Journal of Industrial and Management Optimization

https://www.aimspress.com/journal/jimo

JIMO, 22(1): 1–23.
DOI: 10.3934/jimo.2026001
Received: 15 September 2025
Revised: 22 October 2025
Accepted: 31 October 2025
Published: 12 November 2025

Research article

A class of weighted Tchebycheff preference relations and multi-objective optimization

Hongjie Tan, Kequan Zhao and Yuanmei Xia*

School of Mathematical Sciences, Chongqing Normal University, No. 37 University Town Middle Road, Shapingba District, Chongqing 401331, China

* **Correspondence:** Email: mathymxia@cqnu.edu.cn.

Abstract: In this paper, we have proposed a new class of binary preference relations by utilizing the weighted Tchebycheff scalarization method, and established several related properties in the objective space. Furthermore, we gave the definitions of the (weakly, strictly) efficient solution and (weakly, strictly) nondominated point for multi-objective optimization problems with respect to the proposed preference relations, and investigated the relationship between these solutions (nondominated points) and Pareto solutions (nondominated points). Moreover, we established the theoretical associations between the three types of solutions proposed in this paper and the optimal solutions of the weighted Tchebycheff scalarization model. Finally, two numerical examples were employed to further illustrate the significance of the proposed preference relations. The results indicated that, for certain specific multi-objective optimization problems, the number of weakly (strictly) nondominated points with respect to the relations presented by us is fewer than that with respect to the natural orders and the weighted aggregation preference relations. Meanwhile, the computational time, worst-case computational complexity, and sensitivity analysis for the weight vector were discussed in the second numerical example.

Keywords: Multi-objective optimization; binary preference relation; weighted Tchebycheff scalarization; optimality; (weakly, strictly) nondominated set
Mathematics Subject Classification: 90C26, 90C29, 90C30

1. Introduction

The theory and methods of multi-objective optimization play a very important role in operations research. They have been widely applied to complex problems such as power system scheduling [1], land use in architecture and urban planning [2], therapy selection in medical decision-making [3], traffic network flow optimization [4], and the optimal control of fast sampling and fuzzy Markov jump singularly perturbed systems [5, 6]. In multi-objective optimization problems (MOPs), conflicts often exist among the objective functions, which makes it difficult to maximize or minimize all objectives simultaneously. As an important class of mathematical tools, order relations can effectively characterize "preferences" in the objective space. A considerable amount of research has focused on the theoretical investigations and property analysis of order relations, which can be found in the research articles [7–11] and so on.

Classical order relations in multi-objective optimization include three types of natural orders, lexicographic order, max-order, etc. (see [12–14] and the references therein). These classical order relations play an important role in fields such as addressing practical problems and algorithm design (see, for example, [15–17]). In addition, Kirkwood and Sarin [18] proposed a binary preference relation based on a weighted sum evaluation function, which was applied to the assessment of materials for nuclear waste storage. Based on different order relations, various types of "optimal solutions" for general multi-objective optimization problems have been proposed, such as weakly efficient solutions, efficient solutions, and strictly efficient solutions defined with respect to the natural orders, and "optimal solutions" with respect to the lexicographic order and max-order, etc. (see [14, 19] and the references therein). Furthermore, Ehrgott [14] established some relations among these solutions. In addition, the nondominated set, which is the image of efficient solutions in the objective space with respect to a specific order, has gradually become an important research topic in multi-objective optimization. For example, Holzmann et al. [20] developed a generating algorithm based on the modified augmented weighted Tchebycheff norm, which can produce the entire nondominated set for any number of objectives. Additional related properties and solution methods can be found in [21–23] and so on. Moreover, weakly and strictly nondominated sets have also been studied in [14].

Scalarization methods are categorized into linear scalarization and nonlinear scalarization. Linear scalarization converts multi-objective optimization problems into single-objective problems via linear transformation, that is, the weighted sum scalarization method [24]. Nonlinear scalarization converts multi-objective optimization problems into single-objective problems via nonlinear transformation, which mainly include the weighted Tchebycheff scalarization method, ε-constraint method, and so on (see [14,25] and the references therein). Kasimbeyli et al. [26] presented an analysis, characterizations, and comparison of these scalarization methods. Based on these scalarization methods, some combined scalarization methods have been proposed (see [27–29] and the references therein). Moreover, Wang et al. [30] applied the idea of scalarization methods to propose INSGA-III for efficient of multi-objective cascade reservoir scheduling under different hydrological conditions. As an effective type of tool for solving multi-objective optimization problems, scalarization methods also play an important role in the construction of new preference relations. By using the weighted sum scalarization method, Kaddani et al. [31] introduced a new type of binary preference relation referred to as the weighted aggregation preference relation. Meanwhile, they established the properties of the nondominated set with respect to the new preference relation and explored its relationship with the Pareto nondominated set. Yue et al. [32] applied a special case of this preference relation to the construction of comprehensive evaluation models in the field of management. In addition, Zhao et al. [33] further investigated the partial-order properties and the optimality of three types of the weighted aggregation preference relations.

It is well known that the preferences of different decision-makers vary, meaning that the characterization of "preferences" in the objective space of different multi-objective optimization problems differs. Currently, existing preference relations cannot capture all the "preferences" in the objective spaces of multi-objective optimization problems. For instance, the number of weakly nondominated points with respect to the natural orders is over 1.03 million for the RBTS power system reliability evaluation test system. Meanwhile, the number of (strictly) nondominated points in the image sets of some multi-objective optimization problems may also be very large with respect to the natural orders or the weighted aggregation preference relations. This greatly reduces decision-making efficiency and makes it difficult to reflect the decision-maker's preferences. Existing research on preference relations via scalarization has mainly focused on linear scalarization, while systematic research on preference relations for nonlinear scalarization and their relationship with Pareto solutions remains scarce. Motivated by these observations and inspired by [31], we aim to propose a new class of preference relations via nonlinear scalarization, so that fewer nondominated points can be obtained for some multi-objective optimization problems.

In this paper, we introduce a new class of binary preference relations by utilizing the weighted Tchebycheff scalarization method presented in [25], and establish several associated properties. Furthermore, we define (weakly, strictly) efficient solutions for the (MOP) with respect to our proposed preference relations, examine their relationship to Pareto solutions, and establish fundamental properties of the corresponding nondominated sets. The connections between the optimal solutions of the weighted Tchebycheff scalarization model and the three new types of solutions of the (MOP) are also studied. To validate the performance of the proposed preference relations, two numerical examples are employed to compare the numbers of weakly (strictly) nondominated points with respect to the natural orders, the weighted aggregation preference relations, and the preference relations proposed in this paper. Meanwhile, the computational time, worst-case computational complexity, and sensitivity analysis for the weight vector are discussed in the second numerical example.

2. Preliminaries

In this paper, let $\mathbb{R}$ represent the set of all real numbers and $\mathbb{N}^+$ denote the set of all positive integers. For $p \in \mathbb{N}^+$, we take $[p] = \{1, 2, \cdots, p\}$, $\mathbb{R}^p_{\geqq} = \{y \in \mathbb{R}^p \mid y_i \geq 0, i \in [p]\}$, $\mathbb{R}^p_{>} = \{y \in \mathbb{R}^p \mid y_i > 0, i \in [p]\}$, and $\mathbb{R}^p_{\geq} = \mathbb{R}^p_{\geqq} \setminus \{\mathbf{0}\}$, where $\mathbb{R}^p$ denotes the p-dimensional Euclidean space and $\mathbf{0}$ is the p-dimensional zero vector. For $i \in [p]$, $u(x; i) = (x_1, x_2, \cdots, x_i)^\top$ represents the vector formed by the first i components of x and $\mathbf{e}^i$ denotes the p-dimensional vector with the i-th component being 1 and all other components being 0. Moreover, we let $W_e = \{\mathbf{e}^1, \mathbf{e}^2, \cdots, \mathbf{e}^p\}$. For any given $x, y \in \mathbb{R}^p$, $x \circ y = (x_1y_1, x_2y_2, \cdots, x_py_p)^\top$ represents the Hadamard product of the vectors x and y.

Consider the following multi-objective optimization problem:

$$\begin{aligned} \text{(MOP)} \quad \min \quad & f(x) = (f_1(x), f_2(x), \cdots, f_p(x))^\top \\ \text{s.t.} \quad & x \in X = \{x \in \Omega \mid g_j(x) \leq 0, j \in [m];\ h_k(x) = 0, k \in [l]\}, \end{aligned}$$

where $m \in \mathbb{N}^+, l \in \mathbb{N}^+$, $f_i, g_j, h_k : \Omega \to \mathbb{R}$, $\Omega \subseteq \mathbb{R}^n$, and the functions f_i are bounded below on Ω.

The weighted Tchebycheff scalarization method is an effective approach for solving multi-objective optimization problems. It transforms a multi-objective optimization problem into the following single-objective optimization problem:

$$(\text{WTSOP})_\beta \quad \min_{x \in X} \max_{i \in [p]} \beta_i (f_i(x) - f_i^*),$$

where $\beta \in \mathbb{R}^p_{\geq}$ and $f_i^* = \min_{x\in X} f_i(x) - \varepsilon$ for all $i \in [p]$. In this paper, we assume that ε is an arbitrary small positive constant.

Let $Y = f(X)$ represent the image of the feasible set X under the objective function mapping f. For any $x, x' \in X$, there exist corresponding points y and y' in Y such that $y = f(x)$ and $y' = f(x')$. Consider the following three types of natural orders:

$$y < y' \Leftrightarrow y' - y \in \mathbb{R}^p_{>} \Leftrightarrow y_i < y'_i,\ \forall i \in [p],$$
$$y \leqq y' \Leftrightarrow y' - y \in \mathbb{R}^p_{\geqq} \Leftrightarrow y_i \leq y'_i,\ \forall i \in [p],$$
$$y \leq y' \Leftrightarrow y' - y \in \mathbb{R}^p_{\geq} \Leftrightarrow y \leqq y', y \neq y'.$$

Definition 2.1 ([14]). Let $x' \in X$.

(i) x' is said to be a weakly efficient solution of the (MOP) if there is no $x \in X$ such that $f(x) < f(x')$;

(ii) x' is said to be a efficient solution of the (MOP) if there is no $x \in X$ such that $f(x) \leq f(x')$;

(iii) x' is said to be a strictly efficient solution of the (MOP) if there is no $x \in X \setminus \{x'\}$ such that $f(x) \leqq f(x')$.

The sets of all weakly efficient solutions, efficient solutions, and strictly efficient solutions of the (MOP) are denoted by $WE(X)$, $E(X)$, and $SE(X)$, respectively. If $x' \in X$ is (weakly, strictly) efficient, then $y' = f(x')$ is called a (weakly, strictly) nondominated point of Y. Moreover, the sets of all weakly efficient points, efficient points, and strictly efficient points of Y are denoted by $WN(Y), N(Y)$, and $SN(Y)$, respectively.

