Journal of Industrial and Management Optimization

JIMO, 22(1): 148−181.
DOI: 10.3934/jimo.2026006
Received: 10 July 2025
Revised: 17 October 2025
Accepted: 23 October 2025
Published: 17 November 2025

https://www.aimspress.com/journal/jimo

Research article

Optimal control of persuasive communication for emergency management

Wanglai Li, Hanzhe Yang, Huizhang Shen* and Zhangxue Huang

Antai College of Economics and Management, Shanghai Jiao Tong University, Shanghai 200030, China

* **Correspondence:** Email: hzshen@sjtu.edu.cn; Tel: 86-13512171286.

Abstract: Following emergencies, related information rapidly spreads online, and persuasive communication is essential for reducing negative effects. Based on the elaboration likelihood model (ELM) and observation of real-world cases, there are several heterogeneous cognitive routes among netizens, each with distinct information-processing mechanisms and associated persuasion costs. Thus, the selection of persuasion strategies can be formulated as a constrained optimization problem. To the best of our knowledge, this study is the first to integrate ELM-based cognitive heterogeneity into opinion dynamics modeling and to design the optimal persuasive communication strategy through dynamic optimal control. It was found that the effectiveness of persuasive communication depends on how well the strategy aligns with the major cognitive routes of netizens. Under more complex and dynamic settings, the optimal control solutions indicate that persuasion resource allocation should be adjusted according to the number of netizens and the route-specific persuasion costs. By integrating psychological insights with dynamic modeling and optimal control, this study improves the theoretical rigor and practical applicability of public opinion persuasion.

Keywords: persuasive communication; cognitive routes; opinion dynamics model; dynamic optimal control
Mathematics Subject Classification: 37N35, 37N40

1. Introduction

The Internet has become an indispensable part of daily life and a major channel through which people access information [1,2], which also influences emergency management. An emergency is an urgent, unexpected situation that entails high uncertainty and time pressure, and either threatens health or life or causes widespread negative sentiments, such as anger and panic. Following an emergency, related information spreads online, and netizens express their sentiments about it. For example, panic and worries may spread during natural disasters [3], thus shaping online public opinion.

Dealing with public opinion is critical during emergency management. Without proper crisis communication, public opinion may propagate in a cascading manner, and its influence may increase [4]. Persuasive communication refers to communication activities aimed at changing the views, attitudes, and behaviors of the public [5,6]. For example, after an issue with a product happens, firms may adopt specific persuasion strategies to safeguard their reputation and avoid negative effects. Such strategies typically involve issuing official statements to persuade the public to view the organization more favorably.

The selection of persuasion strategies can be formulated as an optimization problem. The objective is to persuade more netizens while satisfying multiple constraints. According to the elaboration likelihood model (ELM), cognitive routes to process information can be divided into central and peripheral routes. Within public opinion scenarios, different netizens process official responses through these two cognitive routes. When the central route is activated, the corresponding netizens concentrate on essential details, such as causation, remedial actions, and accountability issues. Conversely, netizens under the peripheral route pay more attention to source credibility, leaders' opinions, and other superficial cues. Moreover, persuasion costs vary across the different cognitive routes. For example, conducting a detailed investigation may require additional resources when aiming to persuade netizens through the central route. Thus, the constraints of persuasion are particularly relevant to the cognitive routes employed by netizens.

In an emergency, organizations must pay close attention to the major cognitive routes of netizens and adopt appropriate official responses, thereby minimizing the negative effects of public opinion crisis. An inappropriate response not only wastes resources but also provokes widespread resistance among netizens. Taking the case of BMW MINI's differential treatment of customers at the 2023 Shanghai Auto Show as an example, the company's public apology used inappropriate language and failed to defuse public outrage, intensifying anger toward the brand.

How to more optimally conduct public opinion persuasion efforts based on different cognitive routes constitutes the primary management optimization problem. Different types of events can trigger different cognitive routes among netizens, and the effectiveness and required cost of persuasion strategies may vary significantly depending on the cognitive routes.

The prevailing approach to modeling public opinion is the disease model or epidemic model. Integrating disease dynamics with psychology is an advanced direction in public opinion research [7]. However, existing studies largely ignore the cognitive routes' heterogeneity during mathematical modeling and optimal control. Therefore, this paper develops an opinion dynamics model that captures the heterogeneity of netizens' cognitive routes by categorizing them into central and peripheral routes. Furthermore, based on the dynamics analysis and optimal control method, this paper derives time-dependent, adaptive resource allocation strategies for improving the persuasion effect. The findings offer both theoretical insights and practical guidelines for crisis communication.

To the best of our knowledge, the core innovations of this paper are as follows: (1) ELM-based cognitive heterogeneity is first integrated into opinion dynamics modeling, whereby psychological differences among netizens are captured; (2) enhanced understanding of persuasion mechanisms is developed from a time-varying dynamic resource allocation perspective; and (3) persuasive communication is formulated as a novel optimal control problem where distribution of netizens and cost benefit trade-offs across different persuasion strategies are considered.

The structure of this paper is as follows: Section 2 analyzes recent literature and clearly establishes how this paper addresses existing research gaps. Section 3 constructs an opinion dynamics model based on ELM. Section 4 examines the dynamics property of the model, in which the control variables are fixed (corresponding to strategies that are predetermined, remaining unchanged throughout) and where strategy costs are disregarded. Section 5 addresses cases where the control variables can vary over time, reflecting the practical need for persuasion strategies to be dynamically adjusted and informed by real-time decision-making. Section 6 relaxes several simplifying assumptions of the baseline model and extends the framework to explore whether there are new results. Section 7 provides management implications. Finally, Section 8 summarizes the main conclusions, limitations, and directions for future research.

2. Literature review

This section systematically reviews existing research to identify key limitations and position our contributions. Considerable work has been done on opinion dynamics and persuasive communication across fields such as communication studies, information science, and management optimization [8]. We structure this review around two complementary streams: mathematical modeling of opinion dynamics and cognitive psychological mechanisms of persuasion.

2.1. Mathematical modeling of opinion dynamics research

Effective persuasive communication first requires understanding the dynamics of public opinion.

The epidemic model is the model traditionally used to study the opinion dynamics of information spread [9], with its roots tracing back to the foundational studies of Daley and Kendall [10,11]. Variants of the epidemic model have been widely used. Soodeh Hosseini et al. [12] studied rumor-spreading through a variant of the susceptible–exposed–infectious–recovered (SEIR) epidemic model. Dong et al. [13] constructed an improved two-layer model considering time delay to describe the dynamic process of rumor propagation. Gürbüz et al. [14] investigated the dynamics of a rumor-spreading model derived from an improved SIR framework. Liu et al. [15] proposed a new propagation model SILIHS, to reveal the impact of low-order and higher-order interactions on propagation dynamics. Word-of-mouth (WOM) is a special type of public opinion, and a similar model has been used to simulate its dynamics. For example, Qiao et al. [16] proposed a dynamic model to analyze the influence of access to two information channels on consumers' purchase choices. Mostafa et al. [17] constructed a framework using the susceptible–infected–susceptible model to demonstrate the spread of WOM and derive demand functions.

Because these models can reflect real-world dynamics, some scholars use them to simulate and analyze the persuasion effects of different strategies. Some scholars have also researched how to achieve optimal control based on the epidemic model [18]. Ge et al. [19] introduced a strategy for optimal control to the SI1I2R public opinion propagation model. Sara Bidah et al. [20] investigated optimal control of a new mathematical model that describes agree–disagree opinions during polls. Cheng et al. [21] designed the optimal control problem under pulse vaccination to minimize rumor spreading scale and control cost. Kang et al. [22] constructed a model with transmitter authority and individual acceptance levels and designed an optimal control strategy to facilitate information transmission. Li et al. [23] proposed an uncertain information dissemination model triggered after major emergencies and found the optimal control strategy using the Hamiltonian function. Dong et al. [24] developed an opinion dynamics model with social enhancement, forgetting, and cross-transmission mechanisms, and used it to study how to manage public opinion while promoting fairness and transparency. Zhang et al. [25] proposed an optimal timing selection problem, in which a leader maximizes public opinion at a specific time by strategically selecting intervention times. Luo et al. [26] revealed that controlling propagation speed is more effective in curbing information dissemination than controlling the propagation scope. Nian et al. [27] found that the spreading of one information fragment is influenced by others and proposed means to suppress rumors.

However, there remains a lack of research that comprehensively considers the heterogeneity of netizens' cognitive routes. The BMW MINI's case mentioned in the Introduction illustrates the necessity of capturing the dynamic influence of different persuasion strategies on the cognitive processes of netizens within a modeling framework.

2.2. Cognitive psychological mechanisms of persuasion research

Cognitive psychological mechanisms play an important role in persuasion. The elaboration likelihood model (ELM) explains the dual cognitive routes of people when processing persuasive information [28].

Syrdal et al. [29] employed the ELM to analyze the relationship between the linguistic style of a persuasive post on Instagram and user engagement. Saini et al. [30] found that the spread of vaccination messages was influenced mainly by peripheral route factors, whereas the diffusion of anti-vaccination messages was shaped by both peripheral and central route factors. In addition, anti-vaccination content that is emotionally contagious and objectively concrete persuades netizens more readily and is therefore propagated more widely. Yang et al. [31] found that both central and peripheral route factors influence users' sharing decisions on social networking platforms. Xu et al. [32] studied how sender, signal, feedback, and environment factors can be integrated with ELM mechanisms in affecting consumers' information processing and decision making. Zhang et al. [33] evaluated the impact of review text quality and aesthetic quality of review photos on perceived usefulness. Wu et al. [34] found that the peripheral route of persuasive communication can quickly establish communication trust and change the public's peripheral attitude, but the persuasive effect is unstable. Yang et al. [35] used the ELM to find that the quality, quantity, and source credibility of online reviews, as well as consumers' attitudes toward online reviews, directly influence purchase intentions for energy-saving appliances.

These works used the ELM to examine determinants of public opinion persuasion. However, these models fall short of reflecting the dynamics inherent to public opinion, thereby constraining their practical utility in real-world persuasion efforts.

2.3. Research gap and novelty

This subsection identifies key research gaps and establishes how our study addresses these limitations.

Current literature on control methods has primarily focused on targeting influential individuals or other influencing factors such as transmission rate. However, existing models generally fail to account for cognitive heterogeneity across individuals and do not integrate psychological theories with dynamic optimal control frameworks. Overall, limited attention has been given to adaptive, time-varying, psychology-informed persuasion strategies.

To address these limitations, this study systematically integrates psychological insights with dynamic modeling and optimal control methods. Our approach can enhance both the theoretical rigor and practical effectiveness of public opinion persuasion strategies.

3. Dynamical model construction based on ELM

According to the ELM and observation of public opinion in emergencies, netizens process official responses through central and peripheral routes, forming corresponding attitudes and viewpoints. These attitudes and viewpoints are reflected in whether they are persuaded by the official responses.

This study divides netizens into five categories:

1) **Uninformed individuals** (denoted as U): Netizens who are not yet aware of the emergency.

2) **Informed individual under central route** (denoted as I_1): Netizens who are aware of the emergency and tend to process and analyze official responses via the central route.

3) **Informed individual under peripheral route** (denoted as I_2): Netizens who are aware of the emergency and tend to process and analyze official responses via the peripheral route.

4) **Persuaded individuals** (denoted as P): Netizens who have been persuaded by the official response.

5) **Unpersuaded individuals** (denoted as NP): Netizens who have not been persuaded by the official response.

To describe the dynamic evolution of public opinion following emergencies, this study constructs a model that follows the following dynamic process. It can be regarded as a variation of the typical epidemic model.

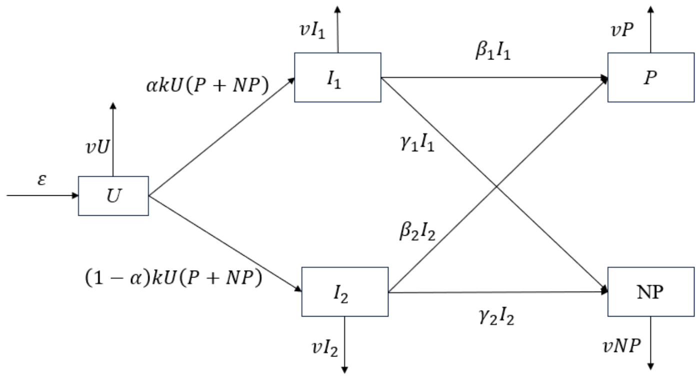

Figure 1. Information propagation process.

The process of opinion dissemination and evolution at each time t can be summarized as follows:

(1) Continuous influx and exit of participants

On social networks, new participants continuously join discussions on various topics. In this study, this is reflected by an increase in the number of uninformed individuals. We denote the inflow rate of unknown individuals as ε. Meanwhile, a portion of netizens may choose to exit the discussion before becoming aware of the emergency. The number of individuals exiting at each time t is represented as vU, where v is the exit rate.

(2) Route selection after exposure to information

From a psychological perspective, once uninformed individuals receive information, they tend to adopt different cognitive routes. Based on ELM, the cognitive routes can be divided into central and peripheral routes. When uninformed individuals possess both cognitive ability and cognitive motivation, they tend to activate the central route, forming their opinions and attitudes through careful thought of a message's core content. In contrast, when cognitive ability or motivation is lacking, they tend to activate the peripheral route, forming their opinions and attitudes by peripheral cues such as source credibility and external evaluations [36]. Computing the tendency to activate the corresponding cognitive route is widely employed in psychological research [37].

Therefore, this study assumes that uninformed individuals will choose the central route with probability α and the peripheral route with probability $1-\alpha$ when processing persuasive information in official responses. The value of α is influenced by various factors, including information reserves, analytical ability, and the nature of the emergency.

According to these assumptions, at each time t, $\alpha kU(P+NP)$ uninformed individuals will transfer to the central-route analyzer group, and $(1-\alpha)kU(P+NP)$ will transfer to the peripheral-route analyzer group, where k is the information propagation rate. Besides, one important term in the expression is $(P+NP)$. Here, P and NP denote the numbers of individuals who have formed and expressed opinions online. The number of $(P+NP)$ can capture the overall scale of public opinion. Recent researches indicate that the scale affects information propagation; in other words, a larger scale of public opinion means more people express views and lead to more potential exposures [38,39]. This structure is also widely adopted in information diffusion models [40].

