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Research article

The irreversible green technology investment strategy of manufacturers under the Cap-and-Trade mechanism

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Abstract: In this paper, we examined the optimal investment strategy in green technology for a monopolist manufacturer under a cap-and-trade mechanism, incorporating demand inertia and market uncertainty. Using a continuous-time stochastic dynamic programming framework, we modeled green technology investment as an irreversible singular control process, with demand following a mean-reverting Ornstein-Uhlenbeck process. Through singular control analysis and free boundary theory, we derived a state-dependent investment threshold that revealed the optimal strategy as sequential rather than a one-off decision. Key findings showed that manufacturers delay investment until carbon emission pressures exceed a critical threshold, emphasizing the "option value of waiting". Numerical simulations, calibrated with data from an electric vehicle manufacturer and China's carbon market, identified factors such as consumer green awareness, carbon price, and emission reduction efficiency as accelerators of investment, while high costs and demand volatility cause delays. We contribute by (1) introducing a dynamic investment threshold strategy for multi-period decisions under uncertainty, (2) quantifying the option value of waiting, and (3) providing actionable insights for policymakers and managers on carbon market design and green technology investment planning.

Keywords: OR in environment and climate change; green technology investment; irreversible investment; Cap-and-Trade; real options
Mathematics Subject Classification: 91B28, 90B50

1. Introduction

Environmental pollution has emerged as a critical global challenge, driving the adoption of market-based mechanisms such as cap-and-trade (C&T) to reduce carbon emissions [1]. Carbon trading markets, including the European Climate Exchange and the Chicago Climate Exchange, have been established worldwide to incentivize emissions reduction. As a key participant in the Paris Agreement, China launched seven carbon trading pilot programs in 2011 and established a national carbon market in 2017, positioning itself as a leader in global carbon mitigation efforts. Under the C&T mechanism, manufacturers exceeding government-allocated emission quotas must purchase additional allowances, while those with surplus allowances can monetize them [2]. This mechanism imposes significant cost pressures on manufacturers while motivating them to adopt greener technologies [3, 4].

The C&T mechanism imposes carbon emission caps and trading allowances, significantly increasing manufacturers' production costs and emission reduction pressure. In response, green technology investment has emerged as a critical strategy for manufacturers under the C&T mechanism [5–7]. Green technology investments not only reduce emission costs but also enhance market influence by attracting green-conscious consumers [8, 9]. However, for many manufacturers, especially those in traditional heavy industries and energy-intensive enterprises, the high costs and uncertain future returns of green technology investments pose significant challenges. As a result, they often adopt a "wait-and-see" strategy, valuing the flexibility to delay investments until market conditions are more favorable. A notable example is NTPC, India's largest power generation company, which once rejected an order worth approximately $2 billion to install emission reduction technologies in its coal-fired power plants. While NTPC cited technical incompatibility, the fundamental reason was the prohibitive cost. In 2016, NTPC estimated that equipping its power plant network with emission reduction technologies would cost $2.4 billion. Even with a potential 25% cost reduction, the expense remains substantial, prompting India to postpone its emission reduction deadlines. Similarly, leading East Asian technology companies, such as Samsung Electronics and Xiaomi, have been criticized for inadequate climate commitments, data disclosure, and actions, with some labeling their efforts as "lacking strong climate actions". These companies operate in the highly competitive consumer electronics market, where cost control and rapid product iteration are paramount, making immediate large-scale investments to transform their supply chains and energy structures unlikely. These cases indicate that manufacturers face numerous challenges when balancing high upfront costs with uncertain returns, which often prompts them to adopt a more cautious and sequential strategy in green technology investments [10]. An example is Ford, which plans to gradually introduce green hydrogen and renewable energy into its new production processes, aiming to reduce carbon dioxide emissions in stages [11]. Similarly, Procter & Gamble is investing in green technology in a phased manner, targeting net-zero greenhouse gas emissions across its product lifecycle, operations, and supply chain by 2040 [12]. Such long-term, sequential investment strategies are favored for their advantages in risk diversification and operational efficiency improvement.

This sequential strategy underscores the role of the "option to wait" in green technology investment, as manufacturers invest only when the marginal benefit exceeds the value of delay. However, the literature has predominantly examined green technology investment through the lenses of supply chain coordination or competitive behaviors under the C&T mechanism [13–16]. Consequently, whether manufacturers should postpone green technology investments and how to plan multi-period investment paths, have received insufficient attention. To address this gap, we investigate the following key questions:

1. What are the structural characteristics of an enterprise's optimal green technology investment strategy under the C&T mechanism and market fluctuations? Is it a multi-period progressive investment or a lump-sum decision?
2. Which key exogenous factors influence the optimal investment threshold and investment value, leading manufacturers to postpone green technology investment?
3. To what extent would neglecting the waiting value, i.e., the option value, of green technology investment lead to decision-making deviation?

We employ singular control analysis, which is well-suited for capturing dynamic decision-making under uncertainty, to investigate the manufacturer's green technology investment strategy under the C&T mechanism. To model demand inertia, we use a first-order autoregressive (AR(1)) process, which is transformed into a mean-reverting Ornstein-Uhlenbeck (O-U) stochastic process for continuous analysis. This approach enables us to capture short-term fluctuations and long-term trends in market demand. Furthermore, we model green technology investment as a dynamic irreversible process, enabling the manufacturer to dynamically adjust investments in response to uncertainties in market demand and carbon emissions. By developing a stochastic dynamic programming model and applying free boundary analysis, we derive the optimal threshold strategy for green technology investment. In the numerical simulation section, we estimate model parameters using Leapmotor's electric vehicle sales data and carbon price data from China's carbon market, further validating the effectiveness of the theoretical results. Additionally, we conduct a static sensitivity analysis for key parameters.

The contributions of this paper are threefold. First, we reveal that manufacturers' green technology investment behavior exhibits an "option to wait" value, enabling them to postpone investment until carbon emission pressure reaches a critical threshold. This strategy mitigates risks stemming from market and regulatory uncertainties, offering a robust framework for risk management. Second, we introduce a dynamic investment threshold strategy grounded in the C&T mechanism, which can guide manufacturers in making decisions regarding multi-period or lump-sum investments under uncertain market conditions. We not only provide a theoretical foundation for corporate managers to formulate green technology investment plans but also offer practical insights for policymakers to design more effective carbon market mechanisms, fostering sustainable economic growth. Third, we identify key factors influencing manufacturers' green technology investment decisions. Our numerical analysis shows that consumer green preferences, carbon prices, and emission reduction efficiency accelerate investment timing, while investment costs and demand volatility delay it.

The rest of the paper is organized as follows. In section 2, we review the relevant literature, followed by the problem formulation in Section 3. In section 4, we present preliminary results and a verification theorem, while in section 5, we analyze the free boundary of green technology investment. Finally, in section 6, we conduct numerical experiments and static analysis based on actual data.

2. Literature review

2.1. Green technology investment

Many researchers have extensively explored strategies for implementing green technology within the C&T mechanisms, with the primary objective of achieving profit and emission targets at the lowest possible cost. We classify these studies into three distinct streams.

In the first stream, we focus on analyzing how the decision to adopt green technology influences the production and carbon trading strategies of monopolistic manufacturers. These researchers explore the implementation of green technology as a multi-period project [17, 18], investigate the dynamics of Stackelberg games [5], evaluate the impact of government low-carbon subsidies [19], and examine the interaction between manufacturers, suppliers, and retailers in a game-theoretic framework [16]. However, these researchers typically assume a static decision-making framework, ignoring the dynamic nature of investment decisions under market uncertainty. In contrast, we incorporate stochastic dynamic programming to capture the evolving market conditions and the option value of waiting, providing a more realistic representation of green technology investment strategies. Compared to researchers [20] who employ real options analysis to examine the timing of green technology investments, we advance this line of research by incorporating the irreversibility of green technology investments and modeling them as a singular control process. This approach enables our analysis to further explore the optimal timing of progressive green technology investments, which aligns more closely with the decision-making practices of real-world managers.

The second stream focuses on analyzing how the implementation of green technology influences the production and sales strategies of supply chain members under different coordinate rules and carbon regulations. These researchers examine the impact of carbon quotas [15], the effects of cost-sharing mechanisms [14], the implications of dual-channel supply chains [21], the concept of green supply chains [22], and the dynamics influenced by different supply chain power structures [16, 23]. However, a common limitation across this stream is the treatment of investment strategies as continuous decision variables, which fails to capture the discrete, lump-sum nature of real-world green technology investments. Our paper addresses this gap by modeling investment decisions as irreversible singular control processes. The third stream focuses on analyzing the effect of government decisions on the production strategies of manufacturers implementing green technology. These researchers employ evolutionary game theory to assess the influence of government carbon pricing policies [24], examine the impacts of government low-carbon subsidies [13, 25], and explore the effects of diverse carbon quota regulations [26, 27].

Overall, researchers have mostly examined investment strategies in green technology within a static decision framework [26]. However, when addressing dynamic investments, research typically treats the strategy as a continuous decision variable, which may not align with practical applications [5, 29, 30]. Therefore, we model the investment strategy for green technology as a time-varying decision variable with singularity and we incorporate the often-neglected aspect of the irreversibility of green technology investments. This enables us to study discrete green technology investment decisions in a dynamic manner, and can also be interpreted as a form of impulse control [31, 32], determining the timing and magnitude of adjustments to the dynamic system.

On the other hand, we find that most researchers consider the product price as a primary decision variable [18, 27]. However, empirical findings suggest that the “cost-plus margin” method is predominantly deployed by firms [33–35]. This method involves establishing the selling price by adding the production costs plus a targeted profit margin, thereby ensuring that enterprises remain profitable while recouping expenses. In aligning with these findings, we also adopt the cost-plus margin approach to pricing products. The efficacy and prevalence of this pricing model have furthermore been the subject of extensive discussion [36].

2.2. Irreversible investment problem

The singular control framework is a suitable approach for addressing the problem of irreversible green technology investment. This framework has been extensively studied within the context of real options and optimal capacity expansion. It was initially introduced in the finite fuel problem, which involved determining the optimal strategy for controlling a spaceship's movement toward a target under the constraint of limited fuel [37]. In the context of singular stochastic control problems, the control strategy is often non-continuous. The strategy is partitioned into regions of waiting and action, and the shape of the value function at different stages is described using variational inequalities. Additionally, within this framework, the admissible cumulative investment level typically has an upper bound. Therefore, understanding the admissible upper limits for each investment decision becomes crucial, leading to the central role of the free boundary problem in such problems [38–42]. In terms of solution approaches, there are generally two options. The first approach is the guess-and-verify approach, where candidate value functions are empirically guessed to obtain a solution for the value function [32, 43, 44]. The second approach involves the use of the viscosity solution approach to obtain some meaningful properties of the value function [45, 46].

In our paper, we model the problem as a two-dimensional degenerate singular stochastic control problem, taking into account the upper bound on the cumulative level of green technology investment. Similar model setups can be found in [47]. One state variable represents the demands of the manufacturer, which are directly linked to production and demand, and follow an O-U process. The other state variable represents the cumulative level of green technology investment. It should be noted that in our model, higher levels of cumulative green technology investment correspond to lower carbon emission levels. To gain further insights into the results, we reference [32] as the main theoretical framework, as it can provide an explicit solution to such a two-dimensional degenerate singular stochastic control problem.

2.3. Summary

Despite the rich insights provided by existing research, such studies exhibits two major limitations. First, most research is confined to static decision frameworks, failing to capture the dynamics and uncertainties inherent in real-world investment decisions. Second, when addressing dynamic investments, researchers typically treat investment strategies as continuous decision variables, contradicting the discrete, phased nature of significant investment decisions in practice. Consequently, these studies cannot accurately reflect the value of "wait-and-see" strategies and the impact of market inertia.

To address these limitations, we employ stochastic dynamic programming to investigate the green technology investment decisions of manufacturers under uncertain market condition. For our model, we aim to explain why manufacturers might delay investments and how they can flexibly plan multi-period investments. Our approach quantifies the option value of waiting and captures the continuous evolution of market states. Additionally, we emphasize the irreversibility of green technology investments as a critical feature. To this end, we model the investment process as a monotonically increasing càdlàg process. This modeling approach treats discrete investment decisions as impulse controls on a dynamic system, thereby enabling the simultaneous determination of optimal investment timing and scale. In summary, the core distinctions between this study and previous literature are: (1) Replacing static models with multi-period dynamic models to capture the waiting value and market inertia. This approach provides a more realistic framework for understanding how manufacturers can strategically delay investments in response to market uncertainty. (2) Treating investment decisions as irreversible singular control processes rather than continuous adjustments. This modeling choice reflects the practical reality of discrete, phased investments in green technology, offering new insights into the timing and scale of such investments. (3) Utilizing stochastic dynamic programming and singular control theory to solve this two-dimensional degenerate stochastic control problem. This methodological innovation enables us to derive optimal investment strategies under dynamic and uncertain conditions, contributing to theoretical and practical advancements in green technology investment research. To more clearly illustrate the similarities and differences between this paper and the literature, we summarize the core features of relevant papers in Table 1.

Table 1. Summary of literature on green technology investment

	Model Dynamics			Influencing Factors			
Papers	**Single-period**	**Multi-period**	**Dynamic**	**Market uncertainty**	**Cost pass-through**	**Consumer green awareness**	**Modelling approach**
Du et al. [2]	×			×			Game theory Profit max. Social welfare
Krass et al. [30]	×				×		Game theory Profit max.
Drake et al. [48]		×		×			Profit max.
Cap et al. [25]	×				×	×	Game theory Profit max. Social welfare
Xu et al. [14]	×						Profit max.
Hassan et al. [49]		×		×			Integer prog. Profit max.
Yang et al. [5]		×					Integer prog. Nonlinear prog.
Li et al. [3]	×						Profit max.
Turken et al. [50]	×						Profit max.
Yang et al. [26]	×				×		Nonlinear prog. Profit max.
Amina et al. [51]		×			×		Integer prog. Profit max. Multi-criteria
Jiang et al. [52]			×		×	×	Profit max. Optimal control
Fan et al. [27]	×			×			Profit max. Stochastic prog.
Wu and Chiu [28]	×			×			Game theory Profit max.
Yang and Chen [18]		×			×	×	Profit max. Social welfare
Liu et al. [20]			×	×			Game theory Profit max. Real options
Our papers		×	×	×	×	×	Dynamic stochastic prog. Profit max. Singular control Real options

3. Problem formulation

In this paper, we focus on a monopolist manufacturer's irreversible green technology (GT) investment strategy in a direct channel under the C&T mechanism. We abstract away from strategic interactions with rivals, such as supply chain coordination or competitive games [5, 14, 30]. This focus on a monopolistic setting is a classic approach in operations management and economics for studying investment timing under uncertainty [53, 54], as it enables us to isolate the intrinsic investment problem from the confounding effects of strategic competition. Consequently, we can precisely examine the interaction between the manufacturer's internal cost structure (including production and investment costs) and the external, market-mediated incentives created by the C&T mechanism.

3.1. Emission costs under the C&T mechanism

Following [15] and [17], we assume that under the C&T mechanism, the government allocates a fixed amount of carbon emission allowances (or quotas), denoted by Q, to the manufacturer for each compliance period t. This approach belongs to a type of "grandfathering" allocation method, where the permit allocation is based on historical emissions [55]. The manufacturer's production activities generate carbon emissions. If the manufacturer's total emissions exceed the quota allocated, it must purchase additional allowances from the carbon trading market at the emission allowances price, which we call carbon price a. Conversely, if the manufacturer's emissions fall below Q, it can sell its surplus allowances on the market, generating additional revenue at the same price a.

Furthermore, while grandfathering is a common initial allocation method, it has been associated with challenges such as over-allocation and weakened abatement incentives in practice [56]. Consequently, alternative schemes such as benchmarking are increasingly employed. Under a benchmarking rule, allowances are allocated based on a sector-specific emissions benchmark and the manufacturer's output level rather than historical emissions. This could fundamentally alter the manufacturer's incentive structure, as the marginal benefit of GT investment becomes more directly tied to production volume to mitigate carbon leakage risks [57]. However, to maintain analytical focus and isolate the intrinsic investment problem from the complexities of dynamic allocation rules, we retain the grandfathering assumption in this paper. This provides a clear baseline understanding, and exploring the implications of benchmarking allocation remains a valuable direction for future research.

The net carbon trading cost (or revenue) becomes an integral part of the profit function. Specifically, without any GT, for a given production quantity q_t, the net carbon trading cost in period t can be formulated as

$$L_t := a(Eq_t - Q), \tag{3.1}$$

where Eq_t represents the total emissions from producing q_t units, and E represents the emission intensity.