Definition 2.2 ([14]). Let Q be a nonempty subset of $\mathbb{R}^p$. A binary relation C on Q is called a partial order if it satisfies the following properties:

(i) Reflexivity: xCx for all $x \in Q$;

(ii) Antisymmetry: xCy and yCx imply $x = y$ for all $x, y \in Q$;

(iii) Transitivity: xCy and yCz imply xCz for all $x, y, z \in Q$.

If C satisfies only reflexivity and transitivity on Q, then C is called a preorder on Q. If for any $x, y \in Q$ with xCy and any positive scalar λ, it holds that $(\lambda x)C(\lambda y)$, then C is said to be compatible with scalar multiplication on Q. If xCy implies $(x + z)C(y + z)$ for any $x, y, z \in Q$, then C is said to be compatible with addition on Q. If C is compatible with both scalar multiplication and addition on Q, then C is called linear on Q; otherwise, it is called nonlinear.

3. Weighted Tchebycheff preference relations and corresponding solutions

In this section, we construct a new class of preference relations, obtain some related properties, and investigate the relationship between the WT (weakly, strictly) efficient solutions of the (MOP) and the classical Pareto solutions. To begin with, we formally define the new preference relations as follows:

Definition 3.1. Let $\beta \in \mathbb{R}^p_{\geq}$, $f^* \in \mathbb{R}^p$. Define the binary preference relations $\leqq_{\|\cdot\|_{\infty\beta,f^*}}$, $<_{\|\cdot\|_{\infty\beta,f^*}}$, and $\leq_{\|\cdot\|_{\infty\beta,f^*}}$ as follows: For any $y, y' \in Y$,

$$y \leqq_{\|\cdot\|_{\infty\beta,f^*}} y' \Leftrightarrow \|u(\beta; i) \circ u(y - f^*; i)\|_\infty \leq \|u(\beta; i) \circ u(y' - f^*; i)\|_\infty,\ \forall i \in [p],$$
$$y <_{\|\cdot\|_{\infty\beta,f^*}} y' \Leftrightarrow \|u(\beta; i) \circ u(y - f^*; i)\|_\infty < \|u(\beta; i) \circ u(y' - f^*; i)\|_\infty,\ \forall i \in [p],$$
$$y \leq_{\|\cdot\|_{\infty\beta,f^*}} y' \Leftrightarrow y \leqq_{\|\cdot\|_{\infty\beta,f^*}} y', y \neq_{\|\cdot\|_{\infty\beta,f^*}} y',$$

where $y \neq_{\|\cdot\|_{\infty\beta,f^*}} y'$ means that there exists some $k \in [p]$ such that $\|u(\beta;k) \circ u(y - f^*;k)\|_\infty \neq \|u(\beta;k) \circ u(y' - f^*;k)\|_\infty$.

In the following, some related order properties for $\leqq_{\|\cdot\|_{\infty\beta,f^*}}$ are discussed.

Theorem 3.2. $\leqq_{\|\cdot\|_{\infty\beta,f^*}}$ *is a preorder on Y.*

Proof. It is obvious that $y \leqq_{\|\cdot\|_{\infty\beta,f^*}} y$ for any $y \in Y$. Therefore, $\leqq_{\|\cdot\|_{\infty\beta,f^*}}$ satisfies reflexivity on Y.

We now prove that $\leqq_{\|\cdot\|_{\infty\beta,f^*}}$ satisfies transitivity on Y. Let $y, y', y'' \in Y$ satisfy $y \leqq_{\|\cdot\|_{\infty\beta,f^*}} y'$ and $y' \leqq_{\|\cdot\|_{\infty\beta,f^*}} y''$. Then, from Definition 3.1, it follows that for any $i \in [p]$,

$$\|u(\beta;i) \circ u(y - f^*;i)\|_\infty \leq \|u(\beta;i) \circ u(y' - f^*;i)\|_\infty, \tag{3.1}$$

$$\|u(\beta;i) \circ u(y' - f^*;i)\|_\infty \leq \|u(\beta;i) \circ u(y'' - f^*;i)\|_\infty. \tag{3.2}$$

Thus, by (3.1) and (3.2), we have

$$\|u(\beta;i) \circ u(y - f^*;i)\|_\infty \leq \|u(\beta;i) \circ u(y'' - f^*;i)\|_\infty, \quad i \in [p].$$

Therefore, $y \leqq_{\|\cdot\|_{\infty\beta,f^*}} y''$. □

Remark 3.3. $\leqq_{\|\cdot\|_{\infty\beta,f^*}}$ may not be antisymmetric.

Example 3.4. Let

$$Y = \left\{(y_1, y_2)^\top \in \mathbb{R}^2 \,\middle|\, 0.1 \leq y_1 \leq 8, -0.9 \leq y_2 \leq 5\right\},$$
$$\beta = (1,1)^\top, \varepsilon = 0.1, \tilde{y} = (2.1, 1.1)^\top, y' = (2.1, 0.1)^\top.$$

Obviously, $\beta \in \mathbb{R}^p_>$. According to the definition of f^*, we obtain $f^* = (0, -1)^\top$. Then

$$\beta \circ (\tilde{y} - f^*) = (2.1, 2.1)^\top, \beta \circ (y' - f^*) = (2.1, 1.1)^\top.$$

Hence, for any $i \in [2]$, we have

$$\|u(\beta;i) \circ u(\tilde{y} - f^*;i)\|_\infty = \|u(\beta;i) \circ u(y' - f^*;i)\|_\infty.$$

This implies that

$$\tilde{y} \leqq_{\|\cdot\|_{\infty\beta,f^*}} y' \text{ and } y' \leqq_{\|\cdot\|_{\infty\beta,f^*}} \tilde{y}.$$

However, $\tilde{y} \neq y'$. Hence, $\leqq_{\|\cdot\|_{\infty\beta,f^*}}$ does not satisfy antisymmetry.

Remark 3.5. $\leqq_{\|\cdot\|_{\infty\beta,f^*}}$ is not necessarily linear on Y.

Example 3.6. Consider $Y, \beta, \tilde{y}, y'$, and ε from Example 3.4. Let $y'' = (1,4)^\top \in Y$. Then $\tilde{y} \leqq_{\|\cdot\|_{\infty\beta,f^*}} y'$. Moreover,

$$\beta \circ (\tilde{y} + y'' - f^*) = (3.1, 6.1)^\top, \beta \circ (y' + y'' - f^*) = (3.1, 5.1)^\top.$$

Thus, we have

$$\|u(\beta;1) \circ u(\tilde{y} + y'' - f^*;1)\|_\infty = \|u(\beta;1) \circ u(y' + y'' - f^*;1)\|_\infty, \tag{3.3}$$

$$\|u(\beta;2) \circ u(\tilde{y} + y'' - f^*;2)\|_\infty > \|u(\beta;2) \circ u(y' + y'' - f^*;2)\|_\infty. \tag{3.4}$$

From (3.3) and (3.4), we know that $\tilde{y} + y'' \leqq_{\|\cdot\|_{\infty\beta,f^*}} y' + y''$ does not hold. Therefore, $\leqq_{\|\cdot\|_{\infty\beta,f^*}}$ is not compatible with addition on Y.

Moreover, if we let $\lambda = \dfrac{1}{2}$, then

$$\beta \circ (\lambda\tilde{y} - f^*) = (1.05, 1.55)^\top, \beta \circ (\lambda y' - f^*) = (1.05, 1.05)^\top.$$

Therefore,

$$\|u(\beta; 1) \circ u(\lambda\tilde{y} - f^*; 1)\|_\infty = \|u(\beta; 1) \circ u(\lambda y' - f^*; 1)\|_\infty, \tag{3.5}$$

$$\|u(\beta; 2) \circ u(\lambda\tilde{y} - f^*; 2)\|_\infty > \|u(\beta; 2) \circ u(\lambda y' - f^*; 2)\|_\infty. \tag{3.6}$$

From (3.5) and (3.6), we know that $\lambda\tilde{y} \leqq_{\|\cdot\|_{\infty\beta,f^*}} \lambda y'$ does not hold. Therefore, $\leqq_{\|\cdot\|_{\infty\beta,f^*}}$ is not compatible with scalar multiplication on Y.

Definition 3.7. Let $x' \in X$.

(i) x' is said to be a WT weakly efficient solution of the (MOP) if there is no $x \in X$ such that $f(x) <_{\|\cdot\|_{\infty\beta,f^*}} f(x')$;

(ii) x' is said to be a WT efficient solution of the (MOP) if there is no $x \in X$ such that $f(x) \leq_{\|\cdot\|_{\infty\beta,f^*}} f(x')$;

(iii) x' is said to be a WT strictly efficient solution of the (MOP) if there is no $x \in X \setminus \{x'\}$ such that $f(x) \leqq_{\|\cdot\|_{\infty\beta,f^*}} f(x')$.

The sets of WT weakly efficient solutions, WT efficient solutions, and WT strictly efficient solutions of the (MOP) are denoted by $WE(X, <_{\|\cdot\|_{\infty\beta,f^*}})$, $E(X, \leq_{\|\cdot\|_{\infty\beta,f^*}})$, and $SE(X, \leqq_{\|\cdot\|_{\infty\beta,f^*}})$, respectively. Moreover, if $x' \in X$ is a WT (weakly, strictly) efficient solution of the (MOP), then $f(x')$ is called a WT (weakly, strictly) nondominated point of Y. Moreover, the sets of all WT weakly nondominated points, WT nondominated points, WT strictly nondominated points of Y are denoted by $WN(Y, <_{\|\cdot\|_{\infty\beta,f^*}})$, $N(Y, \leq_{\|\cdot\|_{\infty\beta,f^*}})$, and $SN(Y, \leqq_{\|\cdot\|_{\infty\beta,f^*}})$, respectively.

The following results are established to analyze the relation between the WT-type optimal solutions and the Pareto solutions.

Theorem 3.8. *Let $x' \in X$. $x' \in WE(X)$ if and only if there exists $\beta \in \mathbb{R}^p_>$ such that $x' \in WE(X, <_{\|\cdot\|_{\infty\beta,f^*}})$.*

Proof. (Sufficiency). Suppose $x' \notin WE(X)$. Then there exists $\tilde{x} \in X$ such that $f(\tilde{x}) < f(x')$, which implies that $f(\tilde{x}) - f^* < f(x') - f^*$. From $\beta \in \mathbb{R}^p_>$ and the definition of f^*, it follows that

$$\mathbf{0} < \beta \circ (f(\tilde{x}) - f^*) < \beta \circ (f(x') - f^*).$$

Then, for any $i \in [p]$, we have

$$\|u(\beta; i) \circ u(f(\tilde{x}) - f^*; i)\|_\infty < \|u(\beta; i) \circ u(f(x') - f^*; i)\|_\infty. \tag{3.7}$$

By (3.7) and Definition 3.1, we have $f(\tilde{x}) <_{\|\cdot\|_{\infty\beta,f^*}} f(x')$, which contradicts $x' \in WE(X, <_{\|\cdot\|_{\infty\beta,f^*}})$.