(3) Attitude formation via different cognitive routes

Netizens following different cognitive routes will analyze the persuasive information in official responses and form corresponding attitudes, either being persuaded or not persuaded. For the central route, individuals are persuaded with probability β_1 and not persuaded with probability γ_1, resulting in $\beta_1 I_1$ central route analyzers becoming persuaded individuals, and $\gamma_1 I_1$ becoming unpersuaded individuals at each time t. For the peripheral route, individuals are persuaded with probability β_2 and not persuaded with probability γ_2, resulting in $\beta_2 I_2$ peripheral route analyzers becoming persuaded individuals, and $\gamma_2 I_2$ becoming unpersuaded individuals at each time t. Meanwhile, some netizens may choose to exit the discussion after becoming aware of the emergency. The number of such individuals at each time t is vI_1 for the central route and vI_2 for the peripheral route, where v is the exit rate.

(4) Exit of netizens who have formed attitudes

Netizens who have already formed their attitudes and viewpoints will eventually exit the discussion. At each time t, the numbers of these individuals are vNP for the unpersuaded group and vP for the persuaded group, where v is the exit rate.

Overall, the main parameters used in the model are presented in Table 1.

It is assumed that $\beta_1+\gamma_1+v=\beta_2+\gamma_2+v=1$. This implies that for individuals in I_1 and I_2, their state is not permanent. In other words, each person must decide to be persuaded, or remain unpersuaded, or exit the system. For simplicity, γ_1, γ_2 can be substituted by $1-\beta_1-v$ and $1-\beta_2-v$.

Table 1. Main parameters.

Parameter	Parameter definition
ε	The inflow rate of unknown individuals
k	The information propagation rate
α	The tendency to activate the central route
β_1	The probability of individuals being persuaded under the central route
β_2	The probability of individuals being persuaded under the peripheral route
γ_1	The probability of individuals not being persuaded under the central route
γ_2	The probability of individuals not being persuaded under the peripheral route
v	The exit rate

Based on the above rules, the dynamical system (1) constructed in this study is as follows:

$$\begin{cases} \dot{U} = \varepsilon - kU(P+NP) - vU \\ \dot{I_1} = \alpha kU(P+NP) - \beta_1 I_1 - (1-\beta_1-v)I_1 - vI_1 = \alpha kU(P+NP) - I_1 \\ \dot{I_2} = (1-\alpha)kU(P+NP) - \beta_2 I_2 - (1-\beta_2-v)I_2 - vI_2 = (1-\alpha)kU(P+NP) - I_2 \\ \dot{P} = \beta_1 I_1 + \beta_2 I_2 - vP \\ \dot{NP} = (1-\beta_1-v)I_1 + (1-\beta_2-v)I_2 - vNP \end{cases} \tag{1}$$

4. Dynamics of public opinion persuasion under fixed strategies

In this section, the equilibrium behavior of dynamical system (1) is analyzed. The results can also reveal the dynamics of public opinion persuasion under fixed strategies, including the evolution process and long-term stability properties. These are applicable to real-world situations in which persuasion strategies are predetermined or remain unalterable once implemented. At these situations, the values of control variables, such as β_1 and β_2, are set as fixed constants.

4.1. Equilibrium points and stability analysis

The equilibrium points of the dynamical system (1) are solved. When the public opinion subsides and an equilibrium point is reached, there are no netizen groups discussing the emergency topic, that is, ${I_1}^* = {I_2}^* = P^* = NP^* = 0$. Regardless of parameter values, system (1) always has a steady-state equilibrium point $Q_0 = \left(\frac{\varepsilon}{v}, 0,0,0,0\right)$.

In addition, another equilibrium point is denoted as $Q^* = (U^*, {I_1}^*, {I_2}^*, P^*, NP^*)$, where ${I_1}^*, {I_2}^*, P^*, NP^* > 0$. To determine the equilibrium point Q^*, the following equations are constructed:

$$\begin{cases} \varepsilon - kU^*(P^* + NP^*) - vU^* = 0 \\ \alpha kU^*(P^* + NP^*) - {I_1}^* = 0 \\ (1-\alpha)kU^*(P^* + NP^*) - {I_2}^* = 0 \\ \beta_1 {I_1}^* + \beta_2 {I_2}^* - vP^* = 0 \\ (1-\beta_1-v){I_1}^* + (1-\beta_2-v){I_2}^* - vNP^* = 0 \end{cases} \tag{2}$$

By solving the above equations, another equilibrium point Q^* of system (1) is shown below:

$$
\begin{cases}
U^* = \frac{v}{k(1-v)} \\
{I_1}^* = \alpha(\varepsilon - vU^*) \\
{I_2}^* = (1-\alpha)(\varepsilon - vU^*) \\
P^* = \frac{(\varepsilon - vU^*)}{v}[\alpha\beta_1 + (1-\alpha)\beta_2] \\
NP^* = \frac{(\varepsilon - vU^*)}{v}[\alpha(1-\beta_1-v) + (1-\alpha)(1-\beta_2-v)]
\end{cases}
\tag{3}
$$

The basic reproduction number R_0 is key in system dynamics. In epidemiology, R_0 is the average number of infected humans caused by one infected human in the course of the infectious period [41]. In the context of information propagation, R_0 is used to indicate the average number of uninformed netizens to whom a netizen, already informed about a piece of information, can pass such information on. Using the next-generation matrix method, this study derives the basic reproduction number for the dynamic system (1): $R_0 = \frac{k\varepsilon(1-v)}{v^2}$.

Then, some mathematical analysis yields the following propositions about the stability of the equilibrium points.

Proposition 1. When $R_0 < 1$, the equilibrium point Q_0 of system (1) is locally asymptotically stable.

The proof of Proposition 1 is presented in Appendix A.1. This equilibrium point can be regarded as the public opinion subsides. No further persuasion efforts are required.

Proposition 2. When $R_0 > 1$, the equilibrium point Q^* of system (1) is locally asymptotically stable.

The proof of Proposition 2 is presented in Appendix A.2. This equilibrium point can be regarded as the situation in which public opinion outbreaks. Therefore, we examine the effects of various persuasion strategies when $R_0 > 1$.

Effective persuasion is reflected in a greater number of persuaded individuals compared to unpersuaded ones. According to the expression for the equilibrium points under $R_0 > 1$, the size relationship between P^* (the number of persuaded individuals) and NP^* (the number of unpersuaded individuals) is given by equation (4).

$$
\Delta = P^* - NP^* = \frac{(\varepsilon - vU^*)}{v}[2\alpha\beta_1 + 2(1-\alpha)\beta_2 + v - 1] \tag{4}
$$

Proposition 3. Under $R_0 > 1$, the size relationship between P^* and NP^* depends on the values of the parameters α, β_1, β_2.

The proof of Proposition 3 is presented in Appendix A.3. β_1 denotes the probability of individuals being persuaded under the central route, and β_2 denotes the probability of individuals being persuaded under the peripheral route. Based on the values of β_1 and β_2, there are several different strategies: 1) persuasion strategies targeting the central route; 2) persuasion strategies targeting the peripheral route; 3) persuasion strategies targeting neither route; 4) persuasion strategies targeting both cognitive routes. The effectiveness of a strategy is relevant with which cognitive route is predominantly adopted by netizens. The major cognitive route is determined by the values of I_1 and I_2. Ultimately, the values of I_1 and I_2 are influenced by the value of α.

4.2. The theoretical and applied basis for parameters

Building on the above propositions, model parameters are categorized into those that directly pertain to the basic reproduction number R_0, and those that do not. Then, in this subsection, we demonstrate the basis for parameters and provide a comprehensive understanding of how parameter values map to real-world scenarios.

The expression of R_0 contains k, ε, v. If $R_0 < 1$ under the specific lower values of k, ε, and v, then the number of people discussing the topic online is zero, and public opinion does not emerge. If $R_0 > 1$ under the specific higher values of k, ε, and v, both persuaded and unpersuaded individuals coexist online. Those parameters reflect the propagation speed and scope of public opinion.

Importantly, the parameters α, β_1, β_2 correspond to the cognitive route selection and processing mechanisms during the evolution of public opinion. This study focuses on public opinion following an emergency. It is therefore essential to clarify how the emergency's type, scale, and other characteristics influence netizens' cognition and attitudes.

As for α, existing studies illustrate that netizens' characteristics [42] and the types of events [43,44] may affect their cognitions and attitudes. Accordingly, we draw on the ELM to map values of α to real-world conditions. When an emergency has little direct bearing on individuals' interests, motivation to engage declines, and people are more likely to rely on peripheral cues. This corresponds to $\alpha < 0.5$. Typical examples include celebrity scandals and natural disasters that occur in other regions. In the limiting case where $\alpha = 0$, the emergency primarily elicits affective and emotional responses, and individuals depend on superficial signals. As α increases, deeper analysis and systematic processing become more prevalent. When $\alpha > 0.5$, the majority of individuals evaluate evidence via central processing. This pattern usually arises for events closely related to netizens' vital interests and practical needs, which foster strong cognitive motivation and a greater willingness to invest effort in analyzing substantive content. Public health crises, product safety scandals, and locally consequential natural disasters are representative cases. In the extreme case where $\alpha = 1$, nearly all individuals process information through the central route [45]. In practice, α can be estimated through two approaches: inference from the event and netizens' characteristics, and real-time monitoring and text mining from social media discussions, which are well-established and widely used in current research about public opinion and public sentiment [46].

Then, for β_1, β_2, the values may indicate different strategies targeting the different cognitive routes. β_1 captures persuasion efforts through the central route, reflecting how well arguments perform in terms of logical structure, pertinent evidence, and the trustworthiness of the source. β_2 captures persuasion efforts through the peripheral route, reflecting the impact of non-argument cues such as the source's attractiveness or status, emotional resonance, message repetition and fluency, and signals of social endorsement. Relevant works on ELM illustrate the factors influencing the public opinion persuasion effect, such as information matching, platform credibility [47], and types of reviews [48]. Thus, these parameters can be determined and optimized by the persuader in practice. For instance, public health agencies may optimize β_1 through evidence-based messaging, while commercial advertisers may improve β_2 through celebrity partnerships and emotional appeals.

Overall, these parameters have clear real-world implications and have been empirically validated in the literature. Consequently, the framework is operationalizable in applied settings when required.

Some little errors are anticipated and acceptable, since the goal is not exact data fitting but the predictive classification of evolutionary trends using qualitatively grounded parameter settings.

4.3. Numerical simulations

The values of parameters need to be selected for numerical simulation based on several principles. First, all parameter values strictly satisfy the theoretical conditions and bounds established in Proposition 3, which is a widely used method [49]. For example, the values need to satisfy a non-negative condition and $R_0 > 1$, which guarantees that $\frac{(\varepsilon - vU^*)}{v} > 0$ in Eq. (4). Thus, the parameters in R_0 are set as follows: $\varepsilon = 0.08$, $k = 0.3$, $v = 0.05$. Besides, Proposition 3 demonstrates that the model's insights are robust across any parameter set that satisfies these conditions. We repeat the same exercises with other parameter values, and the results remain unchanged, which are consistent with the theory. These results can be found in Appendix A.4. The purpose of the numerical simulation presented here is to provide a visualized example of the dynamics proven analytically.

4.3.1. Simulation results of different strategies

The simulations aim to compare the effects of different strategies and identify mechanism-driven stylized trends rather than to fit any single case. Accordingly, we scan a broader parameter space to ensure robustness and generality. Numerical simulations are done using Runge–Kutta methods. To examine the impact of β_1, β_2, and α, the parameters in R_0 remain constant.

(1) Persuasion strategies targeting the central route

In this scenario, the relevant parameter ranges are $\beta_1 > 1 - \beta_1 - v = \gamma_1, \beta_2 < 1 - \beta_2 - v = \gamma_2$. The parameters are set as $\beta_1 = 0.8$, $\beta_2 = 0.2$. For α, according to its definition, uninformed individuals choose the central route with probability α and the peripheral route with probability $1 - \alpha$, thereby processing and analyzing persuasive information in the official response accordingly. Different values of α correspond to different event contexts, natures, and the information processing abilities of the relevant netizen groups. Therefore, simulations are conducted for $\alpha = 0, 0.2, 0.4, 0.6, 0.8, 1.0$, and the phase diagrams of P and NP are presented in Figure 2.

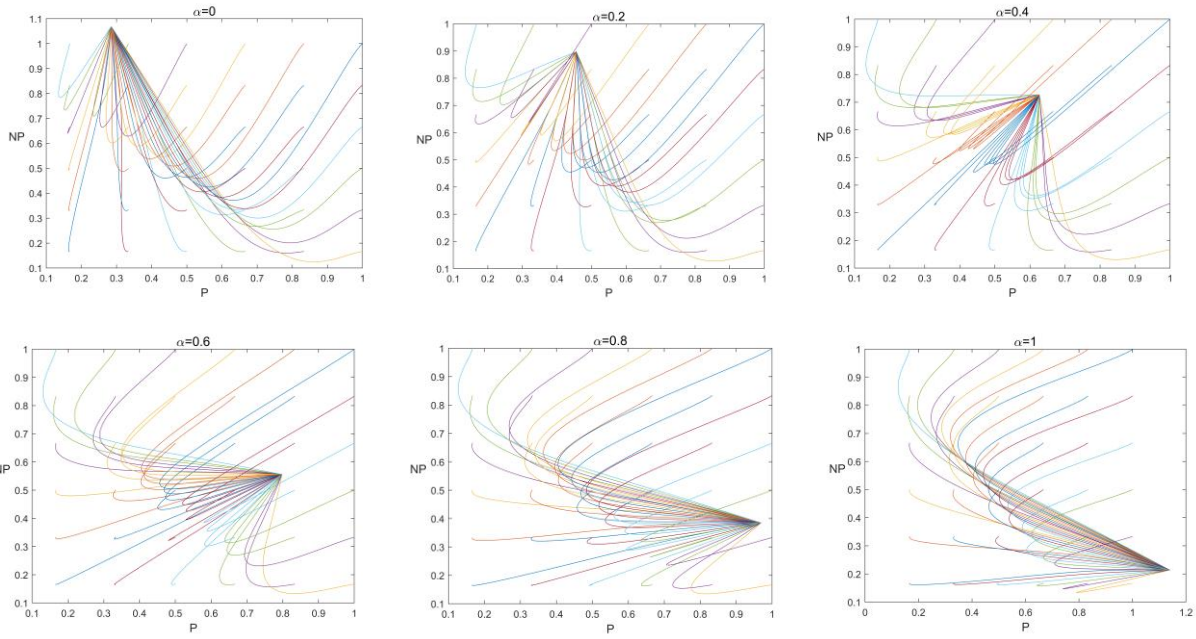

Figure 2. Phase diagram of P and NP for different α when strategies target the central route.

