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Research article

Optimal reinsurance–investment with prevention under common-shock dependence

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Abstract: This paper studies an optimal reinsurance, investment, and prevention problem for an insurer under common shock dependence. The aggregate loss consists of two claim streams with independent claim sizes and dependent claim arrivals driven by a common shock component. The insurer chooses reinsurance, investment, and prevention to maximize the expected utility of terminal wealth, where prevention lowers the arrival intensities at a convex cost. Reinsurance is modeled in a general admissible framework without imposing a specific contract form. Under exponential utility, we derive the Hamilton-Jacobi-Bellman equation and establish a verification theorem. The optimal investment strategy is explicit, the optimal reinsurance policy is shown to be of the excess-of-loss type, and the optimal prevention strategy can be expressed using the inverse marginal cost function. Numerical results further demonstrate the value of prevention and show how prevention effectiveness, prevention cost, and risk aversion affect the optimal strategies.

Keywords: prevention; common shock; verification theorem; optimal strategies

1. Introduction

Reinsurance and investment are two core instruments for insurers to manage underwriting risk and allocate capital. The former helps insurers transfer part of their claim risk and stabilize their liabilities, while the latter enables them to improve asset returns through financial market participation. The joint use of these two instruments also plays a crucial role in maintaining profitability and financial resilience. This model has been widely studied in actuarial science under different objective criteria, including expected-utility maximization, mean–variance criteria, and ruin-based objectives such as minimizing ruin probability or maximizing survival probability; for example, see Liu and Yang [1], Zhao et al. [2], Liang and Bayraktar [3], and Zhu et al. [4] for expected-utility problems, Yi et al. [5], Bai and Zhang [6], Shen and Zeng [7], Bi et al. [8], and Zhang et al. [9] for mean–variance problems, and Schmidli [10], Luo and Taksar [11], and Tan et al. [12] for ruin-based problems. In this paper, we focus on a new risk model that considers the possibility that the insurer implements a prevention plan to reduce claim arrival intensities. Under this model, we further determine the optimal prevention strategy together with the optimal reinsurance and investment strategies.

With the growing complexity of risk environments, prevention strategies have increasingly become an important means for insurers to manage risks before losses occur. Through financial investment and managerial efforts, insurers seek to control risks at their source, with the aim of reducing the claim frequency and alleviating the severity of individual claims. For example, in cyber insurance, an insurer may provide security services, technical support, risk assessment, or employee training to help policyholders identify vulnerabilities, strengthen their defenses, and reduce the likelihood or severity of cyber incidents. Since these prevention measures also bring additional costs, the insurer needs to balance the cost of providing prevention with the benefit of reducing future claim exposure.

The analysis of prevention can be traced back to the seminal work of Ehrlich and Becker [13], who offered a pioneering analysis of market insurance, self-insurance, and self-protection as three basic instruments of risk management, thus laying the theoretical foundation for the study of preventive behavior. Hiebert [14] further examined self-insurance and self-protection in the theory of the competitive firm, thereby clarifying that self-insurance focuses on reducing the size of losses, whereas self-protection aims to reduce the probability of loss occurrence. Following this line of research, prevention has been gradually incorporated into insurance models as an active risk management decision. For example, Gauchon et al. [15] considered a risk model in which the insurer can invest in a prevention plan that lowers the claim intensity, and characterized the optimal prevention level under several risk indicators. Gauchon et al. [16] further identified necessary and sufficient conditions under which an insurer adopts prevention measures in a specific setting. More recently, Chen et al. [17] analyzed the interaction between self-protection effort and insurance under a mean-variance criterion with time inconsistency, while Hu et al. [18] developed a mean-variance Stackelberg framework with a mechanism for sharing prevention costs. These studies show that prevention is not only a risk reduction tool, but also an important decision variable in insurance models.

Different risks often do not occur independently. In many insurance businesses, several lines of claims may be affected by the same event, such as a natural disaster, a large accident, or a market shock. Therefore, it is important to consider the dependence among different risk sources. The common shock structure provides a natural way to describe such dependence. Several studies have used common shocks to model dependent insurance risks. Yuen et al. [19] studied common shock dependence among multiple classes of insurance businesses and obtained optimal proportional reinsurance strategies. Bi et al. [8] considered common shock dependence between two classes of insurance businesses and derived optimal investment and reinsurance strategies under a mean–variance criterion. Liang and Yuen [20] studied common shock dependence between two classes of insurance businesses, where the two claim number processes are correlated through a common shock component, and obtained the optimal proportional reinsurance strategy under the variance premium principle. Bi et al. [21] further studied common shock dependence between two insurance risk classes in a regime-switching financial market. Bi and Chen [22] analyzed common shock dependence between two classes of insurance businesses under different premium principles and obtained closed form expressions for the optimal strategies and value functions. Ceci et al. [23] studied common shocks between aggregate claims and stock prices and characterized the optimal reinsurance and investment strategies. Chen et al. [24] considered common shock dependence between the insurance market and the financial market with a random exit time and derived the explicit optimal reinsurance and investment strategy. Therefore, incorporating common shock dependence into models is important to design more realistic risk management strategies.