The manufacturer can reduce the emission intensity of product production by implementing GT investment [58]. If the level of GT owned by the manufacturer is y, then the emissions per unit of product are $E(1-ry)$, where r represents the relationship between the investment in GT and the emission reduction ratio [30]. The net carbon trading cost in period t can be rewritten as

$$L_t(y) = a(E(1-ry)q_t - Q). \tag{3.2}$$

Equation (3.2) indicates that investing in GT can reduce the costs incurred by emissions during the production process.

3.2. Uncertain market demand and consumer behavior

Next, we model the demand function faced by the monopolist manufacturer. In the real world, consumers' consumption habits usually have dynamic inertia or stickiness. For example, consumers' brand loyalty makes it highly likely that customers with high purchase volumes in the previous period will continue to make purchases in the current period [59–61]. Or the network effect of products enables a large user base from the previous period to drive a further increase in current period demand [62, 63]. This also leads manufacturers to typically predict future demand based on historical data, providing a basis for their operational strategies. In terms of pricing decisions, although the core goal of the manufacturer is to maximize profits, when demand is not only affected by the current period price but also depends on its past values through word-of-mouth effects and habit formation, the decision making becomes more complex. At this time, the manufacturer must take this dynamic of demand into account. This means that the optimal pricing strategy needs to be determined within an inter-temporal framework, that is, the price setting in each period will affect the future demand state, and thus affect long-term profits. Therefore, we consider the temporally-dependent demands, where the manufacturer monitors periodic customer demands. Following [64], these demands follow a stationary first-order autoregressive (AR(1)) process and are given by

$$D_t = \mu_D + \rho_D(D_{t-1} - \mu_D) + \epsilon_t, \tag{3.3}$$

where $|\rho_D| < 1$ ensures a stationary demand process and $\{\epsilon_t\}_{t=-\infty}^{\infty}$ is a sequence of independent and identically distributed random variables with $\mathbf{E}[\epsilon_t] = 0$ and $\mathbf{Var}[\epsilon_t] = \sigma^2$, indicating that current demand is influenced by the prior period's demand. The long-term mean μ_D is regarded as a fixed parameter determined by the external market. Next, we expand this fixed parameter by taking into account the manufacturer's current GT level to capture the impact of GT investment on the manufacturer's supply and demand. It should be noted that due to the normal distribution property of (3.3), the value of D_t may be negative in some cases. Negative values of D_t reflect scenarios where market conditions (e.g., oversupply or economic shocks) temporarily reduce the effective demand below zero, which implies costly inventory disposal [65].

Next, we consider the impact of GT on the long-term demand mean faced by the manufacturer. First, the adoption of GT enhances the environmental-friendly attributes of the product, which can attract environmentally-conscious consumers [9, 14]. Second, the R&D and application of GT are often accompanied by an increase in production costs. This part of the cost is usually transferred to consumers in the form of a green premium, leading to an increase in the product price and thus having a dampening effect on the long-term demand mean [66, 67]. The net effect of these opposing factors may have an impact on the long-term average market demand for the product. Therefore, to capture the net effect of GT investment on the long-term demand mean, we define μ_D as a function of the manufacturer's current GT level y:

$$\mu_D(y) = \hat{\mu} - dp(y) + sy, \tag{3.4}$$

where $\hat{\mu}$ represents the long-term demand mean, $d > 0$ represents the consumers' price sensitivity coefficient, and $s > 0$ represents the consumers' green awareness. The function $p(y)$ represents the unit price of the product and is related to y.

Regarding the setting of $p(y)$, referring to [68], we assume that the unit cost of the product is $c_p + by$. This indicates that the manufacturer's current GT level will lead to an increase in the unit price of the product. Here, by represents the increase in unit production cost, where the increase ratio is related to the coefficient b and c_p is the cost of a unit product. This setup is also commonly observed in practice, as evidenced by instances where organic agricultural products are often priced higher than conventional ones, and electric vehicles are more expensive than traditional gasoline-powered cars. Additionally, it is noteworthy that price-makers heavily rely on cost information to determine the prices of their goods and services [69]. The researchers in [70] argued that cost information plays a pivotal role in setting prices for companies that are unable to employ a competition-based approach. Following this line of thought, we assume that manufacturers typically adhere to the prevalent guideline of seeking to recoup all costs and achieve the desired margin. Therefore, we express the cost-based price as the retail price, denoted as

$$p(y) = (c_p + by)\pi_p, \tag{3.5}$$

where π_p represents the mark-up rate over cost [36]. Consequently, equation (3.4) shows that the long-term demand mean faced by the manufacturer is not only affected by the unit price of the product but also by the manufacturer's current GT level y. Although an increase in the GT level will lead to an increase in the unit price of the product and reduce consumers' willingness to purchase the product, it will also attract an additional group of environmentally conscious consumers to purchase the product. These two effects will jointly affect the long-term demand mean.

3.3. Demand dynamics and production

We assume the manufacturer adopts a make-to-order strategy. This implies that in each period t, the manufacturer can flexibly and promptly adjust its production plan according to the observed current-period demand D_t, and set the production volume q_t to exactly meet the observed demand level in that period, and

$$q_t = D_t. \tag{3.6}$$

Therefore, by combining equations (3.3), (3.4), and (3.5), the demand dynamic process can be expressed as a function of the manufacturer's GT level y. Moreover, the production quantity q_t is endogenously determined by this demand process and will be directly substituted into the carbon trading cost function (3.2) and the profit function. We now transition to a continuous-time framework and denote the demand state variable as X_t:

$$\Delta X_t = (1 - \rho_D)(\hat{\mu} - d(c_p + by)\pi_p + sy - X_{t-1}) + \epsilon_t, \tag{3.7}$$

where $\Delta X_t = X_t - X_{t-1}$.

It is a well-established fact that the regularly sampled stationary O-U process, also known as a continuous-time autoregressive process of the first order (CAR(1)), can be represented as a discrete-time autoregressive model (AR(1)) with independent and identically distributed (i.i.d.) noise [71]. Based on the assumption of $|\rho_D| < 1$ and i.i.d of $\{\epsilon_t\}_{t=-\infty}^{\infty}$, we approximate the discrete-time difference equation (3.7) by a continuous-time differential equation to model the evolution of the variable of interest over time.

Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space with a filtration $\mathbb{F}$ satisfying the usual conditions, and with a standard one-dimensional $\mathbb{F}$-Brownian motion W. We take the infinitesimal limit as the time step approaches zero, replacing discrete changes in X with the differential dX_t, leading to the stochastic differential equation (SDE):

$$dX_t = k(\hat{\mu} - d(c_p + by)\pi_p + sy - X_t)dt + \sigma dW_t, \quad X_{0^-} = x, \tag{3.8}$$

where $k = 1 - \rho_D$, dX_t represents an infinitesimal change in X during an infinitesimal change in time dt, and σdW_t signifies a continuous-time stochastic process with dW_t denoting the increment of a Wiener process (also known as a Brownian motion), capturing the random fluctuations in continuous time. This transition assumes that the discrete random shocks ϵ_t correspond to the increments of the continuous stochastic process σ, dW_t in their impact on X, with σ representing the intensity of the continuous-time random fluctuations. This approximation of discrete systems with continuous-time models is widely adopted, as seen in the pricing of options using the Black-Scholes equations [72].

3.4. Irreversible green technology investment

Next, we introduce the dynamics of the GT level. This means that the manufacturer's GT level is no longer fixed but will increase with GT investment. However, we note that in the real world, GT investment is usually irreversible. As GT investment is typically accompanied by high capital input, its sunk cost is substantial, which means that once GT is implemented, it is usually not dismantled [73]. In addition, the irreversibility of GT investment not only stems from the fact that the technology itself cannot be cancelled, but also from the path-dependence formed by the economic, industrial, and social awareness changes after the successful deployment and operation of GT. This makes it difficult and even undesirable for society to return to the previous state, which is known as the technology lock-in effect [74, 75]. Therefore, manufacturers will demonstrate the "option value of waiting", that is, they will postpone GT investment and take action after the market risks are reduced (refer to [31, 32]). The irreversible GT level is modeled by the process $Y^{y,I}(t)$, which can correspond to the manufacturer's emission reduction skill. It is given by

$$Y_t^{y,I} = y + I_t, \tag{3.9}$$

where $y = Y(0)$ is the initial GT level, and I_t is defined as the amount of GT investment. According to the irreversibility of investment, we assume that $I(t)$ is an $\mathbb{F}$-adapted, nonnegative, and increasing càdlàg process representing the total GT investment over the interval $[0, t]$. When the manufacturer invests I_t in GT at time t, the GT level will accumulate to $Y_t^{y,I}$. Equation (3.9) describes an investment process where GT investment occurs at specific points in time and is irreversible. This better aligns with real-world decision-making.

Regarding the cost of GT investment, we make the following considerations. The implementation of GT typically requires substantial capital expenditure [76–78]. For instance, installing photovoltaic arrays is a typical example. These costs occur at the beginning of the investment and are the major financial hurdles that manufacturers face when considering GT [79]. Referring to the framework of static analysis [50], we set the initial investment cost of GT as a constant, c. The investment cost is linear in the amount of GT implemented. Thus, increasing the technology level by an amount I_t incurs a cost of cI_t.

Additionally, we assume $Y_t^{y,I} \leq \bar{y}$, indicating that the GT adoptable by manufacturers is limited, and the upper bound of the initial investment for GT is $c\bar{y}$. $Y_t^{y,I} = \bar{y}$ represents that the manufacturer's GT level has reached 100%, and it is no longer possible to continue investing in GT. This consideration is based on the following reasons: In general, the production process has physical or chemical "rigid" links. Therefore, it is impossible to completely eliminate the emissions from the consumption of fossil energy through GT [80, 88]. We define the set of admissible GT investment strategies as

$$\begin{aligned} \mathcal{I}(y) = \{ I : \Omega \times [0, \infty) \to [0, \infty) : (I_t)_{t\geq 0} \text{ is } \mathbb{F}\text{-adapted}, \\ t \mapsto I_t \text{ is increasing, càdlàg, with } I_{0^-} = 0, 0 \leq I_t \leq \bar{y} - y \text{ a.s.} \}. \end{aligned} \tag{3.10}$$

Therefore, after introducing (3.9), the demand process $X_t^{x,y,I}$ faced by the manufacturer follows

$$dX_t^{x,y,I} = k\left(\hat{\mu} - d(c_p + bY_t^{y,I})\pi_p + sY_t^{y,I} - X_t^{x,y,I}\right)dt + \sigma dW_t, \quad X_{0^-}^{x,y,I} = x. \tag{3.11}$$

Setting $\mu = \hat{\mu} - dc_p\pi_p$ and $\beta = db\pi_p - s$, we reformulate (3.11) as

$$dX_t^{x,y,I} = k((\mu - \beta Y_t^{y,I}) - X_t^{x,y,I})dt + \sigma dW_t, \quad X_{0^-}^{x,y,I} = x, \tag{3.12}$$

and the uncontrolled demand function that does not invest in GT is defined as X^x, with dynamics given by

$$dX_t^x = k(\mu - X_t^x)dt + \sigma dW_t, \quad X_{0^-}^x = x. \tag{3.13}$$

We assume that $\mu = \hat{\mu} - dc_p\pi_p > 0$ and $\beta = db\pi_p - s > 0$. The condition $\mu > 0$ indicates that the long-term demand is positive, and $\beta > 0$ implies that the act of passing on costs to consumers always reduces consumer demand.

3.5. *Profit function and optimization problem*

We assume the manufacturer is a rational, risk-neutral decision-maker whose objective is to maximize the expected present value of all future cash flows net of the cost of investment, which we call expected profit here. Without considering the manufacturer's survival duration or exit mechanism, the expected profit is given by

$$\begin{aligned} &\mathcal{J}(x,y,I) \\ &= \mathbf{E}\Big[\int_0^\infty e^{-\rho t}\Big(\big(p(Y_t^{y,I}) - (c_p + bY_t^{y,I})\big)\, X_t^{x,y,I} - L_t(Y_t^{y,I})\Big)dt - c\int_0^\infty e^{-\rho t}dI_t\Big] \\ &= \mathbf{E}\Big[\int_0^\infty e^{-\rho t}\Big(\pi(c_p + bY_t^{y,I})\, X_t^{x,y,I} - a\big(E(1 - rY_t^{y,I})X_t^{x,y,I} - Q\big)\Big)dt - c\int_0^\infty e^{-\rho t}dI_t\Big], \end{aligned} \tag{3.14}$$

where ρ is the discount factor and $\pi := \pi_p - 1$ represents the profit margin rate. The expected profit consists of three components: The integrated instantaneous profit, $\big(p(Y_t^{y,I}) - (c_p + bY_t^{y,I})\big)\, X_t^{x,y,I}$, which is revenue from product sales net of production cost; the integrated instantaneous carbon emission cost, $a\big(E(1 - rY_t^{y,I})X_t^{x,y,I} - Q\big)$, which represents the net cost (or revenue if negative) from trading emission allowances under the C&T mechanism; and the total discounted cost of GT investment, defined as:

$$c\int_0^\infty e^{-\rho t}dI_t := \sum_{t\geq 0, \Delta I_t \neq 0} ce^{-\rho t}\Delta I_t + c\int_0^\infty e^{-\rho t}dI_t^c. \tag{3.15}$$

Here, $\Delta I_t := I_t - I_{t-}$ and I_t^c denote the discrete and continuous components of I_t, respectively. To ensure model realism, we assume $\pi c_p - aE > 0$ and $1 - r\bar{y} > 0$. The first assumption implies that the gross profit per unit product exceeds its associated emission cost, ensuring basic profitability. The second assumption implies that GT investment cannot reduce emissions to zero, which is consistent with physical limitations.

The manufacturer's optimization problem is thus

$$V(x,y) = \max_{I \in \mathcal{I}(y)} \mathcal{J}(x,y,I), \quad (x,y) \in \mathbb{R} \times [0,\bar{y}]. \tag{3.16}$$

We note that the optimal strategies $I^*(t)$ derived from (3.16) form a state-feedback control law. That is, they are functions of the current demand shock $X(t)$ and the existing technological level $Y(t)$ observed by the manufacturer. This implies that the strategies are essentially non-pre-determined and are dynamically adjusted as the uncertainties are gradually revealed, which is consistent with the adaptive behavior of decision-makers in the real world. However, our model implicitly rests on the pre-commitment assumption, which is usual in stochastic optimal control theory [81]. This assumes the manufacturer has the ability and commitment to implement throughout the entire planning horizon the feedback strategy that was optimal at time zero. While this approach yields a time-consistent policy (i.e., the future actions remain optimal when viewed from the future), it precludes the possibility of the decision-maker deviating from the pre-specified plan upon re-evaluating the optimization problem at a later date. The purpose of this approach is not to perfectly replicate reality, but to provide a normative benchmark. This benchmark reveals the optimal action rules that a manufacturer should follow when fully considering all possible future risks and values.

3.6. Summary of Model Assumptions

The core assumptions underpinning this framework are summarized as follows:

(A1) We consider a profit-maximizing monopolist, which allows us to isolate the investment timing problem from competitive dynamics, focusing squarely on the interplay between the C&T mechanism and the firm's internal cost structure.

(A2) The manufacturer is a rational, risk-neutral agent that operates with perfect information regarding the model's parameters and dynamics. While the optimal strategy is derived under pre-commitment, it yields a time-consistent feedback control law.

(A3) The product is priced using a cost-plus methodology ($p(y) = (c_p + by)\pi_p$). This is a highly common practice in industry, supported by extensive managerial accounting literature [36, 69, 70], and it simplifies the pricing decision to focus on the core investment problem.

(A4) Market demand is modeled by a continuous-time O-U process, an approximation of a discrete-time AR(1) process. This captures key empirical features such as demand inertia and mean-reversion to a level $\mu_D(y)$ that is endogenously shifted by the manufacturer's GT level y and consumer preferences, s.

(A5) A make-to-order strategy is employed, meaning production exactly matches demand in every period ($q_t = X_t$). This abstracts away from inventory management and enables us to focus solely on the investment and emission trading decisions.

(A6) GT investment is fully irreversible. The control process $I(t)$ is non-decreasing, reflecting the sunk costs and path dependency (e.g., technological lock-in) associated with major environmental upgrades.

(A7) We model the C&T mechanism with a fixed, grandfathered emission quota Q allocated at the beginning of each compliance period and a fixed carbon price a. We abstract from potential policy uncertainties (e.g., changing caps or prices) to establish a clear baseline understanding of firm behavior under a stable regulatory regime.

All parameters associated with these assumptions, along with their economic interpretations, are summarized in Table 2.