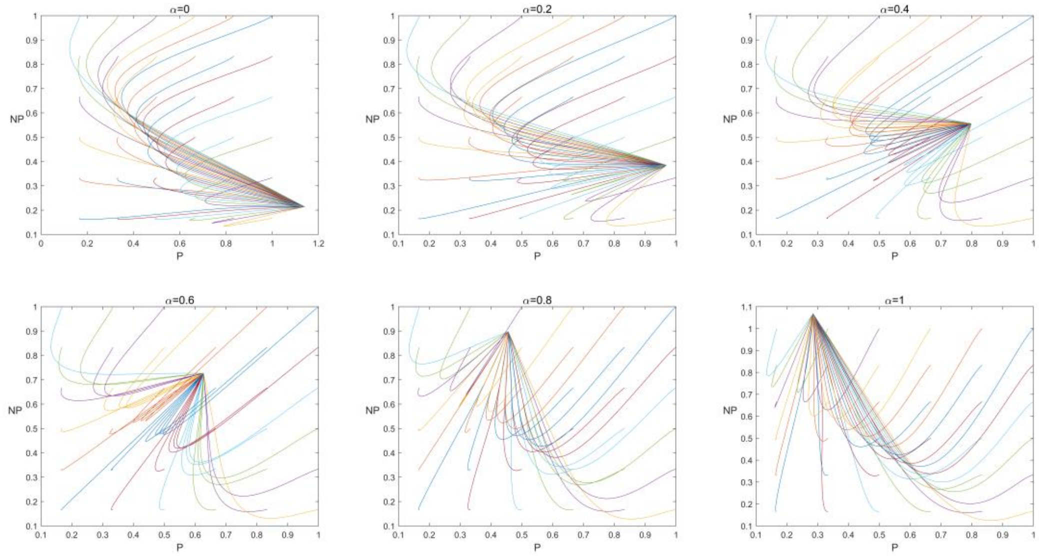

Figure 3. Phase diagram of P and NP for different α when strategies target the peripheral route.

Simulation results show that when $\alpha > 0.5$, most netizens tend to adopt the central route and focus on the core information of the emergency; responses targeting the central route can effectively curb the spread of negative public opinion and persuade more netizen groups, as indicated by $P^* > NP^*$ at the equilibrium. As α increases, the equilibrium value of $(P^* - NP^*)$ also increases, indicating better persuasive effectiveness.

When $\alpha \leq 0.5$, meaning that most netizens tend to adopt the peripheral route and resonate emotionally with others, responses aimed at the central route fail to curb the spread of negative public opinion and cannot persuade netizens, as reflected by $P^* < NP^*$ at the equilibrium. As α decreases, the equilibrium value of $(P^* - NP^*)$ becomes smaller, indicating poorer persuasion effects.

(2) Persuasion strategies targeting the peripheral route

In this scenario, the relevant parameter ranges are $\beta_1 < 1 - \beta_1 - v = \gamma_1, \beta_2 > 1 - \beta_2 - v = \gamma_2$. The parameters are set as $\beta_1 = 0.2$ and $\beta_2 = 0.8$. Simulations are conducted for $\alpha = 0, 0.2, 0.4, 0.6, 0.8, 1.0$, with the phase diagrams of P and NP shown in Figure 3.

Simulation results show that when $\alpha > 0.5$, most netizens prefer the central route, and responses targeting the peripheral route fail to persuade more netizens $P^* < NP^*$. As α increases, the equilibrium value of $(P^* - NP^*)$ further decreases, indicating even less effective persuasion.

When $\alpha \leq 0.5$, most netizens adopt the peripheral route and focus on emotional resonance. In this case, responses targeting the peripheral route effectively address the group's main emotions and succeed to persuade more netizens $P^* > NP^*$. As α decreases, the equilibrium value of $(P^* - NP^*)$ increases, indicating improved persuasion effectiveness.

(3) Persuasion strategies targeting neither route

This scenario corresponds to $\beta_1 < 1 - \beta_1 - v = \gamma_1, \beta_2 < 1 - \beta_2 - v = \gamma_2$. The parameters are set as $\beta_1 = 0.2$ and $\beta_2 = 0.2$. Regarding the parameter α, as defined earlier, uninformed individuals choose to process persuasive information in the official response via the central route with probability α, and via the peripheral route with probability $1 - \alpha$. Different values of α correspond to different event contexts, incident characteristics, and the information-processing abilities of the netizen groups involved. Therefore, this study conducts simulations using a range of values for $\alpha = 0, 0.2, 0.4, 0.6, 0.8, 1.0$, and the phase diagrams for P and NP are shown in Figure 4.

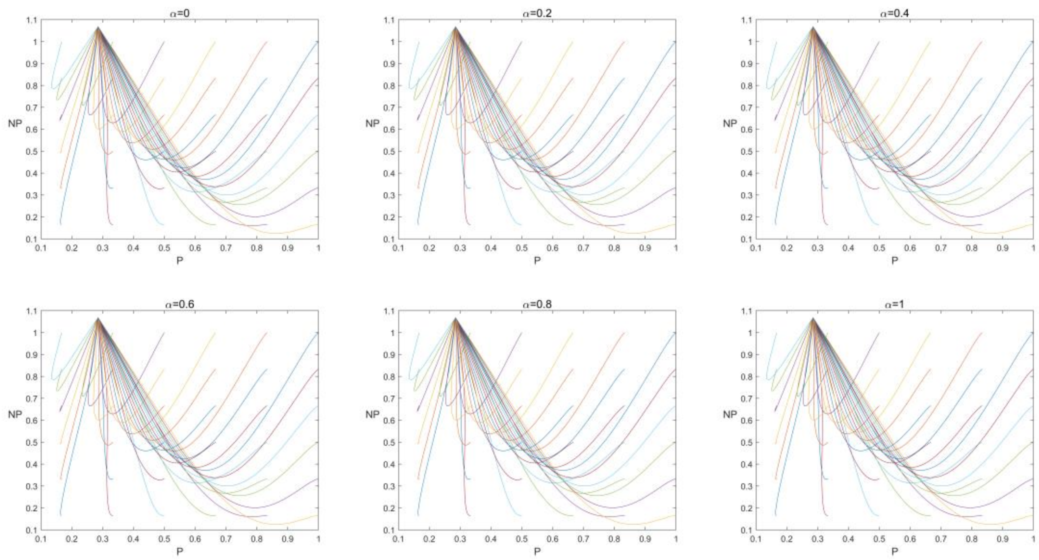

Figure 4. Phase diagrams of P and NP for different α when strategies target neither route.

In management practice, this scenario indicates a highly passive or even counterproductive strategy, further exacerbating the conflict. Here, regardless of the value of α, P^* will always be less than NP^* at equilibrium. This outcome reflects the failure of public opinion persuasion and ineffective response to emergencies and should be avoided.

(4) Persuasion strategies targeting both cognitive routes

This scenario corresponds to $\beta_1 > 1 - \beta_1 - v = \gamma_1, \beta_2 > 1 - \beta_2 - v = \gamma_2$. The parameters are set as $\beta_1 = 0.8$ and $\beta_2 = 0.8$. This scenario reflects a guidance strategy that is comprehensive, targeting both central and peripheral routes. For $\alpha = 0,0.2,0.4,0.6,0.8,1.0$, the phase diagrams of P and NP are shown in Figure 5.

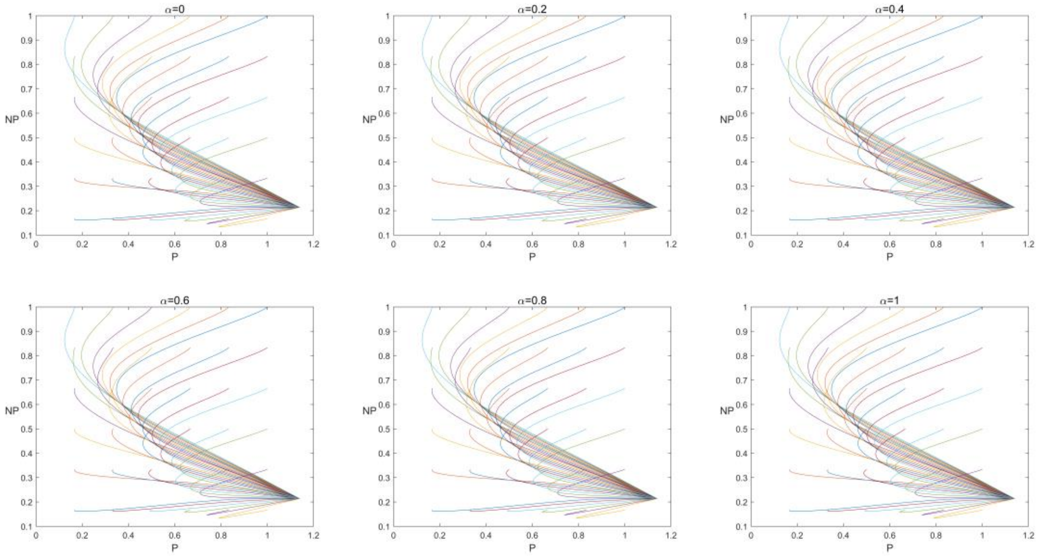

Figure 5. Phase diagrams of P and NP for different α when strategies target both cognitive routes.

This situation represents an all-encompassing persuasion strategy, covering all possible cognitive routes of netizen groups. In this case, regardless of whether α is greater or less than 0.5, or the scale and nature of the incident, P^* will always exceed NP^* at equilibrium. However, such a strategy is often idealized. In real-world practice, it may result in excessive resource allocation. In other words, the resources are limited, so it is challenging to develop persuasion strategies that target both cognitive routes simultaneously. This dilemma is analogous to project scheduling, where decisions must be made regarding how to allocate limited resources between competing tasks [50]. Cost-benefit considerations should be taken into account when adopting such strategies.

4.3.2. Case analysis

Building on the above numerical simulations with different combinations of α, β_1, and β_2, it is observed that the equilibrium of persuasion outcome (P^* versus NP^*) is primarily determined by two relative comparisons: whether α is above or below the benchmark 0.5, and the relative magnitude of β_1 versus β_2. In other words, the effectiveness of persuasive communication is relevant with how well the strategy aligns with the major cognitive routes of netizens.

We now select four representative cases and examine whether the observation in simulation results can be reflected in reality. Within this framework, the cases serve to demonstrate the observability and explanatory power of the classified trends in real-world contexts, rather than to perform accurate time-series fitting.

The cases are representative, covering multiple routes and event types. As for the event types, some events prompt netizens to explore the actual circumstances ($\alpha > 0.5$), while in other events, netizens do not focus on uncovering the facts, but instead pay more attention to factors such as the attitude of official responses and the statements made by experts ($\alpha < 0.5$). This enables a clear comparison of how parameter trends map to distinct dynamical patterns across event types. As for the persuasion strategies, two cases adopted persuasion strategies targeting the central route ($\beta_1 > \beta_2$), and another two cases adopted persuasion strategies targeting the peripheral route ($\beta_1 < \beta_2$).

The first case occurred on February 4, 2025, when a prominent influencer alleged on social media that underwear sold by the well-known Chinese retailer Pangdonglai suffered from serious quality issues. This statement quickly raised concerns among netizens about the quality of products from Pangdonglai. Lots of netizens demanded that Pangdonglai conduct a thorough investigation. This indicates that most netizens were inclined to process information through the central route, corresponding to the scenario where $\alpha > 0.5$. On February 14, Pangdonglai released a response, detailing multiple professional testing reports to demonstrate the acceptable quality of the underwear. This response, centered on the central route, effectively curbed the spread of negative public opinion and persuaded the majority of netizens, that is $P^* > NP^*$.

The second case occurred at a kindergarten in Jiangxi, China, on April 12, 2021. Images showing a teacher asking a boy to smell his feet spread quickly across social media. Lots of netizens condemned the teacher's inappropriate behavior, as anger and dissatisfaction resonated and amplified on social media. In this context, most netizens were more likely to adopt the peripheral route, which corresponds to the scenario where $\alpha \leq 0.5$. The following evening, the local authorities issued an official response, providing an explanation of the incident. However, the response contained some vague wording. Netizens perceived the authorities as attempting to downplay the incident. As a result, the official response failed to persuade the majority of netizens, consistent with the simulation result $P^* < NP^*$.

The third case involves widespread online discussion that erupted in May 2022 over illustrations in a primary school mathematics textbook published by the People's Education Press in China. Many netizens noted that the illustrated characters appeared odd and even contained inappropriate suggestions. This finding led to a surge in public calls for authorities to thoroughly investigate the reasons behind the inclusion of these illustrations. Clearly, most netizens in this case were inclined to use the central route, corresponding to the scenario where $\alpha > 0.5$. On May 26, the publishing house issued an official response that mainly addressed the characteristics of the peripheral route: it expressed a willingness to accept public supervision but failed to address the netizens' primary concerns and did not provide a detailed explanation of the problems. As a result, many netizens criticized the official statement as insincere and unconvincing, that is $P^* < NP^*$.

The fourth case concerns the sudden collapse of a self-built house in Changsha, China, on April 29, 2022, which instantly attracted widespread attention online. Panic spread rapidly, with netizens speculating about the causes of the collapse. In this context, the majority of netizens tended to adopt the peripheral route, which corresponded to $\alpha \leq 0.5$. On the day of the incident, relevant departments promptly launched emergency rescue efforts and issued an official announcement that, while not explaining the specific reasons for the incident, focused on communicating a firm commitment to the rescue and expressing determination to address the situation. This approach, which focused on the peripheral route, effectively eased negative sentiment, and many netizens expressed support and trust in the response. This outcome is $P^* > NP^*$.

