Journal of Industrial and Management Optimization

https://www.aimspress.com/journal/jimo

JIMO, 22 (1): 214–237.
DOI: 10.3934/jimo.2026008
Received: 29 April 2025
Revised: 28 September 2025
Accepted: 27 October 2025
Published: 19 November 2025

Research article

Optimization of differential inclusions with parameter

Elimhan N. Mahmudov[1,2,*], Shakir Sh. Yusubov[3] and Dilara I. Mastaliyeva[4]

[1] Research Center, Azerbaijan National Aviation Academy, Baku, Azerbaijan
[2] Azerbaijan Functional analysis and its applications, Azerbaijan University of Architecture and Construction, Baku, Azerbaijan
[3] Department of Mechanics and Mathematics, Baku State University, Baku, Azerbaijan
[4] Department of Mathematics, Khazar University, Baku, Azerbaijan

* **Correspondence:** Email: elimhan22@yahoo.com; elmkhan.mahmudov@naa.edu.az.

Abstract: The article considered the Bolza optimization problem for differential inclusions (DFIs) with a parameter. To achieve the final goal, the corresponding discrete, discrete-approximate, and continuous problems were studied in parallel. Under a standard interior point condition from convex analysis, optimality conditions for the given parameter-dependent problem were obtained in terms of the Euler–Lagrange equations and the Hamiltonian. The central approach relied on the equivalence results of locally adjoint mappings (LAMs). Specifically, a classical optimal control problem with a parameter was investigated.

Keywords: Euler-Lagrange; parameter; differential inclusion; Hamiltonian; equivalence; necessary and sufficient
Mathematics Subject Classification: 34A60, 34B08, 34A45, 49K15, 90C90

1. Introduction

Optimal control of ordinary [1–14] and partial differential inclusions [15–20] has been expanding during the last few decades. Among them, the qualitative properties of first-order differential inclusions (DFIs) also play an important role in the mathematical theory of optimal processes (see [11], [19], [21–29] and references therein). In [10], the existence of Lyapunov functions for the DFI was analyzed, and the necessary developments on the existence of local and global Lyapunov functions were obtained. The work [23] considered a class of control systems governed by DFIs. For any fixed control policy, the existence of a non-empty solution set for the system was proved. Using this, the existence of optimal controls for certain control problems was then proved, and the paper concluded with some results on relaxed controls and some comments on open related problems. The paper [24] considered a class of control systems governed by differential inclusions in Banach spaces. For any fixed control policy, the existence of a non-empty solution set for the system was proved, followed by proving the existence of optimal controls for some control problems, and showing some results on weakened controls and comments on open problems in the area. In [14], the problems of differential inclusions for fuzzy mappings were introduced, and the existence of solutions to these problems was studied using the continuous selection theorem and the fixed-point theorem.

The main goal of [21] was to provide new explicit criteria for characterizing weak lower semi-continuous Lyapunov pairs or functions associated with first-order DFIs in Hilbert spaces. These inclusions were described by a Lipschitzian perturbation of a maximally monotone operator. The proposed approach was based on advanced tools of variational analysis and generalized differentiation. The work [22] focused on the natural case where the maximal monotone operator governing the inclusion has a domain with non-empty interior, admitting some more explicit criteria for Lyapunov pairs. The book [30] describes several broad topics: duality and optimality conditions, optimization algorithms, optimal control, variational inequalities, and equilibrium problems. Specific topics, covered in individual chapters, include optimal control, unconstrained and constrained optimization, complementarity and variational inequalities, equilibrium problems, and nonsmooth optimization. In [26], in a separable Banach space for a second-order DFI, under certain conditions, the question of the existence of a solution was considered.

In addition to the existing qualitative studies mentioned above, optimal control problems using DFI have also been intensively studied in the literature. Closely related optimality problems for DFIs were considered in [4–6,8,11,13]. In [4], necessary conditions were derived in the form of the Euler–Lagrange and the Pontryagin maximum condition. In [11], the Mayer problem with constraints of a finite number of equalities and inequalities was considered.

In [13], the Lagrange problem of optimal control with boundary constraints defined by polyhedral DFIs was investigated. It was shown that the adjoint Euler–Lagrange inclusion serves simultaneously as a dual relation satisfied by a pair of solutions to the primal and dual problems. In [5], authors addressed optimal control problems governed by second-order bounded hereditary differential inclusions. Particular attention was given to delay-neutral and Hale-type DFIs, which frequently occur in applied sciences, and sufficient optimality conditions together with transversality conditions were established. In [6], a new class of optimal control problems was considered, involving linear second-order self-adjoint Sturm–Liouville-type differential operators with both functional and non-functional endpoint constraints. Sufficient conditions for optimality were obtained, incorporating second-order Euler–Lagrange inclusions as well as Hamiltonian inclusions. Moreover, the presence of functional constraints leads to specific second-order transversality inclusions and complementary slackness conditions characteristic of inequality-type constraints.

In fact, the complexity of problems with both parameters and DFIs lies in the construction of adjoint inclusions of the Euler–Lagrange type and suitable transversality conditions. The paper [31] proposed a new parameter condition for the primal-dual hybrid gradient method. This improvement only requires that either the primal or dual objective function be strongly convex. Preliminary experimental results show that this method with a relaxed parameter condition is more efficient than several state-of-the-art methods. The paper [16] considered the problem of determining the parameters of a system consisting of a large number of dynamic objects connected in an arbitrary order with a single boundary condition. In [32], a new filled function with one parameter was proposed, which is continuously differentiable and always contains local information about the objective function. Then, a new filled function method was developed, incorporating the proposed filled function for unconstrained global optimization problems.

This paper is devoted to the optimization of first-order ordinary DFIs with parameters, aiming to fill the gap in the exploration of the optimization of ordinary first-order DFIs with parameters. Therefore, the present study can be considered as a generalization of first-order DFI optimization to a problem with parameters. The stated problem and the corresponding optimality conditions are new. The paper is organized in the following order:

In Section 2, for the convenience of the reader, the necessary concepts and results, such as set-valued mappings, the properties of LAM in finite-dimensional Euclidean spaces, Hamiltonian functions, argmaximum sets, locally tents, etc., are given, based on Mahmudov's monograph [18] and previous works [6–10,18,19,33]. Then, the main Bolza-type problems with parameters for discrete and DFIs are formulated.

In Section 3, the optimization problem involving discrete inclusions is transformed into a mathematical programming problem, and by employing tools from convex analysis, optimality conditions for these problems are established. To this end, certain properties of set-valued mappings, such as closedness and pointwise closedness, are examined.

In Section 4, the equivalence of LAM for discrete (PD) and discrete-approximate (DAP) problems is demonstrated. Within the framework of classical subdifferential calculus for compositions of functions and local tents, a link between discrete and discrete-approximate problems is derived.

In Section 5, by using first-order difference operators, the problem of first-order discrete inclusions with a parameter is connected to the first-order discrete-approximate problem. Applying the LAM equivalence results obtained in Section 4, the optimality conditions for the PDA problem are formulated.

In Section 6, sufficient optimality conditions for DFI with a parameter are proved.

2. Statement of the problem with parameters and preliminary concepts

In this paragraph, to help readers quickly understand the structure and logical sequence of the article, the main definitions and concepts from the author's monograph [18] are given. We first recall the basic concepts of set-valued mappings: let $\mathbb{R}^n$ be, as usual, an n -dimensional Euclidean space, $\langle x, v\rangle$ be the scalar product of elements x, v, and (x, v) be a pair of vectors x, v. Let us suppose that $F: \mathbb{R}^{n+m} \rightrightarrows \mathbb{R}^n$ is a set-valued mapping. A set-valued mapping F is convex-valued if $F(x, w)$ is a convex set for each $(x, w) \in domF$. A set-valued mapping F is closed if its $gph\, F = \{(x, w, v): v \in F(x, w)\}$ is a closed subset in $\mathbb{R}^{2n+m}$. In contrast to this definition, a set-valued mapping F is pointwise closed if $F(x, w)$ is closed at every fixed point of $domF = \{(x, w): F(x, w) \neq \emptyset\}$. The set-valued mapping F is bounded, if there exists a constant C such that $\|F(x)\| \leq C(1 + \|x\|)$.

A set-valued mapping $F: \mathbb{R}^{n+m} \rightrightarrows \mathbb{R}^n$ is said to be upper semicontinuous at $(\tilde{x}, \widetilde{w})$ if for any neighborhood U of zero in $\mathbb{R}^n$, there exists a neighborhood V of zero in $\mathbb{R}^{n+m}$ such that

$$F(x, w) \subseteq F(\tilde{x}, \widetilde{w}) + U, \forall (x, w) \in (\tilde{x}, \widetilde{w}) + V.$$

The definitions of the Hamiltonian function and the argmaximum set are given as follows:

$$H_F(x,w,v^*)=\sup_v\{\langle v,v^*\rangle : v\in F(x,w)\},\ v^*\in\mathbb{R}^n,$$

$$F_{Arg}(x,w\,;v^*)\equiv F_A(x,w\,;v^*)=\{v\in F(x,w):\langle v,v^*\rangle=H_F(x,w,v^*)\},$$

respectively. We denote by int$M$ and ri$M$ the interior and relative interior of the set M, and the closure of M is denoted by $\overline{M}$.

For a convex mapping $F:\mathbb{R}^{n+m}\rightrightarrows\mathbb{R}^n$, a mapping $F*\mathbb{R}^n\rightrightarrows\mathbb{R}^{n+m}$ defined by

$$F*(v^*;(x,w,v)):=\{(x^*,w^*):(x^*,w^*,-v^*)\in K_F*(x,w,v)\},$$
$$K_F(x,w,v)\equiv K_{gphF}(x,w,v)$$

is called the LAM to F at a point $(x,w,v)\in gphF$, where $K*=\{z^*:\langle\bar{z},z^*\rangle\geq 0,\forall\bar{z}\in K\}$ denotes the dual cone to the cone K, as usual.

The related definition of LAM for a set-valued mapping can be given by the Hamiltonian function, where

$$F*(v^*;(x,w,v)):=\{(x^*,w^*):H_F(x^1,w^1,v^*)-H_F(x,w,v^*)\leq\langle x^*,x^1-x\rangle$$
$$+\langle w^*,w^1-w\rangle,\forall(x^1,w^1)\in\mathbb{R}^{n+m}\},(x,w,v)\in gphF,v\in F_A(x,w\,;v^*).$$

Clearly, for the convex mapping F, the Hamiltonian function $H_F(\cdot,\cdot,v^*)$ is concave, and the latter definition of LAM coincides with the previous definition of LAM [18, Thm. 2.1].

It is appropriate to note here that by rewriting the inequalities associated with the Hamiltonian function, it is easy to verify that LAM is nothing more than a subdifferential of the Hamiltonian function.