Table 2. Model parameters

Parameters	Description	Range
ρ	Discount factor	$\rho > 0$
k	The rate of reversion	$0 < k < 1$
μ	The composite long-term demand parameter	$\mu > 0$
σ	The volatility of demand	$\sigma > 0$
c	The green technology investment cost	$c > 0$
E	Emission intensity	$E > 0$
a	Carbon price	$a > 0$
$\bar{y}$	Green technology investment cap	$\bar{y} > 0$
d	The consumers' sensitivity to price	$d > 0$
b	The additional production cost per unit resulting from green technology investment	$b > 0$
s	The green awareness of consumers	$s > 0$
r	The emission reduction rate	$0 < r < 1$
π_p	The mark-up rate over cost	$\pi_p > 1$
c_p	The unit production cost	$c_p > 0$

4. Dynamic programming approach

In this section, we employ the dynamic programming approach to characterize the manufacturer's value function and optimal GT investment strategy I^*. We first derive the associated Hamilton-Jacobi-Bellman (HJB) equation. Then, a verification theorem establishes that the solution to the HJB equation indeed equals the value function defined in (3.16), and the corresponding strategy I^* is optimal.

We now derive the HJB equation heuristically by considering the manufacturer's marginal decisions in continuous time [32]. The manufacturer continuously monitors demand, deciding at each moment whether to invest immediately or wait. If investing in GT cannot increase the expected profit, the manufacturer will choose to wait for a short period of time Δt. Therefore, the following inequality is satisfied

$$V(x,y) \geq \mathbf{E}\Big[\int_0^{\Delta t} e^{-\rho s}\Big(\pi(c_p+by)X_s^{x,y} - a\left(X_s^{x,y}(1-ry)E - Q\right)\Big)ds + e^{-\rho\Delta t}V(X_{\Delta t}^{x,y}, y)\Big], \tag{4.1}$$

for $(x,y) \in \mathbb{R}\times[0,\bar{y})$. If investing in GT can increase the expected profit, the manufacturer will invest an increment of ϵ in GT to ensure that the expected profit is optimized, and we have

$$V(x,y) \geq V(x,y+\epsilon) - c\epsilon. \tag{4.2}$$

We define the second-order differential operator as

$$\mathcal{L}^y u(x,y) := \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}u(x,y) + k\left((\mu-\beta y)-x\right)\frac{\partial}{\partial x}u(x,y). \tag{4.3}$$

Applying Itô's formula to the last term on the right-hand side of (4.1). Then we divide the result by Δt, and letting $\Delta t \to 0$, we obtain

$$\mathcal{L}^y w(x,y) - \rho w(x,y) + \pi(c_p+by)x - a\left(x(1-ry)E - Q\right) \leq 0,$$

for $(x, y) \in \mathbb{R} \times [0, \bar{y})$, which represents the manufacturer's expected profit when waiting is the optimal strategy. By dividing (4.2) by ϵ and letting $\epsilon \downarrow 0$, we have

$$V_y(x, y) - c \leq 0,$$

which represents the manufacturer's marginal expected profit regarding GT investment. Therefore, the expected profit V should be consistent with the solution w of the following HJB equation

$$\begin{aligned} \max \Big\{ & \mathcal{L}^y w(x, y) - \rho w(x, y) + \pi(c_p + by)x \\ & -a\left(x(1 - ry)E - Q\right), w_y(x, y) - c \Big\} = 0. \end{aligned} \tag{4.4}$$

The HJB equation (4.4) partitions the feasible domain of the optimization problem (3.16) into two subsets: The waiting region

$$\mathbb{W} := \Big\{(x, y) \in \mathbb{R} \times [0, \bar{y}) : \mathcal{L}^y w(x, y) - \rho w(x, y) + P(x, y) = 0, w_y(x, y) - c < 0\Big\}, \tag{4.5}$$

and the investment region

$$\mathbb{I} := \Big\{(x, y) \in \mathbb{R} \times [0, \bar{y}) : \mathcal{L}^y w(x, y) - \rho w(x, y) + P(x, y) \leq 0, w_y(x, y) - c = 0\Big\}, \tag{4.6}$$

where we denote

$$P(x, y) = \pi(c_p + by)x - a\left(x(1 - ry)E - Q\right)$$

for simplicity.

Although the proof of the verification theorem follows ideas similar to [32], the specific form of our value function and state dynamics necessitates adjustments to the analysis. An appropriate solution to the HJB equation (4.4) corresponds to the value function, provided that there exists an admissible GT investment strategy that keeps the state process (X, Y) within the waiting region $\mathbb{W}$ with minimal effort. This is achieved by increasing the level of GT whenever (X, Y) enters the investment region $\mathbb{I}$. Here, $\overline{\mathbb{W}}$ denotes the closure of $\mathbb{W}$. Before presenting the Verification Theorem, we provide some preliminary results.

We denote a non-investment strategy by the function $I^0 \equiv 0$, and denote the GT process implied by I^0 as $(X_t^{x,y})_{t\geq 0}$, where $X_t^{x,y} \equiv X_t^{x,y,I^0}$ and $X_{0-}^{x,y,I^0} = x \in \mathbb{R}$. This represents the manufacturer's strategy of not investing in any GT. In this case, the expected profits of the manufacturer following a non-investment strategy are described by the function $R : \mathbb{R} \times [0, \bar{y}] \mapsto \mathbb{R}$. We can obtain

$$\begin{aligned} R(x, y) &= \mathcal{J}(x, y, I^0) \\ &= \frac{x}{\rho + k}(\pi c_p - aE) + \frac{xy(\pi b + arE)}{\rho + k} + \frac{k(\mu - \beta y)(\pi c_p - aE)}{\rho(\rho + k)} \\ &+ \frac{k(\pi b + arE)(\mu - \beta y)y}{\rho(\rho + k)} + \frac{aQ}{\rho}. \end{aligned} \tag{4.7}$$

The following results provide a growth condition and a monotonicity property of the value function V and establishes its connection to the function R.

Lemma 4.1. *For any initial condition $(x, y) \in \mathbb{R} \times [0, \bar{y}]$, admissible control $I \in \mathcal{I}(y)$ and integer $C \geq 0$, for any $\hat{\rho} > 0$,*

$$\mathbf{E}\left[\sup_{t\geq 0} e^{-\hat{\rho}t}|X_t^x|\right] \leq C(1 + |x|). \tag{4.8}$$

Proof. See Appendix A. □

Proposition 4.2. *There exist a constant $K > 0$ such that for all $(x, y) \in \mathbb{R} \times [0, \bar{y}]$ one has*

$$|V(x, y)| \leq K(1 + |x|). \tag{4.9}$$

Moreover, $V(x, \bar{y}) = R(x, \bar{y})$, and V is increasing in x.

Proof. See Appendix B. □

Next comes the verification theorem.

Theorem 4.3. *(Verification Theorem). Suppose there exists a function $w : \mathbb{R} \times [0, \bar{y}] \mapsto \mathbb{R}$ such that $w \in C^{2,1}(\mathbb{R} \times [0, \bar{y}])$ solves the HJB equation* (4.4) *with boundary condition $w(x, \bar{y}) = R(x, \bar{y})$, and satisfies the growth condition*

$$|w(x, y)| \leq K(1 + |x|), \tag{4.10}$$

for a constant $K > 0$. Then $w \geq V$ on $\mathbb{R} \times [0, \bar{y}]$. Moreover, suppose that for all initial values $(x, y) \in \mathbb{R} \times [0, \bar{y})$, there exists a process $I^* \in \mathcal{I}(y)$ such that

$$(X_t^{x,y,I^*}, Y_t^{y,I^*}) \in \overline{\mathbb{W}},\ \forall t \geq 0,\ \mathbb{P}\text{-}a.s., \tag{4.11}$$

$$I_t^* = \int_{0-}^{t} \mathbf{1}_{\{(X_s^{x,y,I^*}, Y_s^{y,I^*}) \in \mathbb{I}\}} dI_s^*,\ \forall t \geq 0,\ \mathbb{P}\text{-}a.s., \tag{4.12}$$

then we have

$$V(x, y) = w(x, y), \quad (x, y) \in \mathbb{R} \times [0, \bar{y}], \tag{4.13}$$

and I^* is optimal; that is, $V(x, y) = \mathcal{J}(x, y, I^*)$.

Proof. See Appendix C. □

5. Optimal solution to the green technology investment problem

5.1. Free boundary of green technology investment

In this section, we examine the conditions under which a manufacturer will invest in GT and obtain the manufacturer's profit function, i.e., the value function.

The HJB equation (4.4) implies that the manufacturer's optimal decision is characterized by a free boundary $F(y)$, which separates the state space into a waiting region $\mathbb{W}$ and investment region $\mathbb{I}$. Hence, we hypothesize the existence of a boundary $F : [0, \bar{y}] \to \mathbb{R}$ that distinguishes the regions where a manufacturer invests in GT and waits, such that

$$\mathbb{W} := \Big\{(x, y) \in \mathbb{R} \times [0, \bar{y}) : x < F(y)\Big\}, \tag{5.1}$$

$$\mathbb{I} := \Big\{(x, y) \in \mathbb{R} \times [0, \bar{y}) : x \geq F(y)\Big\}. \tag{5.2}$$

Next, we provide the explicit form of this free boundary, the functional form of the manufacturer's expected profit, and how the manufacturer intervenes in the carbon emission process by investing in GT.

In the following, we present the key steps to solve this problem. As the well-posedness of this type of singular control problem is thoroughly discussed in [32] and the verification theorem is provided in Section 4, we omit a repetitive analysis here. We first introduce the following lemma.

Lemma 5.1. *Let $\mathcal{L}$ denote the infinitesimal generator of the uncontrolled O-U process* (3.13)*, that is $\mathcal{L} \equiv \mathcal{L}^0$, where $\mathcal{L}^y$, for $y \geq 0$ be given and fixed, is the generator from* (4.3)*. Then the following results hold. (1) The strictly increasing positive fundamental solution $\psi(\cdot)$, and the strictly decreasing positive fundamental solution $\phi(\cdot)$ to the ordinary differential equation $(\mathcal{L} - \rho)u = 0$ are given by*

$$\psi(x) = e^{\frac{k(x-\mu)^2}{2\sigma^2}} D_{-\frac{\rho}{k}}\left(-\frac{x-\mu}{\sigma}\sqrt{2k}\right), \tag{5.3}$$

$$\phi(x) = e^{\frac{k(x-\mu)^2}{2\sigma^2}} D_{-\frac{\rho}{k}}\left(\frac{x-\mu}{\sigma}\sqrt{2k}\right), \tag{5.4}$$

where

$$D_a(x) := \frac{e^{-\frac{x^2}{4}}}{\Gamma(-a)} \int_0^\infty t^{-a-1} e^{-\frac{t^2}{2} - xt} dt, \quad a < 0, \tag{5.5}$$

is the cylinder function of order a and $\Gamma(\cdot)$ is the Euler's Gamma function.
(2) Denoting by $\psi^{(j)}$ and $\phi^{(j)}$ the j-th derivative of ψ and ϕ, $j \in \mathbb{N}_0$, one has that $\psi^{(j)}$ and $\phi^{(j)}$ are strictly convex and $\psi^{(j)}$ ($\phi^{(j)}$ respectively) identifies with the strictly increasing positive (strictly decreasing positive respectively) fundamental solution (up to a positive constant) to $(\mathcal{L} - (\rho + kj))u = 0$. In particular, it holds

$$\begin{aligned} &\frac{\sigma^2}{2}\psi^{(j+2)}(x + \beta y) + k((\mu - \beta y) - x)\psi^{(j+1)}(x + \beta y) \\ &\qquad\qquad -(\rho + kj)\psi^{(j)}(x + \beta y) = 0, \end{aligned} \tag{5.6}$$

for any $x \in \mathbb{R}$ and $y \geq 0$.
(3) For any $j \in \mathbb{N}_0$, $\psi^{(j)}(x)\psi^{(j+2)}(x) - \psi^{(j+1)}(x)^2 > 0$, for all $x \in \mathbb{R}$.
(4) For any $j \in \mathbb{N}_0$, the function $\Psi_j : \mathbb{R} \mapsto \mathbb{R}$ defined as

$$\Psi_j(x) = \frac{\psi^{(j+1)}(x)^2}{\psi^{(j)}(x)\psi^{(j+2)}(x)}, \tag{5.7}$$

is strictly increasing.
(5) Denote by $\psi(\cdot; y)$ ($\phi(\cdot; y)$ respectively) the strictly increasing (strictly decreasing respectively) positive fundamental solution to $(\mathcal{L}^y - \rho)u = 0$ for $y \geq 0$. Then, one can identify

$$\psi(x; y) = \psi(x + \beta y), \quad \phi(x; y) = \phi(x + \beta y). \tag{5.8}$$

Proof. Please see [82] for details. □

We consider all $(x, y) \in \mathbb{W}$. The candidate value function w should satisfy the equation:

$$\mathcal{L}^y w(x, y) - \rho w(x, y) + P(x, y) = 0. \tag{5.9}$$

Equation (4.7) shows that R is a particular solution to (5.9). The homogeneous part of (5.9) is

$$\mathcal{L}^y w(x, y) - \rho w(x, y) = 0, \tag{5.10}$$

which admits two fundamental strictly positive solutions. These are given by $\psi(x + \beta y)$ and $\phi(x + \beta y)$, with $\psi(\cdot)$ strictly increasing and $\phi(\cdot)$ strictly decreasing, cf. Lemma 5.1-(1),(5). Therefore, the solution to (5.9) can take the following form

$$w(x, y) = A(y)\psi(x + \beta y) + B(y)\phi(x + \beta y) + R(x, y), \quad (x, y) \in \mathbb{W}, \tag{5.11}$$

for some functions $A, B : [0, \bar{y}] \to \mathbb{R}$ to be found. Given a fixed $y \geq 0$, $\phi(x - \beta y)$ grows exponentially to $+\infty$ as x approaches $-\infty$, as shown in Appendix 1 of [83]. Considering the linear growth condition in (4.2) and the structure of the waiting region $\mathbb{W}$, we conclude that $B(y) = 0$ for all $y \in [0, \bar{y}]$. Consequently, the candidate value function w can take the following form

$$w(x, y) = A(y)\psi(x + \beta y) + R(x, y), \text{ for } (x, y) \in \mathbb{W}. \tag{5.12}$$

Next, by the smooth-fit condition

$$\begin{aligned} (w_y(x, y) - c)|_{x=F(y)} &= 0, \\ (w_{xy}(x, y))|_{x=F(y)} &= 0. \end{aligned} \tag{5.13}$$

We can obtain the following equations

$$A'(y)\psi(F(y) + \beta y) + \beta A(y)\psi'(F(y) + \beta y) + R_y(F(y), y) - c = 0, \tag{5.14}$$

and

$$A'(y)\psi'(F(y) + \beta y) + \beta A(y)\psi''(F(y) + \beta y) + R_{xy}(F(y), y) = 0. \tag{5.15}$$

Here, R_y and R_{xy} are the partial derivative of the function R with respect to y and the mixed partial derivative of the function R with respect to y and x, respectively. These derivatives can be obtained from equation (4.7) as follows:

$$R_y(x, y) = \frac{x(\pi b + arE)}{\rho + k} - \frac{\beta k(\pi c_p - aE)}{\rho(\rho + k)} + \frac{k(\pi b + arE)(\mu - 2\beta y)}{\rho(\rho + k)}, \tag{5.16}$$

and

$$R_{xy}(x, y) = \frac{\pi b + arE}{\rho + k}. \tag{5.17}$$

Before solving for $A(y)$, we first define the auxiliary function

$$\widetilde{F}(y) = F(y) + \beta y. \tag{5.18}$$

The following Lemma gives the form and monotonicity of $A(y)$.

Lemma 5.2. *The function A is strictly positive and strictly decreasing. Additionally, $A(y)$ can be expressed by*

$$A(y) = \frac{\pi b + arE}{\beta\rho(\rho + k)} \frac{\frac{\sigma^2}{2}\psi''(F(y) + \beta y) + M^y\psi'(F(y) + \beta y)}{\psi'^2(F(y) + \beta y) - \psi''(F(y) + \beta y)\psi(F(y) + \beta y)}, \tag{5.19}$$

where

$$M^y = (\rho + k)\left[\frac{\beta k}{\rho + k}y + \frac{\beta k(\pi c_p - aE)}{(\pi b + arE)(\rho + k)} + \frac{c\rho}{\pi b + arE} - F(y)\right]. \tag{5.20}$$

Furthermore

$$\begin{aligned} F(y) &> \frac{\beta k}{\rho + k}y + \frac{\beta k(\pi c_p - aE)}{(\pi b + arE)(\rho + k)} + \frac{c\rho}{\pi b + arE} \\ &\geq \frac{\beta k(\pi c_p - aE)}{(\pi b + arE)(\rho + k)} + \frac{c\rho}{\pi b + arE}, \end{aligned} \tag{5.21}$$

for all $y \in [0, \bar{y}]$.

