used them to characterize stock price fluctuations in financial markets, Klamka [13] applied them to energy control systems, and Hu et al. [14] utilized them in ecological systems. Moreover, differential equations involving fuzzy sets [15–18] were investigated by many scholars.

When the distribution and frequency are not close, the Wiener process cannot be used to model "noise". However, we can employ the Liu process from uncertainty theory, which is constructed based on the belief of experts or experienced individuals, to represent "noise". Introducing the Liu process into delay differential equations results in uncertain delay differential equations. In 2010, Barbacioru [19] proposed uncertain delay differential equations to model stock price fluctuations and proved the local existence and uniqueness theorem for a special class of uncertain delay differential equations. In 2012, Ge and Zhu [20] demonstrated the local existence and uniqueness theorem for general uncertain delay differential equations. In 2019, Wang and Ning [21] established the global existence and uniqueness theorem for general uncertain delay differential equations. Additionally, Wang and Ning [22] introduced measure stability, mean stability, and p-th moment stability for uncertain delay differential equations, providing sufficient conditions for each. Wang and Ning [23] also proposed almost sure stability for uncertain delay differential equations and derived sufficient conditions for its validity. Jia and Sheng [24] presented distribution stability for uncertain delay differential equations and provided sufficient conditions for its establishment.

Uncertain multiple-delay differential equations (UMDDEs) have emerged as a critical framework for modeling complex dynamical systems with inherent time delays and environmental noise, which extend traditional delay differential equations by incorporating multiple discrete delays and uncertainties governed by the Liu process, formally expressed as:

$$\mathrm{d}X_t = f(t, X_t, X_{t-\tau_1}, \ldots, X_{t-\tau_m}; \mu)\mathrm{d}t + g(t, X_t, X_{t-\tau_1}, \ldots, X_{t-\tau_m}; \mu)\mathrm{d}C_t,$$

where $\tau_1, \tau_2, \ldots, \tau_m$ represents delays, μ is an unknown parameter vector, and C_t denotes the Liu process. At present, Gao and Tang [25] have proved the existence and uniqueness theorem for the solution of UMDDEs, and the stability in measure for UMDDEs was proposed, then two sufficient theorems for UMDDEs being stable in measure were presented. Moreover, Gao et al. [26] proposed the definition of stability in mean for UMDDEs, and two sufficient theorems for UMDDEs to be stable in measure were presented. In order to apply the UMDDEs to describe some dynamic systems, the method of parameter estimation for UMDDEs is investigated in this paper. Meanwhile, the contribution of this paper are given as below.

- Moment estimation method for uncertain multiple delay differential equations with known delays is proposed.
- Moment estimation method for uncertain multiple delay differential equations with unknown delays is provided.
- Residual method for uncertain multiple delay differential equations with unknown delays is given.
- The paradox of stochastic multiple delay Logistic model is proved.

The paper is structured as follows. Section 2 introduces the method of moment estimation for uncertain multiple delay differential equations with known delays. Section 3 gives the method of moment estimation for uncertain multiple delay differential equations with unknown delays. Section 4 provides the method of residual estimation for uncertain multiple delay differential equations with delays. Section 5 proves the paradox of the stochastic multiple-delay Logistic model and defines the uncertain multiple-delay Logistic model to describe the dynamics of the United States. Finally, Section 6 synthesizes contributions.

2. Moment estimation method for uncertain multiple delay differential equations with known delays

The uncertain multiple delay differential equation is

$$\mathrm{d}X_t = f(t, X_t, X_{t-\tau_1}, X_{t-\tau_2}, \cdots, X_{t-\tau_n}; \mu)\mathrm{d}t + g(t, X_t, X_{t-\tau_1}, X_{t-\tau_2}, \cdots, X_{t-\tau_n}; \mu)\mathrm{d}C_t \tag{2.1}$$

where is an unknown parameter vector, C_t is a Liu process, $\tau_1, \tau_2 ... \tau_m$ are a series of known delays, the uncertain multiple delay differential equation (2.1) has a forward Euler difference format

$$X_{t_{i+1}} = X_{t_i} + f(t_i, X_{t_i}, X_{t_i-\tau_1}, X_{t_i-\tau_2}, \cdots, X_{t_i-\tau_n}; \mu)(t_{i+1} - t_i) + g(t_i, X_{t_i}, X_{t_i-\tau_1}, X_{t_i-\tau_2}, \cdots, X_{t_i-\tau_n}; \mu)(C_{i+1} - C_i).$$

The above formula can also be written

$$\frac{X_{t_{i+1}} - Xt_i - f(t_i, X_t, X_{t-\tau_1}, X_{t-\tau_2}, \cdots, X_{t-\tau_n}; \mu)(t_{i+1} - t_i)}{g(t, X_t, X_{t-\tau_1}, X_{t--\tau_2}, \cdots, X_{t-\tau_n}; \mu)(t_{i+1} - t_i)} = \frac{C_{i+1} - C_i}{t_{i+1} - t_i}.$$

Based on the definition of the Liu process [27], we have

$$\frac{C_{i+1} - C_i}{t_{i+1} - t_i} \sim \mathcal{N}(0, 1).$$

So

$$\frac{X_{t_{i+1}} - Xt_i - f(t_i, X_t, X_{t-\tau_1}, X_{t-\tau_2}, \cdots, X_{t-\tau_n}; \mu)(t_{i+1} - t_i)}{g(t, X_t, X_{t-\tau_1}, X_{t--\tau_2}, \cdots, X_{t-\tau_n}; \mu)(t_{i+1} - t_i)} \sim \mathcal{N}(0, 1). \tag{2.2}$$

At points $t_1, t_2, \cdots, t_n$ with observed data $x_{t_1}, x_{t_2}, \cdots, x_{t_n}$, where $t_{i+1} - t_i = \tau, i = 1, 2, \cdots, n-1$, where τ_i is able to make $\tau_1, \tau_2, \cdots, \tau_n$ and produces recursion. Replacing X_{t_i} with the observed value x_{t_i}, let

$$h_i(\mu) = \frac{x_{t_{i+1}} - x_{t_i} - f(t_i, x_t, x_{t-\tau_1}, x_{t-\tau_2}, \cdots, x_t; \mu)(t_{i+1} - t_i)}{g(t, x_t, x_{t-\tau_1}, x_{t-\tau_2}, \cdots, x_t; \mu)(t_{i+1} - t_i)}, \tag{2.3}$$

which can be seen as a function of the argument μ, $i = 1, 2, \cdots, n-1$. From equation (2.2), we can think of $h_i(\mu)$ as a sample from the standard normal uncertainty distribution. More importantly, the sample moment can be seen as the overall moment, and we obtain the k-order moment.