Alongside LAM, the concept of the coderivative was introduced for set-valued mappings by Mordukhovich [11], defined through the normal cone to their graphs; for smooth and convex mappings, these two notions coincide. As we have observed, in the non-convex case, the tangent cone contains directions corresponding to functions $q(\lambda)$, though this alone may not be sufficient to characterize the properties of the set. Nevertheless, the notion of a local tent provides a way to determine the mapping in M with respect to the nearest tangent directions in relation to each other.

Definition 2.1 [18] *A cone of tangent directions* $K_M(z_0)$ *is called local tent if for any* $\bar{z}_0\in riK_M(z_0)$ *there exists a convex cone* $K\subseteq K_M(z_0)$ *and a continuous function* $\gamma(\cdot)$ *defined in the neighborhood of the origin, such that*

(1) $\bar{z}_0\in$ ri K, $LinK=LinK_M(z_0)$, *where* $LinK$ *is the linear span of* K,

(2) $\gamma(\bar{z})=\bar{z}+r(\bar{z})$, $r(\bar{z})\|\bar{z}\|^{-1}\to 0$ *as* $\bar{z}\to 0$,

(3)$z_0+\gamma(\bar{z})\in A$, $\bar{z}\in K\cap S_\varepsilon(0)$ *for some* $\varepsilon>0$, *where* $S_\varepsilon(0)$ *is the ball of radius* ε.

Definition 2.2 [18] *With respect to the book [18]* $h(\bar{x},x)$ *is called a convex upper approximation (CUA) of the function* $g:\mathbb{R}^n\to\mathbb{R}^1\cup\{\pm\infty\}$ *at a point* $x\in domg=\{x:|g(x)|<+\infty\}$ *if* $h(\bar{x},x)\geq V(\bar{x},x)$ *for all* $\bar{x}\neq 0$, *and* $h(\cdot,x)$ *is a convex closed positive homogeneous function, where*

$$V(\bar{x},x)=\sup_{r(\cdot)}\limsup_{\lambda\downarrow 0}\frac{1}{\lambda}[g(x+\lambda\bar{x}+r(\lambda))-g(x)],\lambda^{-1}r(\lambda)\to 0.$$

Here, the exterior supremum is taken on all $r(\lambda)$ *such that* $\lambda^{-1}r(\lambda)\to 0$ *as* $\lambda\downarrow 0$.

Definition 2.3 [18] *A set defined as follows:*

$$\partial h(0,x)=\{x^*\in\mathbb{R}^n:h(\bar{x},x)\geq\langle\bar{x},x\rangle,\bar{x}\in\mathbb{R}^n\}$$

is called a subdifferential of g *at a point* x *and is denoted by* $\partial g(x)$.

For various classes of functions, the concept of a subdifferential can be introduced in different ways, and for further details, the reader is referred to Mordukhovich [11]. Several important subdifferentials of non-smooth functions belong to the fundamental class of generalized differentials and play a significant role in both pure and applied analysis. In the first part of this paper, we study the following optimization problem with a parameter and an endpoint constraint, denoted by (PD)

$$\text{minimize} \quad \sum_{t=1}^{N} g(x_t, t), \tag{1}$$

$$\text{(PD)} \qquad \text{subject to} \qquad x_{t+1} \in F(x_t, w, t), t = 0,1,\dots,N-1, \tag{2}$$

$$x_0 = \theta_0, x_N \in P_0, \tag{3}$$

where $g(\cdot,t)$ is a real-valued function, $g(\cdot,t): \mathbb{R}^n \to \mathbb{R}^1 \cup \{\pm\infty\}$, $F(\cdot,t)$ is a time-dependent set-valued mapping, $F(\cdot,t): \mathbb{R}^{n+m} \rightrightarrows \mathbb{R}^n$, w is the m-dimensional vector-parameter of vector-space $W = \mathbb{R}^m$ with inner product, N is a fixed natural number, θ_0 is a fixed vector, and $P_0 \subseteq \mathbb{R}^n$ is the endpoint set. A sequence $\{x_t\}_{t=0}^{N} = \{x_t : t = 0,1,\dots,N\}$ is called a feasible trajectory for the stated problem (1)–(3).

In essence, we are considering a model of economic dynamics PD described by discrete inclusions with parameters, where the functioning of a certain economic system occurs at discrete moments in time $t = 0,1,\dots,N$ and it is necessary to select the values of the parameters $w \in W$ in such a way that the corresponding trajectory $\{x_t\}_{t=0}^{N}$ under conditions (2), (3) minimizers the sum (1). Here, at time t, one has a resource vector $x \in \mathbb{R}^n$ that can be transformed at time $t+1$ to one of the vectors $v \in F(x,w,t)$, where it is assumed that all possible amounts of resources are connected by $x_{t+1} \in F(x_t,w,t)$, $t = 0,\dots,N-1$; $x_0 = \theta_0$ is a vector of initial resource. Usually, the sum $\sum_{t=0}^{N} g(x_t, t)$ can be interpreted as the total expenditure.

In the second part of the paper, we investigate optimization problems with parameter and first-order DFI, labeled as (PC):

$$\text{minimie } J[x(\cdot)] = \int_0^T g(x(t),t)dt + f_0(x(T)), \tag{4}$$

$$\text{(PC)} \qquad \text{subject to} \qquad x^{'}(t) \in F(x(t),w,t), a.e t \in [0,T], \tag{5}$$

$$x(0) = \theta, t \in [0,T], x(T) \in P, \tag{6}$$

where $F(\cdot,t): : \mathbb{R}^{n+m} \rightrightarrows \mathbb{R}^n$, $w \in W = \mathbb{R}^m$ is m-dimensional vector-parameter and $g(\cdot,t), f_0$ are continuous proper functions, $P \subseteq \mathbb{R}^n$, θ is a fixed vector. It is necessary to find the solution $x(t), t \in [0,T]$ minimizing the Bolza functional $J[x(\cdot)]$ over a set of feasible trajectories. Here, the feasible trajectory $x(t), t \in [0,T]$ is an absolutely continuous function satisfying almost everywhere (a.e.) the first order DFI (5) and the initial and endpoint conditions (6).

Definition 2.4 *For the convex problem (1)–(3), the regularity condition is satisfied if for points* $x_t \in \mathbb{R}^n$, *one of the following cases is fulfilled:*

(i) $(x_t, x_{t+1}, w) \in ri(gphF(\cdot,t))(t = 0,\dots,N-1)$, $x_t \in ridomg(\cdot,t)$, $x_N \in riP$,

(ii) $(x_t, x_{t+1}, w) \in int(gphF(\cdot,t))(t = 0,\dots,N-1)$, $x_N \in P$ *(with possible exception of one set) and* $g(\cdot,t)$ *are continuous at* x_t.

Condition(C) *Suppose that in the nonconvex problem PD the set-valued mappings are such that the cones of tangent directions* $K_{F(\cdot,t)}$ $(\tilde{x}_t, \tilde{w}, \tilde{x}_{t+1})$, K_{P_0} $(\tilde{x}_N)$ *are local tents, where* $\tilde{x}_t$ *are the points of the optimal trajectory* $\{\tilde{x}_t\}_{t=0}^{N}$ *of problem PD. Further, the functions* $g(\cdot,t)$ *admit a CUA* $h(\bar{x}, \tilde{x}_t)$ *at the points* $\tilde{x}_t$.

3. Optimality conditions for the PD problem with parameter

This section indicates how the result of the main continuous problem PC with the parameter is substantially based on the results obtained for the problem with the discrete inclusion of PD. To solve a convex problem PD with a parameter, we reduce it to a convex minimization problem. Let

$$M_t = \{u : (x_t, w, x_{t+1}) \in gphF(\cdot, t)\},$$

$$\overline{P} = \{u : x_N \in P_0\},\ t = 0,1,\ldots, N-1, \tag{7}$$

where $u = (x_1, \ldots., x_N, w) \in \mathbb{R}^{nN+m}$.
Then our problem is equivalent to the following problem

$$\text{minimize } \Phi(u) = \sum_{t=1}^{N} g(x_t, t) \quad \text{subject to} \quad u \in (\cap_{t=0}^{N-1} M_t) \cap \overline{P}. \tag{8}$$

Theorem 3.1 *Let $g(\cdot, t)$ be continuous at the points $x_t, t = 1, \ldots, N$ of some feasible solution $\{x_t\}_{t=0}^{N}$ of the convex problem PD. Then, for $\{\tilde{x}_t\}_{t=0}^{N}$ to be an optimal solution to the problem PD with initial value $x_0 = \theta_0$ and endpoint condition $x_N \in P_0$, it is necessary that there exist a scalar $\alpha \in \{0,1\}$ and vectors $x_t^*, t = 0, \ldots, N$, not all equal to zero, such that*

(i) $(x_t^*, 0) \in F * (x_{t+1}^*; (\tilde{x}_t, \tilde{w}), t) - \alpha \partial g(\tilde{x}_t, t) \times \{0\}$,
$\partial g(\tilde{x}_0, 0) \equiv \{0\},\ t = 0, \ldots, N-1$,

(ii) $x^* - x_N^* \in \alpha \partial g(\tilde{x}_N, N), x^* \in K_{P_0} * (\tilde{x}_N)$.