Proof. See Appendix D □

Next, we provide the explicit form of the free boundary through the following proposition:

Proposition 5.3. *Define the functions $Q_j : \mathbb{R} \mapsto \mathbb{R}$, $j \in \mathbb{N}_0$, and their first derivatives as*

$$Q_j(z) = \psi^{(j)}(z)\psi^{(j+2)}(z) - \psi^{(j+1)}(z)^2, \tag{5.22}$$

$$Q_j'(z) = \psi^{(j)}(z)\psi^{(j+3)}(z) - \psi^{(j+1)}(z)\psi^{(j+2)}(z). \tag{5.23}$$

There exists a unique solution $\widetilde{x} \in \mathbb{R}$ to the equation $H(x) = 0$ given by

$$H(x) := \frac{\pi b + arE}{\rho + k}\psi(x) + (c - \widetilde{R}(x, \bar{y}))\psi'(x). \tag{5.24}$$

Furthermore, the function $\widetilde{F}(y)$ satisfies the ordinary differential equation

$$\widetilde{F}'(y) = \mathcal{G}(y, \widetilde{F}(y)), \tag{5.25}$$

with the boundary condition $\widetilde{F}(\bar{y}) = \widetilde{x}$, where $\mathcal{G} : (\mathbb{R} \times \mathbb{R}) \setminus \{(y, z) \in \mathbb{R}^2 : D(y, z) = 0\} \to \mathbb{R}$ is given by

$$\mathcal{G}(y, z) := \beta\frac{N(y, z)}{D(y, z)}, \tag{5.26}$$

with

$$N(y, z) = Q_0(z)\left[\frac{2(k + \rho)}{\rho}\psi'(z) + \frac{\rho + k}{\pi b + arE}\left(c - \widetilde{R}(z, y)\right)\psi''(z)\right], \tag{5.27}$$

$$D(y, z) = \psi(z)\left(\frac{\rho + k}{\pi b + arE}\right)(c - \widetilde{R}(z, y))Q_1(z) + \psi(z)Q_0'(z), \tag{5.28}$$

where $\widetilde{R} : \mathbb{R}^2 \to \mathbb{R}$ is defined as

$$\widetilde{R}(x, y) = \frac{(x - \beta y)(\pi b + arE)}{\rho + k} - \frac{\beta k(\pi c_p - aE)}{\rho(\rho + k)} + \frac{k(\pi b + arE)(\mu - 2\beta y)}{\rho(\rho + k)}. \tag{5.29}$$

Proof. See Appendix E. □

Proposition 5.3 establishes that the free boundary $F(y)$ is characterized by the ODE (5.25), where the boundary condition is specified by the solution to the equation $H(x) = 0$ defined in (5.24). Consequently, from the boundary condition $\widetilde{F}(\bar{y}) = \widetilde{x}$, we can obtain the explicit form of the free boundary

$$F(y) = \widetilde{x} - \beta y - \int_y^{\bar{y}} \mathcal{G}(u, \widetilde{F}(u))du.$$

5.2. Value function and optimal green technology investment strategy

The initial demand is denoted by $x_0 := F(0)$. Once the manufacturer observes that the current demand exceeds the free boundary $F(y)$, it will consider investing in GT. At this point, until the available GT investment is exhausted (i.e., $Y_t^{y,I} = \bar{y}$), the manufacturer's profit needs to take into account the cost of investing in GT. Therefore, we present the candidate value function $w : \mathbb{R} \times [0, \bar{y}] \mapsto \mathbb{R}$ as

$$w(x,y) = \begin{cases} A(y)\psi(x + \beta y) + R(x,y), & \text{if } (x,y) \in \mathbb{W} \cup ((-\infty, \overline{x}) \times \{\bar{y}\}), \\ A(F^{-1}(x))\psi(x + \beta F^{-1}(x)) & \\ +R(x, F^{-1}(x)) - c(F^{-1}(x) - y), & \text{if } (x,y) \in \mathbb{I}_1, \\ R(x,\bar{y}) - c(\bar{y} - y), & \text{if } (x,y) \in \mathbb{I}_2 \cup ([\overline{x}, \infty) \times \{\bar{y}\}), \end{cases} \tag{5.30}$$

where

$$\mathbb{I}_1 := \{(x,y) \in \mathbb{R} \times [0, \bar{y}) : x \in [F(y), \overline{x})\},$$
$$\mathbb{I}_2 := \{(x,y) \in \mathbb{R} \times [0, \bar{y}) : x \geq \overline{x}\}.$$

The manufacturer's optimal strategy is characterized by the candidate value function $w(x, y)$ defined over three regions, reflecting different investment behaviors based on current demand x and existing GT level y.

In the waiting region $\mathbb{W} \cup ((-\infty, \overline{x}) \times \{\bar{y}\})$, the manufacturer optimally postpones any further investment in GT. When demand x remains below an endogenously determined investment threshold $F(y)$, the scale of production is sufficiently low that the associated carbon emissions, and thus the corresponding carbon price burden remain manageable. Under these conditions, the substantial sunk cost of a new GT investment cannot be justified by the relatively modest reduction in future emission expenses, rendering the net present value of the investment negative. Alternatively, waiting is also optimal when the GT level has reached its maximum capacity, $\bar{y}$. In this boundary case, the marginal benefit of any additional GT investment is zero, as no further reductions in emissions are feasible. Thus, in both scenarios, insufficient demand pressure or a binding technological constraint, the value of the option to wait outweighs the benefits of immediate action, leading to a period of optimal inactivity.

The partial investment region $\mathbb{I}_1$, defined by $x \geq F(y)$, represents a state where demand pressures have become significant enough to warrant action, but not so extreme as to justify a full GT investment. Here, the manufacturer adopts a strategy of sequential investment. Rising demand increases the marginal cost of operations via higher emission costs, making sequential GT investments profitable. However, given the uncertainty about future demand paths, a lump-sum investment would carry significant risk. The optimal response is a gradual, piecemeal adoption of GT. This strategy strikes a balance: It provides immediate relief from the high marginal cost of emissions while preserving production flexibility. By investing sequentially, the manufacturer retains the option to cease further investment if demand falls or to accelerate it if demand rises.

The manufacturer's strategy shifts to immediate and full investment upon entering the full investment region $\mathbb{I}_2 \cup ([\overline{x}, \infty) \times \{\bar{y}\})$. This action is triggered by one of two scenarios. The first occurs when demand x exceeds a upper threshold $\overline{x}$. At such high demand levels, the marginal cost of emissions becomes so punitive that it severely erodes profitability. The benefit of immediately eliminating this cost through a comprehensive GT upgrade, achieving economies of scale in emission reduction, dwarfs the investment's costs. The opportunity cost of delaying, i.e., continuing to incur prohibitive emission fees, becomes unbearable, and the value of the option to wait diminishes to zero. The second scenario is the attainment of the technological limit $\bar{y}$ again. However, the implication here differs from that in the waiting region. On the boundary $\{y = \bar{y}\}$, the decision problem regarding GT investment is effectively terminated. Since no further technological improvements are available, the manufacturer's optimal course of action on the interval $[\overline{x}, \infty)$ is to invest fully in the best available GT if they have not done so, after which the emission reduction potential is maximized and no further decisions on GT investment remain.

The following lemma illustrates how manufacturers make investments in GT.

Lemma 5.4. *Recall w from* (5.30) *and let* $\Delta := (\bar{y} - y)\mathbf{1}_{\{x\geq\overline{x}\}} + (F^{-1}(x) - y)\mathbf{1}_{\{\overline{x}>x>F(y)\}}$, $K_t = \min\{\sup_{0\leq s\leq t}\{\bar{F}^{-1}(X_s)\}, \bar{y} - (y + \Delta)\}$, $\tau := \inf\{t \geq 0 : K_t = \bar{y} - (y + \Delta)\}$, *and* (X, K) *defined on* $[0, \tau]$ *such that*

$$\begin{aligned} &X_t \leq F(y + \Delta + K_t), \\ &dX_t = k\,(\mu - \beta(y + \Delta + K_t) - X_t)\,dt + \sigma dW_t, \\ &dK_t = {}_{\{X_t=F(y+\Delta+K_t)\}}dK_t, \end{aligned} \tag{5.31}$$

with increasing K, and starting point $(X_0, K_0) = (x, 0)$*, where $\bar{F}^{-1}$ is such that*

$$\bar{F}^{-1}(x) := \begin{cases} 0, & \text{if } x < x_0 \\ F^{-1}(x), & \text{if } x \in [x_0, \overline{x}] \\ \bar{y}, & \text{if } x > \overline{x}. \end{cases} \tag{5.32}$$

Then, the function w identifies with the value function V from (3.16)*, and the optimal investment strategy, denoted by I^*, is given by*

$$\begin{cases} I^*_{0-} = 0 \\ I^*_t = \begin{cases} \Delta + K_t, & t \in [0, \tau), \\ \Delta + K_\tau, & t \geq \tau. \end{cases} \end{cases} \tag{5.33}$$

Proof. The proof of this Lemma essentially relies on the existence and uniqueness results for the Skorokhod reflection problem, as established by the researchers in [84]. After integrating the necessary symbols, the proof proceeds similarly to that of Theorem 4.8 in [32]. For brevity, we refer the reader to the detailed proof provided in that reference. □

Lemma 5.4 provides a formal characterization of the manufacturer's optimal GT investment strategy. The strategy unfolds in two distinct phases:

The process begins with an initial adjustment Δ, the size of which is determined by the observed demand x relative to the endogenous threshold $F(y)$. This threshold represents the demand level at which emission costs begin to outweigh the investment costs given the GT level y. If demand is insufficient, $x < F(y)$, the emission cost burden remains manageable, and the manufacturer optimally chooses to wait. This inaction embodies the value of the option to delay, which avoids incurring irreversible sunk costs in the face of uncertainty. Conversely, if demand is high, the initial investment is executed, yet its magnitude is capped by the available GT investment $\bar{y} - y$.

Following this initial phase, the strategy transitions into a boundary-triggered investment regime. Subsequent investments are not made continuously but are discretely triggered only when the demand process X_t hits a moving boundary $F(y+\Delta+K_t)$. The economic interpretation of this boundary is central to the strategy: it dynamically recalibrates to represent the critical demand level at which emission costs become prohibitively expensive given the cumulative technology stock $y + \Delta + K_t$. The process K_t captures the cumulative investment made during this phase, increasing solely when necessary to maintain the inequality $X_t \leq F(y + \Delta + K_t)$, ensuring that every unit of capital is deployed precisely when the marginal benefit demonstrably outweighs its marginal cost. This mechanism enforces strict cost-effectiveness by preventing premature and excessive investment.

This investment process continues until time τ, when the available GT capacity is exhausted, i.e., K_t reaches the limit $\bar{y} - (y + \Delta)$. Beyond this point, no further GT investment is possible, and the GT level is permanently fixed at $I_t^* = \Delta + K_\tau$, representing the maximum achievable emission reduction under the technological constraint.

6. Numerical experiments

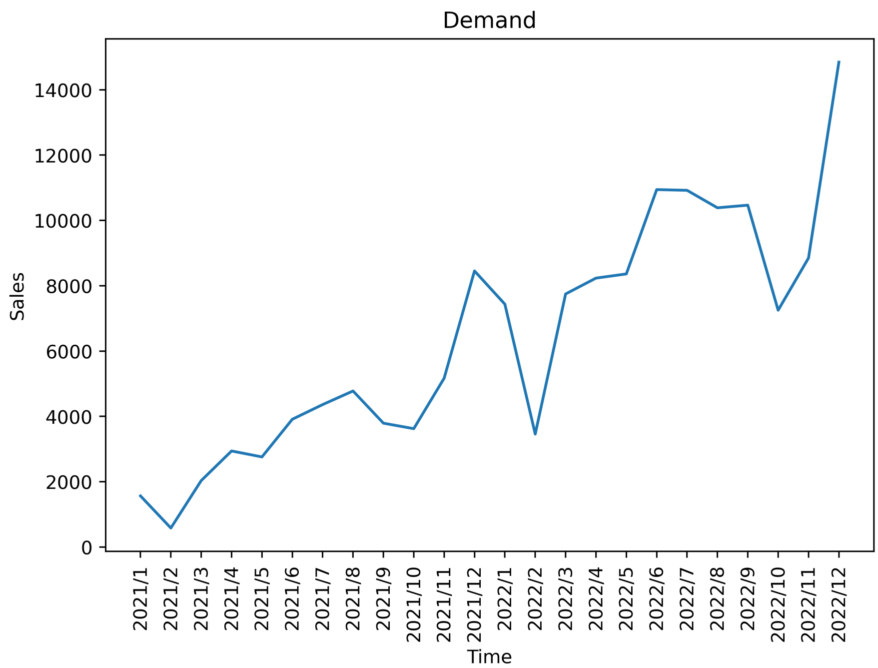


Figure 1. Leapmotor's sales disclosure data from 2021 to 2022.

In this section, we numerically examine the relationship between model parameters and the boundary of GT investment. To ground our model in empirical reality and facilitate numerical illustration, we calibrate a subset of parameters using data from a case: The monthly sales data of electric vehicles from Leapmotor Technology Co., Ltd., a company based in Zhejiang, China, from January 2021 to December 2022, and carbon price data from Beijing's carbon market. This calibration is an illustrative example, while the model is general and can be applied to various manufacturing contexts under C&T mechanism. The core theoretical insights and managerial implications are not limited to this case.

Figure 1 illustrates the monthly sales data of Leapmotor from 2021 to 2022. The demand demonstrates fluctuations within the chosen timeframe, and our empirical analysis indicates a statistically significant mean-reverting characteristic. To estimate the demand process, we employ a discrete approximation method. The approximation is essentially a linear model using the sales data:

$$\Delta \tilde{X}_t = \tilde{k}(\tilde{\mu} - X_t) + \tilde{\sigma}\tilde{\epsilon}_t$$

for $t = 1, 2, ..., 24$, where $\Delta X_t = X_{t+1} - X_t$ and $\{\tilde{\epsilon}_t\}_{t=1,2,...}$ is a sequence of independent standard normal variables.

Table 3. Parameter estimation for the sales data

	Estimate	p-Value
$\tilde{\mu}\tilde{k}$	1263.3239	$8.07542351 \times 10^{-2}$
$1 - \tilde{k}$	0.8857	$2.82596522 \times 10^{-11}$

Table 3 shows the estimates and p-values for the parameters. Both parameters of the mean-reverting rate and level are statistically significant. We rescale the parameters using a linear transformation $X(t) = \tilde{X}(t)/\tilde{\mu}$ and regard X_t as the demand of vehicles.

We refer to empirical research [85] to estimate the carbon emissions produced by the production of electric vehicles, and set the carbon emission intensity as $E = 12.446/tCO2e$ per unit. Based on the average selling price of Leapmotor, which is $(1 + \pi)c_p = 17869.979\$$ per unit, and a profit margin of $\pi = 1.2\%$, we can calculate the production cost per unit as $c_p \approx 17658.0820\$$ [86]. According to the carbon price in Beijing's carbon emission trading market, which is $a = 1.3(\times 10\$/tCO2e)$. Next, we scale the production cost and carbon price by a factor of 1×10^5. We can obtain $c_p = 0.1765$ and $a = 1.3 \times 10^{-4}$. Based on the carbon trust surveys, about 20% of consumers prefer to buy green products [87], it is set that $s = 0.2$. According to the IPCC report, for many industrial sub-sectors, the maximum economic emission reduction potential achievable through energy efficiency measures and the extensive deployment of the best available technologies (BAT) is typically between 10% and 30%. Therefore, we set the upper limit parameter for emission reduction from GT investment at 20%, implies $r = 0.2$ [88]. We refer to [68] for the coefficient of the increase in the unit production cost due to investment in GT, and set it as $b = 0.2$. The other parameters in the objective function as $\rho = 0.050$, $d = 2.5$ and $c = 2.5 \times 10^{-2}$. The corresponding parameters are shown in Table 4.

Table 4. Notation summary

ρ	k	μ	σ	c	E	a
0.05	0.1143	1.0	0.1891	2.5×10^{-2}	12.446	1.3×10^{-4}
$\bar{y}$	d	b	s	r	π	c_p
1	2.5	0.2	0.2	0.2	0.012	0.1765

6.1. The impact of free boundary and carbon emission threshold on green technology investment strategy

In the subsequent proposition, we elucidate the relationship between the demand threshold that triggers GT investment and the average demand. Here, the average demand can be represented by the so-called "mean line" of the O-U process, denoted as $M(y) = \mu - \beta y$, with the mapping of $M : [0, \infty) \to (-\infty, \mu]$, which represent that the demand deviating from the mean will move toward this line.