Reinsurance is usually modeled in a prescribed form to facilitate the analysis and derivation of optimal strategies. Taksar and Markussen [25] considered proportional reinsurance in a large portfolio model and derived optimal dynamic reinsurance policies. Liang and Yuen [20] and Yuen et al. [19] studied proportional reinsurance for dependent insurance risks and obtained optimal proportional reinsurance strategies. Zhao et al. [2] considered excess-of-loss reinsurance and investment under the Heston model and derived explicit optimal strategies and value functions. Centeno and Simões [26] considered quota-share reinsurance, excess-of-loss reinsurance, and their combination for n independent risks, and determined the optimal retentions by maximizing the adjustment coefficient. These studies show the importance of reinsurance in risk management. However, a fixed contract form restricts the structure of the retention policy. Different from these works, we model reinsurance as a general admissible policy and endogenously derive the optimal retention function.

Motivated by the above works, this paper extends the classical joint reinsurance–investment problem by incorporating a prevention control that reduces claim arrival intensities. The aggregate loss consists of two claim streams which correspond to internal and external risks. The claim sizes are independent, while the claim arrivals are dependent through a common shock component. Under this dependence structure, the insurer jointly chooses reinsurance, investment, and prevention to maximize the expected utility of terminal wealth. Using dynamic programming, we derive an explicit optimal investment rule and show that the optimal reinsurance policy admits an excess-of-loss structure. The optimal prevention strategy is characterized by zero at the boundary solution, or by a unique inverse marginal cost relation when an interior solution exists (see Proposition 3.6). We further establish a verification theorem to confirm that the candidate value function and strategies are indeed optimal.

Recently, an increasing number of studies have examined self-protection or prevention alongside dependent insurance risks. Muermann and Kunreuther [27] examined optimal self-protection investment under interdependent risks arising from potential contamination by others and showed that appropriately restricting insurance coverage can partially internalize the associated externality. Franke and Orlando [28] studied interdependent cyber risk by introducing a cross-investment mechanism between insured firms, and argued that insurers should play a more proactive role in cyber risk management. Boyer and Vigneron [29] developed a game-theoretic cyber insurance framework under interdependent losses, where insured firms decide whether to invest in prevention and insurers decide whether to audit their prevention behavior; they showed that moral hazard and network externalities may lead insured firms to underinvest in prevention. Unlike these studies that primarily view prevention as an incentive decision made by insured firms, our paper models prevention as a nonnegative control directly chosen by the insurer within a continuous-time risk framework. We jointly optimize prevention, general reinsurance retention functions, and financial investment under claim arrivals influenced by common shocks. Instead of predefining proportional or excess-of-loss forms, we optimize over all admissible retention functions, thereby deriving the excess-of-loss structure endogenously. Additionally, we demonstrate that the two optimal deductible levels are linked through the common-shock component. Prevention alters the claim arrival intensities, including the common-shock intensity, thereby influencing both the dependence between claim lines and the optimal reinsurance policies. Contrastingly, the optimal investment strategy admits an explicit characterization and is separated from the insurance decisions.

There is also a strand of the literature that derives excess-of-loss optimality without predefining a specific reinsurance form, instead starting from a general admissible class of contracts. For example, Li et al. [30] showed that excess-of-loss reinsurance is the unique equilibrium strategy under a mean–variance criterion and the expected-value premium principle. Han et al. [31] further derived pure excess-of-loss reinsurance for two lines of business coupled through a common-shock component. The key difference between our model and the cited literature is the endogenous role of prevention. Prevention alters both idiosyncratic claim intensities and the common-shock intensity, thus impacting not only each line's risk exposure but also the dependence channel linking the two retention decisions. Thus, prevention reduces the claim frequencies and reshapes how the two optimal reinsurance decisions interact.