$$\frac{1}{n-2} \sum_{i=2}^{n-1} (h_i(\mu))^k, k = 1, 2, \cdots \tag{2.4}$$

And k moments

$$3^{\frac{k}{2}} (2^k - 2) |B_k|, k = 1, 2, \cdots \tag{2.5}$$

where B_k is the number of Bernoulli. So, we have

$$\frac{1}{n-2}\sum_{i=2}^{n-1}(h_i(\mu))^k = 3^{\frac{k}{2}}(2^k-2)\,|B_k|\,, k=1,2,\cdots,K, \tag{2.6}$$

where K is the dimension of the parameter μ.

Example 2.1. *Assume the uncertain multiple delay differential equation*

$$dX_t = (\mu_1 X_{t-0.1} + \mu_2 X_{t-0.2})dt + \mu_3 X_{t-0.3} X_t dC_t, \tag{2.7}$$

According to equation (2.3), we have

$$h_i(\mu) = h_i(\mu) = \frac{x_{t_{i+1}} - x_{t_i} - (\mu_1 x_{t_i-0.1} + \mu_2 x_{t_i-0.2}) * (t_{i+1}-t_i)}{\mu_3 x_{t_i} x_{t_i-0.3}(t_{i+1}-t_i)} \tag{2.8}$$

The observation data for Example 2.1 are shown in Table 1.

Table 1. Observed data of Example 2.1.

i	1	2	3	4	5	6	7	8
t_i	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
x_{t_i}	0.6	0.7	0.8	1.3	1.5	1.4	1.9	2.1
i	9	10	11	12	13	14	15	16
t_i	0.9	1.0	1.1	1.2	1.3	1.4	1.5	1.6
x_{t_i}	2.2	1.7	2.1	2.4	2.6	2.5	2.8	2.7

The values μ_1, μ_2, and τ are estimated from equation (2.4) by using the following equations.

$$\begin{cases} \frac{1}{12}\sum_{i=4}^{15}\left(\frac{x_{t_{i+1}} - x_{t_i} - (\mu_1 x_{t_i-0.1} + \mu_2 x_{t_i-0.2}) * (t_{i+1}-t_i)}{\mu_3 x_{t_i} x_{t_i-0.3}(t_{i+1}-t_i)}\right) = 0 \\ \frac{1}{12}\sum_{i=4}^{15}\left(\frac{x_{t_{i+1}} - x_{t_i} - (\mu_1 x_{t_i-0.1} + \mu_2 x_{t_i-0.2}) * (t_{i+1}-t_i)}{\mu_3 x_{t_i} x_{t_i-0.3}(t_{i+1}-t_i)}\right)^2 = 1 \\ \frac{1}{12}\sum_{i=4}^{15}\left(\frac{x_{t_{i+1}} - x_{t_i} - (\mu_1 x_{t_i-0.1} + \mu_2 x_{t_i-0.2}) * (t_{i+1}-t_i)}{\mu_3 x_{t_i} x_{t_i-0.3}(t_{i+1}-t_i)}\right)^3 = 0. \end{cases} \tag{2.9}$$

Using MATLAB to calculate the equations (2.9), we have

$$\hat{\mu}_1 = 1.8166, \hat{\mu}_2 = -1.0041, \hat{\mu}_3 = 0.3798.$$

Therefore, the uncertain multiple delay differential equation (2.7) is

$$dX_t = (1.8166X_{t-0.1} - 1.0041X_{t-0.2})\,dt + 0.3798X_{t-0.3} * X_t dC_t. \tag{2.10}$$

As shown in Figure 1, we find that the data in Table 1 lies between 0.999-path and 0.001-path, so $\hat{\mu}_1$, $\hat{\mu}_2$, and $\hat{\mu}_3$ are acceptable. In addition, Figure 2 shows X_t^α at different times t and α.

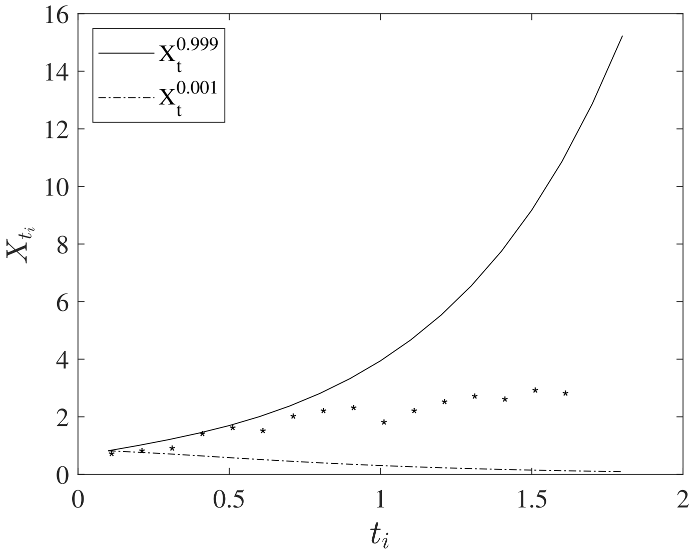

Figure 1. $X_t^{0.001}, X_t^{0.999}$ and observed data.