Proposition 6.1. *Given the upper bound $\bar{y}$ for the GT investment cumulative level, and the corresponding free boundary $F(y)$ starting from $(\bar{x}, \bar{y})$, the line of means $M(y) = \mu - \beta y$:*

(1) Has no intersection with the investment region $\mathbb{I}$ if $F(0) > \mu$;

(2) Intersects the boundary of $\mathbb{I}$ in the free boundary $F(y)$ if $F(0) \leq \mu$ and $\bar{y} \geq y^*$, where

$$y^* = \frac{1}{\beta(2k+\rho)}\left(\mu(\rho+k) - \frac{c\rho(\rho+k)}{\pi b + arE} - \frac{\beta k(\pi c_p - aE)}{\pi b + arE} - \rho\frac{\psi(\mu)}{\psi'(\mu)}\right), \tag{6.1}$$

(3) Intersects the boundary of $\mathbb{I}$ in its upper bound $y = \bar{y}$ if $\bar{y} \leq y^*$.

Proof. See Appendix F. □

In Proposition 6.1, y^* reflects the manufacturer's willingness to invest in GT, and M represents the average level of demand. The closer y^* is to M, the greater the probability that the manufacturer will invest in GT. As can be seen from Figure 2, if condition (1) holds, the dashed sloped red line, which represents the plot of the function M, does not intersect with F and lies within the waiting region $\mathbb{W}$. Since the demand is highly likely to fluctuate around the sloped red line, it is difficult for the demand to reach the threshold that triggers GT investment. This also indicates that the manufacturer's willingness to invest in GT is low at this time.

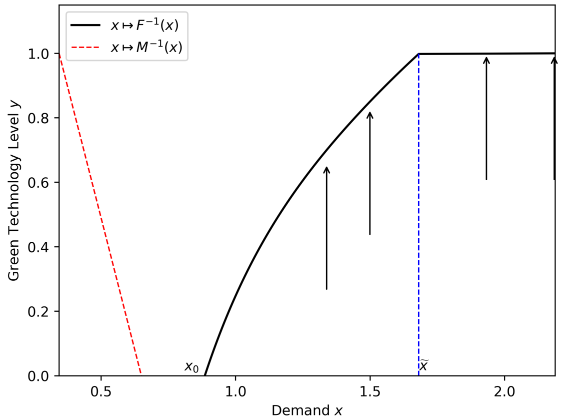


Figure 2. The functions F^{-1} and M^{-1} with $\mu = 0.65$.

If condition (2) holds, as can be seen from Figure 3, the mean line M intersects with the boundary F and lies in the waiting region $\mathbb{W}$. Moreover, $M(y)$ to the right of the intersection point enters the $\mathbb{I}_1$ region. Since the demand is more likely to move towards the mean line, it is highly possible that the demand will reach the manufacturer's GT investment threshold. This also indicates that the manufacturer has a relative strong willingness to invest in GT at this time, and the manufacturer will invest in part of GT to reduce investment risks.

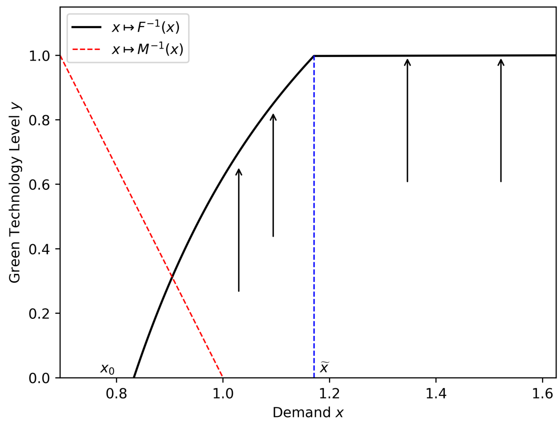


Figure 3. The functions F^{-1} and M^{-1} with $\mu = 1.0$.

Finally, if condition (3) holds, as can be seen from Figure 4, the mean line M not only intersects with F but also lies entirely within the investment region $\mathbb{I}_2$. At this time, there is a very high probability that the demand will reach threshold $\bar{x}$, which will cause the manufacturer to invest in all GTs due to the excessively high emission reduction costs. Moreover, the high demand causes the carbon emission fine cost to far exceed the GT investment cost, and the manufacturer has a strong willingness to make lump sum investments to minimize the total cost.

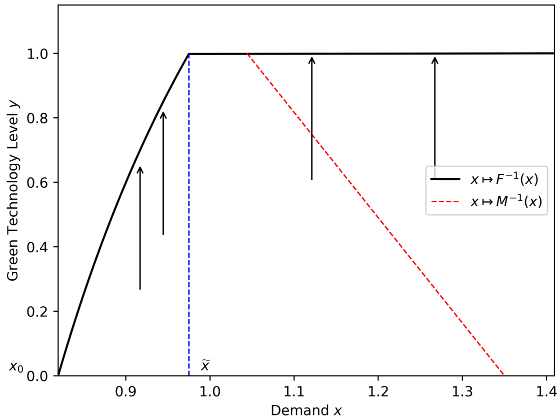


Figure 4. The functions F^{-1} and M^{-1} with $\mu = 1.35$.

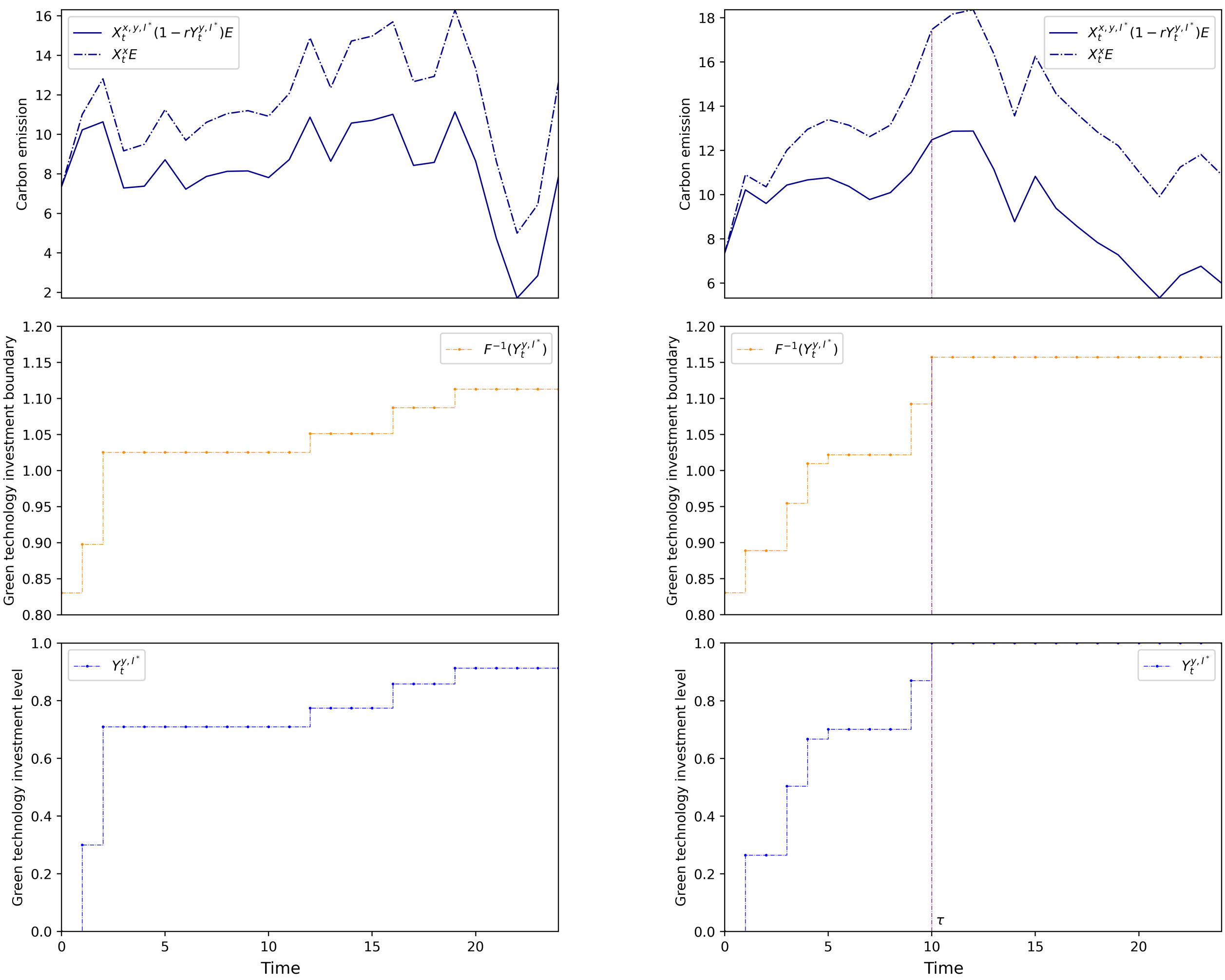
6.2. Simulation

In this section, referring to the simulation method in [43], we conduct a simulation to illustrate how manufacturers determine the investment strategy in GT under stochastic emission. Figure 5 presents two simulation results. In both scenarios, the initial GT level for manufacturers is $y = 0$, and the initial demand is $x = 0.60$. The time horizon is $T = 24$ (2 years). All the figures share a time axis, and other parameters are consistent with Table 4.

The top graph simulates the carbon emission process of manufacturers, as well as the effectiveness of implementing GT in reducing carbon emissions. The solid line indicates the carbon emission process under the optimal GT investment strategy, while the dashed line represents the uncontrolled carbon emission process. The area between the dashed line and the solid line represents the cumulative reduction in carbon emissions achieved through the investment of GT. The middle graph simulates the inverse function of the boundary F, illustrating the emission threshold at different GT levels. By comparing the top and middle graphs, we can observe that once the emission reaches the threshold indicated by the middle graph, the manufacturer will invest in GT, which results in reduced emissions in subsequent periods. Moreover, each investment in GT leads to an increase in the investment threshold. This is because when manufacturers invest in GT, it enhances their emission reduction capabilities. As a result, they can produce fewer emissions at the same production level. This enables manufacturers to more easily achieve their emission reduction targets, thereby weakening their willingness to continue investing in GT. The bottom graph simulates the process of investment in GT. As long as the emissions do not reach the specified threshold, the manufacturer will choose to wait for a better time to invest in GT. Notably, we observe some singularities and an upward trend in the investment strategy for GT. This is because GT investment is irreversible, which is more in line with the real world situation. GT implementation requires substantial financial, resource, and time investments, along with contractual agreements and supplier partnerships, making these commitments difficult to reverse.

As the demand involves stochasticity, we simulate two scenarios. In the first scenario shown in Figure 5a, the cumulative GT investment does not reach the upper limit $\bar{y}$ as the emission remains below the threshold after several investment rounds. This enables the manufacturer to cease GT investment once it achieves its emission reduction target. In Figure 5b, the cumulative GT investment reaches the upper limit $\bar{y}$ at time τ. Consequently, even if the emission level exceeds the threshold after τ, the additional investment is constrained by the upper limit. The threshold then becomes irrelevant, as shown in the upper graph of Figure 5b. Although the emission exceeds the emission reduction target after τ, the manufacturer is unable to invest in GT. Nevertheless, the previously implemented GT persists and continues to reduce emissions throughout the production process.

The numerical solutions obtained from our simulation include a series of timing and numerical values for system adjustments, which can be regarded as impulse control [31]. Our findings demonstrate that decision-makers can determine investment timing and adjustment amounts at discrete time points (every month), enabling manufacturers to adjust their GT strategy in a controlled manner to manage their emissions.



(a) $\tau > 24$ **(b)** $\tau = 10$

Figure 5. Simulated results are obtained using the optimal GT investment rules I_t^*. We generate sample paths, including the optimal carbon emission process $X_t^{x,y,I^*}(1 - rY^{y,I^*})E$, the uncontrolled carbon emission process $X_t^x E$, the free boundary process $F^{-1}(Y_t^{y,I^*})$, and the optimal accumulated GT investment process Y_t^{y,I^*} under two scenarios.

6.3. *Static analysis of the green technology investment boundary*

In this section, we conduct a static analysis of the sensitivity of the GT investment boundary using the parameters in Table 4. We first examine parameters whose increase shifts the boundary leftward, indicating enhanced willingness to invest in GT. Figures 6 and 7 illustrate how emission reduction cost parameters affect the investment threshold. An increase in the carbon trading permit price a raises emission reduction costs, while an increase in the GT efficiency parameter r improves the cost-effectiveness of investment. Higher r values indicate better emission reduction per unit of GT investment. These results align with those in [18], confirming that cost pressures and efficiency gains stimulate GT investment. Although our static analysis yields consistent expectations, the dynamic boundary allows real-time strategy adjustments based on market feedback.

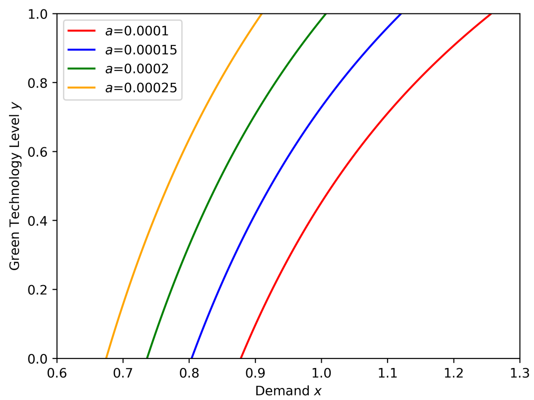


Figure 6. The functions F^{-1} with a.

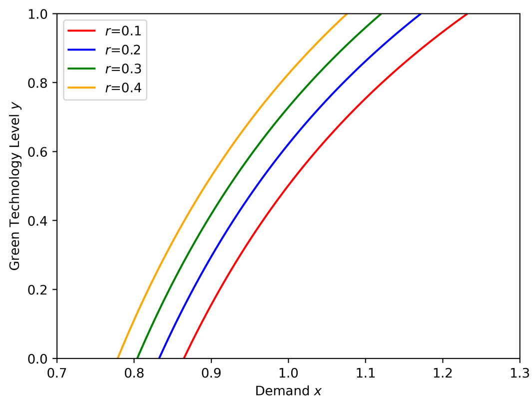


Figure 7. The functions F^{-1} with r.

Figures 8 and 9 show demand related parameters' effects. Increases in average demand μ and consumer green awareness s prompt higher production, raising emission costs and strengthening GT investment incentives. However, when initial GT levels are low, this incentive is attenuated as emission reduction needs only become significant at high production volumes. These findings are consistent with those in [18] regarding consumer sensitivity to carbon emissions. Compared with Figures 6 and 7, direct cost pressures and efficiency gains produce more uniform leftward shifts across scenarios.

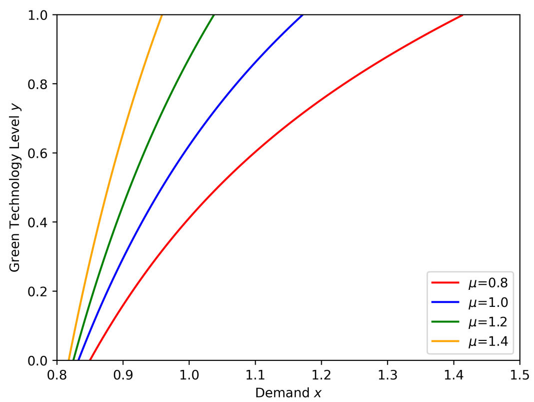


Figure 8. The functions F^{-1} with μ.

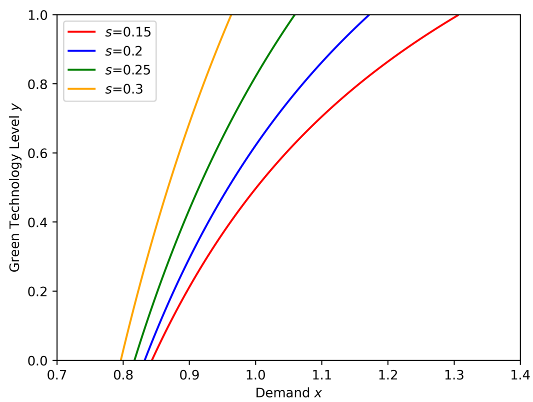


Figure 9. The functions F^{-1} with s.

Figure 10 shows the impact of the profit margin set by the manufacturer for the product on the threshold. It can be seen that the higher the profit margin is set, the higher the willingness to invest in GT. This result seems counterintuitive because a higher profit margin raises the price, which reduces demand and undermines the effectiveness of emission reduction. However, a higher profit margin also increases the profit per unit. This means that when demand is low, the manufacturer can boost profits by reducing emission costs, which encourages GT investment.