(Necessity). Let

$$\beta_i = \frac{1}{f_i(x') - f_i^*}, \quad i \in [p].$$

Suppose $x' \notin WE(X, <_{\|\cdot\|_{\infty\beta,f^*}})$. Then there exists $\tilde{x} \in X$ such that $f(\tilde{x}) <_{\|\cdot\|_{\infty\beta,f^*}} f(x')$. From Definition 3.1, we know that for any $i \in [p]$,

$$\|u(\beta; i) \circ u(f(\tilde{x}) - f^*; i)\|_\infty < \|u(\beta; i) \circ u(f(x') - f^*; i)\|_\infty = 1.$$

It follows that

$$\beta_i(f_i(\tilde{x}) - f_i^*) < 1, \quad i \in [p]. \tag{3.8}$$

Dividing both sides of (3.8) by β_i, we can obtain that

$$f_i(\tilde{x}) - f_i^* < \frac{1}{\beta_i} = f_i(x') - f_i^*, \quad i \in [p].$$

Therefore, for any $i \in [p]$, we have $f_i(\tilde{x}) < f_i(x')$. As a result, $f(\tilde{x}) < f(x')$, which contradicts $x' \in WE(X)$. □

Remark 3.9. Let $x' \in X$. As shown in Theorem 3.8, for any $\beta \in \mathbb{R}^p_>$, if $x' \in WE(X, <_{\|\cdot\|_{\infty\beta,f^*}})$, then it necessarily follows that $x' \in WE(X)$. However, $x' \in WE(X)$ does not necessarily imply $x' \in WE(X, <_{\|\cdot\|_{\infty\beta,f^*}})$ even if $\beta \in \mathbb{R}^p_>$. The following example illustrates this point.

Example 3.10. Consider the multi-objective optimization problem:

$$\begin{aligned} \min \quad & f(x) = (f_1(x), f_2(x))^\top = (x_1, x_2)^\top \\ \text{s.t.} \quad & x \in X = \{(x_1, x_2)^\top \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \geq 16, 0 \leq x_1 \leq 4, 0 \leq x_2 \leq 4\}. \end{aligned}$$

Furthermore, let

$$\beta = (1, 1)^\top, \varepsilon = 0.1.$$

Obviously, $\beta \in \mathbb{R}^2_>$. From the definition of f^*, we can get $f^* = (-0.1, -0.1)^\top$. Taking $x' = (4, 0)^\top$, $\tilde{x} = (2, 2\sqrt{3})^\top$, we can easily verify that $x' \in WE(X)$. However,

$$\beta \circ (f(x') - f^*) = (4.1, 0.1)^\top, \quad \beta \circ (f(\tilde{x}) - f^*) = (2.1, 2\sqrt{3} + 0.1)^\top.$$

Thus we have $f(\tilde{x}) <_{\|\cdot\|_{\infty\beta,f^*}} f(x')$, which implies that $x' \notin WE(X, <_{\|\cdot\|_{\infty\beta,f^*}})$.

Corollary 3.11. *Let $y' \in Y$. $y' \in WN(Y)$ if and only if there exists $\beta \in \mathbb{R}^p_>$ such that $y' \in WN(Y, <_{\|\cdot\|_{\infty\beta,f^*}})$.*

Proof. From Definition 2.1, Definition 3.7, and Theorem 3.8, it follows that the result holds. □

Theorem 3.12. *Let $x' \in X$. $x' \in SE(X)$ if and only if there exists $\beta \in \mathbb{R}^p_\geq$ such that $x' \in SE(X, \leqq_{\|\cdot\|_{\infty\beta,f^*}})$.*

Proof. (Sufficiency). Suppose $x' \notin SE(X)$. Then there exists $\tilde{x} \in X \backslash \{x'\}$ such that $f(\tilde{x}) \leqq f(x')$, which implies that $f(\tilde{x}) - f^* \leqq f(x') - f^*$. From $\beta \in \mathbb{R}^p_\geq$ and the definition of f^*, it follows that

$$\mathbf{0} \leq \beta \circ (f(\tilde{x}) - f^*) \leqq \beta \circ (f(x') - f^*).$$

Then we know that for any $i \in [p]$,

$$\|u(\beta; i) \circ u(f(\tilde{x}) - f^*; i)\|_\infty \leq \|u(\beta; i) \circ u(f(x') - f^*; i)\|_\infty. \tag{3.9}$$

By (3.9) and Definition 3.1, we have $f(\tilde{x}) \leqq_{\|\cdot\|_{\infty\beta,f^*}} f(x')$, which contradicts $x' \in SE(X, \leqq_{\|\cdot\|_{\infty\beta,f^*}})$.

(Necessity). Let

$$\beta_i = \frac{1}{f_i(x') - f_i^*}, \quad i \in [p].$$

Suppose $x' \notin SE(X, \leqq_{\|\cdot\|_{\infty\beta,f^*}})$. Then there exists $\tilde{x} \in X \backslash \{x'\}$ such that $f(\tilde{x}) \leqq_{\|\cdot\|_{\infty\beta,f^*}} f(x')$. From Definition 3.1, we know that for any $i \in [p]$,

$$\|u(\beta; i) \circ u(f(\tilde{x}) - f^*; i)\|_\infty \le \|u(\beta; i) \circ u(f(x') - f^*; i)\|_\infty = 1.$$

It follows that

$$\beta_i(f_i(\tilde{x}) - f_i^*) \le 1, \quad i \in [p]. \tag{3.10}$$

Dividing both sides of (3.10) by β_i, we can obtain that

$$f_i(\tilde{x}) - f_i^* \le \frac{1}{\beta_i} = f_i(x') - f_i^*, \quad i \in [p].$$

Therefore, for any $i \in [p]$, we have $f_i(\tilde{x}) \le f_i(x')$. As a result, $f(\tilde{x}) \leqq f(x')$, which contradicts $x' \in SE(X)$. □

Remark 3.13. Let $x' \in X$. As shown in Theorem 3.12, for any $\beta \in \mathbb{R}^p_{\geq}$, if $x' \in SE(X, \leqq_{\|\cdot\|_{\infty\beta,f^*}})$, then it necessarily follows that $x' \in SE(X)$. However, $x' \in SE(X)$ does not necessarily imply $x' \in SE(X, \leqq_{\|\cdot\|_{\infty\beta,f^*}})$ even if $\beta \in \mathbb{R}^p_{\geq}$. The following example illustrates this point.

Example 3.14. Consider the multi-objective optimization problem:

$$\begin{aligned} \min \quad & f(x) = (f_1(x), f_2(x))^\top = (x_1, x_2)^\top \\ \text{s.t.} \quad & x \in X = \{(x_1, x_2)^\top \in \mathbb{R}^2 \mid x_1 + 6x_2 = 7, 0 \le x_1 \le 1\} \\ & \qquad \cup \{(x_1, x_2)^\top \in \mathbb{R}^2 \mid x_1 + x_2 = 2, 1 \le x_1 \le 2\}. \end{aligned}$$

Furthermore, let

$$\beta = \left(\frac{3}{2}, \frac{1}{2}\right)^\top, \varepsilon = 0.1.$$

Obviously, $\beta \in \mathbb{R}^2_{\geq}$. From the definition of f^*, we can get $f^* = (-0.1, -0.1)^\top$. Taking $x' = (1, 1)^\top$, $\tilde{x} = \left(0, \frac{7}{6}\right)^\top$, we can easily verify that $x' \in SE(X)$. However,

$$\beta \circ (f(x') - f^*) = \left(\frac{33}{20}, \frac{11}{20}\right)^\top, \quad \beta \circ (f(\tilde{x}) - f^*) = \left(\frac{3}{20}, \frac{19}{30}\right)^\top.$$

Thus, $f(\tilde{x}) \leqq_{\|\cdot\|_{\infty\beta,f^*}} f(x')$. This implies that $x' \notin SE(X, \leqq_{\|\cdot\|_{\infty\beta,f^*}})$.

Corollary 3.15. *Let $y' \in Y$. $y' \in SN(Y)$ if and only if there exists $\beta \in \mathbb{R}^p_{\geq}$ such that $y' \in SN(Y, \leqq_{\|\cdot\|_{\infty\beta,f^*}})$.*

Proof. From Definition 2.1, Definition 3.7, and Theorem 3.12, it follows that the result holds. □

Remark 3.16. Let $x' \in X, \beta \in \mathbb{R}^p_{\geq}$. $x' \in E(X, \leq_{\|\cdot\|_{\infty \beta, f^*}})$ does not necessarily imply that $x' \in E(X)$. For example, consider the multi-objective optimization problem:

$$\begin{array}{ll} \min & f(x) = (f_1(x), f_2(x))^\top = (x_1, x_2)^\top \\ \text{s.t.} & x \in X = \{(x_1, x_2)^\top \in \mathbb{R}^2 \mid 3 \leq x_1 \leq 4, 0 \leq x_2 \leq 2\}. \end{array}$$

Furthermore, let

$$\beta = (100, 1)^\top, \varepsilon = 0.1.$$

Obviously, $\beta \in \mathbb{R}^2_{\geq}$. From the definition of f^*, we can get $f^* = (2.9, -0.1)^\top$. Taking $x' = (3, 0)^\top$, we can easily verify that $E(X) = \{x'\}$. Moreover, for any $x \in X$, we have

$$\beta \circ (f(x) - f^*) = (100x_1 - 290, x_2 + 0.1)^\top.$$

Given the range of the feasible set X, it is clear that

$$E(X, \leq_{\|\cdot\|_{\infty \beta, f^*}}) = \{(x_1, x_2)^\top \in \mathbb{R}^2 \mid x_1 = 3, 0 \leq x_2 \leq 2\}.$$