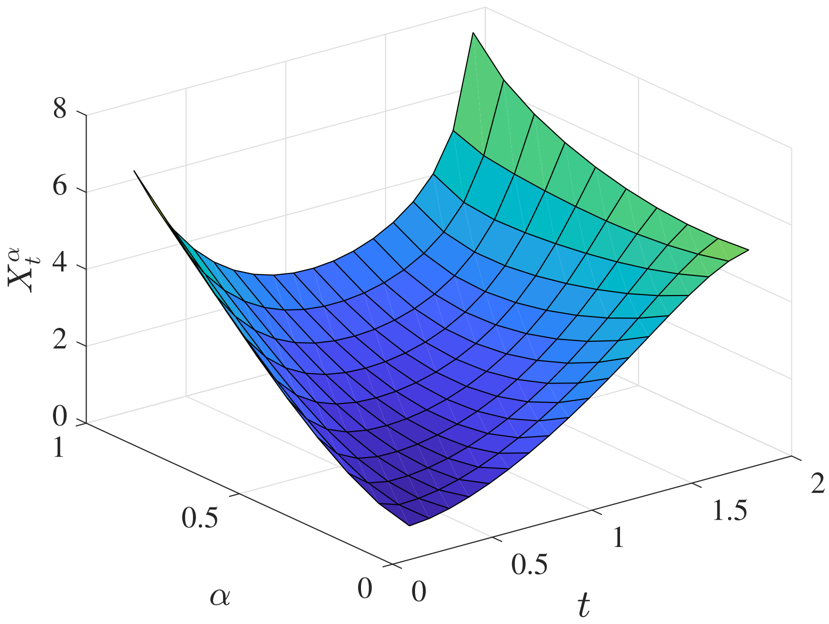

Figure 2. X_t^α of uncertain multiple-delay differential equation (2.10).

3. Moment estimation method for uncertain multiple delay differential equations with unknown delays

Uncertain multiple delay differential equation

$$dX_t = f(t, X_t, X_{t-\tau_1}, X_{t-\tau_2}, \cdots, X_{t-\tau_n}; \mu)dt + g(t, X_t, X_{t-\tau_1}, X_{t-\tau_2}, \cdots, X_{t-\tau_n}; \mu)dC_t, \quad (3.1)$$

where μ is the parameter vector of a position, C_t is the Liu process, τ_1, τ_2, $\cdots\tau_n$ stand for a series of known delays. The uncertain multiple-delay differential equation (3.1) has a forward Euler difference scheme.

$$X_{t_{i+1}} = X_{t_i} + f(t_i, X_{t_i}, X_{t_i-\tau_1}, X_{t_i-\tau_2}, \cdots, X_{t_i-\tau_n}; \mu)(t_{i+1} - t_i) + f(t_i, X_{t_i}, X_{t_i-\tau_1}, X_{t_i-\tau_2}, \cdots, X_{t_i-\tau_n}; \mu)(C_{i+1} - C_i).$$

Expand $X_{t_i-\tau_i}$ at the event point t,

$$X_{t-\tau_i} = X_t - \tau_i \frac{\mathrm{d}X_t}{\mathrm{d}t} + o(\tau_i), \tag{3.2}$$

we get an approximation

$$X_{t-\tau_i} \approx X_t - \tau_i \frac{\mathrm{d}X_t}{\mathrm{d}t}. \tag{3.3}$$

Therefore

$$\begin{aligned}\frac{X_{t_i+1} - X_{t_i}}{t_{i+1} - t_i} \approx{}& f\left(t_i, X_{t_i}, X_{t_i} - \tau_1 \frac{X_{t_i+1} - X_{t_i}}{t_{i+1} - t_i}, X_{t_i} - \tau_2 \frac{X_{t_i+1} - X_{t_i}}{t_{i+1} - t_i}, \cdots, X_{t_i} - \tau_m \frac{X_{t_i+1} - X_{t_i}}{t_{i+1} - t_i}; \mu\right) \\ &+ g\left(t_i, X_{t_i}, X_{t_i} - \tau_1 \frac{X_{t_i+1} - X_{t_i}}{t_{i+1} - t_i}, X_{t_i} - \tau_2 \frac{X_{t_i+1} - X_{t_i}}{t_{i+1} - t_i}, \cdots, X_{t_i} - \tau_m \frac{X_{t_i+1} - X_{t_i}}{t_{i+1} - t_i}; \mu\right) \frac{C_{t_{i+1}} - C_{t_i}}{t_{i+1} - t_i}.\end{aligned}$$

Based on the definition of the Liu process [27], we have

$$\frac{C_{i+1} - C_i}{t_{i+1} - t_i} \sim \mathcal{N}(0, 1).$$

Therefore

$$\frac{\frac{X_{t_{i+1}} - X_{t_i}}{t_{i+1} - t_i} - f\left(t_i, X_{t_i}, X_{t_i} - \tau_1 \frac{X_{t_{i+1}} - X_{t_i}}{t_{i+1} - t_i}, X_{t_i} - \tau_2 \frac{X_{t_{i+1}} - X_{t_i}}{t_{i+1} - t_i}, \ldots, X_{t_i} - \tau_m \frac{X_{t_{i+1}} - X_{t_i}}{t_{i+1} - t_i}; \boldsymbol{\mu}\right)}{g\left(t_i, X_{t_i}, X_{t_i} - \tau_1 \frac{X_{t_{i+1}} - X_{t_i}}{t_{i+1} - t_i}, X_{t_i} - \tau_2 \frac{X_{t_{i+1}} - X_{t_i}}{t_{i+1} - t_i}, \ldots, X_{t_i} - \tau_m \frac{X_{t_{i+1}} - X_{t_i}}{t_{i+1} - t_i}; \boldsymbol{\mu}\right)} \sim \mathcal{N}(0, 1). \tag{3.4}$$

At time points $t_i, t_2, \cdots, t_n$ with observed data $x_{t_1}, x_{t_2}, \cdots, x_{t_n}$, where $t_{i+1} - t_i = \tau, i = 1, 2, \cdots, n-1$, τ_i is able to make $\tau_1, \tau_2, \cdots, \tau_n$ produce recursion. Replace X_{t_i} with the observed value x_{t_i}, let

$$hp_i(\boldsymbol{\mu}) = \frac{\frac{x_{t_{i+1}} - x_{t_i}}{t_{i+1} - t_i} - f\left(t_i, x_{t_i}, x_{t_i} - \tau_1 \frac{x_{t_{i+1}} - x_{t_i}}{t_{i+1} - t_i}, x_{t_i} - \tau_2 \frac{x_{t_{i+1}} - x_{t_i}}{t_{i+1} - t_i}, \ldots, x_{t_i} - \tau_m \frac{x_{t_{i+1}} - x_{t_i}}{t_{i+1}}; \boldsymbol{\mu}\right)}{g\left(t_i, x_{t_i}, x_{t_i} - \tau_1 \frac{x_{t_{i+1}} - x_{t_i}}{t_{i+1} - t_i}, x_{t_i} - \tau_2 \frac{x_{t_{i+1}} - x_{t_i}}{t_{i+1} - t_i}, \ldots, x_{t_i} - \tau_m \frac{x_{t_{i+1}} - x_{t_i}}{t_{i+1} - t_i}; \boldsymbol{\mu}\right)}, \tag{3.5}$$

which can be seen as a function of the parameter μ, $i = 1, 2, \cdots, n-1$, we can think of hp_i as a sample from the standard normal inverse uncertainty distribution. More importantly, the sample moment can be seen as the overall moment, and we obtain the k-order moment.