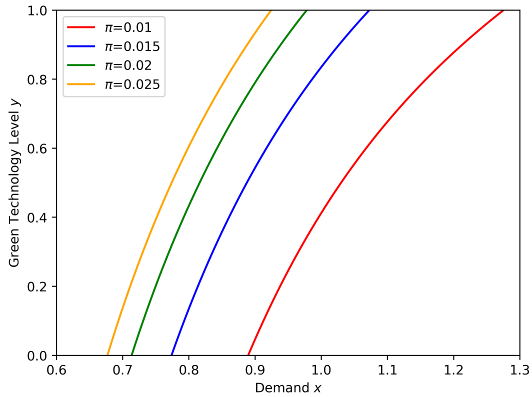


Figure 10. The functions F^{-1} with π.

Next, we present the results for parameters whose increase raises the investment threshold. As shown in Figure 11, an increase in the GT investment cost c reduces manufacturers' willingness to invest, which aligns with our expectations. This situation can be inversely applied to model the effect of government subsidies, because the result of government subsidies for GT is the same as that of a reduction in GT investment costs. This result implies that well-designed subsidies can counteract investment inertia and accelerate GT adoption by triggering the investment decision at lower levels of demand or carbon price pressure. Conversely, in Figure 12, an increase in consumers' price sensitivity d also reduces the manufacturers' willingness to invest in GT. This is because manufacturers pass on a portion of the investment cost to consumers, leading some price-sensitive customers to abandon the product, thereby reducing the benefits of GT investment.

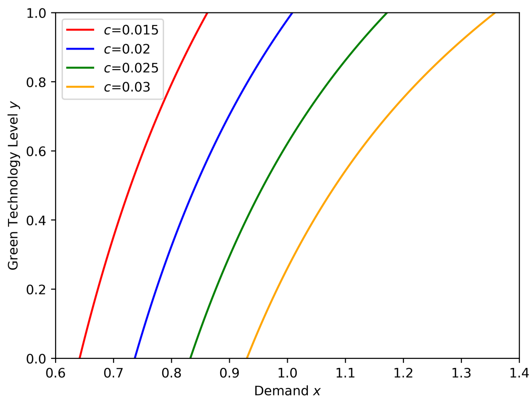


Figure 11. The functions F^{-1} with c.

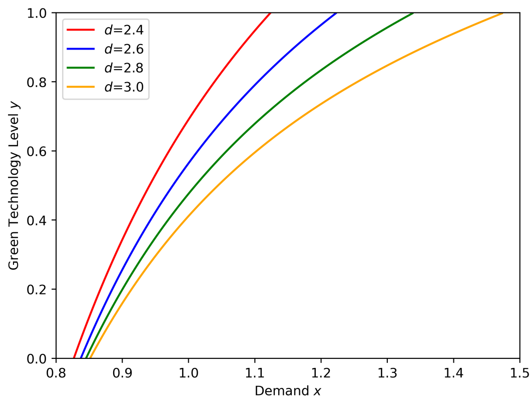


Figure 12. The functions F^{-1} with d.

The results presented in Figure 13 are also in line with expectations. That is, environmental uncertainty generates a value of waiting [54], which leads to a decrease in the manufacturers' willingness to invest in GT. This result aligns with [27] and [20], showing that high volatility in carbon trading policy can reduce the willingness to invest in GT. Similar, our results show that manufacturers will postpone their GT investment due to this. Figure 14 shows the impact of the GT investment cap $\bar{y}$ on the boundary. It is noteworthy that a higher investment cap corresponds to a higher threshold. Due to carbon quotas, manufacturers aim to meet a baseline emission level rather than minimize emissions. Therefore, with a low GT investment cap, they tend to invest earlier to comply with the quota.

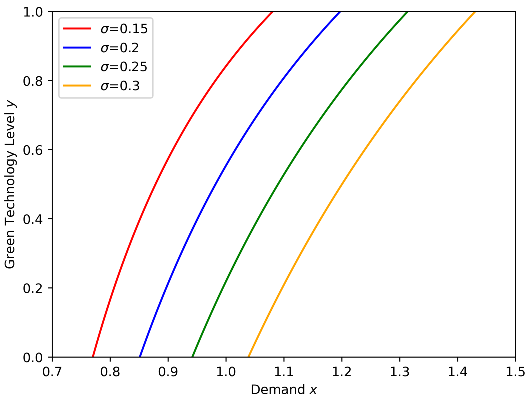


Figure 13. The functions F^{-1} with σ.

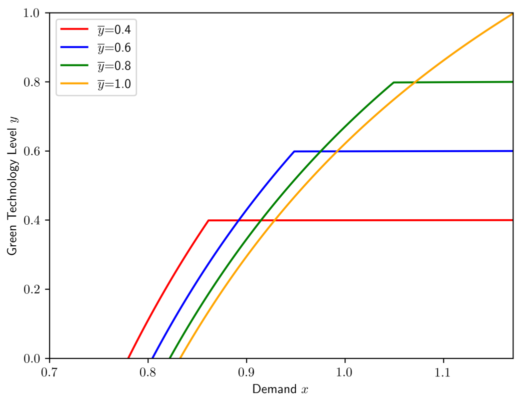


Figure 14. The functions F^{-1} with $\bar{y}$.

Finally, we present the results for the parameter effects on the threshold are uncertain, as shown in Figure 15. An increase in the unit cost b shifts the boundary left under low demand and right under high demand. Our explanation is that GT investment raises production costs, leading manufacturers to raise prices, which in turn reduces demand. Nevertheless, when the external demand is low, the impact of this demand reduction is mitigated. Hence, the manufacturer intends to invest in GT to increase profits. When demand is low, this is achieved by reducing emission costs. Conversely, when demand is high, the resulting price increase can suppress demand so much that it outweighs the benefits of emission reduction, thus discouraging investment.

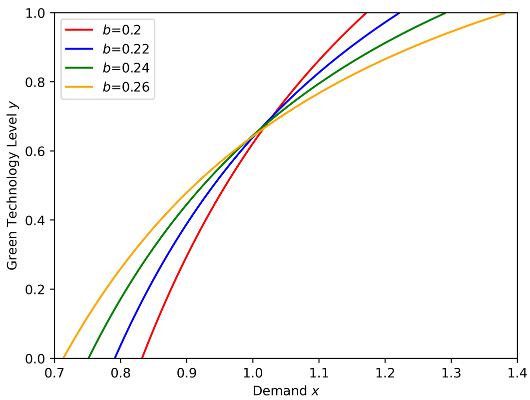


Figure 15. The functions F^{-1} with b.

Remark 6.2. *The sensitivity analysis conducted above demonstrates that while the GT investment threshold vary with parameters, the qualitative insights derived from the model are robust. Specifically, the existence of a dynamic investment threshold and its systematic dependence on key factors, where the increase of carbon price, emission reduction efficiency, and consumer green awareness will encourage the manufacturers' GT investment willing, while investment costs and demand volatility delay GT investment, remains consistent across a wide range of parameter values. This robustness suggests that the core decision-making mechanism uncovered in this paper extends beyond the specific calibration to Leapmotor and the Beijing carbon market, offering general insights for manufacturers operating under C&T mechanisms in diverse industrial contexts.*

Remark 6.3. *The calibration of our model using Leapmotor data is an illustrative case study rather than a restrictive application. Under the modeling framework defined by assumptions A1-A7 in Section 3, re-calibrating the O-U process with data from other manufacturing firms would not alter the fundamental qualitative results. The sensitivity analysis demonstrates that while parameter variations affect the specific numerical values of the investment threshold $F(y)$, they do not change the structural characteristics of the optimal strategy. Specifically:*

- *Parameter changes primarily cause translational shifts or scaling adjustments of the free boundary $F(y)$.*
- *The distance between the upper and lower bounds of $F(y)$ may narrow or widen, but no fundamentally new regions emerge in the state space.*
- *The core insights regarding the existence of a dynamic threshold and the sequential investment pattern remain robust across parameter variations.*

As illustrated in Figure 2-4, the partition of the state space into waiting, partial investment, and full investment regions is structurally stable. Therefore, we select a single representative case to illustrate these robust mechanisms, without loss of generality regarding the theoretical insights. Regarding the reasons leading to this result, we recommend readers refer to the well-posedness study of such models in [32].

6.4. Static analysis of the value function

In this section, we demonstrate how the value function (manufacturer's expected profit) is influenced by the GT level y and other parameters, under the initial time $t = 0$ and a fixed demand $x = 2.0$. Figure 16 shows the impact of the carbon price and the available GT investment $\bar{y}$ on the firm's profit. First, at a low carbon price, GT investment actually reduces profit because the low emission cost offers little benefit from reducing emissions. Furthermore, if consumers lack environmental awareness, the higher price of green products further undermines profitability. Moreover, we also find that an increase in the carbon price actually raises the manufacturer's expected profit, especially when the manufacturer's GT level is high. This seemingly counterintuitive result reflects a realistic market dynamic, a higher carbon price increases the value of the limited emission quota. For manufacturers with mature GT, effective emission control converts their saved quotas into a valuable asset, offsetting the initial cost pressure. By selling these quotas, manufacturers can directly raise their expected profits. This result is similar to that in [27], which shows that under the carbon trading policy, although an increase in the carbon price may increase risks, it may also bring profit opportunities as firms can obtain higher expected profits by controlling production. This finding reveals a firm-level mechanism that contributes to a systemic "lock-in" effect: A manufacturer may perceive high carbon prices not as an investment signal but as a cost burden, leading to investment postponement and prolonged use of polluting technologies. While rational at the firm level, this results in a suboptimal technology mix overall.

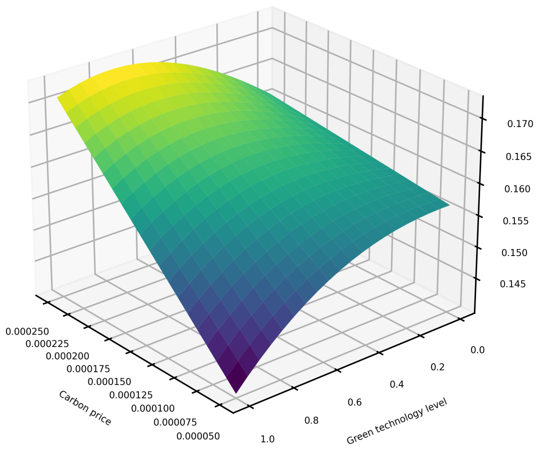


Figure 16. The functions V with a and $\bar{y}$.

As illustrated in Figure 17, the manufacturer's expected profits increase with rising consumer green awareness, a result that aligns with expectations. Notably, when consumer green awareness is low, any investment in GT leads to a decline in the manufacturer's expected profits. This situation improves only when consumer green awareness is enhanced. This highlights that the recognition of the manufacturer's products by environmentally conscious consumers is a critical driver for GT investment. Relying solely on the cost-reduction benefits of GT is insufficient to incentivize manufacturers to invest in such technologies.

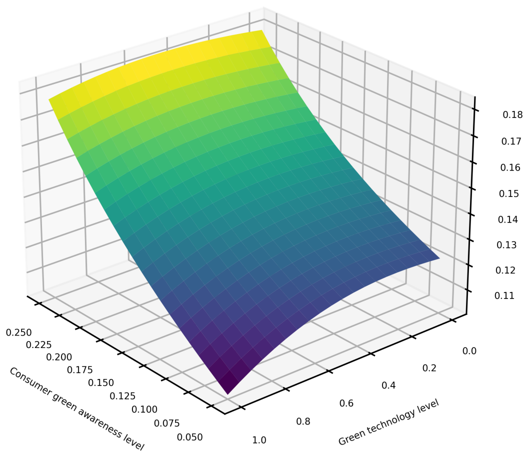


Figure 17. The functions V with s and $\bar{y}$.

Figure 18 demonstrates the impact of the carbon emission reduction rate on the manufacturer's expected profits across different levels of GT. It is evident that increasing the carbon emission reduction rate enhances the manufacturer's expected profits. Furthermore, regardless of whether the emission reduction rate is high or low, manufacturers can boost their expected profits through GT investment. Although the carbon emission reduction rate is treated as exogenous in our analysis, this finding underscores that promoting GT innovation through research and development (R&D) to improve emission reduction efficiency represents a more effective strategy for achieving emission reduction goals. This result also aligns with the findings regarding GT insurance in [28]. When GT can significantly increase manufacturer profits, technology maturity becomes a critical factor in determining the level of profitability.

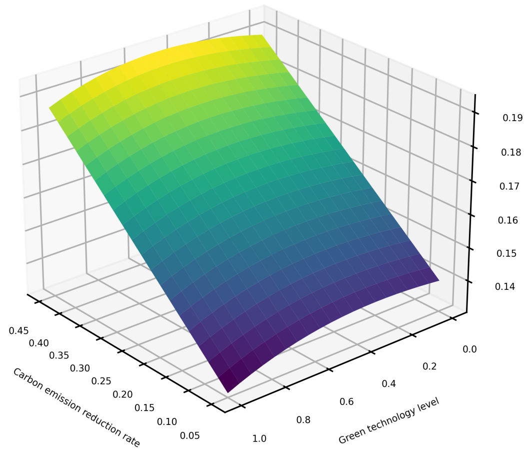


Figure 18. The functions V with r and $\bar{y}$.

7. Conclusion

In this paper, we investigate the optimal irreversible GT investment strategy for a monopolist manufacturer under the C&T mechanism, considering demand inertia and market uncertainty. By formulating the problem within a continuous-time stochastic dynamic programming framework and employing singular control analysis, we derive a variable threshold strategy that dictates the manufacturer's investment decisions. This approach provides a dynamic feedback control rule, enabling managers to determine not only whether to invest but also the precise timing and scale of investment in response to fluctuating market states. Our major findings are as follows: (1) The optimal GT investment is not a one-off event but a sequential and incremental process. The manufacturer will postpone investment until the cumulative pressure from carbon emissions, driven by market demand, exceeds a critical and state-dependent threshold. (2) The option value of waiting is significant. In the face of demand volatility and investment irreversibility, manufacturers optimally choose to "wait and see" until market conditions are sufficiently favorable, rather than investing immediately. (3) Numerical simulations based on real-world data identify key drivers of the investment decision: Factors such as consumer green awareness, permit allowance price, and emission reduction efficiency accelerate investment timing, while high investment costs and demand volatility lead to delays.

7.1. Theoretical and practical insights

We contribute to the literature by integrating irreversible investment theory and singular stochastic control methods into the study of sustainable operations management, extending beyond static models to capture the dynamic interplay between market uncertainty, carbon regulation, and multi-period investment decisions, thereby enriching the application of real options analysis in environmental economics.

For managers, this paper is a practical decision-support tool, advocating for a dynamic and adaptive investment strategy over a static one-time plan. Managers should continuously monitor demand and emission levels against the derived optimal threshold, initiating investment in phases only when emissions persistently breach this threshold, thereby maximizing long-term expected profits while managing risk.

Our findings provide nuanced insights for designing effective carbon markets: First, a stable and sufficiently high carbon price is the most direct signal to incentivize GT investment; second, policies aimed at fostering consumer green awareness can create a pull effect from the demand side, complementing regulatory push measures; third, government support for R&D to improve emission reduction efficiency is more effective than mandating higher technological standards alone. For instance, as our sensitivity analysis shows, merely raising the technological ceiling ($\bar{y}$) without providing subsidies may delay GT investments.

7.2. Limitations and Future Research

Despite its contributions, this paper has limitations that open avenues for future research. First, our model focuses on a monopolist setting. Incorporating strategic competition among firms or within a supply chain could yield new insights into how competitive pressures influence GT adoption. Second, we treat policy parameters as exogenous and constant. In the future, researchers could take policy uncertainties into account, such as changing emission caps, auctioning mechanisms, allowance banking/borrowing, or dynamic adjustments. Furthermore, extending the model to incorporate benchmarking mechanisms is valuable. By linking allowance allocation to sector benchmarks and output levels, benchmarking directly ties GT investment returns to production volume, fundamentally altering investment incentives. Third, the assumption of perfect rationality could be relaxed by introducing behavioral factors to explore how managerial risk preferences affect decisions. Fourth, the numerical calibration primarily relies on data from a single electric vehicle manufacturer and a regional carbon market. This may limit the direct generalizability of our quantitative findings to industries with different emission structures, technology cost curves.

Author contributions

Fanyi Peng: Conceptualization, methodology, formal analysis, investigation, Writing–original draft, Writing–review & editing; Ruoying Shi: Project administration, resources, Writing–review & editing; Zhipeng Zhang: Funding acquisition, software, validation, data curation, Writing–review & editing; Shuhua Zhang: Supervision, Writing–review & editing.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper. Shuhua Zhang is an editorial board member for Journal of Industrial and Management Optimization and was not involved in the editorial review or the decision to publish this article.