Hence, we have $E(X, \leq_{\|\cdot\|_{\infty \beta, f^*}}) \nsubseteq E(X)$. However, if $x' \in E(X, \leq_{\|\cdot\|_{\infty \beta, f^*}})$ for any $\beta \in W_e$, then $x' \in E(X)$. Suppose $x' \notin E(X)$. Then there exists $\tilde{x} \in X$ such that $f(\tilde{x}) \leq f(x')$. Thus we have $f_i(\tilde{x}) \leq f_i(x')$ for all $i \in [p]$, and there exists $k \in [p]$ such that $f_k(\tilde{x}) < f_k(x')$. Let $\beta = \mathbf{e}^k$, and then it follows, from the definition of f^*, that we can obtain

$$e_i^k(f_i(\tilde{x}) - f_i^*) = e_i^k(f_i(x') - f_i^*) = 0, \quad i \neq k,$$

$$0 < e_i^k(f_i(\tilde{x}) - f_i^*) < e_i^k(f_i(x') - f_i^*), \quad i = k.$$

Therefore, for any $i \in [p]$, we have

$$\|u(\mathbf{e}^k; i) \circ u(f(\tilde{x}) - f^*; i)\|_\infty \leq \|u(\mathbf{e}^k; i) \circ u(f(x') - f^*; i)\|_\infty \tag{3.11}$$

and

$$\|u(\mathbf{e}^k; k) \circ u(f(\tilde{x}) - f^*; k)\|_\infty < \|u(\mathbf{e}^k; k) \circ u(f(x') - f^*; k)\|_\infty. \tag{3.12}$$

From (3.11), (3.12), and Definition 3.1, we have $f(\tilde{x}) \leq_{\|\cdot\|_{\infty, \mathbf{e}^k, f^*}} f(x')$, which contradicts the assumption that $x' \in E(X, \leq_{\|\cdot\|_{\infty \beta, f^*}})$ for any $\beta \in W_e$. Correspondingly, in the image space, it follows from Definition 2.1, Definition 3.7, and the above proof that if $y' \in N(Y, \leq_{\|\cdot\|_{\infty \beta, f^*}})$ for any $\beta \in W_e$, then $y' \in N(Y)$.

4. Weighted Tchebycheff scalarization-based optimality

In this section, we establish the theoretical connections between the optimal solutions of the weighted Tchebycheff scalarization model and the WT (weakly, strictly) efficient solutions of the (MOP).

Theorem 4.1. *Let $x' \in X$. If $x' \in WE(X, <_{\|\cdot\|_{\infty \beta, f^*}})$, then x' is an optimal solution of the $(WTSOP)_\beta$, where $\beta_i = \dfrac{1}{f_i(x') - f^*}$ for all $i \in [p]$.*

Proof. From the definition of f^*, we obtain $\beta \in \mathbb{R}^p_>$. By using Theorem 3.8, we can obtain that $x' \in WE(X, <_{\|\cdot\|_{\infty\beta,f^*}})$ implies $x' \in WE(X)$. Moreover, since $\beta_i = \dfrac{1}{f_i(x') - f^*}$ for all $i \in [p]$, then it follows from Theorem 4.24 in [14] that x' is an optimal solution of the $(\text{WTSOP})_\beta$. □

Theorem 4.2. *Let $x' \in X$. For a given $\beta \in \mathbb{R}^p_\geq$, if x' is an optimal solution of the $(WTSOP)_\beta$, then $x' \in WE(X, <_{\|\cdot\|_{\infty\beta,f^*}})$.*

Proof. Suppose $x' \notin WE(X, <_{\|\cdot\|_{\infty\beta,f^*}})$. Then there exists $\tilde{x} \in X$ such that $f(\tilde{x}) <_{\|\cdot\|_{\infty\beta,f^*}} f(x')$. Clearly, $\tilde{x}$ is a feasible solution of the $(\text{WTSOP})_\beta$. From Definition 3.1, we know that for any $i \in [p]$,

$$\|u(\beta; i) \circ u(f(\tilde{x}) - f^*; i)\|_\infty < \|u(\beta; i) \circ u(f(x') - f^*; i)\|_\infty. \tag{4.1}$$

Thus, by (4.1), it follows that

$$\max_{i\in[p]} |\beta_i(f_i(\tilde{x}) - f_i^*)| < \max_{i\in[p]} |\beta_i(f_i(x') - f_i^*)|.$$

From $\beta \in \mathbb{R}^p_\geq$ and the definition of f^*, we have

$$\max_{i\in[p]} \beta_i(f_i(\tilde{x}) - f_i^*) < \max_{i\in[p]} \beta_i(f_i(x') - f_i^*).$$

This contradicts the assumption that x' is an optimal solution of the $(\text{WTSOP})_\beta$. □

Theorem 4.3. *Let $x' \in X$. If $x' \in SE(X, \leqq_{\|\cdot\|_{\infty\beta,f^*}})$, then x' is the unique optimal solution of the $(WTSOP)_\beta$, where $\beta_i = \dfrac{1}{f_i(x') - f^*}$ for all $i \in [p]$.*

Proof. Suppose that x' is not the unique optimal solution of the $(\text{WTSOP})_\beta$. Then there exists $\tilde{x} \in X\backslash\{x'\}$ such that

$$\max_{i\in[p]} \beta_i(f_i(\tilde{x}) - f_i^*) \leq \max_{i\in[p]} \beta_i(f_i(x') - f_i^*).$$

From the definition of f^*, we know that $\beta \in \mathbb{R}^p_>$. Furthermore,

$$\beta_i(f_i(x') - f_i^*) = 1, \quad i \in [p].$$

Thus, for any $i \in [p]$, we have

$$0 < \beta_i(f_i(\tilde{x}) - f_i^*) \leq \beta_i(f_i(x') - f_i^*) = 1. \tag{4.2}$$

From (4.2), we know that

$$\|u(\beta; i) \circ u(f(\tilde{x}) - f^*; i)\|_\infty \leq \|u(\beta; i) \circ u(f(x') - f^*; i)\|_\infty, \quad i \in [p].$$

Therefore, $f(\tilde{x}) \leqq_{\|\cdot\|_{\infty\beta,f^*}} f(x')$, which contradicts $x' \in SE(X, \leqq_{\|\cdot\|_{\infty\beta,f^*}})$. □

Theorem 4.4. *Let $x' \in X$. For a given $\beta \in \mathbb{R}^p_\geq$, if x' is the unique optimal solution of the $(WTSOP)_\beta$, then $x' \in SE(X, \leqq_{\|\cdot\|_{\infty\beta,f^*}})$.*

Proof. Suppose $x' \notin SE(X, \leqq_{\|\cdot\|_{\infty\beta,f^*}})$. Then there exists $\tilde{x} \in X\backslash\{x'\}$ such that $f(\tilde{x}) \leqq_{\|\cdot\|_{\infty\beta,f^*}} f(x')$. Clearly, $\tilde{x}$ is a feasible solution of the (WTSOP)$_\beta$. From Definition 3.1, we know that for any $i \in [p]$,

$$\|u(\beta; i) \circ u(f(\tilde{x}) - f^*; i)\|_\infty \leq \|u(\beta; i) \circ u(f(x') - f^*; i)\|_\infty. \tag{4.3}$$

Thus, by (4.3), it follows that

$$\max_{i\in[p]} |\beta_i(f_i(\tilde{x}) - f_i^*)| \leq \max_{i\in[p]} |\beta_i(f_i(x') - f_i^*)|.$$

From $\beta \in \mathbb{R}^p_\geq$ and the definition of f^*, we have

$$\max_{i\in[p]} \beta_i(f_i(\tilde{x}) - f_i^*) \leq \max_{i\in[p]} \beta_i(f_i(x') - f_i^*).$$

This contradicts that x' is the unique optimal solution of the (WTSOP)$_\beta$. □

Remark 4.5. Let $x' \in X, \beta \in \mathbb{R}^p_\geq$. $x' \in E(X, \leq_{\|\cdot\|_{\infty\beta,f^*}})$ does not necessarily imply that x' is the unique optimal solution of the (WTSOP)$_\beta$. For example, we consider the multi-objective optimization problem:

$$\begin{aligned} \min \quad & f(x) = (f_1(x), f_2(x))^\top = (x_1, x_2)^\top \\ \text{s.t.} \quad & x \in X = \{(x_1, x_2)^\top \in \mathbb{R}^2 \mid x_1 + x_2 \leq 3, 2 \leq x_1 \leq 3,\ 0 \leq x_2 \leq 1\}. \end{aligned}$$

Additionally, let

$$\beta = (0.99, 0.01)^\top, \varepsilon = 0.1.$$

Obviously, $\beta \in \mathbb{R}^2_\geq$. From the definition of f^*, we have $f^* = (1.9, -0.1)^\top$. In this case, the (WTSOP)$_\beta$ can be transformed into

$$\min_{x\in X} \max\{0.99(x_1 - 1.9),\ 0.01(x_2 + 0.1)\}.$$

Considering the feasible set X, we have

$$E(X, \leq_{\|\cdot\|_{\infty\beta,f^*}}) = \{(x_1, x_2)^\top \in \mathbb{R}^2 \mid x_1 = 2, 0 \leq x_2 \leq 1\}.$$

Let $x' = (2, 0)^\top$ and $\tilde{x} = (2, 1)^\top$. It is easy to see that both x' and $\tilde{x}$ are optimal solutions of the (WTSOP)$_\beta$. Fortunately, the following Theorem 4.6 shows that x' being the unique optimal solution of the (WTSOP)$_\beta$ implies $x' \in E(X, \leq_{\|\cdot\|_{\infty\beta,f^*}})$.

Theorem 4.6. *Let $x' \in X$. For a given $\beta \in \mathbb{R}^p_\geq$, if x' is the unique optimal solution of the $(WTSOP)_\beta$, then $x' \in E(X, \leq_{\|\cdot\|_{\infty\beta,f^*}})$.*

Proof. Suppose $x' \notin E(X, \leq_{\|\cdot\|_{\infty\beta,f^*}})$. Then there exists $\tilde{x} \in X$ such that $f(\tilde{x}) \leq_{\|\cdot\|_{\infty\beta,f^*}} f(x')$. Clearly, $\tilde{x}$ is a feasible solution of the (WTSOP)$_\beta$. From Definition 3.1, we know that for any $i \in [p]$,

$$\|u(\beta; i) \circ u(f(\tilde{x}) - f^*; i)\|_\infty \leq \|u(\beta; i) \circ u(f(x') - f^*; i)\|_\infty,$$

and there exists some $k \in [p]$ such that

$$\|u(\beta; k) \circ u(f(\tilde{x}) - f^*; k)\|_\infty < \|u(\beta; k) \circ u(f(x') - f^*; k)\|_\infty.$$

Thus, it follows from $\beta \in \mathbb{R}^p_\geq$ and the definition of f^* that $\tilde{x} \neq x'$ and

$$\max_{i\in[p]} \beta_i(f_i(\tilde{x}) - f_i^*) \leq \max_{i\in[p]} \beta_i(f_i(x') - f_i^*).$$

This contradicts that x' is the unique optimal solution of the (WTSOP)$_\beta$. □

5. Numerical examples

In this section, we present two numerical examples based on the image set of a bi-objective optimization problem and the state space of a power system. For each case, we compute the weakly (strictly) nondominated sets with respect to the natural orders, the weighted aggregation preference relations, and the preference relations proposed by us. Moreover, the computational time, worst-case computational complexity, and sensitivity analysis for β are discussed in the second numerical example. The definitions of three types of the weighted aggregation preference relations [31, 33] are given as follows: For any $y, y' \in \mathbb{R}^p$,

$$y \leqq_{ws} y' \Leftrightarrow \sum_{i=1}^{t} \beta_i y_i \leq \sum_{i=1}^{t} \beta_i y_i', \ \forall t \in [p],$$
$$y <_{ws} y' \Leftrightarrow \sum_{i=1}^{t} \beta_i y_i < \sum_{i=1}^{t} \beta_i y_i', \ \forall t \in [p],$$
$$y \leq_{ws} y' \Leftrightarrow y \leqq_{ws} y' \text{ and } y \neq y',$$

where $\beta \in \mathbb{R}^p_{\geq}$ and $\sum_{i=1}^{p} \beta_i = 1$.