4.3.3. Counterfactual analysis

Now, we select a representative case and use the counterfactual analysis to verify the importance of route adaptation.

Xibei is a Chinese restaurant chain brand. On September 10, 2025, influencer Mr. Luo accused

Xibei of poor taste and reliance on prepared foods. Netizens quickly rallied around the principle of the "right to know". Xibei's founder, Mr. Jia, then flatly denied using any prepared foods and announced plans to sue Luo. Luo responded with a public bounty, offering money for every verifiable sample of prepared food delivered to his team. As a result, the debate remained anchored in evidence-based, central-route processing. Instead of releasing test reports or pledging transparent labeling, Xibei published several stories that implied the high quality of its food. These stories functioned as classic peripheral cues. The tactic backfired, and comments quickly filled with memes mocking the brand for "storytelling instead of truth-telling".

If Xibei had employed route adaptation and matched the public's central-route appetite by publishing ingredient lists, commissioning third-party audits, and committing to clear prepared foods labels, the crisis could have been defused and reputational damage sharply limited.

5. Dynamic optimization of persuasion strategies

While the analysis in Section 4 provides insights about the system's equilibrium behavior, practical persuasion still needs to deal with the time-varying complexities and persuasion costs, which fixed strategies address only imperfectly. This motivates our adoption of optimal control theory in Section 5, treating control variables as functions of time. The results can be regarded as extensions of the discussion in Section 4.

5.1. Problem formulations and optimality conditions

Effective persuasion in public opinion, such as crisis communication and rumor mitigation, requires resource allocation and balancing the resources devoted to interventions and costs against the benefit of averting negative information. To formalize this trade-off, the information propagation model is embedded within a dynamic optimal control framework. Because $\beta_1(t)$ and $\beta_2(t)$ denote the probability that an individual is persuaded via each cognitive routes at each time, $\frac{1}{2}C_1\beta_1^2(t)$ and $\frac{1}{2}C_2\beta_2^2(t)$ represent the resources or efforts on two cognitive routes. $P(t) - NP(t)$ reflects the benefits gained from persuading more netizens. The optimal control model considers an infinite time horizon and incorporates a discount factor ρ, thereby embodying the principle that dealing with the public opinion crisis as early as possible is preferable.

Solving this control problem yields dynamic intervention schedules that minimize deployment costs while maximizing misinformation suppression and cultivating positive public opinion. Under $R_0 > 1$, the optimization problem is described as follows.

$$J_\infty = \min_{\beta_1(t)\geq 0, \beta_2(t)\geq 0} \int_0^{+\infty} e^{-\rho t}\left[NP(t) - P(t) + \frac{1}{2}\left(C_1\beta_1^2(t) + C_2\beta_2^2(t)\right)\right] dt \tag{5}$$

$$s.t.\begin{cases} \dot{U} = \varepsilon - kU(P+NP) - vU \\ \dot{I_1} = \alpha kU(P+NP) - I_1 \\ \dot{I_2} = (1-\alpha)kU(P+NP) - I_2 \\ \dot{P} = \beta_1 I_1 + \beta_2 I_2 - vP \\ \dot{NP} = (1-\beta_1-v)I_1 + (1-\beta_2-v)I_2 - vNP \end{cases}$$

For the sake of simplicity, the current-value Hamiltonian function H_c of the optimization problem is written as follows, which omits the time argument.

$$H_c = NP - P + \frac{1}{2}C_1\beta_1^2 + \frac{1}{2}C_2\beta_2^2 + \lambda_u(\varepsilon - kU(P+NP) - vU) + \lambda_{I_1}(\alpha kU(P+NP) - I_1) + \lambda_{I_2}\big((1-\alpha)kU(P+NP) - I_2\big) + \lambda_p(\beta_1 I_1 + \beta_2 I_2 - vP) + \lambda_{np}\big((1-\beta_1-v)I_1 + (1-\beta_2-v)I_2 - vNP\big) \tag{6}$$

where the co-state variables λ_u, λ_{I_1}, λ_{I_2}, λ_p, and λ_{np} are associated with their state dynamic equations U, I_1, I_2, P, and NP, respectively.

From the Hamiltonian function (6), the first-order conditions and co-state equations can be obtained as follows:

$$\frac{\partial H}{\partial \beta_1} = C_1\beta_1 + \lambda_p I_1 - \lambda_{np} I_1 \tag{7}$$

$$\frac{\partial H}{\partial \beta_2} = C_2\beta_2 + \lambda_p I_2 - \lambda_{np} I_2 \tag{8}$$

$$\dot{\lambda_u} = \rho\lambda_u - \frac{\partial H}{\partial U} = \rho\lambda_u + \lambda_u(k(P+NP)+v) - \lambda_{I_1}\alpha k(P+NP) - \lambda_{I_2}(1-\alpha)k(P+NP) \tag{9}$$

$$\dot{\lambda_{I_1}} = \rho\lambda_{I_1} - \frac{\partial H}{\partial I_1} = \rho\lambda_{I_1} + \lambda_{I_1} - \lambda_p\beta_1 - \lambda_{np}(1-\beta_1-v) \tag{10}$$

$$\dot{\lambda_{I_2}} = \rho\lambda_{I_2} - \frac{\partial H}{\partial I_2} = \rho\lambda_{I_2} + \lambda_{I_2} - \lambda_p\beta_2 - \lambda_{np}(1-\beta_2-v) \tag{11}$$

$$\dot{\lambda_p} = \rho\lambda_p - \frac{\partial H}{\partial P} = \rho\lambda_p + 1 + \lambda_u kU - \lambda_{I_1}\alpha kU - \lambda_{I_2}(1-\alpha)kU + \lambda_p v \tag{12}$$

$$\dot{\lambda_{np}} = \rho\lambda_{np} - \frac{\partial H}{\partial NP} = \rho\lambda_{np} - 1 + \lambda_u kU - \lambda_{I_1}\alpha kU - \lambda_{I_2}(1-\alpha)kU + \lambda_{np} v \tag{13}$$

The associated transversally conditions are $\lim_{t\to\infty}\lambda_u(t)U(t)e^{-\rho t} = 0$, $\lim_{t\to\infty}\lambda_{I_1}(t)I_1(t)e^{-\rho t} = 0$, $\lim_{t\to\infty}\lambda_{I_2}(t)I_2(t)e^{-\rho t} = 0$, $\lim_{t\to\infty}\lambda_p(t)P(t)e^{-\rho t} = 0$, and $\lim_{t\to\infty}\lambda_{np}(t)NP(t)e^{-\rho t} = 0$, respectively.

Some calculations by setting $\frac{\partial H}{\partial \beta_1} = \frac{\partial H}{\partial \beta_2} = 0$ yield the expressions of optimal control. The boundary of $\beta_1^*(t)$ and $\beta_2^*(t)$ need to be specified. To reflect real-world constraints, both $\beta_1^*(t)$ and $\beta_2^*(t)$ are bounded within the interval $[0,1-v]$ for each respective strategy. The boundary means that the persuasion effort must be non-negative. Moreover, exceeding the upper bound $1-v$ may drive the state dynamics to imply I_1 or I_2 populations below zero, which is infeasible and meaningless.

Proposition 4. Under $R_0 > 1$, the dynamic optimal persuasions are expressed as follows:

$$\beta_1^*(t) = min\left\{max\left\{\frac{1}{C_1}\left(\lambda_{np}(t) - \lambda_p(t)\right)I_1(t), 0\right\}, 1-v\right\},$$

$$\beta_2^*(t) = min\left\{max\left\{\frac{1}{C_2}\left(\lambda_{np}(t) - \lambda_p(t)\right)I_2(t), 0\right\}, 1-v\right\}.$$

The results underscore the necessity of real-time, adaptive decision-making in persuasion: strategies must remain responsive to evolving conditions across diverse and dynamic environments to ensure the most effective allocation of resources.

Corollary 1. When $\beta_1^*(t)$ and $\beta_2^*(t)$ are within the interval $[0,1-v]$, at each time t, $\beta_1^*(t)$ is increasing with $I_1(t)$ and $\beta_2^*(t)$ is increasing with $I_2(t)$.

This corollary demonstrates that within the framework of optimal persuasion strategy, the allocation of effort to each persuasion channel should be dynamically adjusted in strict proportion to the size of its corresponding subgroup of netizens. Specifically, resources directed toward the central route should be proportional to the number of central route netizens $I_1(t)$. Similarly, the allocation for peripheral routes should correspond to the number of peripheral route netizens $I_2(t)$. Adhering to this proportional allocation approach minimizes both excessive resource allocation when target groups are small and the risk of insufficient resource deployment when target groups are large.

Corollary 2. When $\beta_1^*(t)$ and $\beta_2^*(t)$ is within the interval $[0,1-v]$, at each time t, the size relationship between $\beta_1^*(t)$ and $\beta_2^*(t)$ is relevant with the unit cost C_1, C_2.

This corollary holds significant relevance for management practices. Official agencies often face a choice between accurate, comprehensive information dissemination through the central route and rapid, easily shared content via the peripheral route. For example, utilizing the central route may sometimes require mobilizing experts for interviews, producing detailed and credible reports, and ensuring information accuracy, which generally involves greater time and resource investment. In contrast, strategies such as creating short videos, using catchy slogans, or enlisting opinion leaders can be implemented more quickly. Therefore, there is a trade-off between costs and benefits. If credibility and truth are essential, it is necessary to allocate higher resources to the central route; if the situation is urgent, it is also advisable to prioritize peripheral approaches in public opinion persuasion.

5.2. Steady-state equilibrium

The steady-state of optimal control refers to the long-term behavior of a controlled dynamical system when, after a period of adjustment or transient dynamics, the state variables and control inputs converge and remain constant under the optimal control policy. In this regime, the system maintains equilibrium, and both the state and the optimal control actions no longer change significantly over time.

Some calculations yield the following proposition, and the procedure is presented in Appendix A.5.

Proposition 5. Under $R_0 > 1$, the steady-state of optimal persuasion is expressed as follows:

$$\beta_1^s = min\left\{max\left\{\frac{2\alpha\left(k\varepsilon(1-v)-v^2\right)}{C_1(\rho+v)k(1-v)}, 0\right\}, 1-v\right\},\ \beta_2^s = min\left\{max\left\{\frac{2(1-\alpha)\left(k\varepsilon(1-v)-v^2\right)}{C_2(\rho+v)k(1-v)}, 0\right\}, 1-v\right\}.$$

β_1^s and β_2^s reflect the long-term resource allocations of persuasion. Within the interval $[0,1-v]$, the value of β_1^s increases with α and the value of β_2^s decreases with α, because $\frac{\partial \beta_1^s}{\partial \alpha} > 0$ and $\frac{\partial \beta_2^s}{\partial \alpha} < 0$. In addition, the size relationship between β_1^s and β_2^s depends on α, C_1, and C_2.

This proposition extends the discussion in Section 4: in addition to accounting for the distinct cognitive routes of netizens, it also systematically addresses how variations in associated costs influence the effectiveness of different persuasion strategies.

Corollary 3. The optimal persuasion strategies adjust dynamically over time: First, during population changes, they follow the expressions $\beta_1^*(t) = min\left\{max\left\{\frac{1}{C_1}\left(\lambda_{np}(t) - \lambda_p(t)\right) I_1(t), 0\right\}, 1-v\right\}$ and $\beta_2^*(t) = min\left\{max\left\{\frac{1}{C_2}\left(\lambda_{np}(t) - \lambda_p(t)\right) I_2(t), 0\right\}, 1-v\right\}$; then they asymptotically converge to internal steady-states $\beta_1^s = \frac{2\alpha\left(k\varepsilon(1-v)-v^2\right)}{C_1(\rho+v)k(1-v)}$, $\beta_2^s = \frac{2(1-\alpha)\left(k\varepsilon(1-v)-v^2\right)}{C_2(\rho+v)k(1-v)}$ or boundary steady-states $\{0,1-v\}$.

This corollary illustrates the general adjustment rule of the optimal persuasion strategy. Persuaders must engage in active adjustments to evolving population dynamics and marginal valuations. Conversely, as the system approaches equilibrium, optimal strategies converge to time-invariant levels fully characterized by exogenous parameters.

5.3. Numerical simulations

Obtaining the closed-form solutions of $\beta_1^*(t)$ and $\beta_2^*(t)$ is difficult. Thus, numerical simulations are employed to analyze the dynamic optimal control strategies. The forward-backward sweep method [51] is a typical numerical algorithm for solving optimal control problems that is widely used in research [52]. Numerical simulations are done using this method. Unless otherwise stated, the parameter values are set as follows: $\varepsilon = 0.08$, $k = 0.3$, $v = 0.05$, $\rho = 0.1$. The parameter justification follows the principles outlined in Section 4.3.

Regarding the parameters C_1 and C_2, they represent the cost-efficiency scaling factor for each persuasion route defined by ELM. The central route involves campaigns built on detailed, expert-vetted content, while the peripheral route relies on simpler, broad-reach emotional cues and rapid responses [53]. Values are shaped by the labor and time invested in achieving information precision, and additional marketing activities [54]. Specifically, C_1 captures costs in producing evidence-based content through comprehensive research, fact verification, expert consultation, and technical documentation. C_2 reflects costs of rapid, wide-reaching communication that prioritizes speed and accessibility over argumentative depth, leveraging heuristic cues, emotional appeals, and authority signals. To provide clear managerial insights, our numerical analysis explores two relevant scenarios: $C_1 = C_2$ and $C_1 > C_2$. The former scenario considers an idealistic situation where both persuasion strategies have identical cost structures. The asymmetric cost case reflects a realistic situation. This asymmetry is illustrated by the 2022 “Zhang Xiaoquan” public opinion crisis in China: after a customer’s cleaver broke, the company targeted the peripheral route, asserting that cleavers “cannot be used to smash garlic” and invoking “Michelin chefs” as authority, which had a lower cost but backfired and intensified the backlash. By contrast, a higher cost central route strategy would have required investment in evidence-based communication, such as publishing technical reports on material. From the time cost perspective alone, the two strategies differ substantially.

Considering the case where $C_1 = C_2 = 1.0$, by setting $\alpha = 0.2$ and $\alpha = 0.8$, the corresponding diagrams of state variables and control variables are shown in Figure 6 and Figure 7.