$$\frac{1}{n-2} \sum_{i=2}^{n-1} (hp_i(\mu))^k, k = 1, 2, \cdots \tag{3.6}$$

And k moments

$$3^{\frac{k}{2}} \left(2^k - 2\right) |B_k|, k = 1, 2, \cdots, K. \tag{3.7}$$

So

$$\frac{1}{n-2}\sum_{i=2}^{n-1}(hp_i(\mu))^k = 3^{\frac{k}{2}}\left(2^k-2\right)|B_k|, \tag{3.8}$$

where k is the dimension of the parameter μ.

Example 3.1. *Assume an uncertain multiple delay differential equation*

$$\mathrm{d}X_t = (a_1X_{t-\tau_1} + a_2X_{t-\tau_2})\,\mathrm{d}t + (a_3X_t + a_4X_{t-\tau_3})\,\mathrm{d}C_t, \tag{3.9}$$

The observation data of Example 3.1 are shown in Table 2.

Table 2. Observed data of Example 3.1.

i	1	2	3	4	5	6	7	8
t_i	0.1	0.3	0.4	0.7	1.0	1.1	1.3	1.4
x_{t_i}	0.58	0.62	0.76	0.82	0.97	1.03	1.32	1.47
i	9	10	11	12	13	14	15	16
t_i	1.54	1.69	1.75	1.81	1.96	2.12	2.27	2.30
x_{t_i}	2.94	2.99	3.05	3.11	3.16	3.22	3.27	3.30

According to equation 2.6, the estimates $a_1, a_2, a_3, a_4, \tau_1, \tau_2$ and τ_3 are solved by the following system of equations.

$$\begin{cases} \frac{1}{15}\sum_{i=1}^{15}\left(\frac{x_{t_{i+1}} - x_{t_i} - \left(a_1\left(x_{t_i} - \tau_1\frac{x_{t_{i+1}}-x_{t_i}}{t_{i+1}-t_i}\right) + a_2\left(x_{t_i} - \tau_2\frac{x_{t_{i+1}}-x_{t_i}}{t_{i+1}-t_i}\right)\right) * (t_{i+1}-t_i)}{\left(a_3x_{t_i} + a_4\left(x_{t_i} - \tau_3\frac{x_{t_{i+1}}-x_{t_i}}{t_{i+1}-t_i}\right)\right)(t_{i+1}-t_i)}\right) = 0 \\ \frac{1}{15}\sum_{i=1}^{15}\left(\frac{x_{t_{i+1}} - x_{t_i} - \left(a_1\left(x_{t_i} - \tau_1\frac{x_{t_{i+1}}-x_{t_i}}{t_{i+1}-t_i}\right) + a_2\left(x_{t_i} - \tau_2\frac{x_{t_{i+1}}-x_{t_i}}{t_{i+1}-t_i}\right)\right) * (t_{i+1}-t_i)}{\left(a_3x_{t_i} + a_4\left(x_{t_i} - \tau_3\frac{x_{t_{i+1}}-x_{t_i}}{t_{i+1}-t_i}\right)\right)(t_{i+1}-t_i)}\right)^2 = 1 \\ \frac{1}{15}\sum_{i=1}^{15}\left(\frac{x_{t_{i+1}} - x_{t_i} - \left(a_1\left(x_{t_i} - \tau_1\frac{x_{t_{i+1}}-x_{t_i}}{t_{i+1}-t_i}\right) + a_2\left(x_{t_i} - \tau_2\frac{x_{t_{i+1}}-x_{t_i}}{t_{i+1}-t_i}\right)\right) * (t_{i+1}-t_i)}{\left(a_3x_{t_i} + a_4\left(x_{t_i} - \tau_3\frac{x_{t_{i+1}}-x_{t_i}}{t_{i+1}-t_i}\right)\right)(t_{i+1}-t_i)}\right)^3 = 0. \end{cases} \tag{3.10}$$

Using MATLAB to calculate the equations (3.10), we have

$$\hat{a}_1 = 0.2451, \hat{a}_2 = 0.2451, \hat{a}_3 = 0.9951, \hat{a}_4 = 0.9472, \hat{\tau}_1 = 1.6257, \hat{\tau}_2 = 1.6257, \hat{\tau}_3 = 0.6337.$$

Therefore, the uncertain multiple delay differential equation (3.1) is

$$\mathrm{d}X_t = (0.2451X_{t-1.6257} + 0.2451X_{t-1.6257})\,\mathrm{d}t + (0.9951X_t + 0.9472X_{t-0.6337})\,\mathrm{d}C_t. \tag{3.11}$$

As the Figure 3 of Example 3.1, we find that the data in Table 2 is located in 0.1-path and 0.9-path. So $(\hat{\mu}_1, \hat{\mu}_2)$ and $\hat{\tau}$ are acceptable, and in addition, X_t^α at different times t and α are given in Figure 4.

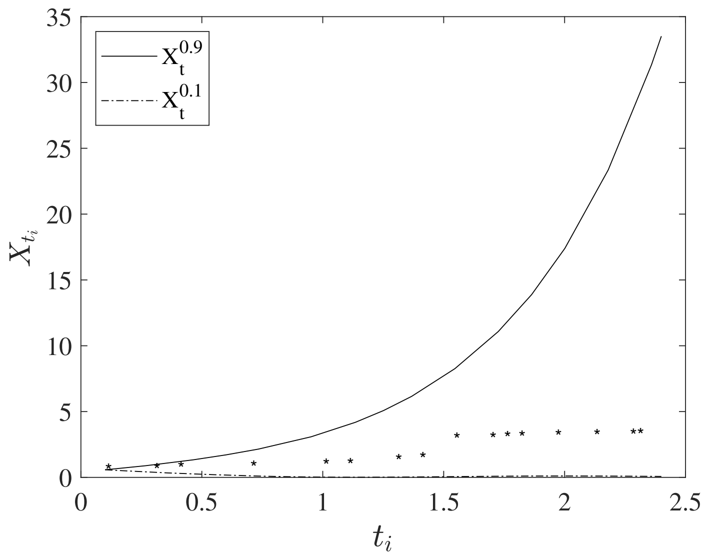

Figure 3. $X_t^{0.1}, X_t^{0.9}$and observed data.