Based on the theory of Ehrgott [14], the weakly and strictly nondominated sets of a multi-objective optimization problem with respect to the weighted aggregation preference relations are defined as follows:

$$WN(Y, <_{ws}) = \{y' \in Y \mid \text{there is no } y \in Y \text{ such that } y <_{ws} y'\},$$
$$SN(Y, \leqq_{ws}) = \{y' \in Y \mid \text{there is no } y \in Y \backslash \{y'\} \text{ such that } y \leqq_{ws} y'.$$

Note that $\beta \in \mathbb{R}^p_{\geq}$ and $\sum_{i=1}^{p} \beta_i = 1$ for the weighted aggregation preference relations, and $\beta \in \mathbb{R}^p_{\geq}$ for the relations proposed by us (see the assumption conditions of Corollary 3.11 and Corollary 3.15). For the convenience of discussion, we can restrict the range of β for the new preference relations to $\beta \in \mathbb{R}^p_{\geq}$ and $\sum_{i=1}^{p} \beta_i = 1$. In fact, for any $\beta \in \mathbb{R}^p_{\geq}$, we let $\hat{\beta} = 1/(\sum_{i=1}^{p} \beta_i)\beta$. Then $\hat{\beta} \in \mathbb{R}^p_{\geq}$, $\sum_{i=1}^{p} \hat{\beta}_i = 1$, and

$$WN(Y, <_{\|\cdot\|_{\infty,\hat{\beta},f^*}}) = WN(Y, <_{\|\cdot\|_{\infty,\beta,f^*}}),$$
$$SN(Y, \leqq_{\|\cdot\|_{\infty,\hat{\beta},f^*}}) = SN(Y, \leqq_{\|\cdot\|_{\infty,\beta,f^*}}).$$

Hence, in the numerical examples, we set $\beta \in \mathbb{R}^p_{\geq}$ and $\sum_{i=1}^{p} \beta_i = 1$, that is, $\beta_i \in [0, 1]$ for all $i \in [p]$ and $\sum_{i=1}^{p} \beta_i = 1$. Moreover, in this section, we let $\varepsilon = 0.1$.

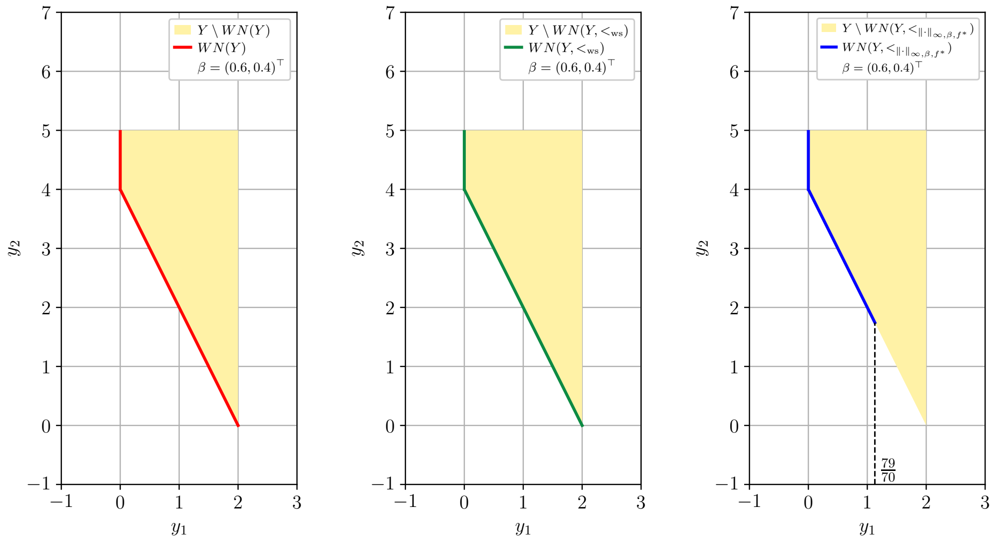
All numerical experiments are conducted under the following computational environment: the hardware configuration consisted of an Intel Core i9-13900H processor (2.60 GHz) and 64 GB of memory, running the 64-bit Windows 11 operating system. The first example is implemented in Python 3.12.4 using the PyCharm 2025.2.2 IDE, and the second is conducted in MATLAB R2022a.

5.1. *A bi-objective optimization problem*

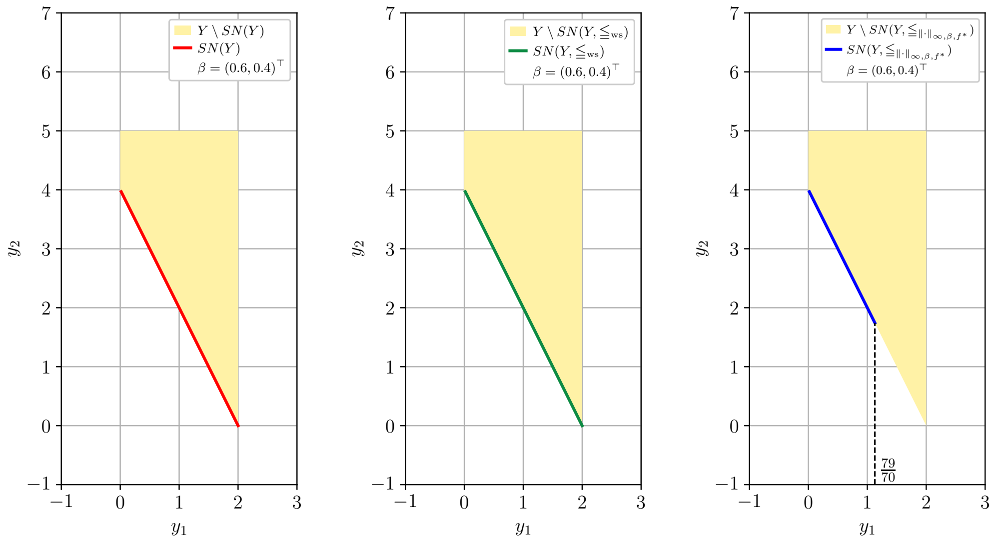
This subsection focuses on the image set of a bi-objective optimization problem. The image set Y is defined as follows:

$$Y = \left\{(y_1, y_2)^\top \in \mathbb{R}^2 \;\middle|\; 2y_1 + y_2 \geq 4, 0 \leq y_1 \leq 2, y_2 \leq 5\right\}.$$

Let β be set to $(0.6, 0.4)^\top$ and $(0.2, 0.8)^\top$. Then for these two cases, we compute the weakly (strictly) nondominated points of Y with respect to the natural orders, the weighted aggregation preference relations, and the new preference relations proposed in this paper. The distributions of weakly (strictly) nondominated points with respect to these preference relations are shown in Figures 1 and 2, respectively.

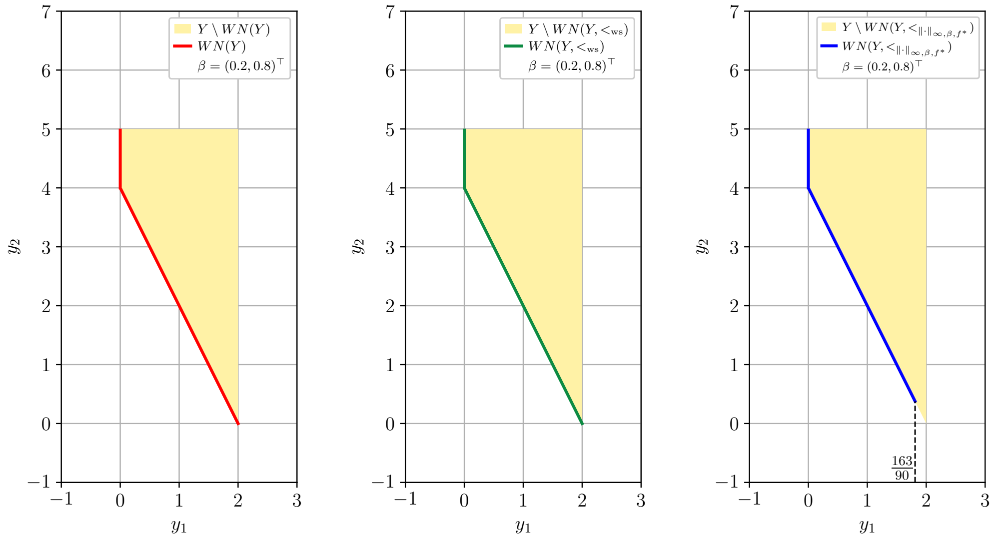


(a) The weakly nondominated points of Y with respect to three relations

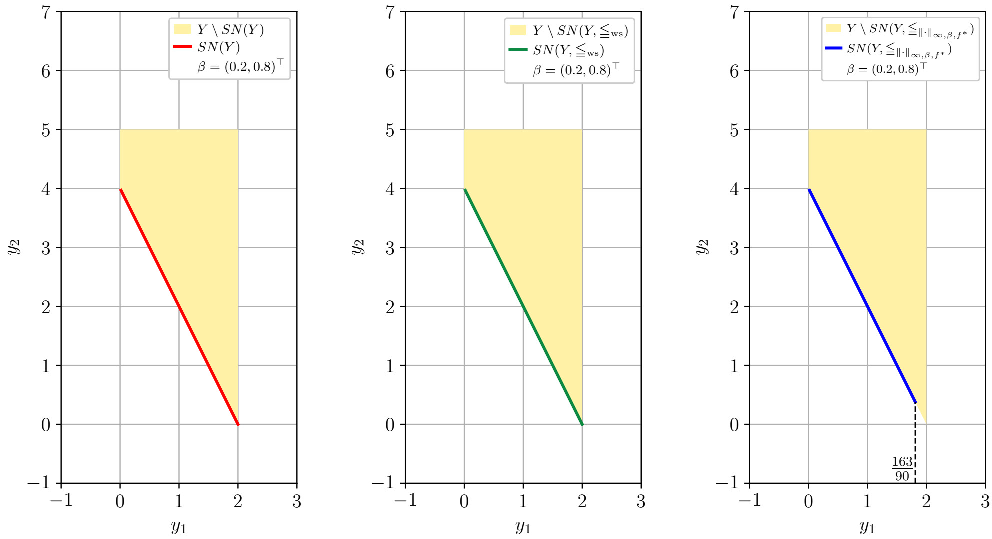


(b) The strictly nondominated points of Y with respect to three relations

Figure 1. The weakly (strictly) nondominated points of Y with respect to three preference relations with $\beta = (0.6, 0.4)^\top$.