Now, we present the main contributions of our paper. First, unlike existing common-shock reinsurance–investment models that either exclude prevention or restrict reinsurance to a specific form, we jointly optimize prevention and reinsurance over a broad class of admissible retention functions. We endogenously derive the optimal reinsurance policies as excess-of-loss contracts, where the two deductible levels are linked through the common-shock component and influenced by the prevention decision. Second, prevention affects not only the idiosyncratic claim intensities but also the common-shock intensity, thereby changing both the individual risk exposures and the strength of the dependence induced by common shocks. We characterize the optimal prevention strategy using a boundary–interior condition: the optimum is zero at the boundary, while an interior optimum is uniquely determined by an inverse marginal cost relation. Third, we integrate prevention, general reinsurance, and investment into a unified continuous-time framework. The optimal investment strategy has an explicit form and is independent of prevention and reinsurance controls. In contrast, the two optimal deductible levels satisfy a coupled fixed-point system in which the common-shock component links each deductible to the retained exposure of the other line and prevention affects the coupling through the claim intensities. Numerical results quantify the value of prevention, its impact on optimal deductible levels and the common-shock dependence, and how sensitive optimal controls are to prevention effectiveness, cost, and risk aversion.

The remainder of this paper is organized as follows: in Section 2, we develop a dependent risk model and formulate the insurer's dynamic wealth process; in Section 3, we derive the associated Hamilton-Jacobi-Bellman (HJB) equation, present the optimal controls with explicit characterizations of the optimal reinsurance, investment, and prevention strategies, and we verify these candidate controls by establishing a verification theorem; and some numerical examples are provided in Section 4, and we conclude the paper in Section 5.

2. The model

Let $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$ be a complete filtered probability space the satisfies the usual conditions, on which all stochastic quantities in this paper are well defined.

2.1. The dependent risk model

In the reinsurance market, we focus on cyber risks that generate small losses per claim but arrive at a high frequency, so that aggregate losses over time can be substantial. In our model, we consider two dependent classes of cyber claims. The first class $X_i,\ i \geqslant 1$ captures losses driven by internal risk, such as contagion within the network and lateral spread across connected nodes. The second class $Y_i,\ i \geqslant 1$ captures losses driven by external risk, such as hacker intrusions, phishing campaigns, DDoS attacks, and ransomware incidents. Let $N_0(t)$ be a Poisson process counting common shocks, and let $N_1(t)$ and $N_2(t)$ be independent Poisson processes counting idiosyncratic arrivals of the two lines.

Assumption 2.1. *(i) The claim sizes $\{X_i\}_{i\geqslant 1}$ are i.i.d. nonnegative random variables with the same distribution as X, and $\{Y_i\}_{i\geqslant 1}$ are i.i.d. nonnegative random variables with the same distribution as Y. The two sequences are mutually independent. Moreover, $\mathbb{E}[X^2] < \infty$, $\mathbb{E}[Y^2] < \infty$. (ii) The counting processes $N_0(t)$, $N_1(t)$, and $N_2(t)$ are mutually independent Poisson processes with intensities λ_0, λ_1, and λ_2, respectively. (iii) The claim sizes are independent of the counting processes.*

The aggregate loss up to time t as follows:

$$L(t) = \sum_{i=1}^{N_1(t)+N_0(t)} X_i + \sum_{i=1}^{N_2(t)+N_0(t)} Y_i, \qquad t \geqslant 0.$$

This common shock component induces dependence between the two lines. The insurer can implement prevention measures to reduce claim frequencies, subject to the associated costs. Under a given defense strategy $p(\cdot)$, let $\{N_k^{(p)}(t)\}_{t\geqslant 0}$ denote the counting process with a predictable intensity of $\lambda_k(p(t))$ at time t, for $k \in \{0, 1, 2\}$.

Assumption 2.2. *For each $k \in \{0, 1, 2\}$, let $\lambda_k : [0, \infty) \to [0, \bar{\lambda}_k]$, where $\bar{\lambda}_k > 0$ denotes the baseline claim intensity in the absence of prevention. Let $C : [0, \infty) \to [0, \infty)$ be the prevention cost function. The functions $\lambda_k(\cdot)$ and $C(\cdot)$ are twice continuously differentiable and satisfy the following conditions:*

1) The common shock intensity satisfies $\lambda_0(0) = \bar{\lambda}_0$, $\lambda_0'(p) \leqslant 0$, $\lambda_0''(p) \geqslant 0$;

2) The idiosyncratic claim intensities satisfy $\lambda_i(0) = \bar{\lambda}_i$, $\lambda_i'(p) < 0$, $\lambda_i''(p) \geqslant 0$;

3) The prevention cost function satisfies $C(0) = 0$, $C'(p) \geqslant 0$, $C''(p) > 0$;

4) The marginal prevention cost becomes arbitrarily large as the prevention level tends to infinity: $\lim_{p\to\infty} C'(p) = \infty$.