Moreover, if $\alpha = 1$, *these conditions are sufficient for optimality.*
Proof. Obviously, if $\{\tilde{x}_t, \tilde{w}\}_{t=0}^{N} = (\tilde{x}_1, .., \tilde{x}_N, \tilde{w})$ is an optimal solution to problem (1)–(3), then we claim that $\tilde{u} = (\tilde{x}_1, .., \tilde{x}_N, \tilde{w})$ is a solution to the convex mathematical programming problem (8). Due to the continuity $g(\cdot, t)$ at the points of some feasible solution $\{x_t, w\}_{t=0}^{N}$ from Theorem 3.4 [18], it follows that there are vectors

$$u^*(t) \in K *_{M_t} (\tilde{u}), t = 0, \ldots, N-1;\ \hat{u}^* \in K_{\overline{P}} * (\tilde{u}),\ \bar{\bar{u}}^* \in \partial \Phi(\tilde{u})$$

and the scalar $\alpha \in \{0,1\}$, not all equal to zero, such that

$$\hat{u}^* + \sum_{t=0}^{N-1} u^*(t) = \alpha \bar{\bar{u}}^*,\ \bar{\bar{u}}^* \in \partial \Phi(\tilde{u}). \tag{9}$$

The calculation of the dual cone shows that

$$\mathrm{K} *_{\mathrm{M_t}} (\mathrm{u}) = \{\mathrm{u}^* = (\mathrm{x}_1^*, \ldots., \mathrm{x}_\mathrm{N}^*, \mathrm{w}^*) : (\mathrm{x}_\mathrm{t}^*, \mathrm{w}^*, \mathrm{x}_{\mathrm{t}+1}^*) \in \mathrm{K} *_\mathrm{F} (\mathrm{x}_\mathrm{t}, \mathrm{w}, \mathrm{x}_{\mathrm{t}+1}),$$

$$w^* = 0,\ x_k^* = 0, k \neq t, t+1\},\ t = 0, \ldots, N-1;$$

Indeed, if

$$u + \lambda \bar{u} \in M_t,\ t = 0,1, \ldots, N-1,$$

for sufficiently small $\lambda > 0$, i.e.,

$$(x_t + \lambda \bar{x}_t, w + \lambda \bar{w}, x_{t+1} + \lambda \bar{x}_{t+1}) \in gphF(\cdot, t),$$

then $\bar{u} \in K_{M_t}(\tilde{u})$. Therefore

$$K_{M_t}(u) = \{\bar{u}: (\bar{x}_t, \bar{w}, \bar{x}_{t+1}) \in K_F(x_t, w, x_{t+1})\}.$$

Then, due to the arbitrariness of $\bar{w}$, from the definition of the dual cone, it's easy to see that

$$K *_{M_t} (u) = \{u^*: (x_t^*, 0, x_{t+1}^*) \in K *_F (x_t, w, x_{t+1}), x_k^* = 0, k \neq t, t+1\}, \tag{10}$$

where $u^* \in \mathbb{R}^{nN+m}$. In this case,

$$\langle \bar{u}, u^* \rangle = \sum_{k=1}^{N} \langle \bar{x}_k, x_k^* \rangle + \langle \bar{w}, w^* \rangle \geq 0$$

has a form

$$\langle \bar{x}_t, x_t^* \rangle + \langle \bar{w}, 0 \rangle + \langle \bar{x}_{t+1}, x_{t+1}^* \rangle \geq 0, (\bar{x}_t, \bar{w}, \bar{x}_{t+1}) \in K_F(x_t, w, x_{t+1}),$$

that is, $(x_t^*, 0, x_{t+1}^*) \in K *_F (x_t, w, x_{t+1})$.
On the other hand,

$$K *_{\overline{P}} (u) = \{u^* = (x_1^*, \dots, x_N^*, 0): x_N^* \in K *_{P_0} (x_N), x_t^* = 0, t < N\}, \tag{11}$$

Therefore, it is clear to see that

$$u^*(t) = \left(x_1^*(t), \dots, x_N^*(t), \underbrace{0, \dots, 0}_{m} \right), x_k^*(t) = 0, k \neq t, t+1, \; t = 0, \dots, N-1,$$

$$\hat{u}^* = (0,0,\dots,0, \hat{x}_N^*, \underbrace{0, \dots, 0}_{m}), \hat{x}_N^* \in K *_{P_0} (\tilde{x}_N),$$

$$\tilde{u} = (\tilde{x}_1, \dots, \tilde{x}_N, \widetilde{w}_1, \dots, \widetilde{w}_m). \tag{12}$$

Indeed, when the regularity condition (i) or (ii) of Definition 2.4 holds, the dual cone of the intersections

$$\left[K_{M_t}(\tilde{u}) \cap K_{\overline{P}}(\tilde{u})\right]^* = K *_{M_t} (\tilde{u}) + K *_{\bar{P}} (\tilde{u}), \; K *_{\bar{P}} (\tilde{u})$$

and by Theorems 1.30 and 3.4 [18], for the problem (8), equality (9) holds with $\alpha = 1$. Moreover, the conditions of Moreau-Rockafellar theorem [2,18] are satisfied. Finally, if $t = 0$, then, since $\tilde{x}_0 = \theta_0$ is fixed, $K_0(u) = \{\bar{u}: \bar{x}_1 \in K_{F(x_0,w)}(x_1)\}$ and

$$K_0 * (u) = \{u^*: x_1^* \in K *_{F(x_0,w)} (x_1), x_t^* = 0, t = 2, \dots, N\}. \tag{13}$$

Let us return to the relation (9); it follows from $\Phi(u) \equiv \sum_{t=1}^{t=N} g(x_t, t)$ that the vector $\bar{\bar{u}}^*$ has the form $\bar{\bar{u}}^* = (\bar{\bar{x}}_1^*, \dots, \bar{\bar{x}}_N^*, 0)$, where $\bar{\bar{x}}_t^* \in \partial g(\tilde{x}_t, t), t = 1, \dots, N$. Then, considering (10)–(12), the relation (9) can be written as follows:

$$x_t^*(t-1) + x_t^*(t) = \alpha \bar{\bar{x}}_t^*, t = 1, \dots, N. \tag{14}$$

Using (10), we can write

$$x_t^*(t) \in F * (-x_{t+1}^*(t); (\tilde{x}_t, \widetilde{w}), t), t = 0, \ldots, N-1. \tag{15}$$

Hence, from (14) and (15), we have

$$\lambda \bar{\bar{x}}_t^* - x_t^*(t-1) \in F^*(-x_{t+1}^*(t); (\tilde{x}_t, \widetilde{w}), t), t = 1, \ldots, N-1. \tag{16}$$

Thus, introducing the notations

$$x_t^* \equiv -x_t^*(t-1), t = 1, \ldots, N,$$

from (16), we have inclusions (i) of the theorem.
Denoting $x^* \equiv x_N^*(N)$ from (14) for $t = N$, we have

$$-x_N^* + x^* = \alpha \bar{\bar{x}}_N^*, \ x^* \in K *_{P_0} (\tilde{x}_N). \tag{17}$$

Further, it is easy to see from (13) that the relation (16) can be extended to the case $t = 0$. Thus, by virtue of (16) and (17), we have proved this theorem.

□

Remark 3.1 *It is well established [18] that the non-separability of tangent direction cones* $K_{M_t}(\tilde{u})$ *and* $K_{\overline{P}}(\tilde{u})$ *play a significant role in the theory of extremal problems of type (8). Furthermore, when the regularity condition from Definition 2.4 holds, the dual cone of the intersection of tangent direction cones equals the algebraic sum of their dual cones; that is, these cones are non-separable. The following example demonstrates the essential importance of regularity conditions in solving such problems.*
Let D_1 and D_2 be two closed disks of radius r around $(r, 0)$ and $(-r, 0)$, respectively, on the plane $\mathbb{R}^2 = \mathbb{R}^1 \times \mathbb{R}^1$:

$$D_1 = \{(x_1, x_2) \in \mathbb{R}^2 : (x_1 - r)^2 + {x_2}^2 \le r^2\},$$

$$D_2 = \{(x_1, x_2) \in \mathbb{R}^2 : (x_1 + r)^2 + {x_2}^2 \le r^2\}.$$

Then for $(\bar{x}_1, \bar{x}_2) = (0,0) \in D_1 \cap D_2$, we have $K_{D_1 \cap D_2}(\bar{x}_1, \bar{x}_2) = \{(0,0)\}$, $K_{D_1}(\bar{x}_1, \bar{x}_2) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \ge 0\}$, $K_{D_2}(\bar{x}_1, \bar{x}_2) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \le 0\}$, and $K *_{D_1 \cap D_2} (\bar{x}_1, \bar{x}_2) = \mathbb{R}^2$, while $K *_{D_1} (\bar{x}_1, \bar{x}_2) = \{(x_1, 0) : x_1 \ge 0\}$, $K *_{D_2} (\bar{x}_1, \bar{x}_2) = \{(x_1, 0) : x_1 \le 0\}$, and $K *_{D_1} (\bar{x}_1, \bar{x}_2)$ $+ K *_{D_2} (\bar{x}_1, \bar{x}_2) = \{(x_1, 0) : x_1 \in \mathbb{R}^1\}$. Thus, $K *_{D_1} (\bar{x}_1, \bar{x}_2) + K *_{D_2} (\bar{x}_1, \bar{x}_2) \subset K *_{D_1 \cap D_2} (\bar{x}_1, \bar{x}_2)$, in other words, $K *_{D_1 \cap D_2} (\bar{x}_1, \bar{x}_2) \neq K *_{D_1} (\bar{x}_1, \bar{x}_2) + K *_{D_2} (\bar{x}_1, \bar{x}_2)$. □
If $F(\cdot, t)$ is "pointwise" closed (the set $F(\cdot, t)$ is closed for each (x, w)), then the conditions of this theorem can be rewritten in a more symmetrical form. On the other hand, it is obvious that the usual closedness of $F(\cdot, t)$ implies "pointwise" closedness, but not vice versa, i.e., "pointwise" closedness is weaker than the usual one. First of all, we need the following proposition.
Proposition 3.1 *If* $F(\cdot, t): \mathbb{R}^{2n} \rightrightarrows \mathbb{R}^n$ *is convex and* $F(x, w, t)$ *is a closed set for each* $(x, w) \in domF(\cdot, t)$*, then*

$$\partial_{v^*} H_F(x, w, \bar{v}^*) = F_A(x, w; \bar{v}^*, t).$$

Proof. If $v \in F_A(x, w; \bar{v}^*, t)$ or if $v \in F(x, w, t)$ and $\langle v, \bar{v}^* \rangle = H_F(x, w, \bar{v}^*)$, then $H_F(x, w, v^*) -$

$H_F(x,w,\bar{v}^*) \geq \langle v, v^* \rangle - \langle v, \bar{v}^* \rangle$ or $H_F(x,w,v^*) - H_F(x,w,\bar{v}^*) \geq \langle v, v^* - \bar{v}^* \rangle$, that is, $v \in \partial_{v^*} H_F(x,w,\bar{v}^*)$. Conversely, assume that $\bar{v} \in \partial_{v^*} H_F(x,w,\bar{v}^*)$. First, we prove that $\bar{v} \in F(x,w,t)$. Suppose that $\bar{v} \notin F(x,w,t)$. Then, by the separation theorems [18], there is a vector c such that

$$\sup_v \{\langle v, c \rangle : v \in F(x,w,t)\} < \langle \bar{v}, c \rangle. \tag{18}$$

On the other hand, we have $\sup_v \{\langle v, v^* - \bar{v}^* \rangle : v \in F(x,w,t)\} \geq H_F(x,w,v^*) - H_F(x,w,\bar{v}^*) \geq \langle \bar{v}, v^* - \bar{v}^* \rangle$. Setting now in this inequality $v^* = \bar{v}^* + c$, we have $\sup_v \{\langle v, a \rangle : v \in F(x,w,t)\} \geq \langle \bar{v}, a \rangle$, which contradicts the previous inequality (18). It means that $\bar{v} \in F(x,w,t)$. Then, $\bar{v} \in \partial_{v^*} H_F(x,w,\bar{v}^*)$ we derive $H_F(x,w,v^*) - \langle v^*, \bar{v} \rangle \geq H_F(x,w,\bar{v}^*) - \langle \bar{v}^*, \bar{v} \rangle$ and setting $v^* = 0$, we have $\langle \bar{v}^*, \bar{v} \rangle \geq H_F(x,w,\bar{v}^*)$. Moreover, since $\bar{v} \in F(x,w,t)$, it follows that $\langle \bar{v}^*, \bar{v} \rangle \leq H_F(x,w,\bar{v}^*)$. Therefore, $\langle \bar{v}^*, \bar{v} \rangle = H_F(x,w,\bar{v}^*)$, which means that $\bar{v} \in F_A(x,w\,;\bar{v}^*,t)$. □

Below we prove that an upper semi-continuous set-valued mapping $F(\cdot,t)$ (not necessarily convex) with closed values is closed ($gphF(\cdot,t)$ is closed).