(a) The weakly nondominated points of Y with respect to three relations



(b) The strictly nondominated points of Y with respect to three relations

Figure 2. The weakly (strictly) nondominated points of Y with respect to three preference relations with $\beta = (0.2, 0.8)^\top$.

In Figures 1 and 2, the red, green, and blue solid lines represent the weakly (strictly) nondominated set with respect to the natural orders, the weighted aggregation preference relations, and the new preference relations, respectively. Meanwhile, the yellow region represents the dominated points. Hence, we can find that the newly proposed preference relations yield fewer weakly (strictly) nondominated points of Y than the other relations. This implies that the preference relations we proposed have more comparable elements in such cases.

5.2. The RBTS power system reliability evaluation test system

Electrical components are the basic units of a power system, typically operating in two states: normal and fault. The two states are represented by a binary variable, where 0 denotes normal and 1 denotes fault. Therefore, a power system can be abstracted as a set of vectors characterized by normal and fault states. Power system reliability assessment is an important area of study in the field of power systems. Billinton [34] proposed the Roy Billinton Test System (RBTS), which has been widely used in power system reliability assessment testing. The RBTS system consists of 11 generators, 9 transmission lines, and 6 buses, with generators and transmission lines labeled as 20 system components in order, as shown in Table 1. Therein, the second and third components of $(G_i, -, -)$ represent the maximum generation capacity and the bus location of the i-th generator, respectively. The second and third components of $(L_i, -, -)$ represent the maximum transmission capacity and the bus location of the i-th transmission line, respectively.

Table 1. Component order of the RBTS power system reliability evaluation test system.

Index	1	2	3
Component	$(G_1, 40MW, B_1)$	$(G_2, 40MW, B_1)$	$(G_3, 40MW, B_2)$
Index	4	5	6
Component	$(G_4, 20MW, B_1)$	$(G_5, 20MW, B_2)$	$(G_6, 20MW, B_2)$
Index	7	8	9
Component	$(G_7, 20MW, B_2)$	$(G_8, 20MW, B_2)$	$(G_9, 10MW, B_1)$
Index	10	11	12
Component	$(G_{10}, 5MW, B_2)$	$(G_{11}, 5MW, B_2)$	$(L_1, 195.5MW, B_1 - B_3)$
Index	13	14	15
Component	$(L_2, 195.5MW, B_1 - B_3)$	$(L_3, 163.3MW, B_1 - B_2)$	$(L_4, 163.3MW, B_2 - B_4)$
Index	16	17	18
Component	$(L_5, 163.3MW, B_2 - B_4)$	$(L_6, 163.3MW, B_3 - B_4)$	$(L_7, 163.3MW, B_3 - B_5)$
Index	19	20	
Component	$(L_8, 163.3MW, B_4 - B_5)$	$(L_9, 163.3MW, B_5 - B_6)$	

The state space of the RBTS can be represented by Y, that is,

$$Y = \left\{ y = (y_1, y_2, \cdots, y_{20})^\top \in \mathbb{R}^{20} \mid y_i = 0 \text{ or } 1,\ i \in [20] \right\}.$$

The i-th element of the state vector y corresponds to the operational state of the i-th system component. We use $\{k_1, k_2, \cdots, k_l\}$ to represent a vector $y \in Y$, which satisfies the following conditions: $\sum_{k=1}^{20} y_k = l$ and $y_{k_m} = 1$ for any $m \in [l]$. For example, $\{14, 17, 19\}$ represents

$$(0, \cdots, 0, \underset{\substack{\uparrow \\ 14}}{1}, 0, 0, \underset{\substack{\uparrow \\ 17}}{1}, 0, \underset{\substack{\uparrow \\ 19}}{1}, 0)^\top.$$

Furthermore, let Y_F denote the set of all fault states of the RBTS system. The set Y_F is obtained by solving an optimal load shedding model based on DC power flow in [35], and it contains 1,033,302 elements with 509,756 of them having the first component as 0. Taking Y_F as the image set under consideration, we compute its Pareto weakly nondominated set and Pareto strictly nondominated set, which contain 1,033,302 and 62 points, respectively. Table 2 presents seven cases of the weight vector. For these seven cases, we compute the weakly (strictly) nondominated sets of Y_F with respect to both the weighted aggregation preference relations and the proposed preference relations. The corresponding results are shown in Table 3 and Table 4, and the worst-case computational complexities for calculating the weakly (strictly) nondominated sets with respect to these three types of preference relations are all $O(n^2)$.

Table 2. Weight vector settings.

Instance	$\beta^\top$
Case 1	(0.006, 0.084, 0.094, 0.072, 0.079, 0.074, 0.043, 0.065, 0.022, 0.070, 0.005, 0.030, 0.007, 0.012, 0.082, 0.073, 0.034, 0.095, 0.004, 0.047)
Case 2	(0.004, 0.048, 0.054, 0.042, 0.046, 0.043, 0.025, 0.038, 0.125, 0.040, 0.003, 0.017, 0.004, 0.071, 0.047, 0.042, 0.020, 0.055, 0.002, 0.274)
Case 3	(0.001, 0.015, 0.016, 0.013, 0.014, 0.013, 0.008, 0.011, 0.004, 0.012, 0.001, 0.005, 0.001, 0.002, 0.014, 0.013, 0.006, 0.017, 0.001, 0.833)
Case 4	(0.032, 0.053, 0.075, 0.021, 0.032, 0.053, 0.032, 0.053, 0.021, 0.075, 0.032, 0.021, 0.075, 0.097, 0.053, 0.021, 0.032, 0.075, 0.097, 0.053)
Case 5	(0.021, 0.035, 0.009, 0.008, 0.004, 0.016, 0.007, 0.047, 0.038, 0.093, 0.042, 0.061, 0.072, 0.069, 0.076, 0.077, 0.085, 0.090, 0.096, 0.053)
Case 6	(0.014, 0.023, 0.006, 0.006, 0.003, 0.011, 0.005, 0.032, 0.026, 0.063, 0.028, 0.041, 0.049, 0.047, 0.052, 0.052, 0.057, 0.061, 0.065, 0.359)
Case 7	(0.013, 0.048, 0.057, 0.045, 0.044, 0.043, 0.029, 0.038, 0.026, 0.059, 0.006, 0.021, 0.003, 0.010, 0.052, 0.047, 0.023, 0.056, 0.051, 0.328)

Table 3. The numbers of weakly nondominated points of Y_F with respect to three preference relations.

Instance	$WN(Y_F)$	$WN(Y_F, <_{ws})$	Time (s)	$WN(Y_F, <_{\|\cdot\|_{\infty,\beta,f^*}})$	Time (s)
Case 1	Y_F: 1,033,302	509,756	83.0	509,916	35.4
Case 2		509,766	139.9	510,128	43.3
Case 3		509,757	94.0	510,548	46.1
Case 4		509,757	91.5	513,788	56.3
Case 5		509,756	65.0	509,756	49.3
Case 6		509,756	155.9	509,756	42.3
Case 7		509,756	133.0	510,030	27.8

Table 4. The strictly nondominated sets of Y_F with respect to three preference relations.

Instance	$SN(Y_F)$	$SN(Y_F, \leqq_{ws})$	Time (s)	$SN(Y_F, \leqq_{\|\cdot\|_{\infty,\beta,f^*}})$	Time (s)
Case 1	{20}, {18, 19}, {3, 8}, {3, 7}, {3, 6}, {3, 5}, {3, 4}, {2, 8}, {2, 7}, {2, 6}, {2, 5}, {2, 4}, {2, 3}, {1, 8}, {1, 7}, {1, 6}, {1, 5}, {1, 4}, {1, 3}, {1, 2}, {14, 17, 19}, {14, 15, 16}, {12, 13, 16}, {12, 13, 15}, {12, 13, 14}, {6, 7, 8}, {5, 7, 8}, {5, 6, 8}, {5, 6, 7}, {4, 7, 8}, {4, 6, 8}, {4, 6, 7}, {4, 5, 8}, {4, 5, 7}, {4, 5, 6}, {15, 16, 17, 19}, {15, 16, 17, 18}, {12, 13, 17, 19}, {12, 13, 17, 18}, {3, 14, 17, 18}, {3, 9, 10, 11}, {2, 14, 17, 18}, {2, 9, 10, 11}, {1, 14, 17, 18}, {1, 9, 10, 11}, {7, 8, 14, 17, 18}, {7, 8, 9, 10, 11}, {6, 8, 14, 17, 18}, {6, 8, 9, 10, 11}, {6, 7, 14, 17, 18}, {6, 7, 9, 10, 11}, {5, 8, 14, 17, 18}, {5, 8, 9, 10, 11}, {5, 7, 14, 17, 18}, {5, 7, 9, 10, 11}, {5, 6, 14, 17, 18}, {5, 6, 9, 10, 11}, {4, 9, 14, 17, 18}, {4, 8, 9, 10, 11}, {4, 7, 9, 10, 11}, {4, 6, 9, 10, 11}, {4, 5, 9, 10, 11}	{20}	6.4	{20}, {14, 17, 19}	4.8e-3
Case 2		{20}, {18, 19}, {12, 13, 17, 19}, {1, 8}, {1, 7}	9.9	{20}, {18, 19}, {15, 16, 17, 19}, {12, 13, 17, 19}	4.5e-3
Case 3		{20}, {18, 19}, {14, 17, 19}, {12, 13, 14}	6.8	{20}, {18, 19}, {15, 16, 17, 19}, {14, 17, 19}	3.7e-3
Case 4		{20}, {1, 4}	8.2	{20}	1.1e-2
Case 5		{20}, {5, 6, 7}, {4, 5, 7}, {3, 7}, {3, 5}	7.2	{20}, {5, 6, 7}, {4, 5, 7}	4.9e-2
Case 6		{20}, {18, 19}, {6, 7, 8}, {5, 7, 9, 10, 11}, {5, 7, 8}, {5, 6, 7}, {4, 5, 7}, {3, 9, 10, 11}, {3, 8}, {3, 7}, {3, 5}	12.4	{20}, {18, 19}, {15, 16, 17, 19}, {15, 16, 17, 18}, {14, 17, 19}, {14, 15, 16}, {12, 13, 14}, {6, 7, 14, 17, 18}, {5, 7, 14, 17, 18}, {5, 6, 7}, {4, 5, 7}	9.2e-2
Case 7		{20}, {18, 19}, {14, 17, 19}, {12, 13, 16}, {12, 13, 14}	6.0	{20}, {18, 19}, {15, 16, 17, 19}, {14, 17, 19}, {12, 13, 14}	2.9e-3

From Tables 3 and 4, we can find that the number of WT weakly (strictly) nondominated points is smaller than that of Pareto weakly (strictly) nondominated points. Moreover, the number of WT weakly (strictly) nondominated points is comparable to weakly (strictly) nondominated points with respect to the weighted aggregation preference relations, but the time required for the former is less. Moreover, since there are 509,756 elements with the first component being 0 in Y_F, then by using the definitions of the weakly nondominated points with respect to the weighted aggregation preference relations and the preference relations proposed in this paper, we can obtain that both sets $WN(Y_F, <_{ws})$ and $WN(Y_F, <_{\|\cdot\|_{\infty,\beta,f^*}})$ contain at least 509,756 elements.