Remark 2.3. Assumption 2.2 allows the common shock intensity to be identically zero, (i.e., $\lambda_0(p) \equiv 0$), which corresponds to the benchmark case without common shock dependence. In contrast, the idiosyncratic intensities are assumed to be strictly positive so that both claim classes remain active.

This assumption states that prevention can reduce the arrival intensities of both idiosyncratic and common shock claims. The convexity of the intensity functions reflects the diminishing marginal effectiveness of prevention, while $C'(p) \geqslant 0$ and $C''(p) > 0$ mean that prevention is costly and has a strictly increasing marginal cost. Finally, the condition $\lim_{p\to\infty} C'(p) = \infty$ prevents the optimizer of the prevention problem from being only attained at infinity.

Given a prevention strategy $p = \{p(t)\}_{t\geqslant 0}$, the aggregate loss process becomes the following:

$$L^{(p)}t) = \sum_{i=1}^{N_1^{(p)}(t)+N_0^{(p)}t)} X_i + \sum_{i=1}^{N_2^{(p)}(t)+N_0^{(p)}t)} Y_i, \qquad t \geqslant 0.$$

Here, the superscript p indicates that the counting processes are controlled by the entire prevention process $p = \{p(t)\}_{t\geqslant 0}$. Specifically, the instantaneous intensity of $N_k^{(p)}$ at time t is $\lambda_k(p(t))$.

2.2. Wealth dynamics

The insurer adopts a prevention strategy $p = \{p(t)\}_{t\geqslant 0}$ and purchases per-loss reinsurance for each line. The per-loss reinsurance strategies l_1 and l_2 are not restricted to proportional or excess-of-loss forms, but are modeled by general measurable functions $l_i(t, \cdot) : [0, \infty) \to [0, \infty)$, $i \in \{1, 2\}$. Let $u_0 > 0$ be the initial surplus and let $c > 0$ be the premium income rate. The surplus of the insurer at time t under the retention strategy $l = (l_1, l_2)$ and defense level p is given by the following:

$$\mathcal{W}^{l,p}(t) = u_0 + ct - \int_0^t q^{l,p}(s)\,\mathrm{d}s - \int_0^t C(p(s))\,\mathrm{d}s - \sum_{i=1}^{N_1^{(p)}(t)+N_0^{(p)}(t)} l_1(X_i) - \sum_{i=1}^{N_2^{(p)}(t)+N_0^{(p)}(t)} l_2(Y_i), \quad t \geqslant 0.$$

where $q^{l,p}(t)$ denotes the reinsurance premium rate at time t. For brevity, $l_1(X_i)$ and $l_2(Y_i)$ denote the retained losses from claims, which are determined by the respective retention strategies active at each claim's arrival time. According to the expected-value principle, the reinsurance premium rate at time t is as follows:

$$\begin{aligned} q^{l,p}(t) = {} & (1 + \eta_1)(\lambda_0(p(t)) + \lambda_1(p(t)))\mathbb{E}[X - l_1(t, X)] \\ & + (1 + \eta_2)(\lambda_0(p(t)) + \lambda_2(p(t)))\mathbb{E}[Y - l_2(t, Y)], \end{aligned} \tag{2.1}$$

where $\eta_1 \geqslant 0$ and $\eta_2 \geqslant 0$ are the reinsurer's safety loadings of the two classes of the insurance business, respectively. We impose the following net-profit and no-arbitrage condition:

$$\big(\lambda_0(p(t)) + \lambda_1(p(t))\big)\mathbb{E}[X] + \big(\lambda_0(p(t)) + \lambda_2(p(t))\big)\mathbb{E}[Y] < c < q^{0,p}(t), \tag{2.2}$$

where

$$q^{0,p}(t) = (1 + \eta_1)(\lambda_0(p(t)) + \lambda_1(p(t)))\mathbb{E}[X] + (1 + \eta_2)(\lambda_0(p(t)) + \lambda_2(p(t)))\mathbb{E}[Y],$$

is the reinsurance premium rate when the insurer transfers all claims to the reinsurer, that is, $l_1 \equiv 0$ and $l_2 \equiv 0$.