Proposition 3.2 *Let $F(\cdot,t): \mathbb{R}^{2n} \rightrightarrows \mathbb{R}^n$ be an upper semi-continuous pointwise closed set-valued mapping. Then $F(\cdot,t)$ is closed.*

Proof. Suppose that $(x_i, w_i, v_i) \in gphF(\cdot,t)$ is a convergent sequence and $(x_i, w_i, v_i) \rightarrow (x_0, w_0, v_0)$, but $v_0 \notin F(x_0, w_0, t)$. Then there exists an open set Ω containing $F(x_0, w_0, t)$ such that $v_0 \notin . \overline{\Omega}$. On the other hand, $F(\cdot,t)$ is upper semi-continuous and there exists a positive integer i_0 such that $v_i \in \Omega, \forall i > i_0$, i.e., $v_0 \in \overline{\Omega}$, a contradiction. □

Corollary 3.1 *Let the optimization problem PD satisfy the conditions of Theorem 3.1. Moreover, let $F(\cdot,t)$ be a closed set for each fixed (x,w). Then condition (i) of Theorem 3.1 can be rewritten as follows*

$$(x_t^*, 0) \in \partial_{(x,w)} H_F(\tilde{x}_t, \widetilde{w}, x_{t+1}^*) - \alpha \partial g(\tilde{x}_t, t) \times \{0\},$$

$$\tilde{x}_{t+1} \in \partial_{v^*} H_F\left(\tilde{x}_t, \tilde{w}, x_{t+1}^*\right), \partial g(\tilde{x}_0, 0) = \{0\}, \; t = 0, \ldots, N-1.$$

Proof. Recall [18] that $F * (v^*, (x,w,v), t) = \partial_{(x,w)} H_F(x,w,v^*)$ is non-empty if so is the argmaximum set $F_A(x,w\,;v^*,t)$. Further, from the corollary assumption and from Lemma 3.1, it follows that $\partial_{v^*} H_F(x,w,v^*) = F_A(x,w\,;v^*,t)$. Then, considering these in Theorem 3.1, we will verify the validity of the corollary statements. □

Theorem 3.2 *Let condition (C) be satisfied for a non-convex problem PD, i.e., the cones of tangent directions $K_F(\tilde{x}_t, \widetilde{w}, \tilde{x}_{t+1})$, $K_{P_0}(\tilde{x}_N)$ are local tents, $g(\cdot,t)$ admit CUA at points $\tilde{x}_t$. Then, for the trajectory $\{\tilde{x}_t, \widetilde{w}\}_{t=0}^N$ to be optimal, it is necessary that there exist a number $\alpha \in \{0,1\}$ and vectors $\{x_t^*\}$, satisfying the conditions of Theorem 3.1.*

Proof. In this case, the Condition (C) guarantees the assumptions of Theorem 3.25 [18] for the convex minimization problem (8). Consequently, by this theorem, we obtain the necessary condition in the same form as Theorem 3.1, starting from relation (9) formulated for the non-convex problem. □

4. Equivalence of LAMs for discrete-approximate problems

In this section, based on the results of the previous section and an auxiliary equivalence result, necessary and sufficient optimality conditions for the discrete-approximate problem are obtained.

Suppose L is a positive natural scalar and $T/L=\delta$. Clearly, δ is a step on the t-axis, and $x(t)\equiv x_\delta(t)$ is a grid function on a uniform grid on $[0,T]$. We introduce the following first-order difference operators:

$$\Delta x(t)=\frac{1}{\delta}[x(t+\delta)-x(t)].$$

Note that the discrete approximate inclusions according to problem (PC) are the following inclusions

$$\Delta x(t)\in F(x(t),w,t),\ t=0,\delta,\ldots,T-\delta, \tag{19}$$

which can be rewritten in a more relevant form

$$x(t+\delta)\in x(t)+\delta F(x(t),w,t),\ t=0,\delta,\ldots,T-\delta.$$

Denoting $x(t)\equiv x, x(t+\delta)\equiv v$, we define the following new mapping $G(x,w,t)=x+\delta F(x,w,t)$. Obviously, the discrete approximate inclusions associated with (19) are equivalent to

$$x(t+\delta)\in G(x(t),w,t),\ t=0,\delta,\ldots,T-\delta.$$

It can easily be seen that these optimization conditions will be expressed in terms of LAM $G*$. Therefore, we will establish a connection between LAMs $G*$ and $F*$.
The following two lemmas play a decisive role in the following results based on discrete first-order approximations.
Lemma 4.1 *The following relation exists between the Hamiltonians* H_G *and* H_F

$$H_G(x,w,v^*)=\langle x,v^*\rangle+\delta H_F(x,w,v^*),$$

where

$$G(x,w,t)=x+\delta F(x,w,t).$$

Proof. Indeed, by the definition of the Hamiltonians of set-valued mappings F,G, we have

$$H_G(x,w,v^*)=sup\{\langle v,v^*\rangle : v\in G(x,w,t)\}$$
$$=\langle x,v^*\rangle+\delta\, sup\{\langle v_1,v^*\rangle : v_1\in F(x,w,t)\}=\langle x,v^*\rangle+\delta H_F(x,w,v^*). \qquad \square$$

Although the following lemma can be proved in another way, given the growing interest in the upcoming transition from a discrete problem to a discrete-approximative one, it makes sense to use the technique of proving the following lemma.
Lemma 4.2 *The following relationship holds between the subdifferentials of the Hamiltonians* H_F,H_G*:*

$$\partial_{(x,w)}H_G(x,w,v^*)=\{v^*\}\times\{0\}+\delta A*\partial_{(x,w)}H_F(x,w,v^*),$$

where

$$A=\begin{bmatrix} I_n & 0_{n\times m}\\ 0_{m\times n} & I_m\end{bmatrix}$$

is the $(n+m)\times(n+m)$ *identity matrix partitioned into identity submatrices* I_n, I_m *and zero matrices* $0_{n\times m}, 0_{m\times n}$*, where* $A*$ *is the transpose of* A.
***Proof*.** Our aim is to express the subdifferential $\partial_{(x,w)}H_F(x,w,v^*)$ in terms of the subdifferential $\partial_{(x,w)}H_G(x,w,v^*)$. Let $Q:\mathbb{R}^{2n}\to\mathbb{R}^1$ be a convex function continuous at a point

$(\varphi_1(x,w),\varphi_2(x,w))$, where $\varphi_i:\mathbb{R}^{2n}\to\mathbb{R}^n$, $i=1,2$ are Frechet differentiable functions at a point (x,w). Then, by Theorem 2 [3], the subdifferential $\partial f_0(x,w)$ of function

$$f_0(x,w)=Q(\varphi_1(x,w),\varphi_2(x,w))$$

can be calculated as follows:

$$\partial f_0(x,w)=A*\partial Q(\varphi_1(x,w),\varphi_2(x,w)) \tag{20}$$

where

$$A=\begin{bmatrix}\frac{\partial\varphi_1(x,w)}{\partial x} & \frac{\partial\varphi_1(x,w)}{\partial w}\\ \frac{\partial\varphi_2(x,w)}{\partial x} & \frac{\partial\varphi_2(x,w)}{\partial w}\end{bmatrix} \tag{21}$$

Is a $2n\times 2n$ matrix, $\frac{\partial\varphi_i(x,w)}{\partial x}$, $\frac{\partial\varphi_i(x,w))}{\partial w}$, $i=1,2$ are Jacobi matrices. Setting $\varphi_1(x,w)\equiv x$, $\varphi_2(x,w)\equiv w$, in (21), it is easy to compute that

$$A=\begin{bmatrix}I_n & 0_{n\times m}\\ 0_{m\times n} & I_m\end{bmatrix}.$$

Then, setting $f_0(x,w=Q(\varphi_1(x,w),\ \varphi_2(x,w))=H_F(x,w,\ v^*)$ from formula (20) and Lemma 4.1, we obtain the needed result.

□

Theorem 4.1 *For the mapping* $G(x,w,t)=x+\delta F(x,w,t)$, *the following inclusions are equivalent:*

(1) $(x^*,0)\in G*(v^*;(x,w,v),t),\ v\in G_A(x,w;v^*,t)$,

(2) $\left(\frac{x^*-v^*}{\delta},0\right)\in F*(v^*;(x,w),t),\quad \frac{v-x}{\delta}\in F_A(x,w;v^*,t),\ v^*\in\mathbb{R}^n$

where $G_A(x,w;v^*,t)$ *is the argmaximum set for mapping* G*:*

$$G_A(x,w;v^*,t)=\{v\in G(x,w,t):\langle v,v^*\rangle=H_G(x,w,v^*)\}.$$

*Proof***.** Let us prove $(1)\Rightarrow(2)$. By Lemma 4.2, it is easy to see that if

$$(x^*,w^*)\in\partial_{(x,w)}H_G(x,w,v^*) \tag{22}$$

then

$$(A*)^{-1}\left(\frac{x^*-v^*}{\delta},\frac{w^*}{\delta}\right)\in\partial_{(x,w)}H_F(x,w,v^*). \tag{23}$$

But since $A*=I_{n+m}$ is the identity matrix, so is $(A*)^{-1}=I_{n+m}$.
Then, by substituting $(A*)^{-1}$ into (20), we derive that

$$\left(\frac{x^*-v^*}{\delta},\frac{w^*}{\delta}\right)\in\partial_{(x,w)}H_F(x,w,v^*). \tag{24}$$

On the other hand, by Theorem 2.1 [18]

$$G*(v^*;(x,w,v),t)=\partial_{(x,w)}H_G(x,w,v^*), v\in G_A(x,w;v^*,t),$$

$$F*\left(v^*;\left(x,w,\frac{v-x}{\delta}\right),t\right)=\partial_{(x,w)}H_F(x,v^*),\quad \frac{v-x}{\delta}\in F_A(x,w;v^*,t). \tag{25}$$

Now, with the help of formula (25), it is necessary to reformulate (22) and (23) in term of LAMs. In addition, recall that $v\in G_A(x,w;v^*,t)$, $\frac{(v-x}{\delta}\in F_A(x,w;v^*,t)$ guarantees that LAMs are nonempty. Hence, considering (25) in (22) and (24), we have $(1)\Rightarrow(2)$. By a similar way, it can be proved that $(2)\Rightarrow(1)$. It remains to show in conditions (1), (2) of theorem $w^*=0$. Indeed, since $w\in W$ is arbitrary, it follows directly from (22) of the Hamiltonian that $w^*=0$. ☐

For the non-convex problem PD, Theorem 4.1 can be generalized to the case of local tents.