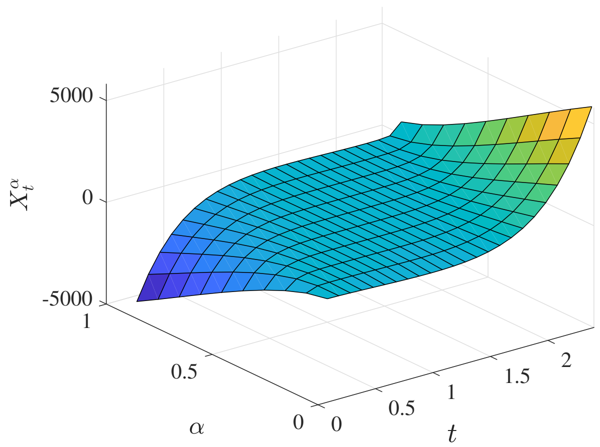

Figure 4. X_t^α of uncertain multiple delay differential equation (3.11).

4. Residual-based method of moment estimation

When the time interval of the observed data is particularly large, the parameters estimated by the above moment estimation method are not very good. In order to overcome this shortcoming, this section first introduces the concept of residual, and then uses residual estimation to generate parameters from uncertain multiple delay differential equations. Consider the uncertain multiple delay differential equation (2.1). Suppose that $x_{t_1}, x_{t_2}, \cdots, x_{t_n}$ is the observed data of the uncertain process X_t in the time interval $t_1, t_2, \cdots, t_n$. For any given index i $(2 \leq i \leq n)$, we solve the following updated uncertain multiple delay differential equations.

$$\begin{cases} \mathrm{d}X_t = f(t, X_t, X_{t-\tau_1}, \cdots, X_{t-\tau_n})\mathrm{d}t + g\left(t, X_t, X_{t-\tau_1}, \cdots, X_{t-\tau_n}\mathrm{d}t\right)\mathrm{d}C_t \\ X_{t_{i-1}} = x_{t_{i-1}} \\ X_{t_{i-1}-\tau_1} = x_{t_{i-1}-\tau_1} \\ \vdots \\ X_{t_{i-2}-\tau_1} = x_{t_{i-2}-\tau_1} \\ X_{t_{i-n}-\tau_1} = x_{t_{i-1}-\tau_n} \end{cases} \tag{4.1}$$

where $x_{t_{i-1}}$ is the new initial value at the time point t_{i-1}, the uncertainty distribution of the uncertain variable X_{t_i} is expressed as Φ_{t_i}, and for any x $(0 < x < 1)$, we have

$$\mathcal{M}\{\Phi_{t_i}(X_{t_i}) \le x\} = \mathcal{M}\left\{X_{t_i} \le \Phi_{t_i}^{-1}(x)\right\} = \Phi_{t_i}\left(\Phi_{t_i}^{-1}(x)\right) = x.$$

Therefore, Φ_{t_i} is a linear uncertain variable whose distribution function is

$$F(x) = \begin{cases} 0, x \le 0 \\ x, 0 < x < 1 \\ 1, x > 1 \end{cases} \tag{4.2}$$

Using the observation x_{t_i} instead of X_{t_i} , we have

$$\varepsilon_i = \Phi_{t_i}(x_{t_i}) \tag{4.3}$$

Therefore, ε_i is a sample of the linear uncertain variable $\Phi_{t_i}(X_{t_i})$.

Definition 4.1. *For any index $i(2 \le i \le n)$, ε_i in the formula (4.3) is called the i th residual of the uncertain multiple delay differential equation (2.1).*

The uncertain multiple delay differential equation (2.1) generally has no analytical solution, so we propose a numerical method to calculate the residual of the uncertain multiple delay differential equation (2.1). Firstly, the difference form of the uncertain multiple delay differential equation (2.1) is obtained by Euler.

$$X_{t_i} = X_{t_{i-1}} + f(t_{i-1}, X_{t_{i-1}}, X_{t_{i-1}-\tau_1}, \cdots, X_{t_i-\tau_m}; \mu)(t_i - t_{i-1}) + g(t_{i-1}, X_{t_{i-1}}, X_{t_{i-1}-\tau_1}, \cdots, X_{t_{i-1}-\tau_m}; \mu)(C_i - C_{i-1}) \tag{4.4}$$

approximate $X_{t_{i-1}-\tau_j}$ as

$$X_{t_{i-1}-\tau_j} = X_{t_{i-1}} - \frac{\tau_j}{t_{i-1} - t_{i-2}}(x_{t_{i-1}} - x_{t_{i-2}}),$$

For more convenience, let

$$U_j = X_{t_{i-1}} - \frac{\tau_j}{t_{i-1} - t_{i-2}}(x_{t_{i-1}} - x_{t_{i-2}}),$$

where $j = 1, 2, \cdots, m$. So the difference formula (4.4) becomes

$$X_{t_i} = X_{t_{i-1}} + f(t_{i-1}, X_{t_{i-1}}, U_1, U_2, \cdots, U_m; \mu)(t_i - t_{i-1}) + g(t_{i-1}, X_{t_{i-1}}, U_1, U_2, \cdots, U_m; \mu)(C_i - C_{i-1}),$$

replace $X_{t_{i-1}}, X_{t_{i-2}}$ with the observation data $x_{t_{i-1}}, x_{t_{i-2}}$, and make

$$V_j = x_{t_{i-1}} - \frac{\tau_j}{t_{i-1} - t_{i-2}} \left(x_{t_{i-1}} - x_{t_{i-2}} \right),$$

we have

$$X_{t_i} = X_{t_{i-1}} + f(t_{i-1}, X_{t_{i-1}}, V_1, V_2, \cdots, V_m; \mu)(t_i - t_{i-1}) + g(t_{i-1}, X_{t_{i-1}}, V_1, V_2, \cdots, V_m; \mu)(C_i - C_{i-1}).$$