To investigate the impact of the weight vector β on the number of WT weakly (strictly) nondominated points, we perform a sensitivity analysis with respect to β. To ensure that $\beta \in \mathbb{R}^p_{\geq}$ with

$\sum_{i=1}^{p} \beta_i = 1$, and to examine the impact of each β_i, for any $l \in [20] \setminus \{i\}$, we let

$$\frac{\beta_l}{\sum_{k \in [20] \setminus \{i\}} \beta_k} = \frac{\bar{\beta}_l}{\sum_{k \in [20] \setminus \{i\}} \bar{\beta}_k} \text{ and } \sum_{k \in [20] \setminus \{i\}} \beta_k = 1 - \beta_i,$$

where $\beta_i \in [0, 1]$ and

$$\begin{aligned}
&\bar{\beta}_1 = 0.006,\ \bar{\beta}_2 = 0.084,\ \bar{\beta}_3 = 0.094,\ \bar{\beta}_4 = 0.072,\ \bar{\beta}_5 = 0.079,\\
&\bar{\beta}_6 = 0.074,\ \bar{\beta}_7 = 0.043,\ \bar{\beta}_8 = 0.065,\ \bar{\beta}_9 = 0.022,\ \bar{\beta}_{10} = 0.070,\\
&\bar{\beta}_{11} = 0.005,\ \bar{\beta}_{12} = 0.030,\ \bar{\beta}_{13} = 0.007,\ \bar{\beta}_{14} = 0.012,\ \bar{\beta}_{15} = 0.082,\\
&\bar{\beta}_{16} = 0.073,\ \bar{\beta}_{17} = 0.034,\ \bar{\beta}_{18} = 0.095,\ \bar{\beta}_{19} = 0.004,\ \bar{\beta}_{20} = 0.047.
\end{aligned}$$

For any $i \in [20]$, when β_i varies from 0 to 1 in steps of 0.05, the trends of the numbers of WT weakly nondominated points and WT strictly nondominated points with respect to β_i are shown in Figure 3 and Figure 4, respectively.

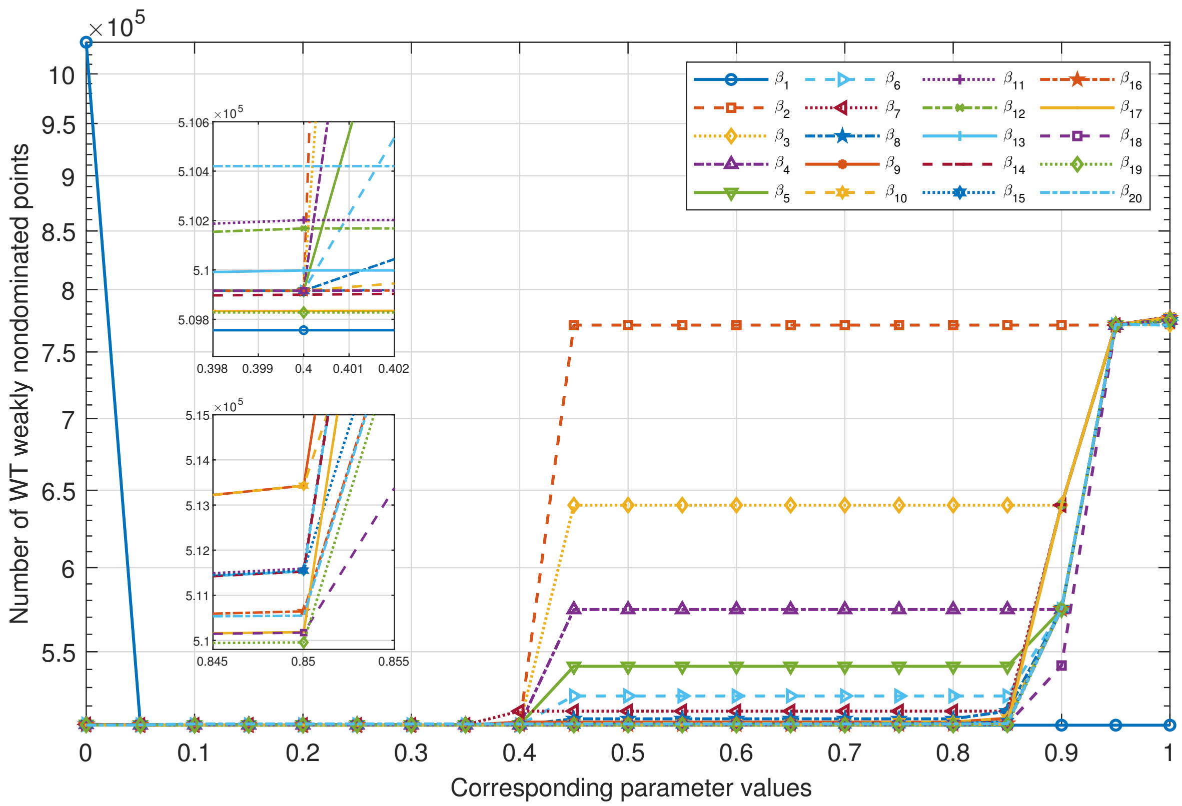


Figure 3. The variation trend of the number of WT weakly nondominated points of Y_F with respect to each β_i.

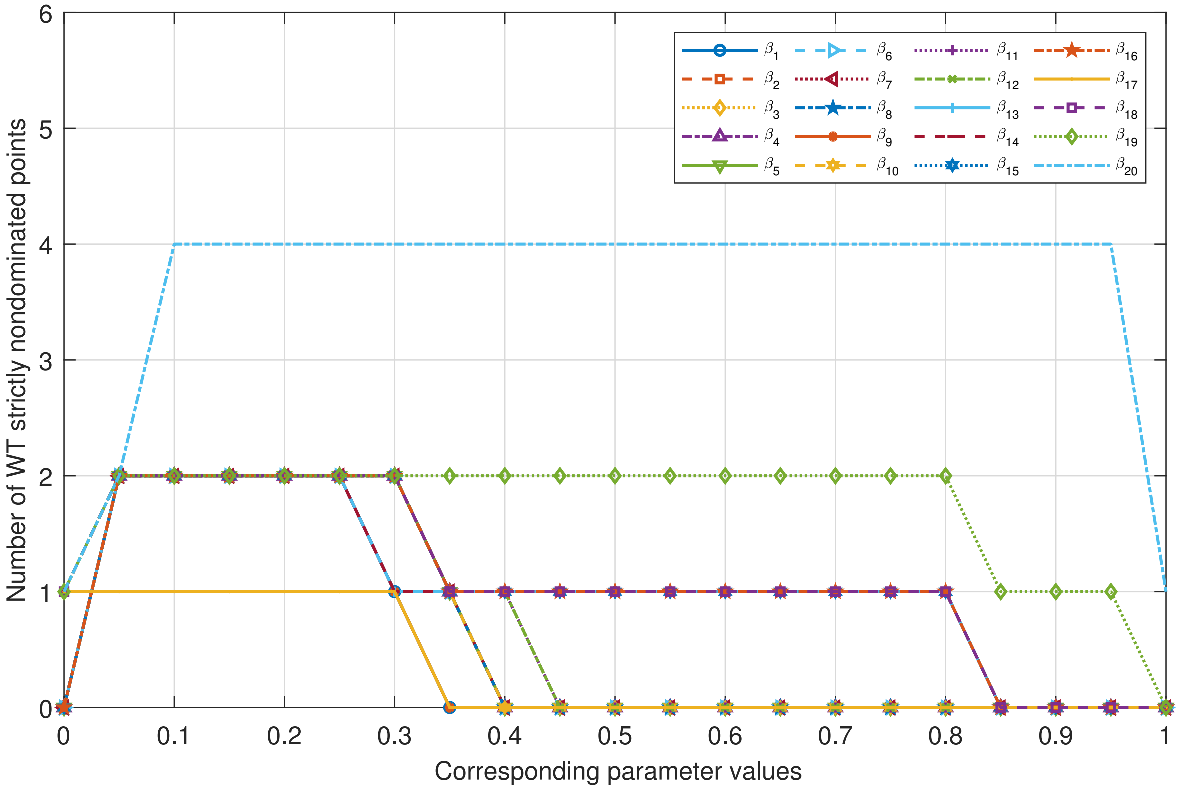


Figure 4. The variation trend of the number of WT strictly nondominated points of Y_F with respect to each β_i.

As shown in Figures 3 and 4, β_1 induces the greatest fluctuations in the number of WT weakly nondominated points, and β_{20} induces the greatest fluctuations in the number of WT strictly nondominated points.

6. Conclusions

We propose a class of weighted Tchebycheff preference relations and establish some related properties in the objective space. Moreover, WT (weakly, strictly) efficient solutions of the (MOP) are defined with respect to the preference relations presented in this paper. The relationships of these solutions with Pareto solutions and the optimal solutions of the weighted Tchebycheff scalarization model are also analyzed. Finally, two numerical examples are employed to further illustrate the significance of the proposed preference relations. Unfortunately, we find that the computational complexity of WT weakly (strictly) nondominated points in solving the power system reliability evaluation test system is $O(n^2)$. Thus, how to propose an algorithm for solving WT weakly (strictly) nondominated points in such problems to reduce computational complexity stands as a crucial topic for future research.