Theorem 4.2 *Let* $K_{G(\cdot,t)}(x,w,v)$ *define a local tent to the mapping* $G(x,w,t)=x+\delta F(x,w,t)$. *Then, the statements of the previous theorem are also true in the non-convex case.*

*Proof***.** By conditions of the theorem, there are functions $r(\bar z), r_i(\bar z), \bar z=(\bar x,\bar w,\bar v), r_i(\bar z)\|\bar z\|^{-1}\to 0 (i=1,2)$, $r(\bar z)\|\bar z\|^{-1}\to 0$ as $\bar z\to 0$, such that

$$v+\bar v+r(\bar z)\in x+\bar x+r_1(\bar z)$$
$$+\delta F(x+\bar x+r_1(\bar z),w+\bar w+r_2(\bar z),t)$$

for sufficiently small $\bar z\in K$, where $K\subseteq riK_{G(\cdot,t)}(z)$, $z=(x,w,v)$ is a convex cone. Dividing by δ this relation, we have

$$\frac{v-x}{\delta}+\frac{\bar v-\bar x}{\delta}+\frac{r(\bar z)-r_1(\bar z)}{\delta}$$
$$\in F(x+\bar x+r_1(\bar z),w+\bar w+r_2(\bar z),t).$$

Note that the cone $K_{F(\cdot,t)}(x,w)$ represents a local tent and

$$\left(\bar x,\bar w,\frac{\bar v-\bar x}{\delta}\right)\in K_{F(\cdot,t)}\left(x,w,\frac{v-x}{\delta}\right). \tag{26}$$

By analogy with relation (26) in the opposite direction, it is easy to verify that

$$(\bar x,\bar w,\bar v)\in K_{G(\cdot,t)}(x,w,v). \tag{27}$$

Hence, we get the equivalence of (26) and (27). Now,

$$(x^*,w^*)\in G*(v^*;(x,w,v,),t),\ v\in G_A(x,w;v^*,t),$$

whereas

$$\langle\bar x,x^*\rangle+\langle\bar w,w^*\rangle-\langle\bar v,v^*\rangle\geq 0,(\bar x,\bar w,\bar v)\in K_{G(\cdot,t)}(x,w,v). \tag{28}$$

At once, we note that for arbitrariness of $\bar w$ in (28), it follows that $w^*=0$.

Let us rewrite cone $K_{F(\cdot,t)}\left(x, w, \frac{(v-x)}{\delta}\right)$ (see (26)) in the form

$$\langle \bar{x}, \theta_1 \rangle + \langle \bar{w}, \theta_2 \rangle - \left\langle \frac{\bar{v}-\bar{x}}{\delta}, v^* \right\rangle \geq 0, \tag{29}$$

$$\left(\bar{x}, \bar{w}, \frac{\bar{v}-\bar{x}}{\delta}\right) \in K_{F(\cdot,t)}\left(x, w, \frac{v-x}{\delta}\right).$$

Multiplying (29) by δ as a result of a rearrangement, we have

$$\langle \bar{x}, \delta\theta_1 + v^* \rangle + \langle \bar{w}, \delta\theta_2 \rangle - \langle \bar{v}, v* \rangle \geq 0.$$

Comparing this inequality with (28) means that

$$\theta_1 = \frac{x^* - v^*}{\delta}, \theta_2 = \frac{w^*}{\delta} = 0.$$

Then, from the equivalence of (28) and (29), it follows that

$$\left(\frac{x^* - v^*}{\delta}, 0\right) \in F * \left(v^*; \left(x, w, \frac{(v-x)}{\delta}\right), t\right).$$

Further, since $G * (v^*; (x, w, v), t) \neq \emptyset$, $v \in G_A(x, w; v^*, t)$ and

$$F * \left(v^*; \left(x, w, \frac{v-x}{\delta}\right), t\right) \neq \emptyset, \ \frac{(v-x)}{\delta} \in F_A(x, w; v^*, t)$$

we have the needed result □

Theorem 4.3 *Let* $F(\cdot, t): \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ *be a parameter-independent set-valued mapping such that the cone* $K_{G(\cdot,t)}(x, v)$, $(x, v) \in gphG(\cdot, t)$ *of tangent directions to* $G(x, t) = x + \delta F(x, t)$ *defines a local tent. Then the following inclusions are equivalent:*

(a) $\quad x^* \in G(v^*; (x, v), t), \ v \in G_A(x; v^*, t), \ \ v^* \in \mathbb{R}^n,$

(b) $\quad \frac{x^* - v^*}{\delta} \in F^*\left(v^*; \left(x, \frac{v-x}{\delta}\right), t\right), \ \frac{v-x}{\delta} \in F_A(x; v^*, t).$

Proof. It is clear that there are functions $r(\bar{z}), r_i(\bar{z})$, $r_i(\bar{z})\|\bar{z}\|^{-1} \to 0$, $\bar{z} = (\bar{x}, \bar{v})$ $(i = 1,2)$, $r(\bar{z})\|\bar{z}\|^{-1} \to 0$ as $\bar{z} \to 0$, such that

$$v + \bar{v} + r(\bar{z}) \in x + \bar{x} + r_1(\bar{z}) + \delta F(x + \bar{x} + r_1(\bar{z}), t)$$

for small $\bar{z} \in K \subseteq riK_{G(\cdot,t)}(z)$, $z = (x, v)$. Then, from the following convenient form of this relation,

$$\frac{v - x}{\delta} + \frac{\bar{v} - \bar{x}}{\delta} + \frac{r(\bar{z}) - r_1(\bar{z})}{\delta} \in F(x + \bar{x} + r_1(\bar{z}), t)$$

it is easy to see that the cone $K_{F(\cdot,t)}\left(x, (\frac{v-x}{\delta}\right)$ is a local tent to $gphF$. Furthermore, the inclusions

$$\left(\bar{x}, \frac{\bar{v}-\bar{x}}{\delta}\right) \in K_{F(\cdot,t)}\left(x, \frac{v-x}{\delta}\right) \text{ and } (\bar{x}, \bar{v}) \in K_{G(\cdot,t)}(x, v)$$

are equivalent. Then, from a more convenient form of inequality

$$\langle \bar{x}, x^* \rangle - \langle \bar{v}, v^* \rangle \geq 0, (\bar{x}, \bar{v}) \in K_{G(\cdot,t)}(x, v)$$

in the form

$$\langle \bar{x}, \theta \rangle - \left\langle \frac{\bar{v}-\bar{x}}{\delta}, v * \right\rangle \geq 0,\ \left(\bar{x}, \frac{\bar{v}-\bar{x}}{\delta}\right) \in K_{F(\cdot,t)}\left(x, \frac{v-x}{\delta}\right)$$

we find $\theta = \frac{(x^*-v^*)}{\delta}$, which means that

$$\frac{x^*-v^*}{\delta} \in F * \left(v^* ; \left(x, \frac{v-x}{\delta}\right), t\right).$$

Furthermore, we find that LAMs $G * (v^* ;(x, v), t)$ and $F * \left(v^* ; \left(x, \frac{(v-x)}{\delta}\right), t\right)$ are nonempty, if $v \in G_A(x\,; v^*, t)$ and $\frac{v-x}{\delta} \in F_A(x\,; v^*, t)$, respectively. □

5. Optimality conditions for the PDA problem with parameter

In this section, we will consider the PDA problem with the parameter, where we will rely on the results of Section 4. In terms of the former notations, we associate the discrete-approximate problem, in accordance with (4)–(6):

$$\text{minimize } J_\delta[x(\cdot)] = \textstyle\sum_{t=0,\dots,T-\delta} \delta g(x(t), t) + f_0(x(T)), \tag{30}$$

$$\Delta x(t) \in F(x(t), w, t), t = 0, \delta, \dots, T - \delta, \tag{31}$$

$$x(0) = \theta, x(T) \in P, w \in W.$$

Now, using the mapping $G(x, w, t) = x + \delta F(x, w, t)$, we reduce the problem (30) and (31) to the following form, denoted as (PDA):

$$\text{minimize } J_\delta[x(\cdot)] = \textstyle\sum_{t=0,\dots,T-\delta} \delta g(x(t), t) + f_0(x(T)), \tag{32}$$

(PDA) $$x(t + \delta) \in G(x(t), w, t), \quad t = 0, \delta, \dots, T - \delta,$$

$$x(0) = \theta, x(T) \in P. \tag{33}$$

From Theorems 3.1 and 3.2, we obtain that for the optimal pair $\{(\tilde{x}(t), \widetilde{w}) : t = 0, \delta, \dots, T\}$ of problem (32), (33), there are vectors $\{x^*(t) : t = 0, \delta, \dots, T\}$ and the number $\alpha = \alpha_\delta \in \{0,1\}$ such that

$$(x^*(t), 0) \in G * (x^*(t + \delta), (\tilde{x}(t), \widetilde{w}, \tilde{x}(t + \delta)), t)$$

$$-\alpha\delta\partial g(\tilde{x}(t), t) \times \{0\}; \partial g(\tilde{x}(0), 0) = \{0\},\ t = 0, \dots, T - \delta; \tag{34}$$

$$x_e^* - x^*(T) \in \alpha\partial f_0(\tilde{x}(T)), x_e^* \in K_P * (\tilde{x}(T)). \tag{35}$$

Theorem 5.1. *Let PDA be a convex problem, and* $g(\cdot, t), f_0(\cdot)$ *be proper convex functions. Then, for a pair* $\{\tilde{x}(t), \widetilde{w}\}$ *to be optimal, it is necessary that there are vectors* x_e^*, $\{x^*(t) : t = 0, \delta, \dots, T\}$ *and a number* $\alpha = \alpha_\delta \in \{0,1\}$, *satisfying inclusions (i) and transversality conditions (ii):*

$(i) - (\Delta x^*(t),0) \in F * (x^*(t+\delta)\,;(\tilde{x}(t), \widetilde{w}, \Delta\tilde{x}(t)), t)$

$-\alpha\partial g(\tilde{x}(t),t) \times \{0\}; \partial g(\tilde{x}(0),0) = \{0\}, t = 0, \dots, T-\delta;$

(ii) $x_e^* - x^*(T) \in \alpha\partial f_0(\tilde{x}(T)), x_e^* \in K_P * (\tilde{x}(T))$.

And these conditions are also sufficient if the regularity condition is satisfied.