Define

$$\begin{aligned} e_i =& X_{t_{i-1}} + f(t_{i-1}, X_{t_{i-1}}, V_1, V_2, \cdots, V_m; \mu)(t_i - t_{i-1}), \\ \sigma_i =& g(t_{i-1}, X_{t_{i-1}}, V_1, V_2, \cdots, V_m; \mu)(t_i - t_{i-1}). \end{aligned}$$

It can be seen that X_{t_i} follows a normal distribution with mean e_i and standard deviation σ_i, then

$$\varepsilon_i \approx \left(1 + \exp\left(\frac{\pi\,(e_i - x_i)}{\sqrt{3}\sigma_i} \right) \right)^{-1}. \tag{4.5}$$

For any given parameter μ and n observation data $x_{t_1}, x_{t_2}, \cdots, x_{t_n}$, we can get the following residual of $n-2$ uncertain multiple delay differential equation 2.1,

$$\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{n-2}.$$

For any given positive integer k, the k-order sample moment of the linear uncertainty distribution is

$$\frac{1}{n-1} \sum_{i=1}^{n-2} \varepsilon_i^k(\mu).$$

And the k-order global moment of the linear uncertainty distribution is

$$\frac{1}{k+1}.$$

Assuming that the sample moments of order k and the population moments of order k are equal, we have

$$\frac{1}{n-1} \sum_{i=1}^{n-2} \varepsilon_i^k(\mu) = \frac{1}{k+1}, k = 1, 2, \cdots, p. \tag{4.6}$$

Solve the equation (4.6) to obtain the positional parameter μ. Here, p means that p parameters need to be estimated.

Example 4.1. *Consider the uncertain multiple delay differential equation*

$$\mathrm{d}X_t = (a_1 X_{t-\tau_1} + a_2 X_{t-\tau_2})\,\mathrm{d}t + (a_3 X_t t - \tau_3)\,\mathrm{d}C_t, \tag{4.7}$$

The observation data of Example 4.1 are shown in Table 3.

Table 3. Observed data of Example 4.1.

i	1	2	3	4	5	6	7	8
t_i	0.2	0.5	0.8	1.1	1.4	1.7	2.0	2.3
x_{t_i}	0.37	0.53	0.65	0.71	0.87	1.13	1.23	1.34
i	9	10	11	12	13	14	15	16
t_i	2.6	2.9	3.2	3.5	3.8	4.1	4.4	4.7
x_{t_i}	1.47	1.55	1.67	1.76	1.87	1.92	2.12	2.30

The estimates of $a_1, a_2, a_3, \tau_1, \tau_2$ and τ_3 from equation (4.6) are obtained from the following system of equations,

$$\begin{cases} \frac{1}{14}\sum_{i=1}^{14} \varepsilon_i^1(\mu) = \frac{1}{2} \\ \frac{1}{14}\sum_{i=1}^{14} \varepsilon_i^2(\mu) = \frac{1}{3} \\ \frac{1}{14}\sum_{i=1}^{14} \varepsilon_i^3(\mu) = \frac{1}{4} \\ \frac{1}{14}\sum_{i=1}^{14} \varepsilon_i^4(\mu) = \frac{1}{5} \\ \frac{1}{14}\sum_{i=1}^{14} \varepsilon_i^5(\mu) = \frac{1}{6} \\ \frac{1}{14}\sum_{i=1}^{14} \varepsilon_i^6(\mu) = \frac{1}{7} \end{cases} \tag{4.8}$$

Using MATLAB to calculate the equations (4.8) we have

$$\hat{\tau}_1 = 0.0171, \hat{\tau}_2 = 0.0038, \hat{\tau}_3 = 0.0001, \hat{a}_1 = 0.5785, \hat{a}_2 = -0.1812, \hat{a}_3 = 0.3769.$$

Therefore, the uncertain multiple delay differential equation (4.7) is

$$\mathrm{d}X_t = (0.5785 X_{t-0.0171} + 0.1812 X_{t-0.0038})\,\mathrm{d}t + 0.3769 X_{t-0.0001}\mathrm{d}C_t. \tag{4.9}$$

According to the formula (4.5), we can obtain the residual plot of the uncertain multiple delay differential equation (4.9) as shown in Figure 5.

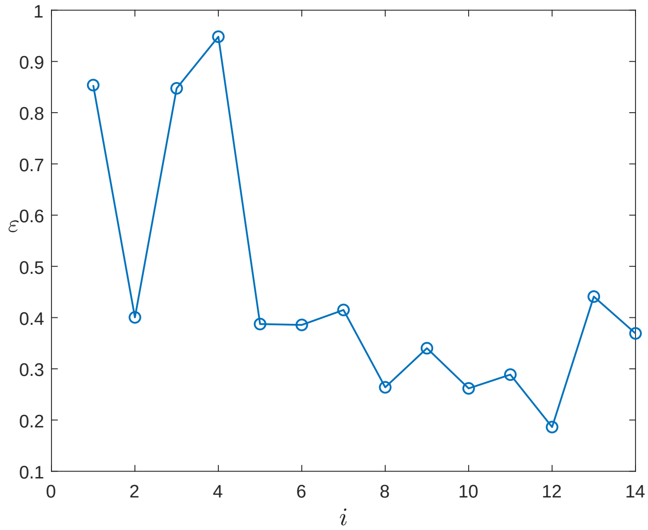

Figure 5. Picture of residuals for uncertain multiple delay differential equation (4.9).

According to the hypothesis testing method proposed by Ye and Liu [28] , in order to determine whether the uncertain multiple delay differential equation 4.9 is suitable for the observed data in Table 3, we should test whether the residual data in Table 4 is from the linear uncertainty distribution.

Table 4. Table of residuals of uncertain multiple delay differential equations (4.9).

i	1	2	3	4	5	6	7
ε_i	0.8538	0.4004	0.8475	0.9483	0.3874	0.3857	0.4149
i	8	9	10	11	12	13	14
ε_i	0.2639	0.3402	0.2618	0.2887	0.1863	0.4410	0.3691

Test whether the residual in Table 4 satisfies

$$W = \{(z_1, z_2, \cdots, z_{10}) : \textit{At least one indicator }_i (1 \le i \le 14) \textit{ must be } z_i < 0.0251 \textit{ or } z_i > 0.975\}$$

by setting the signal level $\alpha = 0.05, \alpha \times 14 = 0.7$. According to the residual data in Table 4, we have $(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{14}) \notin W$. Therefore, the uncertain multiple-delay Logistic model 4.9 is consistent with the observed data.