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Appendix

A. Proof of Lemma 4.1

Recall the uncontrolled carbon emission process X^x from (3.13), and notice that by an application of Itô's formula we find, for any $\hat{\rho} > 0$,

$$|e^{-\hat{\rho}t}X_t^x| \leq |x| + \hat{\rho}\int_0^t e^{-\hat{\rho}u}|X_u^x|du + \int_0^t e^{-\hat{\rho}u}k(|\mu| + |X_u^x|)du + \left|\int_0^t e^{-\hat{\rho}u}\sigma dW_u\right| \tag{A.1}$$

which implies

$$\begin{aligned}\mathbf{E}\Big[\sup_{t\geq 0} e^{-\hat{\rho}t}|X_t^x|\Big] &\leq |x| + C_1\left(1 + \int_0^\infty e^{-\hat{\rho}u}\mathbf{E}[|X_u^x|]du\right)\\ &\quad + \mathbf{E}\left[\sup_{t\geq 0}\left|\int_0^t e^{-\hat{\rho}u}\sigma dW_u\right|\right],\end{aligned} \tag{A.2}$$

for some $C_1 > 0$. An application of the Burkholder-Davis-Gundy inequality yields

$$\begin{aligned}\mathbf{E}\Big[\sup_{t\geq 0} e^{-\hat{\rho}t}|X_t^x|\Big] &\leq |x| + C_1\left(1 + \int_0^\infty e^{-\hat{\rho}t}\mathbf{E}[|X_u^x|]du\right)\\ &\quad + C_2\mathbf{E}\left[\left(\int_0^\infty e^{-2\hat{\rho}u}du\right)^{\frac{1}{2}}\right],\end{aligned} \tag{A.3}$$

for a constant $C_2 > 0$, and therefore

$$\mathbf{E}\Big[\sup_{t\geq 0} e^{-\hat{\rho}t}|X_t^x|\Big] \leq C(1 + |x|), \tag{A.4}$$

for some constant $C > 0$, since it follows from standard calculations that $\mathbf{E}\left[|X_u^x|\right] \leq C_3(1 + |x|)$ for a constant $C_3 > 0$.

B. Proof of Proposition 4.2

In order to prove the lower bound of V, we take the admissible (non-)investment strategy I^0 to obtain for all $y \in [0, \bar{y}]$.

$$V(x, y) \geq R(x, y) > -K_1(1 + |x|). \tag{B.1}$$

Now, for any $I \in \mathcal{I}(y)$, we find by Lemma 4.1 that

$$\begin{aligned}
\mathcal{J}(x,y,I) &\leq \mathbf{E}\left[\int_0^\infty e^{-\rho t}\left((\pi c_p - aE)X_t^{x,y,I} + (\pi b + arE)X_t^{x,y,I}Y_t^{y,I} + aQ\right)dt\right] \\
&\leq \mathbf{E}\left[\int_0^\infty e^{-\rho t}\left((\pi c_p - aE)X_t^{x} + (\pi b + arE)X_t^{x}Y_t^{y,I} + aQ\right)dt\right] \\
&\leq \mathbf{E}\left[\int_0^\infty e^{-\rho t}\left(\left((\pi c_p - aE) + \bar{y}(\pi b + arE)\right)X_t^{x} + aQ\right)dt\right] \\
&\leq \mathbf{E}\left[\int_0^\infty e^{-\rho t}\left(\left((\pi c_p - aE) + \bar{y}(\pi b + arE)\right)|X_t^{x}| + aQ\right)dt\right] \\
&\leq \mathbf{E}\left[\int_0^\infty \left(\left((\pi c_p - aE) + \bar{y}(\pi b + arE)\right)e^{-\frac{\rho}{2}t}|e^{-\frac{\rho}{2}t}X_t^{x}|\right)dt\right] + \frac{aQ}{\rho} \\
&\leq K_2(1+|x|),
\end{aligned} \tag{B.2}$$

for some $K_2 \geq 0$, and upon observing that since $\beta > 0$, we have $X^{x,y,I} \leq X^x$ $\mathbb{P}$-a.s. for any $I \in \mathcal{I}(y)$. Finally, from (B.1) and (B.2), we have that (4.9) holds with $K = \max(K_1, K_2)$.

If $y = \bar{y}$, then the only permissible strategy is I^0; therefore, $V(x,\bar{y}) = R(x,\bar{y})$. To demonstrate the monotonicity of $x \mapsto V(x,y)$, consider $x_2 > x_1$, and observe that $X_t^{x_2,y,I} \geq X_t^{x_1,y,I}$ almost surely for any $t \geq 0$ and $I \in \mathcal{I}(y)$. Hence, $\mathcal{J}(x_2,y,I) \geq \mathcal{J}(x_1,y,I)$, leading to $V(x_2,y) \geq V(x_1,y)$ when $\pi c_p - aE > 0$.

C. Proof of Theorem 4.3

Step 1: Let $(x,y) \in \mathbb{R} \times [0,\bar{y})$ be fixed, and let $I \in \mathcal{I}(y)$ be given. For $N > 0$, we define $\tau_{R,N} := \tau_R \wedge N$, where $\tau_R := \inf\left\{s > 0 : X_s^{x,y,I} \notin (-R,R)\right\}$. Here, $\Delta I_s := I_s - I_{s^-}$ for $s \geq 0$, and I^c represents the continuous part of $I \in \mathcal{I}(y)$. By utilizing Itô-Tanaka-Meyer's formula and the differentiability of w, we can derive

$$\begin{aligned}
&e^{-\rho\tau_{R,N}}w(X_{\tau_{R,N}}^{x,y,I}, Y_{\tau_{R,N}}^{y,I}) - w(x,y) \\
&= \int_0^{\tau_{R,N}} e^{-\rho s}\left(\mathcal{L}^y w(X_s^{x,y,I}, Y_s^{y,I}) - \rho w(X_s^{x,y,I}, Y_s^{y,I})\right)ds \\
&\quad + \sigma\int_0^{\tau_{R,N}} e^{-\rho s}w_x(X_s^{x,y,I}, Y_s^{y,I})dW_s \\
&\quad + \sum_{0\leq s\leq\tau_{R,N}} e^{-\rho s}\left[w(X_s^{x,y,I}, Y_s^{y,I}) - w(X_s^{x,y,I}, Y_{s^-}^{y,I})\right] \\
&\quad + \int_0^{\tau_{R,N}} e^{-\rho s}w_y(X_s^{x,y,I}, Y_s^{y,I})dI_s^c,
\end{aligned} \tag{C.1}$$

observing that $t \mapsto X_t^{x,y,I}$ is almost surely continuous for any $I \in \mathcal{I}(y)$, we find

$$\begin{aligned}
w(X_s^{x,y,I}, Y_s^{y,I}) - w(X_s^{x,y,I}, Y_{s^-}^{y,I}) &= w(X_s^{x,y,I}, Y_{s^-}^{y,I} + \Delta I_s) - w(X_s^{x,y,I}, Y_{s^-}^{y,I}) \\
&= \int_0^{\Delta I_s} w_y(X_s^{x,y,I}, Y_{s^-}^{y,I} + u)du,
\end{aligned} \tag{C.2}$$

which substituted back into (C.1) gives the equivalence

$$\begin{aligned}
&\int_0^{\tau_{R,N}} e^{-\rho s}\Big((\pi c_p - aE)X_s^{x,y,I} + (\pi b + arE)X_s^{x,y,I}Y_s^{y,I} + aQ\Big)ds - c\int_0^{\tau_{R,N}} e^{-\rho s} dI_s \\
&= w(x,y) - e^{-\rho\tau_{R,N}} w(X_{\tau_{R,N}}^{x,y,I}, Y_{\tau_{R,N}}^{y,I}) \\
&+ \int_0^{\tau_{R,N}} e^{-\rho s}\Big(\mathcal{L}^y w(X_s^{x,y,I}, Y_s^{y,I}) - \rho w(X_s^{x,y,I}, Y_s^{y,I}) \\
&+ (\pi c_p - aE)X_s^{x,y,I} + (\pi b + arE)X_s^{x,y,I}Y_s^{y,I} + aQ\Big)ds \\
&+ \sigma\int_0^{\tau_{R,N}} e^{-\rho s} w_x(X_s^{x,y,I}, Y_s^{y,I})dW_s \\
&+ \sum_{0\le s\le\tau_{R,N}} e^{-\rho s}\left[\int_0^{\Delta I_s} w_y(X_s^{x,y,I}, Y_{s^-}^{y,I} + u) - c\right]du \\
&+ \int_0^{\tau_{R,N}} e^{-\rho s}\left[w_y(X_s^{x,y,I}, Y_s^{y,I}) - c\right]dI_s^c,
\end{aligned} \tag{C.3}$$

by adding $\int_0^{\tau_{R,N}} e^{-\rho s}\Big((\pi c_p - aE)X_s^{x,y,I} + (\pi b + arE)X_s^{x,y,I}Y_s^{y,I} + aQ\Big)ds - c\int_0^{\tau_{R,N}} e^{-\rho s} dI_s$ on both sides of (C.1). Since w satisfies (4.10), by taking expectations on both sides of the latter equation, and using that $\mathbf{E}[\sigma\int_0^{\tau_{R,N}} e^{-\rho s} w_x(X_s^{x,y,I}, Y_s^{y,I})dW_s] = 0$, we have

$$\begin{aligned}
&\mathbf{E}\left[\int_0^{\tau_{R,N}} e^{-\rho s}\Big((\pi c_p - aE)X_s^{x,y,I} + (\pi b + arE)X_s^{x,y,I}Y_s^{y,I} + aQ\Big)ds - c\int_0^{\tau_{R,N}} e^{-\rho s} dI_s\right] \\
&\le \mathbf{E}\left[w(x,y) - e^{-\rho\tau_{R,N}} w(X_{\tau_{R,N}}^{x,y,I}, Y_{\tau_{R,N}}^{y,I})\right] \le w(x,y) + \mathbf{E}\left[K e^{-\rho\tau_{R,N}}\Big(1 + |X_{\tau_{R,N}}^{x,y,I}|\Big)\right],
\end{aligned} \tag{C.4}$$

To utilize the dominated convergence theorem in (C.4), we observe that since $\beta > 0$ and $Y_t^{y,I} \ge 0$, $X_t^{x,y,I} \le X_t^x$ $\mathbb{P}$-a.s. for all $t \ge 0$. Consequently, we have

$$\begin{aligned}
X_t^{x,y,I} &= x + \int_0^t k\Big((\mu - \beta Y_t^{y,I}) - X_t^{x,y,I}\Big)ds + \sigma W_t \\
&\ge x + \int_0^t k(\mu - X_s^x)ds + \sigma W_t - k\beta\bar{y}t \\
&= X_t^x - k\beta\bar{y}t \ge -|X_t^x| - k\beta\bar{y}t,
\end{aligned} \tag{C.5}$$

where we have used that $Y_t^{y,I} \le \bar{y}$ $\mathbb{P}$-a.s. for all $t \ge 0$. Also, one has $X_t^{x,y,I} \le X_t^x \le |X_t^x| + k\beta\bar{y}t$. Hence,

$$|X_t^{x,y,I}| \le |X_t^x| + k\beta\bar{y}t. \tag{C.6}$$

Now, we find

$$\begin{aligned}
&\left|\int_0^{\tau_{R,N}} e^{-\rho s}\Big((\pi c_p - aE)X_s^{x,y,I} + (\pi b + arE)X_s^{x,y,I}Y_s^{y,I} + aQ\Big)ds - c\int_0^{\tau_{R,N}} e^{-\rho s} dI_s\right| \\
&\le \left|\int_0^{\infty} e^{-\rho s}\Big((\pi c_p - aE)X_s^{x,y,I} + (\pi b + arE)X_s^{x,y,I}Y_s^{y,I} + aQ\Big)ds\right| + c\bar{y} \\
&\le \left|\int_0^{\infty} e^{-\rho s}\Big(\Big((\pi c_p - aE) + (\pi b + arE)\bar{y}\Big)|X_s^{x,y,I}| + aQ\Big)ds\right| + c\bar{y} \\
&\le \Big((\pi c_p - aE) + (\pi b + arE)\bar{y}\Big)\int_0^{\infty} e^{-\rho s}\left(|X_s^x| + k\beta\bar{y}s + aQ\right)ds + c\bar{y}
\end{aligned} \tag{C.7}$$

and the first expression on the right-hand side of (C.7) is integrable by Lemma 4.1. To handle the expectation appearing on the right-hand side of (C.4), we utilize (C.6) to obtain, for a certain constant $C_1 > 0$,

$$\begin{aligned}&\mathbf{E}\left[Ke^{-\rho\tau_{R,N}}\left(1+|X^{x,y,I}_{\tau_{R,N}}|\right)\right]\\&\leq C_1\mathbf{E}\left[e^{-\rho\tau_{R,N}}\left(1+\tau_{R,N}\right)\right]+\mathbf{E}\left[e^{-\frac{\rho}{2}\tau_{R,N}}\sup_{t\geq 0}e^{-\frac{\rho}{2}t}|X^x_t|\right]\\&\leq C_1\mathbf{E}\left[e^{-\rho\tau_{R,N}}\left(1+\tau_{R,N}\right)\right]+\mathbf{E}\left[e^{-\rho\tau_{R,N}}\right]^{\frac{1}{2}}\mathbf{E}\left[\sup_{t\geq 0}e^{-\rho t}(X^x_t)^2\right]^{\frac{1}{2}}\end{aligned} \tag{C.8}$$

where we apply the Hölder's inequality in the last step. From the last expectation in (C.8), we can utilize Itô's formula to derive the following inequality:

$$\begin{aligned}e^{-\rho t}(X^x_t)^2 &\leq x^2+\int_0^t e^{-\rho u}\left[\rho(X^x_u)^2+\sigma^2\right]\\&+\int_0^t 2e^{-\rho u}|X^x_u|(k(|\mu|+|X^x_u|))du+2\sigma\sup_{t\geq 0}\left|\int_0^t e^{-\rho u}X^x_u dW_u\right|.\end{aligned} \tag{C.9}$$

Subsequently, applying the Burkholder-Davis-Gundy inequality yields

$$\mathbf{E}\left[\sup_{t\geq 0}\left|\int_0^t e^{-\rho u}\sigma X^x_u dW_u\right|\right]\leq C_2(1+|x|), \tag{C.10}$$

where $C_2 > 0$ is a constant. Moreover, standard calculations show that $\mathbf{E}\left[|X^x_u|^q\right] \leq \widetilde{C}(1+|x|^q)$ for $q \in \{1,2\}$ and some $\widetilde{C} > 0$. Combining (C.9) and (C.10), we obtain:

$$\mathbf{E}\left[\sup_{t\geq 0}e^{-\rho t}(X^x_t)^2\right]\leq C_3(1+x^2), \tag{C.11}$$

where $C_3 > 0$ is a constant. Consequently, it follows from (C.8) that:

$$\lim_{N\uparrow\infty}\lim_{R\uparrow\infty}\mathbf{E}\left[e^{-\rho\tau_{R,N}}(1+|X^{x,y,I}_{\tau_{R,N}}|)\right]=0. \tag{C.12}$$

Hence, the dominated convergence theorem can be invoked to take limits as $R \to \infty$ and then as $N \to \infty$, resulting in

$$\mathcal{J}(x,y,I)\leq w(x,y). \tag{C.13}$$

Since $I \in \mathcal{I}(y)$ is arbitrary, we have

$$V(x,y)\leq w(x,y). \tag{C.14}$$

This yields $V \leq w$ due to the arbitrariness of (x,y) in $\mathbb{R}\times[0,\bar{y})$.

Step 2: Let $I^* \in \mathcal{I}(y)$ be such that (4.11) and (4.12) are satisfied, and let $\tau^*_{R,N} := \inf\left\{s>0 : X^{x,y,I^*}_s \notin (-R,R)\right\}\wedge N$. By employing the same arguments as in Step 1, all the inequalities become equalities. Consequently, we obtain

$$\begin{aligned}&\mathbf{E}\Big[\int_0^{\tau^*_{R,N}} e^{-\rho s}\left((\pi c_p-aE)X^{x,y,I^*}_s+(\pi b+arE)X^{x,y,I^*}_s Y^{y,I^*}_s+aQ\right)ds-c\int_0^{\tau^*_{R,N}}e^{-\rho s}dI^*_s\Big]\\&+\mathbf{E}\left[e^{-\rho\tau^*_{R,N}}w(X^{x,y,I^*}_{\tau^*_{R,N}},Y^{y,I^*}_{\tau^*_{R,N}})\right]=w(x,y)\end{aligned} \tag{C.15}$$

Now, as I^* is admissible, and by employing (4.10) and (C.12), we proceed as in Step 1 and take limits as $R \uparrow \infty$ and $N \uparrow \infty$ in (C.15) to find $\mathcal{J}(x, y, I^*) \geq w(x, y)$. Since $V(x, y) \geq \mathcal{J}(x, y, I^*)$, then $V(x, y) \geq w(x, y)$ for all $(x, y) \in \mathbb{R} \times [0, \bar{y})$. Hence, utilizing (C.14), we have $V = w$ on $\mathbb{R} \times [0, \bar{y})$, and I^* is optimal.