*Proof***.** In fact, by virtue of the relations (1), (2) of Theorem 4.1, conditions (34) takes the form

$$\left(\frac{x^*(t)-x^*(t+\delta)}{\delta}, 0\right) \in F * (x^*(t+\delta)\,;(\tilde{x}(t), \widetilde{w}, \Delta\tilde{x}(t)), t) \tag{36}$$

$$-\alpha\partial g(\tilde{x}(t),t) \times \{0\}, \partial g(\tilde{x}(0),0) = \{0\}, t = 0, \dots, T-\delta.$$

We only note that the LAM is positively homogeneous in the first variable and, therefore, in (36), $\delta x^*(t)$, it is again denoted by $x^*(t)$. Considering (35) and (36) in the first-order discrete conditions, we obtain the required result. □

Theorem 5.2 *If the problem PDA is non-convex and Condition (C) is satisfied for it, then for a pair* $\{\tilde{x}(t), \widetilde{w}\}$ *to be an optimal pair of the problem, it is necessary that there exist a number* $\alpha \in \{0,1\}$ *and vectors* x^*, $\{x^*(t)\}$, *satisfying conditions (i)-(ii) of Theorem 5.1 in the non-convex case.*

Theorem 5.3 *Let the problem PDA be independent of the parameter w, i.e.,* $F\left(x, \frac{v-x}{\delta}, t\right) \equiv F(x,t)$. *Then, the adjoint discrete inclusion (i) and the transversality condition (ii) of Theorem 5.1 are transformed as follows:*

$$-\Delta x^*(t) \in F * (x^*(t+\delta)\,;(\tilde{x}(t), \Delta\tilde{x}(t)), t) - \alpha\partial g(\tilde{x}(t),t), \ t = 0, \delta, \dots, T-\delta;$$

$$\partial g(\tilde{x}(0),0) = \{0\}; \quad x_e^* - x^*(T) \in \alpha\partial f_0(\tilde{x}(T)), x_e^* \in K_P * (\tilde{x}(T)),$$

respectively.

Proof. According to condition (b) of Theorem 4.3, it is enough to recall that

$$\frac{x^*(t)-x^*(t+\delta)}{\delta} \in F * (x^*(t+\delta)\,;(\tilde{x}(t), \Delta\tilde{x}(t)), t) - \alpha\partial g(\tilde{x}(t),t),$$

$$\partial g(\tilde{x}(0),0) = \{0\}, t = 0, \dots, T-\delta,$$

where $\frac{(x^*(t+\delta)-x^*(t))}{\delta} = \Delta x^*(t)$. The proof of the theorem is completed. □

6. Optimization of first-order DFIs with parameter

In this section, the proof of sufficient optimality conditions for the parameter-dependent problem PC is mainly based on the results obtained in the preceding section. Specifically, in conditions (i) and (ii) of Theorem 5.1, we set $\alpha = 1$ and formally pass to the limit as $\delta \to 0$, thereby establishing the adjoint DFI for the PC problem. The final step is to derive the sufficient optimality conditions.

First, we formulate the adjoint DFIs for the convex problem (PC):

(i) $\quad -\left(\frac{dx^*(t)}{dt}, 0\right) \in F * (x^*(t)\,;(\tilde{x}(t), \widetilde{w}, \tilde{x}'(t)), t)$

$$-\partial g(\tilde{x}(t),t) \times \{0\}, \text{ a.e. } t \in [0,T],$$

(ii) transversality conditions at a point $t = T$:

$$x_e^* - x^*(T) \in \partial f(\tilde{x}(T)), x_e^* \in K_P * (\tilde{x}(T)).$$

Suppose that $x^*(t)$, $t \in [0, T]$ is an absolutely continuous function and $x^{*\prime}(\cdot) \in L_1^n[0, T]$. Besides, let the following condition be satisfied:

(iii) $\frac{d\tilde{x}(t)}{dt} \in F_A(\tilde{x}(t), \widetilde{w}\,; x^*(t), t)$, a.e. $t \in [0, T]$.

Theorem 6.1 *Suppose that* $g : \mathbb{R}^n \times [0, T] \to \mathbb{R}^1$ *and* $f(\cdot)$ *are continuous and convex functions of* x. *Moreover,* $F(\cdot, t) : \mathbb{R}^{n+m} \rightrightarrows \mathbb{R}^n$ *is a convex and bounded mapping. Then, for a trajectory* $\tilde{x}(t)$ *to be optimal in the PC problem, it suffices that there exists an adjoint function* $x^*(t), t \in [0, T]$ *satisfying conditions (i)–(iii).*

*Proof***.** By Theorem 2.1 [18] $F * (v^*, (x, w, v), t) = \partial_{(x,w)} H_F(x, w, v^*)$, $v \in F_A(x, w\,; v^*, t)$. Then, in terms of Hamiltonian function

$$-\left(\frac{dx^*(t)}{dt}, 0\right) \in \partial_{(x,w)}[H_F(\tilde{x}(t), \widetilde{w}, x^*(t))]$$

$$-\partial g(\tilde{x}(t), t) \times \{0\}, \text{a.e. } t \in [0, T]. \tag{37}$$

Next, by Proposition 2.315 [2], using the following subdifferential relation of the Hamiltonian function H_F, we have

$$\partial_{(x,w)}[H_F(\tilde{x}(t), \widetilde{w}, x^*(t))]$$

$$\subseteq \partial_x H_F(\tilde{x}(t), \widetilde{w}, x^*(t)) \times \partial_w H_F(\tilde{x}(t), \widetilde{w}, x^*(t)).$$

Then the relation (37) means that

$$-\frac{dx^*(t)}{dt} \in \partial_x\left[H_F\left(\tilde{x}(t), \widetilde{w}, x^*(t)\right)\right] - \partial g(\tilde{x}(t), t), \text{a.e. } t \in [0, T];$$

$$0 \in \partial_w[H_F(\tilde{x}(t), \widetilde{w}, x^*(t))], \text{a.e. } t \in [0, T]. \tag{38}$$

The first formula (38) using the definition of subdifferential has the form

$$H_F(x(t), \widetilde{w}, x^*(t)) - H_F(\tilde{x}(t), \widetilde{w}, x^*(t)) - g(x(t), t) + g(\tilde{x}(t), t) \tag{39}$$

$$\leq -\left\langle \frac{dx^*(t)}{dt}, x(t) - \tilde{x}(t) \right\rangle, t \in [0, T].$$

Therefore, given the definition, the Hamiltonian inequality (39) can be transformed into a new equality

$$\left\langle \frac{dx(t)}{dt}, x^*(t) \right\rangle - \left\langle \frac{d\tilde{x}(t)}{dt}, x^*(t) \right\rangle - g(x(t), t) + g(\tilde{x}(t), t)$$

$$\leq -\left\langle \frac{dx^*(t)}{dt}, x(t) - \tilde{x}(t) \right\rangle, t \in [0,T]$$

or

$$g(x(t),t) - g(\tilde{x}(t),t) \geq \left\langle \frac{d(x(t)-\tilde{x}(t))}{dt}, x^*(t) \right\rangle + \left\langle \frac{dx^*(t)}{dt}, x(t) - \tilde{x}(t) \right\rangle.$$

Finally

$$g(x(t),t) - g(\tilde{x}(t),t) \geq \frac{d}{dt}\langle x(t) - \tilde{x}(t), x^*(t) \rangle, \ t \in [0,T].$$

Then it can be easily seen that

$$\int_0^T [g(x(t),t) - g(\tilde{x}(t),t)]\, dt \geq \int_0^T d\langle x^*(t), x(t) - \tilde{x}(t) \rangle$$

$$= \langle x^*(T), x(T) - \tilde{x}(T) \rangle - \langle x^*(0), x(0) - \tilde{x}(0) \rangle.$$

Therefore, given that $x(0) = \tilde{x}(0) = \theta$, we obtain

$$\int_0^T [g(x(t),t) - g(\tilde{x}(t),t)]\, dt \geq \langle x^*(T), x(T) - \tilde{x}(T) \rangle. \tag{40}$$

But it is easy to see that, from the transversality condition (ii) of the theorem, we can write

$$f_0(x(T)) - f_0(\tilde{x}(T)) \geq \langle x_e^* - x^*(T), x(T) - \tilde{x}(T) \rangle; \quad \langle x_e^*, x(T) - \tilde{x}(T) \rangle \geq 0$$

for all feasible trajectories $x(\cdot)$ at a point $t = T$. From here, we have

$$f_0(x(T)) - f_0(\tilde{x}(T)) \geq -\langle x^*(T), x(T) - \tilde{x}(T) \rangle. \tag{41}$$

Hence, summing up inequalities (40), (41), we obtain the desired result:

$$\int_0^T [g(x(t),t) - g(\tilde{x}(t),t)]\, dt + f_0(x(T)) - f_0(\tilde{x}(T)) \geq 0$$

or $J[x(\cdot)] \geq J[\tilde{x}(\cdot)]$ for an arbitrary feasible trajectory $x(\cdot)$.

□

Corollary 6.1 *Let $F(\cdot,t) \equiv G(\cdot,t): \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, and the conditions of Theorem 6.1 are satisfied. Then the adjoint DFI and the condition (iii) consist of the following*

(i) $-\frac{dx^*(t)}{dt} \in G * (x^*(t); (\tilde{x}(t), \tilde{x}'(t)), t) - \partial g(\tilde{x}(t),t)$ *a.e.* $t \in [0,T]$,

(ii) $x_e^* - x^*(T) \in \partial f_0(x(T)), x_e^* \in K_P * (\tilde{x}(T))$,

(iii) $\frac{d\tilde{x}(t)}{dt} \in G_A(\tilde{x}(t); x^*(t), t)$, *a.e.* $t \in [0,T]$.

Proof. The proof is elementary and therefore skipped. □

Corollary 6.2. *If F is a closed mapping, then conditions (i)–(iii) of Theorem 6.1 can be rewritten in a more symmetric form.*

Proof. It is enough to prove that

$$\frac{d\tilde{x}(t)}{dt} \in \partial_{v^*} H_F(\tilde{x}(t), \widetilde{w}, x^*(t)), \text{ a.e. } t \in [0, T]. \tag{42}$$

Indeed, by Lemma 3.1, the set argmaximum is a subdifferential $\partial_{v^*} H_F(\tilde{x}(t), \widetilde{w}, x^*(t))$, and inclusion (iii) is nothing other than inclusion (42). Therefore, the assertions of the corollary are equivalent to conditions (i)–(iii) of Theorem 6.1. □

Remark 6.1 *In this remark, we will try to compare the obtained results with Pontryagin's maximum principle [34]. Assume that we have the following classical optimal control problem:*

$$\text{minimie } J[x(\cdot)] = \int_0^T g(x(t), t) dt,$$

$$\frac{dx(t)}{dt} = f(x(t), u(t), w), a.e t \in [0, T],$$

$$x(0) = \theta, x(T) \in P, u = u(t) \in U,$$

where U is a convex compact and $w \in W = \mathbb{R}^m$, w is m-dimensional vector-parameter, $u(t)$ - *from the class of piecewise continuous functions, $g(\cdot, t)$ is continuous, $f(x, u, w)$ is vector-function with coordinates $f^i, i = 1, \dots, n$, which are continuous with respect to the set of variables, and continuously differentiable with respect to x^i, $i = 1, \dots, n$. In other words, the functions $f^i(x, u, w), \frac{\partial f^i(x,u,w)}{x^j}, i, j = 1, \dots, n$ are defined and continuous on the direct product $\mathbb{R}^n \times U \times W$.* Additionally, assume that $\frac{\partial f^i(x,u,w)}{w_j}, i, j = 1, \dots, m$ exist.