5. Uncertain multiple delay Logistic model

In characterizing the Australian fly population, Litching [4] pointed out in 1977 that the life cycle of Australian flies is characterized by multiple delays. Based on this observation, Braddock and Driessche [5] proposed two delayed Logistic models in 1983.

$$\frac{\mathrm{d}p_t}{\mathrm{d}t} = rp_t \left(1 - a_1 p_{t-\tau_1} - a_2 p_{t-\tau_2}\right),$$

where r, a_1, a_2, τ_1 and τ_2 are normal numbers, considering the effect of environmental fluctuations on the system, if we characterize with the Wiener process under probability theory, $\mathrm{d}p_t/\mathrm{d}t = W_t$, therefore, replacing r with $r + \sigma W_t$, we have

$$\mathrm{d}p_t/\mathrm{d}t = \left(r + \sigma \dot{W}_t\right) p_t \left(1 - a_1 p_{t-\tau_1} - a_2 p_{t-\tau_2}\right). \tag{5.1}$$

The unreasonableness of the uncertain multiple delay Logistis model (5.1) will be proved below. First of all, from the definition of the Wiener process, we know the expected value and variance of W_t are 0 and $\frac{1}{\mathrm{d}t}$, respectively, which is

$$W_t \sim N\left(0, \frac{1}{\mathrm{d}t}\right).$$

By means of the stochastic multiple delayed Logistic model (5.1), we have

$$\frac{\mathrm{d}p_t}{\left(p_t\left(1 - a_1 p_{t-\tau_1} - a_2 p_{t-\tau_2}\right)\right)\mathrm{d}t} = r + \sigma \dot{W}_t \sim N\left(r, \sigma^2 \frac{1}{\mathrm{d}t}\right).$$

For any given normal number M as $\mathrm{d}t$ goes to 0, we have

$$\begin{aligned}
&\Pr\left\{\left|\frac{\mathrm{d}p_t}{\left(p_t\left(1 - a_1 p_{t-\tau_1} - a_2 p_{t-\tau_2}\right)\right)\mathrm{d}t}\right| \geq M\right\} \\
&= \Pr\left\{\left(\frac{\mathrm{d}p_t}{\left(p_t\left(1 - a_1 p_{t-\tau_1} - a_2 p_{t-\tau_2}\right)\right)\mathrm{d}t} \geq M\right) \bigcup \left(\frac{\mathrm{d}p_t}{\left(p_t\left(1 - a_1 p_{t-\tau_1} - a_2 p_{t-\tau_2}\right)\right)\mathrm{d}t} \leq -M\right)\right\} \\
&= \Pr\left\{\frac{\mathrm{d}p_t}{\left(p_t\left(1 - a_1 p_{t-\tau_1} - a_2 p_{t-\tau_2}\right)\right)\mathrm{d}t} \geq M\right\} + \Pr\left\{\frac{\mathrm{d}p_t}{\left(p_t\left(1 - a_1 p_{t-\tau_1} - a_2 p_{t-\tau_2}\right)\right)\mathrm{d}t} \leq -M\right\} \\
&= \Pr\left\{\frac{\frac{\mathrm{d}p_t}{\left(p_t\left(1 - a_1 p_{t-\tau_1} - a_2 p_{t-\tau_2}\right)\right)\mathrm{d}t} - r}{\sigma / \sqrt{\mathrm{d}t}} \geq \frac{M - r}{\sigma}\sqrt{\mathrm{d}t}\right\} + \Pr\left\{\frac{\frac{\mathrm{d}p_t}{\left(p_t\left(1 - a_1 p_{t-\tau_1} - a_2 p_{t-\tau_2}\right)\mathrm{d}t\right.} - r}{\sigma / \sqrt{\mathrm{d}t}} \leq -\frac{M + r}{\sigma}\sqrt{\mathrm{d}t}\right\} \\
&= 1 - \Phi\left(\frac{M - r}{\sigma}\sqrt{\mathrm{d}t}\right) + \Phi\left(-\frac{M + r}{\sigma}\sqrt{\mathrm{d}t}\right) \to 1,
\end{aligned}$$

where $\Phi(\cdot)$ represents a random normal distribution, it also means

$$\Pr\left\{\left|\frac{\mathrm{d}p_t}{\left(p_t\left(1 - a_1 p_{t-\tau_1} - a_2 p_{t-\tau_2}\right)\right)\mathrm{d}t}\right| \geq M\right\} = 1,$$

That is, at least one of the terms in $\mathrm{d}p_t/\mathrm{d}t$ and $1/(p_t(1 - a_1 p_{t-\tau_1} - a_2 p_{t-\tau_2}))$ is positive infinity. However, K is a normal number that does not exceed the limit of the resource, that is, at any time t, $p_t < K$, so $1/(p_t(1 - a_1 p_{t-\tau_1} - a_2 p_{t-\tau_2}))$ is bounded. So $\mathrm{d}p_t/\mathrm{d}t$ is $+\infty$. In other words, the instantaneous increase rate of population $\mathrm{d}p_t/\mathrm{d}t$ is positive infinity, and it is impossible to reach positive infinity according to a finite population. Therefore, the stochastic multiple delayed Logistic model (5.1) is not reasonable.

We can take $\mathrm{d}C_t/\mathrm{d}t$ to represent the noise term, and the uncertain multiple-delay logistic model is

$$\frac{\mathrm{d}p_t}{\mathrm{d}t} = rp_t(1 - a_1 p_{t-\tau_1} - a_2 p_{t-\tau_2}) + \sigma p_t(1 - a_1 p_{t-\tau_1} - a_2 p_{t-\tau_2})\frac{\mathrm{d}C_t}{\mathrm{d}t} \tag{5.2}$$

Where C_t represents the Liu process, according to the definition of the Liu process, the expected value and variance of $\mathrm{d}C_t/\mathrm{d}t$ are 0 and 1, respectively. Therefore, there is a limit to the instantaneous growth rate of population size that we can obtain. In other words, the theory satisfies the practical situation. The data in Table 5 comes from the data in Table I in [29], which shows the continuous development of American society and economy and the characteristics of multiple delays in female childbearing age. Fan [29] use differential equation with two delays to describe the population dynamics of the United States from 1900 to 2010. When we consider the effect of "noise" on population dynamics in the United States, it can be characterized by a Logistic model 5.2 with uncertain multiple delays.