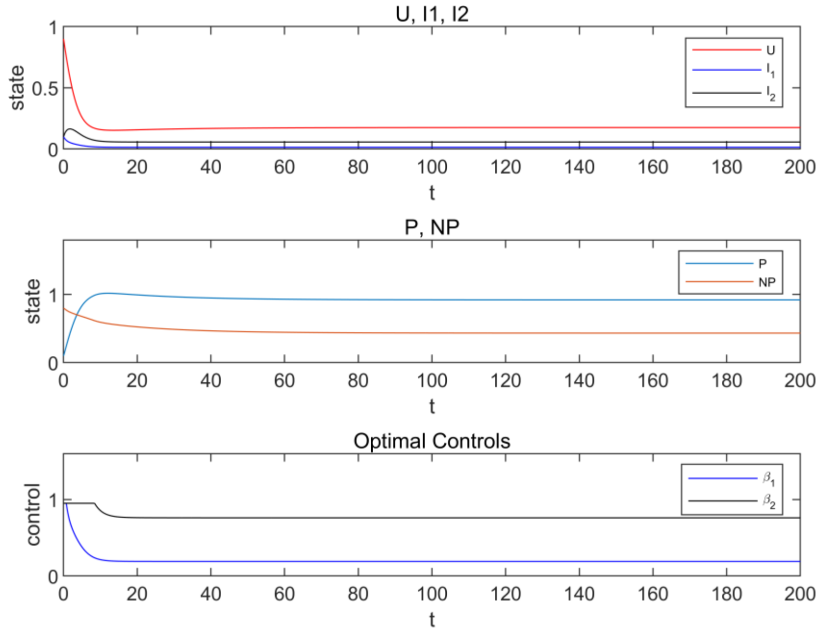

Figure 6. Diagrams of state variables and control variables when $\alpha = 0.2$, $C_1 = C_2 = 1.0$.

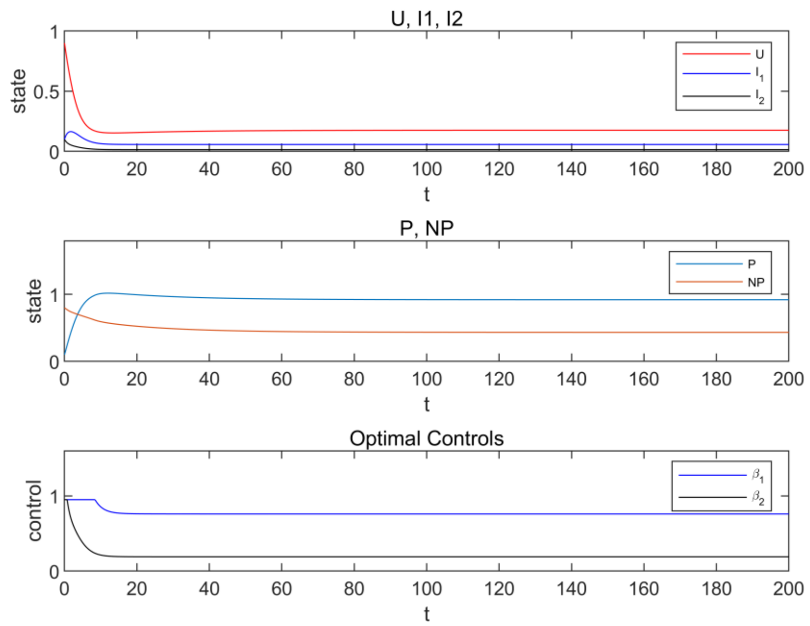

Figure 7. Diagrams of state variables and control variables when $\alpha = 0.8$, $C_1 = C_2 = 1.0$.

Observation 1. When $C_1 = C_2$, the size relationship between $\beta_1^*(t)$ and $\beta_2^*(t)$ is relevant with α. With time going by, after a period of adjustment and evolution, the state and control variables converge to the steady state and remain constant.

This observation is consistent with the discussion in Section 4. Critically, this observation provides a dynamic validation, as it is derived directly from the numerical simulation of the system's evolution. Moreover, the optimal persuasion strategies adjust dynamically over time: first, $\beta_1^*(t)$, $\beta_2^*(t)$ adjust while the population varies, and then asymptotically converge to internal steady-states or boundary steady-states.

When $C_1 \neq C_2$, the dynamics may exhibit new patterns. For example, considering the case where $C_1 = 1.0$ and $C_2 = 0.2$, by setting $\alpha = 0.2$ and $\alpha = 0.8$, the corresponding diagrams of state variables and control variables are shown in Figure 8 and Figure 9.

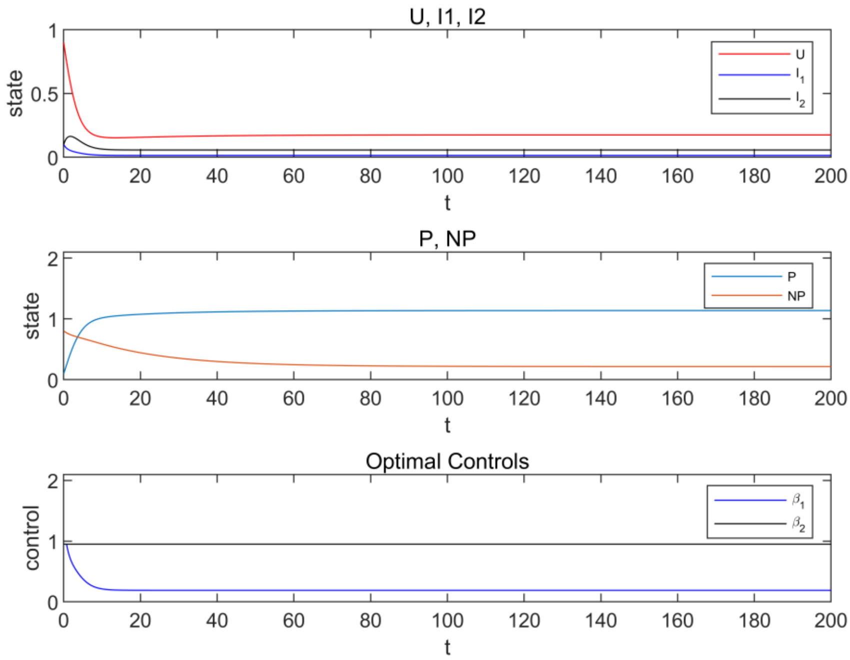

Figure 8. Diagrams of state variables and control variables when $\alpha = 0.2$, $C_1 = 1.0$, $C_2 = 0.2$.

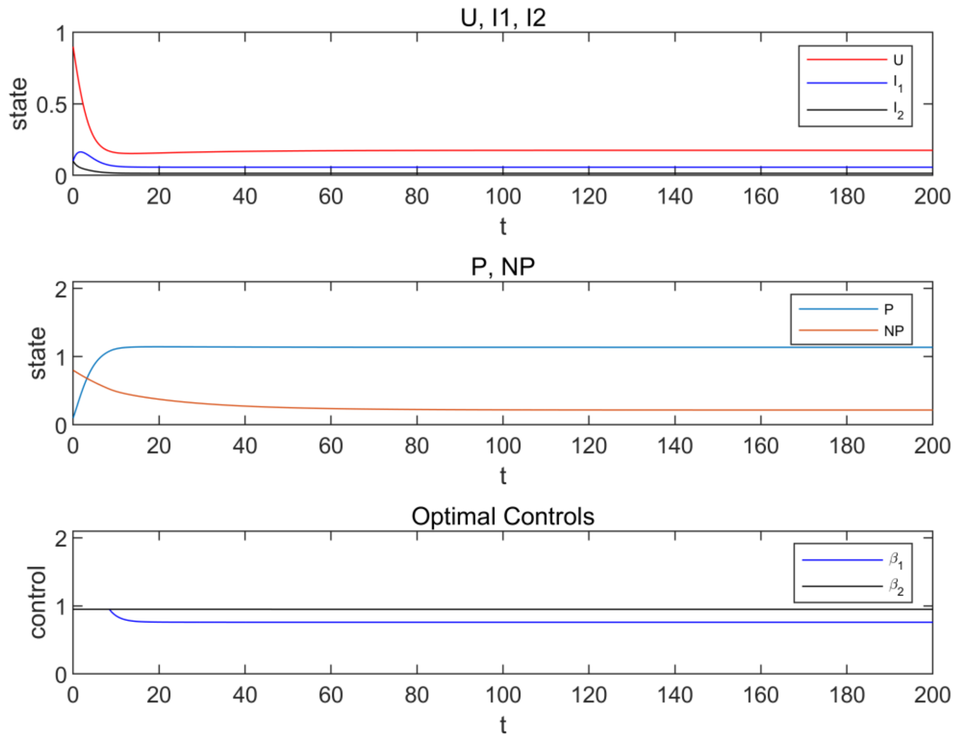

Figure 9. Diagrams of state variables and control variables when $\alpha = 0.8$, $C_1 = 1.0$, $C_2 = 0.2$.

Observation 2. When $C_1 \neq C_2$, if the associated costs are prohibitively high, it may be optimal to adopt a lower-cost persuasion strategy that does not directly align with the dominant cognitive route of the current netizen group.

This observation is consistent with Corollary 2 and Proposition 5. During certain emergencies, although some netizens seek access to the truth, such information may be difficult to obtain quickly. In these cases, alleviating negative public emotions rather than conducting investigations may become a more effective persuasion strategy. Besides, the result extends Proposition 3 and numerical simulations in Section 4: in addition to accounting for the distinct cognitive routes of netizens, it also systematically addresses how variations in associated costs influence the effectiveness of different persuasion strategies.

Observation 3. The optimal controls $\beta_1^*(t)$ and $\beta_2^*(t)$ are confined to the limited intervals $[0,1-v]$, leading to saturation at the bounds where they remain constant over time intervals despite evolving system dynamics.

This observation illustrates that sometimes, the optimal control may not adjust over time and can resemble a fixed strategy. One reason is that it can be optimal to allocate more resources to a lower-cost persuasion strategy according to dynamic optimization conditions. Indeed, only the dynamic optimal control method can identify the conditions under which saturating the controls at their bounds constitutes the truly optimal solution.

6. Extension

In this section, we relax several simplifying assumptions of the baseline model and extend the framework to explore whether there are new results.

6.1. Heterogeneous exit rates

For the sake of simplicity, we initially assume a uniform exit rate v for all groups in the model, representing the rate at which individuals lose interest in the topic. However, in reality, exit rates may vary across groups. Therefore, to test the robustness of our conclusions against this assumption, we now use heterogeneous exit rates. While assigning a unique exit rate to each of the five groups may introduce excessive complexity, we assume that there are three types of exit rates to strike a balance between realism and tractability: the exit rate for uninformed individuals (U) ν_U; the exit rate for individuals (I_1, I_2) still processing information ν_I; and the exit rate for individuals $(P,\ NP)$ with formed opinions ν_P. This captures their higher persistence and continued engagement with the topic. We assume that the relative magnitudes of the exit rates follow the order $\nu_U > \nu_I > \nu_P$. This can reflect that individuals with strong formed opinions might be more persistent, while uninformed individuals or those still processing information might exit more readily.

$$\begin{cases} \dot{U} = \varepsilon - kU(P+NP) - \nu_U U \\ \dot{I_1} = \alpha kU(P+NP) - \beta_1 I_1 - \gamma_1 I_1 - \nu_I I_1 = \alpha kU(P+NP) - I_1 \\ \dot{I_2} = (1-\alpha)kU(P+NP) - \beta_2 I_2 - \gamma_2 I_2 - \nu_I I_2 = (1-\alpha)kU(P+NP) - I_2 \\ \dot{P} = \beta_1 I_1 + \beta_2 I_2 - \nu_P P \\ \dot{NP} = \gamma_1 I_1 + \gamma_2 I_2 - \nu_P NP \end{cases} \tag{14}$$

It is obvious that the system has two equilibrium points. One corresponds to the public opinion subsiding equilibrium $\left(\frac{\varepsilon}{\nu_U}, 0,0,0,0\right)$. The other one corresponds to the public opinion outbreak equilibrium, which is calculated as follows:

$$\begin{cases} U^* = \dfrac{\nu_P}{k(1-\nu_I)} \\ {I_1}^* = \alpha(\dfrac{\nu_P \nu_U}{k(-1+\nu_I)} + \varepsilon) \\ {I_2}^* = \dfrac{(-1+\alpha)(\nu_P\nu_U + k(-1+\nu_I)\varepsilon)}{k(1-\nu_i)} \\ P^* = \dfrac{(\alpha(\beta_1-\beta_2)+\beta_2)(\nu_P\nu_U + k(-1+\nu_i)\varepsilon)}{k(-1+\nu_I)\nu_p} \\ NP^* = \dfrac{(-1+\nu_I+\alpha(\beta_1-\beta_2)+\beta_2)(\nu_P\nu_U + k(-1+\nu_I)\varepsilon)}{k(1-\nu_I)\nu_P} \end{cases} \tag{15}$$

The basic reproduction number of this system is $R_0 = \frac{k\varepsilon(1-\nu_I)}{\nu_P \nu_U}$.

Proposition 6. When $R_0 < 1$, the public opinion subsiding equilibrium is locally asymptotically stable. When $R_0 > 1$, the public opinion outbreak equilibrium is locally asymptotically stable.

According to this proposition, R_0 varies with the parameters ν_P, ν_U, and ν_I. Specifically, R_0 increases as ν_P, ν_U decreases, or as ν_I increases. This implies that different exit rates lead to different likelihoods of public opinion outbreak emergence.