In this case,

$$F(x, w, t) \equiv F(x, w) = f(x, U, w), \; H_F(x, w, v^*) = \max_{u \in U} \langle v^*, f(x, u, w) \rangle,$$

$$F_A(x, w\,; v^*) = \{v = f(x, u, w) : \langle v^*, f(x, u, w) \rangle = H_F(x, w, v^*)\}.$$

If the maximum of the scalar product $\langle v^*, f(x, u, w) \rangle$ in the Hamiltonian function $H_F(x, w, v^*)$ for fixed values x, v^* and w is denoted by $M(x, w, v^*)$, that is,

$$M(x, w, v^*) = \max_{u \in U} \langle v^*, f(x, u, w) \rangle,$$

then, under condition (iii) of Theorem 6.1, we obtain that for any $t \in [0, T]$, the function $\langle x^*(t), f(\tilde{x}(t), u, \widetilde{w}) \rangle$ of the variable $u \in U$ reaches its maximum at the point $u = u(t)$:

$$H_F(\tilde{x}(t), \widetilde{w}, x^*(t)) = M(\tilde{x}(t), \widetilde{w}, x^*(t))$$

or in more detail

$$H_F(\tilde{x}(t), \widetilde{w}, x^*(t)) = \sum_{k=1}^n x_k^*(t) f^k(\tilde{x}(t), \tilde{u}(t), \widetilde{w}), \text{ that } x^*(t) = (x_1^*(t), \dots, x_n^*(t)).$$

Thus, since on the definition of LAM, $x^* = f'_x * (x, u, w)v^*$, where the matrix $f'_x *$ is the transpose of the matrix f'_x, then using the first equation (38), the adjoint equation has the form

$$\frac{dx^*(t)}{dt} = -f'_x * (\tilde{x}(t), \tilde{u}(t), \widetilde{w})x^*(t) + \partial g(\tilde{x}(t), t), \text{a.e. } t \in [0, T].$$

On the other hand, by the second relation of (38)

$$0 \in \partial_w\left[H_F\left(\tilde{x}(t), \widetilde{w}, x^*(t)\right)\right], \text{ a.e. } t \in [0, T],$$

where $\partial_w H_F(\tilde{x}(t), \widetilde{w}, x^*(t)) = \{\nabla_w H_F(\tilde{x}(t), \widetilde{w}, x^*(t))\}$, it follows that in a more expanded form

$$\nabla_w H_F(\tilde{x}(t), \tilde{u}(t), \widetilde{w}, x^*(t))$$

$$= \left(\sum_{k=1}^{n} x_k^*(t) \frac{\partial f^k(\tilde{x}(t), \tilde{u}(t), \widetilde{w})}{\partial w_1}, \ldots, \sum_{k=1}^{n} x_k^*(t) \frac{\partial f^k(\tilde{x}(t), \tilde{u}(t), \widetilde{w})}{\partial w_m}\right).$$

Now, since $\nabla_w H_F(\tilde{x}(t), \tilde{u}(t), \widetilde{w}, x^*(t))$ is the only subgradient of $\partial_w H_F(\tilde{x}(t), \widetilde{w}, x^*(t))$, it follows that $\nabla_w H_F(\tilde{x}(t), \tilde{u}(t), \widetilde{w}, x^*(t)) = 0$ or it is easy to see that, equivalently

$$\sum_{k=1}^{n} x_k^*(t) \frac{\partial f^k(\tilde{x}(t), \tilde{u}(t), \widetilde{w})}{\partial w_j} = 0, j = 1, 2, \ldots, m.$$

Theorem 6.2 *Let* $g : \mathbb{R}^n \times [0, T] \to \mathbb{R}^1$ *and* $f(\cdot)$ *be non-convex functions with respect to x, and let* F *be a non-convex mapping, and* $K_{F(\cdot,t)}(\tilde{x}(t), \widetilde{w}, \tilde{x}'(t))$, $K_P(\tilde{x}(T)), \tilde{x}(T) \in P$ *be local tents. Then for the trajectory* $\tilde{x}(t), t \in [0, T]$ *to be optimal, it is sufficient that there exists an absolutely continuous function* $x^*(t), t \in [0, T]$ *satisfying the conditions of Theorem 6.1 in this case:*

(i) $-\left(\frac{dx^*(t)}{dt} - x^*(t), 0\right) \in F * (x^*(t); (\tilde{x}(t), \widetilde{w}, \tilde{x}'(t)), t), \text{a.e.} t \in [0, T],$

(ii) $\frac{d\tilde{x}(t)}{dt} \in F_A(\tilde{x}(t), \widetilde{w}; x^*(t), t)$, *a.e.* $t \in [0, T]$,

(iii) $g(x, t) - g(\tilde{x}(t), t) \geq \langle x^*(t), x - \tilde{x}(t)\rangle, t \in [0, T]$, $\forall x \in \mathbb{R}^n$,

(iv) $f_0(x) - f_0(\tilde{x}(T)) \geq -\langle x^*(T), x - \tilde{x}(T)\rangle, x_e^* \in K_P * (\tilde{x}(T))$.

***Proof*.** By condition (i) and the definition of LAM, we can write

$$\left\langle \frac{dx(t)}{dt}, x^*(t) \right\rangle - \left\langle \frac{d\tilde{x}(t)}{dt}, x^*(t) \right\rangle$$

$$\leq -\left\langle \frac{dx^*(t)}{dt}, x(t) - \tilde{x}(t) \right\rangle + \langle x^*(t), x(t) - \tilde{x}(t)\rangle, t \in [0, T],$$

from where, according to condition (i), for $x = x(t)$, we obtain

$$g(x(t), t) - g(\tilde{x}(t), t) \geq \left\langle \frac{d(x(t) - \tilde{x}(t))}{dt}, x^*(t) \right\rangle + \left\langle \frac{dx^*(t)}{dt}, x(t) - \tilde{x}(t) \right\rangle. \quad (43)$$

The proof is a direct consequence of the last inequality (43). □

Clearly, for a convex problem, condition (iii) of Theorem 6.2 is equivalent to condition $x^*(t) \in \partial_x g(\tilde{x}(t), t)$ together with (iii) of Theorem 6.1. Furthermore, it is straightforward to verify that conditions (i) and (ii) of Theorem 6.1 correspond exactly to conditions (i) and (ii) of Theorem 6.2, respectively. To conclude, let us examine the following example of optimal control:

$$\text{minimize } J[x(\cdot)] = \int_0^T g(x(t), t)dt,$$

(PL) $$x'(t) = A_0 x(t) + A_1 w + Bu(t), \; a.e. \; t \in [0, T],$$

$$x(0) = \theta, x(T) \in P,$$

where g is continuously differentiable with respect to x, A_i, $i = 0,1$ and B are matrices of dimensions $n \times n$ and $n \times r$, respectively, and $U \subseteq \mathbb{R}^r$ is a convex compact. It is required to find the trajectory $\tilde{x}(t)$ corresponding to the control function $\tilde{u}(t) \in U$.

Let us rewrite this problem as follows:

$$\text{minimize } J[x(\cdot)] = \int_0^T g(x(t), t)dt,$$

$$x'(t) \in F(x(t), w), \; a.e. \; t \in [0, T], \tag{44}$$

$$x(0) = \theta, x(T) \in P,$$

$$F(x, w) = A_0 x + A_1 w + BU,$$

where an admissible arc $x(\cdot)$ is an absolutely continuous function, together with the first-order derivatives for which $x^{*\prime}(\cdot) \in L_1^n([0, T])$.

Theorem 6.3 *An arc* $\tilde{x}(t)$ *corresponding to a control function* $\tilde{u}(t)$ *and a parameter* $\tilde{w}$ *minimizes* $J[x(\cdot)]$ *in a linear problem (PL) with a parameter if there exists a function* $x^*(t)$ *satisfying the adjoint differential equation, transversality and Pontryagin conditions with respect to the control function* $\tilde{u}(t)$ *and the parameter* $\tilde{w}$*:*

$$\frac{dx^*(t)}{dt} = -A_0 * x^*(t) + g'(\tilde{x}(t), t), \text{ a.e. } t \in [0, T];$$

$$x^*(T) \in K *_P \left(\tilde{x}(T)\right); \; A_1 * x^*(t) = 0,$$

$$\langle B\tilde{u}(t), x^*(t) \rangle = \max_{u \in U} \langle Bu, x^*(t) \rangle, t \in [0, T].$$

*Proof***.** Considering that $F(x, w) \equiv A_0 x + A_1 w + BU$ for the problem (44), we calculate that

$$H_F(x, w, v^*) = \sup_v \{\langle v, v^* \rangle : v \in F(x, w)\}$$

$$= \sup_v \{\langle A_0 x + A_1 w + Bu, v^* \rangle : v \in F(x, w)\}$$

$$= \langle x, A_0 * v^* \rangle + \langle w, A_1 * v^* \rangle + \sup_u \{\langle Bu, v^* \rangle : u \in U\},$$

where $A *$ is the transpose of matrix A. Furthermore, by Theorem 2.1 [18], we have

$$F * (v^*; (\tilde{x}, \widetilde{w}, \tilde{v})) = \begin{cases} (A_0 * v^*, A_1 * v^*), & -B * v^* \in K_U * (\tilde{u}), \\ \emptyset, & -B * v^* \notin K_U * (\tilde{u}), \end{cases} \quad (45)$$

where $\tilde{v} = A_0 \tilde{x} + A_1 \widetilde{w} + B\tilde{u}, \tilde{u} \in U$. Hence, applying (45) and relations (i), (ii) of Theorem 6.1, we obtain the linear adjoint equation

$$-\frac{dx^*(t)}{dt} = A_0 * x^*(t) - g'(\tilde{x}(t), t), \text{a.e.} t \in [0, T]. \quad (46)$$

Since $f_0(\cdot) \equiv 0$ and so $x_e^* = x^*(T)$, transversality conditions (ii) of Theorem 6.1 are as follows:

$$x^*(T) \in K_P * (\tilde{x}(T)). \quad (47)$$

Moreover, the Pontryagin maximum principle of theorem is an immediate consequence of the conditions (iii) of Theorem 6.1 and formula (45). Indeed, the condition $-B * v^* \in K_U * (\tilde{u})$ means that $\sup_{u \in U} \langle Bu, v^* \rangle = \langle B\tilde{u}, v^* \rangle$, and finally,

$$\langle B\tilde{u}(t), x^*(t) \rangle = \sup_{u \in U} \langle Bu, x^*(t) \rangle, t \in [0, T].$$

Let us interpret the condition $0 \in \partial_w [H_F(\tilde{x}(t), \widetilde{w}, x^*(t))]$. From the form of the Hamiltonian function, it is clear that its subdifferential contains a single subgradient and therefore $\nabla_w H_F(x, u, w, v^*) = A_1 * v^* = 0$ or $\nabla_w H_F(\tilde{x}(t), \tilde{u}(t), \widetilde{w}, x^*(t)) = 0$ and $A_1 * x^*(t) = 0$.