Author contributions

Conceptualization: K. Q. Zhao, Y. M. Xia; Formal analysis: H. J. Tan, K. Q. Zhao, Y. M. Xia; Funding acquisition: K. Q. Zhao, Y. M. Xia; Investigation: H. J. Tan, K. Q. Zhao, Y. M. Xia; Methodology: H. J. Tan, K. Q. Zhao, Y. M. Xia; Resources: K. Q. Zhao, Y. M. Xia; Supervision: K. Q. Zhao, Y. M. Xia; Validation: H. J. Tan, K. Q. Zhao, Y. M. Xia; Visualization: H. J. Tan, K. Q. Zhao, Y. M. Xia; Writing – original draft: H. J. Tan, Y. M. Xia; Writing – review and editing: K. Q. Zhao, Y. M. Xia.

Use of Generative-AI tools declaration

During the preparation of this work, the authors used AI tools such as DeepSeek and DouBao to improve the clarity and readability of the language. All AI-assisted content was critically reviewed and revised by the authors, who take full responsibility for the final version of the manuscript.

Acknowledgments

The authors are deeply grateful to the handling editor and reviewers for their constructive comments and suggestions. This work was supported by the National Key R&D Program of China (Grant No. 2023YFA1011302), National Natural Science Foundation of China (Grant No. 12171063), Natural Science Foundation of Chongqing (Grant No. CSTB2025NSCQ-GPX0994), and Science and Technology Research Program of Chongqing Municipal Education Commission (Grant Nos. KJQN202500510, KJZD-K202300509).

Conflict of interest

This study does not have any conflicts to disclose.

References

1. E. Zio, P. Baraldi, N. Pedroni, Optimal power system generation scheduling by multi-objective genetic algorithms with preferences, *Reliab. Eng. Syst. Saf.*, **94** (2009), 432–444. https://doi.org/10.1016/j.ress.2008.04.004
2. M. Shaygan, A. Alimohammadi, A. Mansourian, Z. S. Govara, S. M. Kalami, Spatial multi-objective optimization approach for land use allocation using NSGA-II, *IEEE J. Sel. Top. Appl. Earth Obs. Remote Sens.*, **7** (2013), 906–916. https://doi.org/10.1109/JSTARS.2013.2280697
3. S. Breedveld, D. Craft, R. Van Haveren, B. Heijmen, Multi-criteria optimization and decision-making in radiotherapy, *Eur. J. Oper. Res.*, **277** (2019), 1–19. https://doi.org/10.1016/j.ejor.2018.08.019
4. J. B. Mou, Intersection traffic control based on multi-objective optimization, *IEEE Access*, **8** (2020), 61615–61620.

5. H. Shen, C. J. Peng, H. C. Yan, S. Y. Xu, Data-driven near optimization for fast sampling singularly perturbed systems, *IEEE Trans. Autom. Control*, **69** (2024), 4689–4694. https://doi.org/10.1109/TAC.2024.3352703
6. J. Wang, Y. L. Huang, X. P. Xie, H. C. Yan, H. Shen, Optimal control for fuzzy Markov jump singularly perturbed systems: A hybrid zero-sum game iteration approach, *IEEE Trans. Fuzzy Syst.*, **32** (2024), 6388–6398. https://doi.org/10.1109/TFUZZ.2024.3449322
7. A. M. Mármol, J. Puerto, F. R. Fernández, The use of partial information on weights in multicriteria decision problems, *J. Multi-Criteria Decis. Anal.*, **7** (1998), 322–329. https://doi.org/10.1002/(SICI)1099-1360(199811)7:6<322::AID-MCDA203>3.0.CO;2-4
8. A. D. Athanassopoulos, V. V. Podinovski, Dominance and potential optimality in multiple criteria decision analysis with imprecise information, *J. Oper. Res. Soc.*, **48** (1997), 142–150. https://doi.org/10.1057/palgrave.jors.2600345
9. A. Dehnokhalaji, P. J. Korhonen, M. Köksalan, N. Nasrabadi, J. Wallenius, Convex cone-based partial order for multiple criteria alternatives, *Decis. Support Syst.*, **51** (2011), 256–261. https://doi.org/10.1016/j.dss.2010.11.019
10. H. Wang, M. Olhofer, Y. Jin, A mini-review on preference modeling and articulation in multi-objective optimization: current status and challenges, *Complex Intell. Syst.*, **3** (2017), 233–245. https://doi.org/10.1007/s40747-017-0053-9
11. R. Szlapczynski, J. Szlapczynska, W-dominance: Tradeoff-inspired dominance relation for preference-based evolutionary multi-objective optimization, *Swarm Evol. Comput.*, **63** (2021), 100866. https://doi.org/10.1016/j.swevo.2021.100866
12. Y. M. Xia, D. D. Xia, K. Q. Zhao, Generalized Tchebycheff norm scalarization in multi-objective optimization, *Oper. Res. Trans.*, **29** (2025), 175–183.
13. X. B. Li, S. J. Li, Z. M. Fang, A minimax theorem for vector-valued functions in lexicographic order, *Nonlinear Anal.*, **73** (2010), 1101–1108. https://doi.org/10.1016/j.na.2010.04.047
14. M. Ehrgott, *Multicriteria Optimization*, 2 Eds., Berlin: Springer, 2005.
15. Y. M. Zhang, B. Atasoy, R. R. Negenborn, Preference-based multi-objective optimization for synchromodal transport using adaptive large neighborhood search, *Transp. Res. Rec.*, **2676** (2022), 71–87. https://doi.org/10.1177/03611981211049148
16. Q. Z. Wang, Y. H. Li, Z. L. Hou, J. Zou, J. H. Zheng, A novel multi-population evolutionary algorithm based on hybrid collaboration for constrained multi-objective optimization, *Swarm Evol. Comput.*, **87** (2024), 101581. https://doi.org/10.1016/j.swevo.2024.101581
17. L. Lai, L. Fiaschi, M. Cococcioni, K. Deb, Pure and mixed lexicographic-paretian many-objective optimization: state of the art, *Nat. Comput.*, **22** (2023), 227–242. https://doi.org/10.1007/s11047-022-09911-4
18. C. W. Kirkwood, R. K. Sarin, Ranking with partial information: A method and an application, *Oper. Res.*, **33** (1985), 38–48. https://doi.org/10.1287/opre.33.1.38
19. J. Dutta, C. Y. Kaya, A new scalarization and numerical method for constructing the weak Pareto front of multi-objective optimization problems, *Optimization*, **60** (2011), 1091–1104. https://doi.org/10.1080/02331934.2011.587006

20. T. Holzmann, J. C. Smith, Solving discrete multi-objective optimization problems using modified augmented weighted Tchebychev scalarizations, *Eur. J. Oper. Res.*, **271** (2018), 436–449. https://doi.org/10.1016/j.ejor.2018.05.036

21. D. Foroutannia, F. Ahmadi, On the structure of the equitably nondominated set of multi-objective optimization problems, *Int. J. Nonlinear Anal. Appl.*, **15** (2024), 289–298.

22. L. H. Lee, E. P. Chew, S. Teng, D. Goldsman, Finding the non-dominated Pareto set for multi-objective simulation models, *IIE Trans.*, **42** (2010), 656–674. https://doi.org/10.1080/07408171003705367

23. I. Doğan, B. Lokman, M. Köksalan, Representing the nondominated set in multi-objective mixed-integer programs, *Eur. J. Oper. Res.*, **296** (2022), 804–818. https://doi.org/10.1016/j.ejor.2021.04.005

24. S. Gass, T. Saaty, The computational algorithm for the parametric objective function, *Nav. Res. Logist. Q.*, **2** (1955), 39–45. https://doi.org/10.1002/nav.3800020106

25. V. J. Bowman, On the relationship of the Tchebycheff norm and the effcient frontier of multiple-criteria objectives, *Proceedings of the Springer Conference on Multiple Criteria Decision Making*, Jouy-en-Josas, France, 1976, 76–86.

26. R. Kasimbeyli, Z. K. Ozturk, N. Kasimbeyli, G. D. Yalcin, B. I. Erdem, Comparison of some scalarization methods in multiobjective optimization, *Bull. Malays. Math. Sci. Soc.*, **42** (2019), 1875–1905.

27. Y. M. Xia, X. M. Yang, K. Q. Zhao, A combined scalarization method for multi-objective optimization problems, *J. Ind. Manag. Optim.*, **17** (2021), 2669–2683. https://doi.org/10.3934/jimo.2020088

28. K. Q. Zhao, X. M. Yang, Y. M. Xia, Complete scalarization of multi-objective optimization problems, *Sci. Sin. Math.*, **51** (2021), 411–424. https://doi.org/10.1360/SCM-2019-0004

29. Y. M. Xia, K. Q. Zhao, Y. Gao, Generalized Pascoletti-Serafini scalarization for multi-objective optimization, *Sci. Sin. Math.*, **54** (2024), 231–244. https://doi.org/10.1360/SSM-2023-0047

30. Z. C. Wang, H. F. Zhao, Q. Lu, T. H. Wu, Improved non-dominated sorting genetic algorithm III for efficient of multi-objective cascade reservoirs scheduling under different hydrological conditions, *J. Hydrol.*, **656** (2025), 132998. https://doi.org/10.1016/j.jhydrol.2025.132998

31. S. Kaddani, D. Vanderpooten, J. M. Vanpeperstraete, H. Aissi, Weighted sum model with partial preference information: Application to multi-objective optimization, *Eur. J. Oper. Res.*, **260** (2017), 665–679. https://doi.org/10.1016/j.ejor.2017.01.003

32. L. Z. Yue, C. Lu, Z. J. Zhang, Partial ordered set representation of comprehensive evaluation model, *Oper. Res. Manag. Sci.*, **31** (2022), 101–106.

33. K. Q. Zhao, M. H. Liu, Y. M. Xia, Partial order properties and optimality of a special class of vector sets, *J. Chongqing Norm. Univ. (Nat. Sci. Ed.)*, **41** (2024), 35–41.

34. R. Billinton, S. Kumar, N. Chowdhury, K. Chu, K. Debnath, L. Goel, et al., A reliability test system for educational purposes-basic data, *IEEE Trans. Power Syst.*, **4** (1989), 1238–1244. https://doi.org/10.1109/59.32623

35. Z. Y. Liu, P. T. Tang, K. Hou, L. W. Zhu, J. B. Zhao, H. J. Jia, et al., A Lagrange-multiplier-based reliability assessment for power systems considering topology and injection uncertainties. *IEEE Trans. Power Syst.*, **39** (2024), 1178–1189. https://doi.org/10.1109/TPWRS.2023.3258319

AIMS Press

©2026 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (https://creativecommons.org/licenses/by/4.0)