Then, the optimization problem under $R_0 > 1$ is described as follows:

$$J_\infty = \min_{\beta_1(t)\geq 0, \beta_2(t)\geq 0} \int_0^{+\infty} e^{-\rho t}\left[NP(t) - P(t) + \frac{1}{2}\left(C_1\beta_1^2(t) + C_2\beta_2^2(t)\right)\right]dt$$

$$\begin{cases} \dot{U} = \varepsilon - kU(P+NP) - \nu_U U \\ \dot{I_1} = \alpha kU(P+NP) - \beta_1 I_1 - \gamma_1 I_1 - \nu_I I_1 = \alpha kU(P+NP) - I_1 \\ \dot{I_2} = (1-\alpha)kU(P+NP) - \beta_2 I_2 - \gamma_2 I_2 - \nu_I I_2 = (1-\alpha)kU(P+NP) - I_2 \\ \dot{P} = \beta_1 I_1 + \beta_2 I_2 - \nu_P P \\ \dot{NP} = \gamma_1 I_1 + \gamma_2 I_2 - \nu_P NP \end{cases} \tag{16}$$

For the sake of simplicity, the current-value Hamiltonian function H_c of the optimization problem is as follows, which omits the time argument:

$$\begin{aligned} H_c = NP - P + \tfrac{1}{2}C_1\beta_1^2 + \tfrac{1}{2}C_2\beta_2^2 + \lambda_u(\varepsilon - kU(P+NP) - \nu_U U) + \lambda_{I_1}(\alpha kU(P + \\ NP) - I_1) + \lambda_{I_2}\big((1-\alpha)kU(P+NP) - I_2\big) + \lambda_p(\beta_1 I_1 + \beta_2 I_2 - \nu_P P) + \\ \lambda_{np}\big((1-\beta_1-\nu_I)I_1 + (1-\beta_2-\nu_I)I_2 - \nu_P NP\big) \end{aligned} \tag{17}$$

where the co-state variables λ_u, λ_{I_1}, λ_{I_2}, λ_p, and λ_{np} are associated with their state dynamic equations U, I_1, I_2, P, and NP, respectively.

For the analysis of our extension, we consolidate the optimal control functions $\beta_1^*(t)$, $\beta_2^*(t)$ and their corresponding steady states β_1^s, β_2^s into a single proposition. This enhances clarity and sharpens the focus on the impact of introducing heterogeneous exit rates.

Proposition 7. Under $R_0 > 1$, when exit rates are heterogeneous across population groups, the optimal persuasions are changed as follows:

The functions of optimal controls are: $\beta_1^*(t) = min\left\{max\left\{\frac{1}{C_1}\left(\lambda_{np}(t) - \lambda_p(t)\right)I_1(t), 0\right\}, 1 - \nu_I\right\}$, $\beta_2^*(t) = min\left\{max\left\{\frac{1}{C_2}\left(\lambda_{np}(t) - \lambda_p(t)\right)I_2(t), 0\right\}, 1 - \nu_I\right\}$.

The steady states are: $\beta_1^s = min\left\{max\left\{\frac{2\alpha(k(1-\nu_I)\varepsilon - \nu_P\nu_U)}{C_1(\nu_P+\rho)k(1-\nu_I)}, 0\right\}, 1-\nu_I\right\}$, $\beta_2^s = min\left\{max\left\{\frac{2(1-\alpha)(k(1-\nu_I)\varepsilon - \nu_P\nu_U)}{C_2(\nu_P+\rho)k(1-\nu_I)}, 0\right\}, 1-\nu_I\right\}$.

This proposition highlights both the robustness of our core framework and the new insights gained from introducing heterogeneous exit rates.

First, setting uniform exit rates $\nu_U = \nu_I = \nu_P = \nu$ causes the expressions to reduce exactly to those of the baseline model, which confirms the model's consistency and robustness.

Moreover, the heterogeneous rates $\nu_U > \nu_I > \nu_P$ provide new insights. The expressions of the optimal controls $\beta_1^*(t)$ and $\beta_2^*(t)$ still depend on the cost parameters C_1, C_2, the size of the target populations $I_1(t)$, $I_2(t)$, and the co-state variables $\lambda_p(t)$, $\lambda_{np}(t)$. This structure remains analogous to the baseline model. Moreover, the boundary difference emerges from the extension. In the extension model, the upper bound becomes $1-\nu_I$ instead of $1-v$. $\frac{\partial\beta_1^s}{\partial\nu_P}<0$, $\frac{\partial\beta_2^s}{\partial\nu_P}<0$, indicating a decrease in the exit rate ν_P, increases the steady-state β_1^s, β_2^s. In other words, the more persistent individuals with formed opinions will increase the need to the persuasion effort. Conversely, $\frac{\partial\beta_1^s}{\partial\nu_U}>0$, $\frac{\partial\beta_2^s}{\partial\nu_U}>0$, indicating an increase in the uninformed exit rate ν_U, decreases the steady-state of optimal control. It means that a faster outflow of uninformed individuals diminishes the need to the persuasion effort.

6.2. Gradual nature of cognitive processing

The assumption that $\beta_1+\gamma_1+v=1$ and $\beta_2+\gamma_2+v=1$ implies that individuals must immediately decide their state at each time step. However, in reality, some individuals will not immediately decide their state. To relax this assumption, we introduce a stay probability $\delta\in[0,1]$ and assume that $\beta_1+\gamma_1+v=\beta_2+\gamma_2+v=1-\delta$. When $\delta>0$, it can reflect the gradual nature of cognitive processing and attitude formation.

The dynamical system can be changed as follows:

$$\begin{cases} \dot{U}=\varepsilon-kU(P+NP)-vU \\ \dot{I_1}=\alpha kU(P+NP)-\beta_1 I_1-\gamma_1 I_1-vI_1=\alpha kU(P+NP)-(1-\delta)I_1 \\ \dot{I_2}=(1-\alpha)kU(P+NP)-\beta_2 I_2-\gamma_2 I_2-vI_2=(1-\alpha)kU(P+NP)-(1-\delta)I_2 \\ \dot{P}=\beta_1 I_1+\beta_2 I_2-vP \\ \dot{NP}=\gamma_1 I_1+\gamma_2 I_2-vNP \end{cases} \tag{18}$$

It is obvious that the system has two equilibrium points. One corresponds to the public opinion subsiding equilibrium $\left(\frac{\varepsilon}{v},0,0,0,0\right)$. The other one corresponds to the public opinion outbreak equilibrium, which is calculated as follows:

$$\begin{cases} U^*=\dfrac{v(1-\delta)}{k(1-v-\delta)} \\ {I_1}^*=\dfrac{\alpha(v^2(1-\delta)+kv\varepsilon+k(-1+\delta)\varepsilon)}{k(1-\delta)(1-v-\delta)} \\ {I_2}^*=\dfrac{(-1+\alpha)(v^2(1-\delta)+kv\varepsilon+k(-1+\delta)\varepsilon)}{k(1-\delta)(1-v-\delta)} \\ P^*=\dfrac{(\alpha(\beta_1-\beta_2)+\beta_2)(v^2(-1+\delta)-kv\varepsilon+k(1-\delta)\varepsilon)}{kv(1-\delta)(1-v-\delta)} \\ NP^*=\dfrac{(-1+v+\alpha(\beta_1-\beta_2)+\beta_2+\delta)(v^2(1-\delta)+kv\varepsilon+k(-1+\delta)\varepsilon)}{kv(1-\delta)(1-v-\delta)} \end{cases} \tag{19}$$

The basic reproduction number of this system is $R_0=\frac{k\varepsilon(1-v-\delta)}{v^2(1-\delta)}=\frac{k\varepsilon}{v^2}-\frac{k\varepsilon}{v(1-\delta)}$.

Proposition 8. When $R_0<1$, the equilibrium point of the system is locally asymptotically stable.

When $R_0 > 1$, the equilibrium point of the system is locally asymptotically stable.

According to this proposition, $\frac{\partial R_0}{\partial \delta} < 0$, indicating that R_0 increases as δ decreases. This implies that a lower stay probability leads to a higher probability of public opinion outbreak emergence.

Then, the optimization problem under $R_0 > 1$ is described as follows:

$$J_\infty = \min_{\beta_1(t)\geq 0, \beta_2(t)\geq 0} \int_0^{+\infty} e^{-\rho t} \left[NP(t) - P(t) + \frac{1}{2}\left(C_1\beta_1^2(t) + C_2\beta_2^2(t)\right)\right] dt$$

$$\begin{cases} \dot{U} = \varepsilon - kU(P+NP) - vU \\ \dot{I_1} = \alpha kU(P+NP) - \beta_1 I_1 - \gamma_1 I_1 - vI_1 = \alpha kU(P+NP) - (1-\delta)I_1 \\ \dot{I_2} = (1-\alpha)kU(P+NP) - \beta_2 I_2 - \gamma_2 I_2 - vI_2 = (1-\alpha)kU(P+NP) - (1-\delta)I_2 \\ \dot{P} = \beta_1 I_1 + \beta_2 I_2 - vP \\ \dot{NP} = \gamma_1 I_1 + \gamma_2 I_2 - vNP \end{cases} \tag{20}$$

For sake of simplicity, the current-value Hamiltonian function H_c of the optimization problem is as follows, which omits the time argument:

$$H_c = NP - P + \frac{1}{2}C_1\beta_1^2 + \frac{1}{2}C_2\beta_2^2 + \lambda_u(\varepsilon - kU(P+NP) - vU) + \lambda_{I_1}(\alpha kU(P+ NP) - I_1) + \lambda_{I_2}\left((1-\alpha)kU(P+NP) - I_2\right) + \lambda_p(\beta_1 I_1 + \beta_2 I_2 - vP) + \lambda_{np}\left((1-\beta_1-\nu-\delta)I_1 + (1-\beta_2-\nu-\delta)I_2 - \nu NP\right) \tag{21}$$

where the co-state variables λ_u, λ_{I_1}, λ_{I_2}, λ_p, and λ_{np} are associated with their state dynamic equations U, I_1, I_2, P, and NP, respectively.

We also consolidate the optimal control functions $\beta_1^*(t)$, $\beta_2^*(t)$ and their corresponding steady states β_1^s, β_2^s into a single proposition, focusing on the impact of introducing δ.

Proposition 9. Under $R_0 > 1$, when $\beta_1 + \gamma_1 + v = \beta_2 + \gamma_2 + v = 1 - \delta$, the optimal persuasions are changed as follows:

The functions of optimal controls are: $\beta_1^*(t) = min\left\{max\left\{\frac{1}{C_1}\left(\lambda_{np}(t) - \lambda_p(t)\right)I_1(t), 0\right\}, 1 - \nu - \delta\right\}$, $\beta_2^*(t) = min\left\{max\left\{\frac{1}{C_2}\left(\lambda_{np}(t) - \lambda_p(t)\right)I_2(t), 0\right\}, 1 - \nu - \delta\right\}$.

The steady states are: $\beta_1^s = min\left\{max\left\{\frac{2\alpha\left(k\varepsilon(1-v-\delta) - v^2(1-\delta)\right)}{C_1 k(1-\delta)(1-v-\delta)(v+\rho)}, 0\right\}, 1 - \nu - \delta\right\}$, $\beta_2^s = min\left\{max\left\{\frac{2(1-\alpha)\left(k\varepsilon(1-v-\delta) - v^2(1-\delta)\right)}{C_2 k(1-\delta)(1-v-\delta)(v+\rho)}, 0\right\}, 1 - \nu - \delta\right\}$.

This proposition highlights both the robustness of our core framework and the new insights gained from introducing δ.

Setting $\delta = 0$ causes the expressions to reduce exactly to those of the baseline model, which confirms the model's consistency and robustness.

Moreover, when $\delta > 0$, it provides new insights. The expressions of the optimal controls $\beta_1^*(t)$ and $\beta_2^*(t)$ still depend on the cost parameters C_1, C_2, the size of the target populations $I_1(t)$, $I_2(t)$, and the co-state variables $\lambda_p(t)$, $\lambda_{np}(t)$. This structure remains analogous to the baseline model. Then, the boundary difference emerges. In the extension model, the upper bound becomes $1 - v - \delta$ instead of $1 - v$, so δ limits the feasible set. Within the interval $[0, 1 - v - \delta]$, $\frac{\partial \beta_1^s}{\partial \delta} = 2\alpha \frac{k(-1+v+\delta)^2 \varepsilon - v^2(-1+\delta)^2}{C_1 k(-1+\delta)^2(-1+v+\delta)^2(v+\rho)} > 0$, $\frac{\partial \beta_2^s}{\partial \delta} = 2(1-\alpha) \frac{k(-1+v+\delta)^2 \varepsilon - v^2(-1+\delta)^2}{C_2 k(-1+\delta)^2(-1+v+\delta)^2(v+\rho)} > 0$, thus an increase in δ increases the optimal steady-state effort β_1^s, β_2^s. This means that more individuals remain in thinking and information processing, which may diminish the need of persuasion efforts and resources.

6.3. Different cost functions

In this subsection, we consider the different cost functions. The equilibrium points remain unchanged. Therefore, we proceed directly to the optimization analysis.

First, $\frac{1}{2} C_1 \beta_1^2(t)$ and $\frac{1}{2} C_2 \beta_2^2(t)$ can be considered as the increasing marginal costs. We consider another similar cost function $\frac{1}{2} C_1 \beta_1^2(t) + C_1 \beta_1(t)$ and $\frac{1}{2} C_2 \beta_2^2(t) + C_2 \beta_2(t)$, which impose a larger cost on persuasion effort than the baseline model.

The optimization problem under $R_0 > 1$ is formulated as follows:

$$J_\infty = \min_{\beta_1(t) \ge 0, \beta_2(t) \ge 0} \int_0^{+\infty} e^{-\rho t} \left[NP(t) - P(t) + \frac{1}{2} C_1 \beta_1^2(t) + C_1 \beta_1(t) + \frac{1}{2} C_2 \beta_2^2(t) + C_2 \beta_2(t) \right] dt$$

$$s.t. \begin{cases} \dot{U} = \varepsilon - kU(P + NP) - vU \\ \dot{I_1} = \alpha k U(P + NP) - I_1 \\ \dot{I_2} = (1 - \alpha) k U(P + NP) - I_2 \\ \dot{P} = \beta_1 I_1 + \beta_2 I_2 - vP \\ \dot{NP} = (1 - \beta_1 - v) I_1 + (1 - \beta_2 - v) I_2 - vNP \end{cases} \tag{22}$$

For the sake of simplicity, the current-value Hamiltonian function H_c of the optimization problem is as follows, which omits the time argument:

$$H_c = NP - P + \frac{1}{2} C_1 \beta_1^2 + C_1 \beta_1 + \frac{1}{2} C_2 \beta_2^2 + C_2 \beta_2 + \lambda_u(\varepsilon - kU(P + NP) - vU) + \lambda_{I_1}(\alpha k U(P + NP) - I_1) + \lambda_{I_2}\big((1 - \alpha) k U(P + NP) - I_2\big) + \lambda_p(\beta_1 I_1 + \beta_2 I_2 - vP) + \lambda_{np}\big((1 - \beta_1 - v) I_1 + (1 - \beta_2 - v) I_2 - vNP\big) \tag{23}$$

where the co-state variables λ_u, λ_{I_1}, λ_{I_2}, λ_p, and λ_{np} are associated with their state dynamic equations U, I_1, I_2, P, and NP, respectively.