Then, by this maximum principle and relations (46), (47), we have the desired result. In this case, the condition regarding the parameter w has the following form (see the second relation (38)) $0 \in \partial_w \left[H_F\left(\tilde{x}(t), \widetilde{w}, x^*(t)\right)\right]$, a.e. $t \in [0, T]$. The proof is completed. ☐

Remark 6.2. In general, problems with parameters are widely used in optimization problems in mathematics. The paper considers the optimization of the first-order DFI with a parameter. To obtain a sufficient optimality condition in the form of Euler–Lagrange, a discretized method is used. To transition from the optimality condition of a discrete problem to the optimality of a continuous problem, the key idea is the results on the equivalence of LAM of these problems. Thus, it is concluded that the proposed method is reliable for solving the various optimization problems with "higher-order" DFIs and parameters. At the same time, one might superficially think that by replacing variables, one can always move to a first-order system. However, this operation can cause side effects; performing a classical transformation to first-order form can destroy some properties of controllability and observability. As an example, recall the time-optimal control problem expressing Newton's second law or "The fastest train stop at a station", described by the equation $x'' = u$, $|u| \leq 1$. This problem, due to the lack of "higher-order optimization" with the help of additional variables, is reduced to a system of equations, the reason for which is that the Pontryagin maximum principle is valid only for first-order controllable systems. Moreover, this example shows that it is advisable to study the optimal control problem with a second/higher-order differential equation without reducing it to a system of first-order differential equations. Thus, unlike the traditional method developed to solve such a problem, it makes sense to conduct a study not in the phase space $x^1 0 x^2$ but in the plane xt; in both cases, the member of the family of solutions consists of pieces of two parabolas. In this case, since this equation is of even order, then without changing the sign, the adjoint equation in Pontryagin's terms has the form $\psi'' = 0$. Moreover, the main results of this work can be generalized to the case of partial differential inclusions. □

Author contributions

E.N.M.: Conceptualization; Formal analysis; Writing – original draft; Y.Sh.Sh.: Resources; Formal analysis; Supervision; Writing – review & editing; D.I.M.: Data curation; Visualization; Writing – review & editing.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors wish to express their sincere thanks to Assistant Editor, Yan Qi of the Journal of Industrial and Management Optimization and anonymous reviewers for their valuable suggestions that improved the final manuscript.

Conflict of interest

No conflict interest.

References

1. P. Cannarsa, H. Frankowska, T. Scarinci, Sensitivity relations for the Mayer problem with differential inclusions, *ESAIM: COCV*, **21** (2015), 789–814. https://doi.org/10.1051/cocv/2014050
2. F. H. Clarke, *Optimization and non smooth analysis*, John Wiley and Sons Inc. 1983.
3. A. D. Ioffe, V. Tikhomirov, *Theory of extremal problems*, "Nauka", Moscow, 1974; English transl., North-Holland, Amsterdam, 1978.
4. P. D. Loewen, R. T. Rockafellar, Optimal control of unbounded differential inclusions, *SIAM J. Control Optim.,* **32** (1994), 442–470. https://doi.org/10.1137/S0363012991217494
5. E. N. Mahmudov, D. I. Mastaliyeva, Optimal control of second order hereditary functional-differential inclusions with state constraints, *J. Indus. Man. Optim.*, **20** (2024), 3562–3579. https://doi.org/10.3934/jimo.2024065
6. E. N. Mahmudov, Optimal control of Sturm-Liouville type evolution differential inclusions with endpoint constraints, *J. Indus. Man. Optim.*, **16** (2020), 2503–2520. https://doi.org/10.3934/jimo.2019066

7. E. N. Mahmudov, Infimal convolution and duality in problems with third-order discrete and differential inclusions, *J. Optim. Theory Appl.*, **184** (2020), 781–809. https://doi.org/10.1007/s10957-019-01630-8
8. E. N. Mahmudov, Optimal control of higher order differential inclusions with functional constraints, *ESAIM: COCV*, **26** (2020), 1–23. https://doi.org/10.1051/cocv/2019018
9. E. N. Mahmudov, Optimal Control of Evolution Differential Inclusions with Polynomial Linear Differential Operators, *Evol. Equ. Contr. Theory*, **8** (2019), 603–619. https://doi.org/10.5772/intechopen.90888
10. L. Marco, J. A. Murillo, Lyapunov functions for second order differential inclusions: A viability approach, *J. Math. Anal. Appl.,* **262** (2001), 339–354. https://doi.org/10.1006/jmaa.2001.7583
11. B. S. Mordukhovich, D. Wang, Optimal Control of Semilinear Unbounded Evolution Inclusions with Functional Constraints, *J. Optim Theory Appl.*, **167** (2015), 821–841. https://doi.org/10.1007/s10957-013-0301-0
12. M. Raap, M. Zsifkovits, S. W. Pickl, Trajectory optimization under kinematical constraints for moving target search, *Comput. Oper. Res.*, **88** (2017), 324–331. https://doi.org/10.1016/j.cor.2016.12.016
13. S. D. Sağlam, E. N. Mahmudov, The Lagrange problem for differential inclusions with boundary value conditions and duality, *Pacific J. Optim.*, **17** (2021), 209–225.
14. Y. Zhu, L. Rao, Differential inclusions for fuzzy maps, *Fuzzy Sets Syst.*, **112** (2000), 257–261. https://doi.org/10.1016/S0165-0114(98)00077-3
15. N. U. Ahmed, K. L. Teo, *Optimal Control of Distributed Parameter Systems*, New York: Elsevier North Holland, 1981.
16. K. Aida-zade, Y. Ashrafova, Determination of the parameters of dynamic objects, arbitrarily related by boundary conditions, *J. Indus. Man. Optim.*, **20** (2024), 2477–2499. https://doi.org/10.3934/jimo.2024010
17. E. N. Mahmudov, Approximation and optimization of Darboux type differential inclusions with set-valued boundary conditions, *Optim. Letters*, **7** (2013), 871–891. https://doi.org/10.1007/s11590-012-0460-1
18. E. N. Mahmudov, *Approximation and Optimization of Discrete and Differential Inclusions*, Elsevier, Boston, USA 2011.
19. E. N. Mahmudov, Optimal control of hyperbolic type discrete and differential inclusions described by the Laplace operator, *ESAIM: COCV*, **28** (2022), 1–21. https://doi.org/10.1051/cocv/2022061
20. S. S. Yusubov, E. N. Mahmudov, Some necessary optimality conditions for systems with fractional Caputo derivatives, *J. Indust. Manag. Optim.*, **19** (2023), 8831–8850. https://doi.org/10.3934/jimo.2023063
21. S. Adly, A. Hantoute, M. Théra, Nonsmooth Lyapunov pairs for infinite dimensional first-order differential inclusions, *Nonlin. Anal.: Theory, Methods Appl.*, **75** (2012), 985–1008. https://doi.org/10.1016/j.na.2010.11.009
22. S. Adly, A. Hantoute, M. Théra, Local nonsmooth Lyapunov pairs for first-order evolution differential inclusions, arXiv preprint arXiv:1305.3737(2013). https://doi.org/10.48550/arXiv.1305.3737
23. N. U. Ahmed, X. Xiang, Optimal relaxed controls for differential inclusions on Banach spaces, *Dynam. Syst. Appl.*, **12** (2003), 235–250.

24. N. U. Ahmed, X. Xiang, Differential inclusions on Banach spaces and their optimal control, *Nonlin. Funct. Anal. Appl.*, **8** (2003), 461–488.
25. N. U. Ahmed, Existence of Optimal Relaxed Controls for Differential Inclusions on Banach Space, In V. Lakshmikantham, edited, *Nonlinear Analysis and Applications*, Marcel Dekker, Lecture Notes in Pure and Applied Mathematics, Vol. 109, Marcel Dekker Inc., New York, 1987.
26. D. L. Azzam, A. Makhlouf, L. Thibault, Existence and relaxation theorem for a second order differential inclusion, *Numer. Funct. Anal. Optim.*, **31** (2010), 1103–1119. https://doi.org/10.1080/01630563.2010.510982
27. J. P. Aubin, A. Cellina, *Differential Inclusions*, Springer, Berlin, Heidelberg, 1984. https://doi.org/10.1007/978-3-642-69512-4
28. A. Auslender, J. Mechler, Second order viability problems for differential inclusions, *J. Math. Anal. Appl.,* **181** (1994), 205–218. https://doi.org/10.1006/jmaa.1994.1015
29. S. S. Yusubov, E. N. Mahmudov, Optimality conditions of singular controls for systems with Caputo fractional derivatives, *J. Indust. Manag. Optim.*, **19** (2023), 246-264. https://doi.org/10.3934/jimo.2021182
30. Q. Liqun, K. L. Teo, X. Yang, *Optimization and Control with Applications*, Springer, 2005.
31. X. Zhang, Y. Kong, S. Liu, Y. Shen, A relaxed parameter condition for the primal-dual hybrid gradient method for saddle-point problem, *J. Indus. Man. Optim.*, **19** (2023), 1595–1610. https://doi.org/10.3934/jimo.2022008
32. Q. Yan, W. Chen, X. Yang, A novel one-parameter filled function method with an application to pathological analysis, *Optim. Lett.*, **18** (2023), 1–22. https://doi.org/10.1007/s11590-023-02010-y
33. E. N. Mahmudov, D. I. Mastaliyeva, Optimization of the Dirichlet problem for gradient differential inclusions, *Nonlin. Diff. Equ. Appl.*, **31** (2024), 1–18. https://doi.org/10.1007/s00030-023-00904-5
34. L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, E. F. Mishchenko, *The mathematical theory of optimal processes*. New York, London, Sydney: John Wiley Sons, Inc., 1965.