Table 5. The number of American population (Unit:Billion).

i	1	2	3	4	5	6	7	8	9	10	11	12
t_i	1900	1910	1920	1930	1940	1950	1960	1970	1980	1990	2000	2010
x_{t_i}	0.76	0.92	1.06	1.23	1.32	1.51	1.79	2.03	2.27	2.49	2.81	3.09

For a Logistic model (5.2) with uncertain multiple delays, we have parameters r, ϵ, a_1 and a_2 to estimate.

$$\begin{cases} \frac{1}{10}\sum_{i=1}^{10}\varepsilon_i^1(\mu) = \frac{1}{2} \\ \frac{1}{10}\sum_{i=1}^{10}\varepsilon_i^2(\mu) = \frac{1}{3} \\ \frac{1}{10}\sum_{i=1}^{10}\varepsilon_i^3(\mu) = \frac{1}{4} \\ \frac{1}{10}\sum_{i=1}^{10}\varepsilon_i^4(\mu) = \frac{1}{5} \end{cases} \tag{5.3}$$

Using MATLAB to calculate the equations (5.3), we have

$$\hat{a}_1 = 0.0129, \hat{a}_2 = 0.0668, \hat{\tau}_1 = 0.22, \hat{\tau}_2 = 0.127. \hat{r} = 0.1516, \hat{\sigma}_2 = 0.0029.$$

Therefore, based on US population data, the uncertain multiple delayed Logistic model (5.2) is

$$\mathrm{d}p_t = 0.1516p_t(1 - 0.0129p_{t-0.22} - 0.0668p_{t-0.127})\,\mathrm{d}t + 0.0029p_t(1 - 0.0129p_{t-0.22} - 0.0668p_{t-0.127})\,\mathrm{d}C_t. \tag{5.4}$$

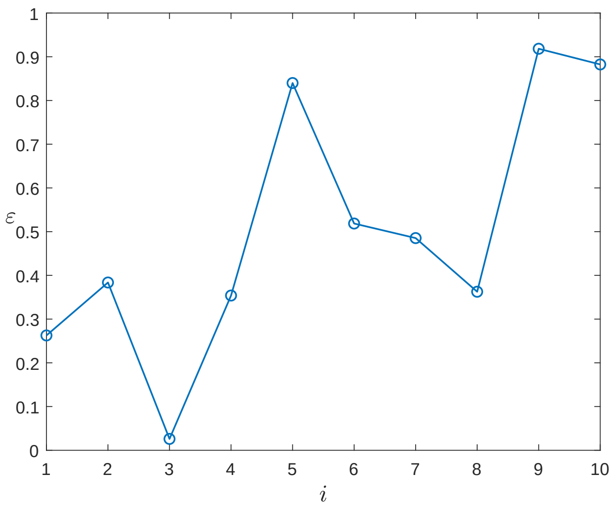

Figure 6. The residual plot for uncertain multiple delay Logistic model (5.4).

According to formula (4.5), we can obtain the residual graph of the uncertain multiple delay Logistic model (5.4) as shown in Figure 6.

Based on the hypothesis testing method proposed by Ye and Liu [28], in order to determine whether the uncertain multiple delay differential equation (5.4) is suitable for the observed data, we should test whether the residual in Table 6 is from the linear uncertainty distribution.

Table 6. The residual table for uncertain multiple delay Logistic model (5.4).

i	1	2	3	4	5
ε_i	0.2626	0.3837	0.0260	0.3539	0.8399
i	6	7	8	9	10
ε_i	0.5186	0.4853	0.3626	0.9183	0.8824

Set the signal level $\alpha = 0.05, 0.05 \times 10 = 0.5$, that is, test whether the residual data is satisfied.

$$W = \{(z_1, z_2, \cdots, z_{10}) : \textit{At least one indicator } z_i (1 \leq i \leq 10) \textit{ must be } z_i < 0.0251 \textit{ or } z_i > 0.975\}$$

According to the residual data in Table 6, we have $(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{10}) \notin W$. Therefore, the uncertain multiple-delay Logistic model (5.4) is consistent with the observed data. On the other hand, if it is uncertain that the Liu process C_t in the multiple-delay Logistic model (5.4) is replaced by the Wiener process, we can obtain a stochastic multiple-delay Logistic model.

$$\mathrm{d}p_t = 0.1516 p_t \left(1 - 0.0129 p_{t-0.22} - 0.0668 p_{t-0.127}\right) \mathrm{d}t + 0.0029 p_t \left(1 - 0.0129 p_{t-0.22} - 0.0668 p_{t-0.127}\right) \mathrm{d}W_t. \tag{5.5}$$

Select $t = 1910$, $\triangle t = 10^{-4}$, we have

$$\Pr\{X_{1910+\Delta t} < X_{1910}\} = 49.93\%.$$

This means that the population of the United States will decline with a probability of 49.93%. In real terms, however, the population of the United States is increasing. Therefore, the random multiple delayed Logistic model (5.5) is not reasonable.

6. Conclusion

In order to apply the uncertain multiple delay differential equation to describe the dynamic systems in our world, this paper presented the method of moment estimation to estimate the parameters of uncertain multiple delay differential equations with known and unknown delays. Some numerical examples were discussed to confirm the effectiveness of the above method. When the time intervals of data are relatively large, the above method was invalid, the method of moment estimation based on residuals was provided to estimate the parameters of uncertain multiple delay differential equations. The numerical example was given to confirm the effectiveness of the above method by hypothesis testing. Meanwhile, the paradox of stochastic multiple delay Logistic model was proved, an uncertain multiple delay Logistic model was defined and applied to describe the population dynamics of the United States.

Author contributions

Yilin Yang: Writing–original draft; Yin Gao: Conceptualization, Methodology, Software, Supervision, Validation, Writing–review and editing.

Use of Generative-AI tools declaration

The authors declares they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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