For the analysis of our extension, we consolidate the optimal control functions $\beta_1^*(t)$, $\beta_2^*(t)$ and their corresponding steady states β_1^s, β_2^s into a single proposition.

Proposition 10. Under $R_0 > 1$, when another increasing marginal cost function is considered, the optimal persuasions are changed as follows:

The functions of optimal controls are: $\beta_1^*(t) = min\left\{max\left\{\frac{1}{C_1}\left(\lambda_{np}(t) - \lambda_p(t)\right)I_1(t) - 1{,}0\right\}, 1 - \nu\right\}$, $\beta_2^*(t) = min\left\{max\left\{\frac{1}{C_2}\left(\lambda_{np}(t) - \lambda_p(t)\right)I_2(t) - 1{,}0\right\}, 1 - \nu\right\}$.

The steady states are: $\beta_1^s = min\left\{max\left\{\frac{2\alpha\left(k(1-\nu)\varepsilon-\nu^2\right)}{C_1k(1-\nu)(\nu+\rho)} - 1{,}0\right\}, 1 - \nu\right\}$, $\beta_2^s = min\left\{max\left\{\frac{2(1-\alpha)\left(k(1-\nu)\varepsilon-\nu^2\right)}{C_2k(1-\nu)(\nu+\rho)} - 1{,}0\right\}, 1 - \nu\right\}$.

This proposition reveals that the structures are similar with the ones in the Section 5. In addition, because this alternative cost function imposes higher penalties on persuasion effort, it raises total costs and consequently reduces the optimal persuasion effort.

Second, we consider another cost function $C_1\beta_1(t)$ and $C_2\beta_2(t)$ that reflects constant marginal cost. The optimization problem under $R_0 > 1$ is formulated as follows:

$$J_\infty = \min_{\beta_1(t)\geq 0, \beta_2(t)\geq 0} \int_0^{+\infty} e^{-\rho t}[NP(t) - P(t) + C_1\beta_1(t) + C_2\beta_2(t)]\,dt$$

$$s.t.\begin{cases} \dot{U} = \varepsilon - kU(P + NP) - \nu U \\ \dot{I_1} = \alpha kU(P + NP) - I_1 \\ \dot{I_2} = (1 - \alpha)kU(P + NP) - I_2 \\ \dot{P} = \beta_1 I_1 + \beta_2 I_2 - \nu P \\ \dot{NP} = (1 - \beta_1 - \nu)I_1 + (1 - \beta_2 - \nu)I_2 - \nu NP \end{cases} \tag{24}$$

For the sake of simplicity, the current-value Hamiltonian function H_c of the optimization problem is as follows, which omits the time argument:

$$H_c = NP - P + C_1\beta_1 + C_2\beta_2 + \lambda_u(\varepsilon - kU(P + NP) - \nu U) + \lambda_{I_1}(\alpha kU(P + NP) - I_1) + \lambda_{I_2}\left((1 - \alpha)kU(P + NP) - I_2\right) + \lambda_p(\beta_1 I_1 + \beta_2 I_2 - \nu P) + \lambda_{np}\left((1 - \beta_1 - \nu)I_1 + (1 - \beta_2 - \nu)I_2 - \nu NP\right) \tag{25}$$

where the co-state variables λ_u, λ_{I_1}, λ_{I_2}, λ_p, and λ_{np} are associated with their state dynamic equations U, I_1, I_2, P, and NP, respectively.

Computing the partial derivatives of the Hamiltonian with respect to the β_1 and β_2:

$$\frac{\partial H_c}{\partial \beta_1} = C_1 + \lambda_p I_1 - \lambda_{np} I_1,\ \frac{\partial H_c}{\partial \beta_2} = C_2 + \lambda_p I_2 - \lambda_{np} I_2 \tag{26}$$

The values of $\frac{\partial H_c}{\partial \beta_1}$ and $\frac{\partial H_c}{\partial \beta_2}$ do not depend on β_1 and β_2. This is the defining characteristic that leads to bang-bang control.

Proposition 11. Under $R_0 > 1$, when constant marginal cost is considered, the optimal persuasions are changed as bang-bang control: $\beta_1^*(t) = \begin{cases} 1-v, & if\ C_1 + \lambda_p I_1 - \lambda_{np} I_1 \le 0 \\ 0, & if\ C_1 + \lambda_p I_1 - \lambda_{np} I_1 > 0 \end{cases}$, $\beta_2^*(t) = \begin{cases} 1-v, & if\ C_1 + \lambda_p I_1 - \lambda_{np} I_1 \le 0 \\ 0, & if\ C_1 + \lambda_p I_1 - \lambda_{np} I_1 > 0 \end{cases}$.

This proposition reveals that the optimal strategy switches between these extremes as the system evolves. Bang-bang structure also reflects the trade-off between the benefits and costs. If the marginal benefit exceeds the cost, full effort is optimal; otherwise, zero effort is optimal.

Finally, we consider another cost function $2C_1\sqrt{\beta_1(t)}$ and $2C_2\sqrt{\beta_2(t)}$ that reflects decreasing marginal cost. The optimization problem under $R_0 > 1$ is formulated as follows:

$$J_\infty = \min_{\beta_1(t)\ge 0, \beta_2(t)\ge 0} \int_0^{+\infty} e^{-\rho t}\left[NP(t) - P(t) + 2C_1\sqrt{\beta_1(t)} + 2C_2\sqrt{\beta_2(t)}\right] dt$$

$$s.t. \begin{cases} \dot{U} = \varepsilon - kU(P+NP) - vU \\ \dot{I_1} = \alpha kU(P+NP) - I_1 \\ \dot{I_2} = (1-\alpha)kU(P+NP) - I_2 \\ \dot{P} = \beta_1 I_1 + \beta_2 I_2 - vP \\ \dot{NP} = (1-\beta_1-v)I_1 + (1-\beta_2-v)I_2 - vNP \end{cases} \tag{27}$$

For the sake of simplicity, the current-value Hamiltonian function H_c of the optimization problem is as follows, which omits the time argument:

$$\begin{aligned} H_c = NP - P + 2C_1\sqrt{\beta_1} + 2C_2\sqrt{\beta_2} + \lambda_u(\varepsilon - kU(P+NP) - vU) + \\ \lambda_{I_1}(\alpha kU(P+NP) - I_1) + \lambda_{I_2}\big((1-\alpha)kU(P+NP) - I_2\big) + \lambda_p(\beta_1 I_1 + \beta_2 I_2 - \\ vP) + \lambda_{np}\big((1-\beta_1-v)I_1 + (1-\beta_2-v)I_2 - vNP\big) \end{aligned} \tag{28}$$

where the co-state variables λ_u, λ_{I_1}, λ_{I_2}, λ_p, and λ_{np} are associated with their state dynamic equations U, I_1, I_2, P, and NP, respectively.

Computing the first-order derivatives of the Hamiltonian with respect to the β_1 and β_2:

$$\frac{\partial H_c}{\partial \beta_1} = \frac{C_1}{\sqrt{\beta_1}} + \lambda_p I_1 - \lambda_{np} I_1,\ \frac{\partial H_c}{\partial \beta_2} = \frac{C_2}{\sqrt{\beta_2}} + \lambda_p I_2 - \lambda_{np} I_2 \tag{29}$$

Computing the second-order derivatives of the Hamiltonian with respect to the β_1 and β_2:

$$\frac{\partial^2 H_c}{\partial {\beta_1}^2} = -\frac{C_1}{2{\beta_1}^{\frac{3}{2}}} < 0,\ \frac{\partial^2 H_c}{\partial {\beta_2}^2} = -\frac{C_2}{2{\beta_2}^{\frac{3}{2}}} < 0 \tag{30}$$

Thus, the Hamiltonian is strictly concave in the control variables. The optimal can be obtained on the boundary. This characteristic also leads to bang-bang control.

Proposition 12. Under $R_0 > 1$, when decreasing marginal cost is considered, the optimal persuasions are also adjusted as bang-bang control: $\beta_1^*(t) = \begin{cases} 1-v & if\ 2C_1 + \sqrt{1-v}\left(\lambda_p - \lambda_{np}\right) I_1 \le 0 \\ 0 & if\ 2C_1 + \sqrt{1-v}\left(\lambda_p - \lambda_{np}\right) I_1 > 0 \end{cases}$,

$$\beta_2^*(t) = \begin{cases} 1-v & if\ 2C_2 + \sqrt{1-v}(\lambda_p - \lambda_{np})I_2 \le 0 \\ 0 & if\ 2C_2 + \sqrt{1-v}(\lambda_p - \lambda_{np})I_2 > 0 \end{cases}.$$

This bang-bang structure also reflects the trade-off between the benefits and costs. If the marginal benefit exceeds the cost, full effort is optimal; otherwise, zero effort is optimal.

7. Management implication

The insights derived from our theoretical and numerical analysis yield several actionable implications for practitioners in marketing, public relations, and crisis communication who seek to influence public opinion with greater effectiveness and resource efficiency. Effective persuasion is reflected in a greater number of persuaded individuals compared to unpersuaded ones.

First, there are several different strategies: 1) persuasion strategies targeting the central route; 2) persuasion strategies targeting the peripheral route; 3) persuasion strategies targeting neither route; 4) persuasion strategies targeting both cognitive routes. The effectiveness of persuasive communication is relevant to how well the strategy aligns with the major cognitive routes of netizens. Huolala's public opinion crisis is a good example to illustrate that. As a logistics service platform, a fatal accident involving a female passenger of Huolala occurred in 2021, followed by an immediate public attention surge. However, the company's official statement was issued several days after the incident, and although it offered detailed remedial measures aimed at persuading through the central route, the delayed response failed to persuade netizens on the peripheral route. As a result, public anger persisted, and the apology was perceived as insincere.

Persuasion strategies targeting both cognitive routes cover all possible cognitive routes of netizen groups. However, in real-world practice, it may result in excessive resource allocation while resources are limited, so it is challenging to develop persuasion strategies that target both cognitive routes simultaneously. Therefore, cost–benefit analysis should be considered when adopting such strategies.

Then, considering the more complex and dynamic settings, optimal control solutions indicate that persuasion resource allocation should be adaptively adjusted according to the size of netizens and the route-specific persuasion costs. This finding is also important in management. In practice, official agencies frequently confront a trade-off between disseminating accurate, comprehensive information via the central route and distributing rapid, easily shareable content via the peripheral route. For instance, targeting the central route often necessitates mobilizing subject-matter experts for interviews, producing detailed and verifiable reports, and instituting robust fact-checking. These activities usually entail higher time and budget outlays. In contrast, peripheral-route tactics can scale quickly but may yield shallower processing and shorter-lived attitude change. The results underscore the need for dynamic allocation across routes to maximize persuasion effect subject to resource constraints.

We introduce three extension models, which consider heterogeneous exit rates, gradual nature of cognitive processing, and different cost functions. The baseline model's consistency and robustness are confirmed. Moreover, some new insights are obtained. For example, greater persistence among individuals with formed opinions may increase the optimal persuasion effort. By contrast, a faster outflow of uninformed individuals reduces the need for persuasion resources. Besides, when more individuals remain in thinking and information processing, the need for persuasion efforts and resources may decrease. Different cost functions may lead to different optimal control forms, but the trade-off between the benefits and costs still exists.

8. Conclusion

This study develops a dynamical model for public opinion persuasion based on ELM and employs optimal control to design the optimal strategy for persuasive communication in emergency management. Compared with previous research, it examines how different responses impact various groups of netizens, each with distinct cognitive routes, thus shaping the effectiveness of public opinion persuasion over time.

Some conclusions are briefly presented. The equilibria of the model suggest that the effectiveness of persuasion is relevant to whether the selected approach corresponds to the predominant cognitive routes of the target netizen groups. In some cases, even substantial efforts by opinion leaders may have only a limited impact if the chosen method does not align with these cognitive tendencies. Considering situations that need to deal with time-varying complexities and persuasion costs, the optimal control solution reveals that resource allocation to each strategy should be continuously adjusted according to the number of netizens and the persuasion cost associated with each cognitive route.

Some areas still require further exploration. The model does not capture conversions between cognitive routes. In reality, individuals initially processing information via the peripheral route may shift to the central route when their involvement increases or the message becomes personally relevant. In addition, the influence of network structure is very important, such as the role of KOL [55] and community clustering in shaping information diffusion, but this study ignores it.

These limitations might affect the practical implementation of the proposed strategies. Specifically, the current framework remains confined to persuasion strategies operating within established cognitive routes by setting the parameter α. Integrating the mechanisms of route conversion can introduce an alternative perspective on public opinion management. For example, consistent publicity efforts that promote route conversion may encourage netizens to process information through the central route, leading to more rational engagement. Such an effect could be more effective than relying on persuasion after an emergency occurs and information has already spread. Moreover, given the influence of network structure, targeting a limited number of highly influential individuals is sometimes far more cost-effective and efficient than attempting to influence the entire population simultaneously. Therefore, future work should incorporate route conversion dynamics and integrate complex network simulations to reflect real-world mechanisms and propose optimal strategies.

Author contributions

Wanglai Li: Conceptualization, Writing–original draft; Wanglai Li and Hanzhe Yang: Methodology; Hanzhe Yang and Zhangxue Huang: Software; Huizhang Shen: Supervision; Zhangxue Huang: Validation; Wanglai Li and Huizhang Shen: Writing–review & editing.

Acknowledgments

This work is supported by National Social Science Fund of China (Grant No. 20AZD059).

Use of Generative-AI tools declaration

During the preparation of this work, the authors used generative AI tools such as ChatGPT to assist in translating the text into English and improving the clarity of the language. All AI-assisted content was carefully reviewed and revised by the authors, who take full responsibility for the final version of the manuscript.

Conflict of interest

All authors declare no conflicts of interest in this paper.