D. Proof of Lemma 5.2

From (5.15), we have

$$A'(y) = -\beta \frac{\psi''(\widetilde{F}(y))}{\psi'(\widetilde{F}(y))} A(y) - \frac{R_{xy}(F(y), y)}{\psi'(\widetilde{F}(y))} = \mathcal{H}(\widetilde{F}(y), A(y)), \tag{D.1}$$

where $\mathcal{H} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined as

$$\begin{aligned} \mathcal{H}(\widetilde{F}(y), A(y)) &= -\beta \frac{\psi''(\widetilde{F}(y))}{\psi'(\widetilde{F}(y))} A(y) - \frac{R_{xy}(F(y), y)}{\psi'(\widetilde{F}(y))} \\ &= -\frac{R_{xy}(F(y), y)}{\psi'(\widetilde{F}(y))} \left(\frac{\psi''(\widetilde{F}(y)) A(y)}{\beta R_{xy}(F(y), y)} + 1 \right). \end{aligned} \tag{D.2}$$

By the boundary condition $w(x, \bar{y}) = R(x, \bar{y})$, as in Theorem 4.3, it follows that

$$A(\bar{y}) = 0, \tag{D.3}$$

which indicates that when all available GT investments are fully utilized (i.e., $y = \bar{y}$), the value associated with making additional investments in GT becomes zero. Moreover, given (D.3) and the fact that $\mathcal{H}|_{\mathbb{R}\times[0,\infty)}$ is strictly negative due to the strict positivity of $\psi^{(j)}$ for any $j \in \mathbb{N}_0$, as shown in Lemma 5.1-(2). Since $A(\bar{y}) = 0$ and $\mathcal{H}(z, A) < 0$ for all $z \in \mathbb{R}$ and $A \geq 0$ (due to the strict positivity of $\psi^{(j)}$), by the theory of backward ODEs, there exists a unique solution $A(y)$ on $[0, \bar{y}]$ with $A(y) > 0$ for $y < \bar{y}$ and $A'(y) = \mathcal{H}(\widetilde{F}(y), A(y)) < 0$. A is strictly positive and strictly decreasing on $[0, \bar{y})$.

Combining (5.14) and (5.15), we can solve for

$$A(y) = \frac{R_{xy}(F(y), y)\psi(\widetilde{F}(y)) - (R_y(F(y), y) - c)\psi'(\widetilde{F}(y))}{\beta \left(\psi'^2(\widetilde{F}(y)) - \psi''(\widetilde{F}(y))\psi(\widetilde{F}(y)) \right)}, \tag{D.4}$$

where Lemma 5.1-(3) ensures that the denominator of $A(y)$ is nonzero. Now, the numerator on the right-hand side of (D.4) can be written as

$$\begin{aligned} &\frac{R_{xy}(F(y), y)\psi(\widetilde{F}(y)) - (R_y(F(y), y) - c)\psi'(\widetilde{F}(y))}{\beta \left(\psi'^2(\widetilde{F}(y)) - \psi''(\widetilde{F}(y))\psi(\widetilde{F}(y)) \right)} \\ &= \frac{R_{xy}(F(y), y)}{\rho} \left(\frac{\rho \psi(\widetilde{F}(y)) - \frac{\rho}{R_{xy}} (R_y(F(y), y) - c)\psi'(\widetilde{F}(y))}{\beta \left(\psi'^2(\widetilde{F}(y)) - \psi''(\widetilde{F}(y))\psi(\widetilde{F}(y)) \right)} \right) \\ &= \frac{R_{xy}(F(y), y)}{\rho} \left(\frac{\frac{\sigma^2}{2} \psi''(\widetilde{F}(y)) + M^y \psi'(\widetilde{F}(y))}{\beta \left(\psi'^2(\widetilde{F}(y)) - \psi''(\widetilde{F}(y))\psi(\widetilde{F}(y)) \right)} \right), \end{aligned} \tag{D.5}$$

where

$$M^y = (\rho + k)\left[\frac{\beta k}{\rho + k}y + \frac{\beta k(\pi c_p - aE)}{(\pi b + arE)(\rho + k)} + \frac{c\rho}{\pi b + arE} - F(y)\right], \tag{D.6}$$

upon using the differential equation from Lemma 5.1-(2) with $j = 0$. Hence,

$$A(y) = \frac{R_{xy}(F(y), y)}{\rho}\left(\frac{\frac{\sigma^2}{2}\psi''(\widetilde{F}(y)) + M^y\psi'(\widetilde{F}(y))}{\beta\left(\psi'^2(\widetilde{F}(y)) - \psi''(\widetilde{F}(y))\psi(\widetilde{F}(y))\right)}\right). \tag{D.7}$$

Since the denominator on the right-hand side of (D.7) is strictly negative by Lemma 5.1-(3), we must have from the results A that both are strictly positive and strictly decreasing. Therefore the numerator on the right-hand side of (D.7) is also strictly negative: This is possible only if

$$\frac{\beta k}{\rho + k}y + \frac{\beta k(\pi c_p - aE)}{(\pi b + arE)(\rho + k)} + \frac{c\rho}{\pi b + arE} - F(y) < 0, \tag{D.8}$$

as $\psi^{(k)}$ is strictly positive for any $k \in \mathbb{N}_0$. Hence, F satisfies

$$\begin{aligned} F(y) &> \frac{\beta k}{\rho + k}y + \frac{\beta k(\pi c_p - aE)}{(\pi b + arE)(\rho + k)} + \frac{c\rho}{\pi b + arE} \\ &\geq \frac{\beta k(\pi c_p - aE)}{(\pi b + arE)(\rho + k)} + \frac{c\rho}{\pi b + arE}, \end{aligned} \tag{D.9}$$

for all $y \in [0, \bar{y}]$.

E. Proof of Proposition 5.3

We introduce a new function $\widetilde{R} : \mathbb{R}^2 \to \mathbb{R}$ as follows:

$$\widetilde{R}(x, y) = \frac{(x - \beta y)(\pi b + arE)}{\rho + k} - \frac{\beta k(\pi c_p - aE)}{\rho(\rho + k)} + \frac{k(\pi b + arE)(\mu - 2\beta y)}{\rho(\rho + k)}. \tag{E.1}$$

Note that by definition $\widetilde{F}(y) = F(y) + \beta y$, and from the expression of R_y, we have

$$\begin{aligned} R_y(F(y), y) &= \frac{(\widetilde{F}(y) - \beta y)(\pi b + arE)}{\rho + k} - \frac{\beta k(\pi c_p - aE)}{\rho(\rho + k)} + \frac{k(\pi b + arE)(\mu - 2\beta y)}{\rho(\rho + k)} \\ &= \widetilde{R}(\widetilde{F}(y), y), \end{aligned} \tag{E.2}$$

It can be observed that

$$\widetilde{R}_x(\widetilde{F}(y), y) = \frac{\pi b + arE}{\rho + k} = R_{xy}(F(y), y). \tag{E.3}$$

By substituting $\widetilde{F}(y)$ for $F(y)$ in (5.14) and (5.15), and solving the system of equations $A(y)$ and $A'(y)$, we have

$$A(y) = \frac{\widetilde{R}_x(\widetilde{F}(y), y)\psi(\widetilde{F}(y)) + (c - \widetilde{R}(\widetilde{F}(y), y))\psi'(\widetilde{F}(y))}{-\beta Q_0(\widetilde{F}(y))}, \tag{E.4}$$

and

$$A'(y) = \frac{\widetilde{R}_x(\widetilde{F}(y), y)\psi'(\widetilde{F}(y)) + (c - \widetilde{R}(\widetilde{F}(y), y))\psi''(\widetilde{F}(y))}{Q_0(\widetilde{F}(y))}, \tag{E.5}$$

where $\widetilde{R}_x(\widetilde{F}(y), y)$ denotes the partial derivative of $\widetilde{R}(x, y)$ with respect to x.

It is guaranteed by Lemma 5.1-(3) that Q_j is strictly positive for all $j \in \mathbb{N}_0$, ensuring non-zero denominators in (E.4) and (E.5). Consequently, the right-hand sides of (E.4) and (E.5) cannot be zero. Furthermore, considering the boundary condition $w(x, \bar{y}) = R(x, \bar{y})$ from Theorem 4.3, it follows that $A(\bar{y}) = 0$, which implies the existence of a point $\widetilde{x} = \widetilde{F}(\bar{y})$. To ensure that $\widetilde{x}$ can be solved, recall the function $H(x)$ defined in (5.24)

$$H(x) = \frac{\pi b + arE}{\rho + k}\psi(x) + (c - \widetilde{R}(x, \bar{y}))\psi'(x). \tag{E.6}$$

By analogous arguments to those in the proof of Lemma 4.2 in [32], we conclude that there exists a unique solution $\widetilde{x} \in \mathbb{R}$ to $H(x) = 0$.

Next, we can use the above results to obtain an ODE that the free boundary F satisfies. By differentiating (E.4) with respect to y, we obtain

$$\begin{aligned} A'(y) &= \frac{d}{dy}\left[\frac{\widetilde{R}_x(\widetilde{F}(y), y)\psi(\widetilde{F}(y)) + (c - \widetilde{R}(\widetilde{F}(y), y))\psi'(\widetilde{F}(y))}{-\beta Q_0(\widetilde{F}(y))}\right] \\ &= \frac{1}{-\beta Q_0^2}\left[\left(\frac{d}{dy}[\widetilde{R}_x\psi + (c - \widetilde{R})\psi']\right)Q_0 - \left(\widetilde{R}_x\psi + (c - \widetilde{R})\psi'\right)\frac{dQ_0}{dy}\right] \end{aligned}$$

Now, compute the derivatives term by term:

$$\begin{aligned} \frac{d}{dy}[\widetilde{R}_x\psi] &= \widetilde{R}_{xx}\widetilde{F}'(y)\psi + \widetilde{R}_{xy}\psi + \widetilde{R}_x\psi'\widetilde{F}'(y) \\ \frac{d}{dy}[(c - \widetilde{R})\psi'] &= -\widetilde{R}_x\widetilde{F}'(y)\psi' - \widetilde{R}_y\psi' + (c - \widetilde{R})\psi''\widetilde{F}'(y) \\ \frac{dQ_0}{dy} &= Q_0'(\widetilde{F}(y))\widetilde{F}'(y) \end{aligned}$$

Note that $\widetilde{R}_{xx} = 0$ and $\widetilde{R}_{xy} = 0$ since $\widetilde{R}_x$ is constant. After straightforward though lengthy algebraic manipulations, we obtain

$$A'(y) = \frac{\pi b + arE}{\beta(\rho + k)}\frac{M(y, \widetilde{F}(y), \widetilde{F}'(y))}{Q_0(\widetilde{F}(y))^2}, \tag{E.7}$$

where $M : \mathbb{R}^3 \mapsto \mathbb{R}$ is defined as

$$M(y, z, \omega) = -\frac{\beta(2k + \rho)}{\rho}\psi'(z)Q_0(z) + \omega D(y, z), \tag{E.8}$$

with $D : \mathbb{R}^2 \mapsto \mathbb{R}$ given by

$$D(y, z) = \psi(z)\left(\frac{\rho + k}{\pi b + arE}\right)(c - \widetilde{R}(z, y))Q_1(z) + \psi(z)Q_0'(z). \tag{E.9}$$

To keep the proof structure clear, we omit the trivial derivation steps here.

By equating (E.5) and (E.7), we find

$$M(y,\widetilde{F}(y),\widetilde{F}'(y)) = \beta Q_0(\widetilde{F}(y))\Big[\frac{\pi b + arE}{\rho + k}\left(c - \widetilde{R}(\widetilde{F}(y),y)\right)\psi''(\widetilde{F}(y)) + \psi'(\widetilde{F}(y))\Big]. \tag{E.10}$$

We define $N : \mathbb{R}^2 \mapsto \mathbb{R}$ as

$$N(y,z) = Q_0(z)\Big[\frac{2(k+\rho)}{\rho}\psi'(z) + \frac{\rho + k}{\pi b + arE}\left(c - \widetilde{R}(z,y)\right)\psi''(z)\Big]. \tag{E.11}$$

From (E.10) and (E.11), we can obtain the ODE

$$\widetilde{F}'(y) = \mathcal{G}(y,\widetilde{F}(y)), \tag{E.12}$$

with the boundary condition $\widetilde{F}(\bar{y}) = \widetilde{x}$, and where $\mathcal{G} : (\mathbb{R}\times\mathbb{R})/\{(y,z) \in \mathbb{R}^2 : D(y,z) = 0\} \mapsto \mathbb{R}$ is such that

$$\mathcal{G}(y,z) = \beta\frac{N(y,z)}{D(y,z)}. \tag{E.13}$$

F. Proof of Proposition 6.1

For case (1), if $\mu - \beta \times 0 = \mu < F(0)$, the line of means $x = \mu - \beta y$ is decreasing in y, and the free boundary F is increasing, resulting in no intersection.

Let us assume $\mu \geq F(0)$ and differentiate between cases (2) and (3). The line of means $x = \mu - \beta y$ and the free boundary $x = F(y)$ have either one or zero intersections based on whether $\overline{x} = F(\bar{y}) > \mu - \beta\bar{y}$ or not, respectively. In other words, it depends on whether $F(\bar{y}) + \beta\bar{y} = \widetilde{x}(\bar{y}) > \mu$, where $\widetilde{x}(\bar{y})$ represents the dependence of $\widetilde{x}$ on $\bar{y}$. By using Proposition 5.3 and the implicit function theorem, we can obtain:

$$\widetilde{x}'(\bar{y}) = \frac{\psi'(\widetilde{x})\widetilde{R}_y(\widetilde{x},\bar{y})}{\psi''(\widetilde{x})\left(c - \widetilde{R}(\widetilde{x},\bar{y})\right)} = -\frac{\beta(2k+\rho)(\pi b + arE)\psi'(\widetilde{x})}{\rho(\rho+k)\psi''(\widetilde{x})\left(c - \widetilde{R}(\widetilde{x},\bar{y})\right)}, \tag{F.1}$$

where the strict inequality holds. Since $c - \widetilde{R}(\widetilde{x},\bar{y}) < 0$ by Lemma 5.2 and equation (E.4), the denominator $\psi''(\widetilde{x})(c - \widetilde{R}(\widetilde{x},\bar{y}))$ is negative as $\psi'' > 0$. This ensures that $\widetilde{x}'(\bar{y}) > 0$. Hence, $F(\bar{y}) + \beta\bar{y}$ increases with respect to $\bar{y}$, indicating the existence of a point y^*, where there is an intersection for $\bar{y} > y^*$ and no intersection for $\bar{y} < y^*$. The point y^* is characterized by the fact that the line of means $x = \mu - \beta y$ intersects the free boundary $x = F(y)$ and the upper bound of the domain $y = \bar{y}$ at the same point $(\overline{x}, y^*)$. Based on Lemma 5.2 and its conclusion, the point $\overline{x}$ is identified as $\overline{x} = \widetilde{x} - \beta\bar{y}$, where $\widetilde{x}$ is the solution to $H(\widetilde{x}) = 0$, with H defined in Equation (5.24). Thus, to find y^*, we need to simultaneously satisfy the following conditions:

$$\begin{cases} \overline{x} = \mu - \beta y^*, \\ \overline{x} = \widetilde{x} - \beta y^*, \\ \psi'(\widetilde{x})\left(c - \widetilde{R}(\widetilde{x},y^*)\right) + \dfrac{\pi b + arE}{k+\rho}\psi(\widetilde{x}) = 0. \end{cases}$$

In this case, $\widetilde{x} = \mu$, and the third equation can be rewritten as:

$$\psi'(\mu)\left(c + \frac{\beta(\rho + 2k)(\pi b + arE)\bar{y}}{\rho(\rho + k)} + \frac{\beta k(\pi c_p - aE)}{\rho(\rho + k)} - \frac{\mu}{\rho}(\pi b + arE)\right)$$
$$+ \frac{\pi b + arE}{\rho + k}\psi(\mu) = 0.$$

This is a first-order algebraic equation for y^*, and the solution can be obtained as shown in Equation (6.1).