References

1. X. Wang, N. Zhang, Uncertain spreading model of internet rumors on social media platforms, *J. Ind. Manag. Optim.*, **20** (2024), 1–14. https://dx.doi.org/10.3934/jimo.2023066
2. E. Bakshy, S. Messing, L. A. Adamic, Exposure to ideologically diverse news and opinion on Facebook, *Science*, **348** (2015), 1130–1132. https://dx.doi.org/10.1126/science.aaa1160
3. Z. S. Dong, L. Meng, L. Christenson, L. Fulton, Social media information sharing for natural disaster response, *Nat. Hazards*, **107** (2021), 2077–2104. https://dx.doi.org/10.1007/s11069-021-04528-9
4. L. Luo, F. Nian, Cascading propagation of events in hierarchical supply chain networks, *Nonlinear Dyn*, **113** (2025), 17057–17080. https://dx.doi.org/10.1007/s11071-025-10881-2
5. W. Wood, Attitude change: Persuasion and social influence, *Annu. Rev. Psychol.*, **51** (2020), 539–570. https://dx.doi.org/10.1146/annurev.psych.51.1.539
6. T. Ghofrani, The impact of new media technologies on persuasive communication in the time of global crisis, *Int. J. Technol. Manage.,* **94** (2024), 419–435. https://dx.doi.org/10.1504/IJTM.2024.136420
7. F. Nian, X. Guo, J. Li, The model of information dissemination on social networks under epidemic-related panic, *Int. J. Mod. Phys. B*, **36** (2022), 2250005. https://dx.doi.org/10.1142/S0217979222500059
8. J. Jia, X. Liu, Y. Zhang, Z. Li, Y. Xu, J. Yan, Rumor propagation controlling based on finding important nodes in complex network, *J. Ind. Manag. Optim,* **16** (2020), 2521–2529. https://dx.doi.org/10.3934/jimo.2019067
9. D. Centola, *How Behavior Spreads: The Science of Complex Contagions*, Princeton University Press, 2018. https://dx.doi.org/10.2307/j.ctvc7758p.4
10. D. J. Daley, D. G. Kendall, Epidemics and rumours, *Nature*, **204** (1964), 1118–1118. https://dx.doi.org/10.1038/2041118a0
11. D. J. Daley, D. G. Kendall, Stochastic rumours, *IMA J. Appl. Math.*, **1** (1965), 42–55. https://dx.doi.org/10.1093/imamat/1.1.42
12. S. Hosseini, M. Abdollahi Azgomi, A. Torkaman Rahmani, Dynamics of a rumor‐spreading model with diversity of configurations in scale-free networks, *Int. J. Commun. Syst.*, **28** (2015), 2255–2274. https://dx.doi.org/10.1002/dac.3016

13. Y. Dong, L. Huo, L. Zhao, An improved two-layer model for rumor propagation considering time delay and event-triggered impulsive control strategy, *Chaos Solitons Fractals,* **164** (2022), 112711. https://dx.doi.org/10.1016/j.chaos.2022.112711
14. B. Gürbüz, H. Mawengkang, I. Husein, G. W. Weber, Rumour propagation: an operational research approach by computational and information theory, *Cent. Europ. J. Oper. Res*., **30** (2022), 345–365. https://dx.doi.org/10.1007/s10100-020-00727-0
15. G. Liu, F. Nian, Impact of peer pressure on higher-order network dynamics, *Chaos Solitons Fractals*, **200** (2025), 116958. https://dx.doi.org/10.1016/j.chaos.2025.116958
16. R. Qiao, Y. Hu, Dynamic analysis of a SI 1 I 2 AD information dissemination model considering the word of mouth, *Nonlinear Dyn*, **111** (2023), 22763–22780. https://dx.doi.org/ 10.1007/s11071-023-09021-5
17. M. Pazoki, H. Samarghandi, Word-Of-Mouth and estimating demand based on network structure and epidemic models, *Eur. J. Oper. Res.*, **291** (2021), 323–334. https://dx.doi.org/10.1016/j.ejor.2020.09.004
18. L. Zhu, W. Zheng, X. Zhang, Bifurcation Analysis of a Reaction–Diffusion Rumor Spreading Model with Nonsmooth Control, *Int. J. Bifurcation Chaos*, **32** (2022), 2250109. https://dx.doi.org/10.1142/S0218127422501097
19. Q. Ge, C. Huang, H. Xu, J. Zhang, H. Wang, A Novel SIR Model-Based Public Opinion Propagation and Optimal Control, In *2024 China Automation Congress (CAC)*, IEEE, 2024. https://dx.doi.org/10.1109/CAC63892.2024.10865290
20. S. Bidah, O. Zakary, M. Rachik, Modeling and Control of the Public Opinion: An Agree - Disagree Opinion Model, *Int. J. Differ. Equat.,* **2020** (2020), 5864238. https://dx.doi.org/10.1155/2020/5864238
21. Y. Cheng, L. Huo, L. Zhao, Stability analysis and optimal control of rumor spreading model under media coverage considering time delay and pulse vaccination, *Chaos Solitons Fractals*, **157** (2022), 111931. https://dx.doi.org/10.1016/j.chaos.2022.111931
22. S. Kang, X. Hou, Y. Hu, H. Liu, Dynamical analysis and optimal control of the developed information transmission model, *PLoS One*, **17** (2022), e0268326. https://dx.doi.org/10.1371/journal.pone.0268326
23. B. Li, H. Li, Q. Sun, R. Lv, H. Yan, Dynamics analysis and optimal control study of uncertain information dissemination model triggered after major emergencies, *Front. Physics,* **12** (2024), 1349284. https://dx.doi.org/10.3389/fphy.2024.1349284
24. S. Dong, L. Xu, Z. Lan, Y. A, F. Bu, W. Hua, Q. Chunlai, L.Yifei, G. Minjie, G. Kai, Multilingual SEIR public opinion propagation model with social enhancement mechanism and cross transmission mechanisms, *Sci Rep*, **14** (2024), 31081. https://dx.doi.org/10.1038/s41598-024-82024-3
25. Q. Zhang, L. Wang, X. Wang, G. Chen, Optimal Selection of Intervention Timing in Opinion Dynamics, *IEEE Trans. Autom. Control,* **70** (2025), 4392–4407. https://dx.doi.org/10.1109/TAC.2025.3528350
26. L. Luo, F. Nian, F. Li, Y. Cui, Information dissemination and phase transition in fractal social networks, *Commun. Nonlinear Sci. Numer. Simul.*, **146** (2025), 108812. https://dx.doi.org/10.1016/j.cnsns.2025.108812
27. F. Nian, X. Liu, Y. Zhang, X. Yu, Human flesh search based on information coupling, *Int. J. Mod. Phys. B,* **36** (2022), 2250025. https://dx.doi.org/10.1142/S0217979222500254

28. H. Cheng, C. Tan, C. Chiu, H. Huang, Y. Lee, Using the ELM to Explore the Impact of Fake News on Panic Vaccination Intention: Taiwan's COVID-19 Vaccination Phenomenon, *J. Glob. Inf. Manag.*, **32** (2024), 1–39. https://dx.doi.org/10.4018/JGIM.335487
29. H. A. Syrdal, S. Myers, S. Sen, P. J Woodroof, W. C. McDowell, Influencer marketing and the growth of affiliates: The effects of language features on engagement behavior, *J. Bus. Res.*, **163** (2023), 113875. https://dx.doi.org/10.1016/j.jbusres.2023.113875
30. V. Saini, L. L. Liang, Y. C. Yang, H. M. Le, C. Y. Wu, The association between dissemination and characteristics of pro-/anti-COVID-19 vaccine messages on Twitter: application of the elaboration likelihood model, *JMIR infodemiology,* **2** (2022), e37077. https://dx.doi.org/10.2196/37077
31. B. Yang, C. Liu, X. Cheng, X. Ma, Understanding users' group behavioral decisions about sharing articles in social media: An elaboration likelihood model perspective, *Group Decis. Negot.*, **31** (2022), 819–842. https://dx.doi.org/10.1007/s10726-022-09784-z
32. X. Y. Xu, Q. D. Jia, A new exploration of signaling theory in social commerce facilitated cross-border retailing: A four-stage approach, *Inf. Manage.,* **62** (2025), 104154. https://dx.doi.org/10.1016/j.im.2025.104154
33. Y. Zhang, Q. Li, J. Yan, Exploring multimodal factors in online reviews: A machine learning approach to evaluating content effectiveness, *J. Retail. Consum. Serv.*, **84** (2025), 104261. https://dx.doi.org/10.1016/j.jretconser.2025.104261
34. T. Wu, G. Yu, Persuasive Effects of Crisis Communication during Public Health Emergency Outbreaks in China, *Behav. Sci.*, **14** (2024), 885. https://dx.doi.org/10.3390/bs14100885
35. C. Yang, M. Chen, W. Zhou, X. Zhang, D. Zhang, The role of online consumer reviews in shaping Chinese consumers' purchase intentions for energy-saving household appliances: a social communication theory perspective, *Discov. Sustain.*, **5** (2024), 440. https://dx.doi.org/10.1007/s43621-024-00669-6
36. R. E. Petty, J.T. Cacioppo, The Elaboration Likelihood Model of Persuasion. In: *Communication and Persuasion. Springer Series in Social Psychology,* Springer, New York, NY, 1986. https://dx.doi.org/10.1007/978-1-4612-4964-1_1
37. M. B. Ferreira, L. Garcia-Marques, S. J. Sherman, J. W. Sherman, Automatic and controlled components of judgment and decision making, *J. Pers. Soc. Psychol.*, **91** (2006), 797–813. https://dx.doi.org/10.1037/0022-3514.91.5.797
38. M. Z. Al-Taie, S. Kadry, Information Diffusion in Social Networks. In: *Python for Graph and Network Analysis. Advanced Information and Knowledge Processing,* Springer, Cham, 2017. https://dx.doi.org/10.1007/978-3-319-53004-8_8
39. F. Nian, R. Liu, A. Cong, An incentive mechanism model based on the correlation between neighbor behavior and distance, *Int. J. Mod. Phys. C,* **31** (2020), 2050161. https://dx.doi.org/10.1142/S0129183120501612
40. W. Li, H. Shen, Z. Huang, H. Yang, J. Zhao, Research on the Dynamics of Word-of-Mouth Influencing Stock Prices, *Systems*, **12** (2024), 344. https://dx.doi.org/10.3390/systems12090344
41. C. Ding, W. Liu, Y. Sun, Y. Zhu, A delayed Schistosomiasis transmission model and its dynamics, *Chaos Solitons Fractals*, **118** (2019), 18–34. https://dx.doi.org/10.1016/j.chaos.2018.11.005
42. F. Nian, J. Feng, Phase transition in infection scale and the impact of user characteristics under the herd effect in information propagation, *Eur. Phys. J. Plus*, **140** (2025), 380. https://dx.doi.org/10.1140/epjp/s13360-025-06330-z

43. S. Li, Z. Liu, Y. Li, Temporal and spatial evolution of online public sentiment on emergencies, *Inf. Process. Manage.*, **57** (2020), 102177. https://dx.doi.org/10.1016/j.ipm.2019.102177
44. W. Li, H. Yang, H. Shen, Z. Huang, BNTree for Predicting Persuasion Effect in Digital Era Crisis Communication, *J. Theor. Appl. Electron. Commer. Res.*, **20** (2025), 276. https://dx.doi.org/10.3390/jtaer20040276
45. R. E. Petty, J. T. Cacioppo, Issue involvement can increase or decrease persuasion by enhancing message-relevant cognitive responses, *J. Pers. Soc. Psychol.,* **37** (1979), 1915. https://dx.doi.org/10.1037/0022-3514.37.10.1915
46. J. C. Hu, M. S. Modi, B. K. Szymanski, Using LLMs to Infer Non-Binary COVID-19 Sentiments of Chinese Microbloggers, *Entropy*, **27** (2025), 290. https://dx.doi.org/10.3390/e27030290
47. J. Song, Y. Li, X. Guo, K. N. Shen, X. Ju, Making mobile health information advice persuasive: an elaboration likelihood model perspective, *J. Organ. End User Comput.*, **34** (2022), 1–22. https://dx.doi.org/10.4018/JOEUC.287573
48. Y. Li, M. Zhang, G. A. Wang, N. Zhang, The effect of different types of comparative reviews on product sales, *Decis. Support Syst.,* **184** (2024), 114287. https://dx.doi.org/10.1016/j.dss.2024.114287
49. L. A. Huo, C. Ma, Dynamical analysis of rumor spreading model with impulse vaccination and time delay, *Phys A,* **471** (2017), 653–665. https://dx.doi.org/10.1016/j.physa.2016.12.024
50. C. Wu, X. Wang, J. Lin, Optimizations in Project Scheduling: A State-of-Art Survey, In: Xu, H., Wang, X. (eds) Optimization and Control Methods in Industrial Engineering and Construction. Intelligent Systems, Control and Automation: Science and Engineering, vol 72, Springer, Dordrecht, 2014. https://dx.doi.org/10.1007/978-94-017-8044-5_10
51. M. McAsey, L. Mou, W. Han, Convergence of the forward-backward sweep method in optimal control, *Comput. Optim. Appl.*, **53** (2012), 207–226. https://dx.doi.org/10.1007/s10589-011-9454-7
52. S. Ben Rhila, M. Karim, M. Rachik, Three Measures to Control a Spatiotemporal Epidemic Model: An Optimal Control Approach, *Optim. Control Appl. Methods*, **46** (2025), 937–947. https://dx.doi.org/10.1002/oca.3239
53. M. Moradi, F. Zihagh, A meta‐analysis of the elaboration likelihood model in the electronic word of mouth literature, *Int. J. Consum. Stud.*, **46** (2022), 1900–1918. https://dx.doi.org/10.1111/ijcs.12814
54. A. Degan, M. Li, Persuasion with costly precision. *Econ. Theory*, **72** (2021), 869–908. https://dx.doi.org/10.1007/s00199-021-01346-9
55. F. Nian, Z. Zhang, The influence of opinion leaders on public opinion spread and control strategies in online social networks, In: *IEEE Transactions on Computational Social Systems,* Ahead-of-print. https://dx.doi.org/10.1109/TCSS.2025.3537701