The lower bound in (2.2) ensures that the insurer's premium income rate exceeds the expected loss rate of the two underlying risk lines, thus making the insurance business profitable before reinsurance. The upper bound ensures that the premium income rate does not exceed the full-reinsurance premium rate; otherwise, the insurer could transfer all claims to the reinsurer, thus creating a risk-free strategy and rendering the optimization problem economically meaningless. In addition, since the prevention level reduces the claim intensities, both the expected loss rate and the full-reinsurance premium rate decrease as the prevention level increases. Therefore, prevention lowers both bounds of the inequality. Moreover, we require Condition (2.2) to hold for all prevention levels $p(t)$ at every time $t \in [0, T]$. Without loss of generality, we suppose that $\eta_1 \geqslant \eta_2$, since the results for the case of $\eta_1 < \eta_2$ can be obtained along the same lines.

To obtain a tractable continuous-time control model, we employ a diffusion approximation for the retained aggregate loss process. The diffusion approximation is particularly appropriate when cyber claims arise with a sufficiently high frequency and the retained loss from each individual event is moderate relative to the aggregate exposure. Nevertheless, it may be less accurate when cyber losses are dominated by rare but extremely severe events, strongly heavy-tailed claim sizes, or large systemic incidents. First, we recall the classical diffusion approximation for a compound Poisson risk process. Let

$$L(t) = \sum_{i=1}^{N(t)} Z_i,$$

where $N(t)$ is a Poisson process with an intensity of $\zeta > 0$, and $\{Z_i\}_{i\geqslant 1}$ are i.i.d. claim sizes independent of $N(t)$. It is well known from the diffusion approximation theory of collective risk models, see Iglehart [32] and Grandell [33], that the compound Poisson process can be approximated by a Brownian diffusion whose drift and volatility are determined by the first two moments of the claim sizes. Specifically, over a short time interval dt,

$$\mathbb{E}[\mathrm{d}L(t)] = \zeta\mathbb{E}[Z]\,\mathrm{d}t, \qquad \mathrm{Var}(\mathrm{d}L(t)) = \zeta\mathbb{E}[Z^2]\,\mathrm{d}t.$$

Hence, the corresponding diffusion approximation is

$$\mathrm{d}L(t) = \zeta\mathbb{E}[Z]\,\mathrm{d}t - \sqrt{\zeta\mathbb{E}[Z^2]}\,\mathrm{d}B(t),$$

where $B(t)$ is a standard Brownian motion.

Now, we apply this principle to the retained aggregate loss in our model. At time t, we denote the current prevention level by $p = p(t)$ for simplicity. The instantaneous intensities of the common shock and the two idiosyncratic shocks are $\lambda_0(p)$, $\lambda_1(p)$, and $\lambda_2(p)$, respectively. Thus, the retained aggregate loss can be written formally as follows:

$$L^{l,p}(t) = \sum_{j=1}^{N_0^p(t)}(l_1(X_j^0) + l_2(Y_j^0)) + \sum_{j=1}^{N_1^p(t)} l_1(X_j^1) + \sum_{j=1}^{N_2^p(t)} l_2(Y_j^2),$$

where N_0^p denotes the common-shock arrival process, while N_1^p and N_2^p denote the idiosyncratic arrival processes of the two lines of business. The random variables X_j^0, X_j^1 have the same distribution as X, and Y_j^0, Y_j^2 have the same distribution as Y.

Over a short time interval dt, the conditional means of the retained losses of the two lines are as follows:

$$\mathbb{E}[\mathrm{d}L_1^{l,p}(t) \mid \mathcal{F}_t] = (\lambda_0(p) + \lambda_1(p))\mathbb{E}[l_1(X)]\,\mathrm{d}t, \quad \mathbb{E}[\mathrm{d}L_2^{l,p}(t) \mid \mathcal{F}_t] = (\lambda_0(p) + \lambda_2(p))\mathbb{E}[l_2(Y)]\,\mathrm{d}t.$$

Similarly, the conditional infinitesimal variance of the total retained aggregate loss is as follows:

$$\mathrm{Var}(\mathrm{d}L^{l,p}(t) \mid \mathcal{F}_t) = \sigma_L^2(p, l_1, l_2)\,\mathrm{d}t,$$

where

$$\begin{aligned}\sigma_L^2(p, l_1, l_2) : = &(\lambda_0(p) + \lambda_1(p))\mathbb{E}[(l_1(X))^2] + (\lambda_0(p) + \lambda_2(p))\mathbb{E}[(l_2(Y))^2] \\ &+ 2\lambda_0(p)\mathbb{E}[l_1(X)]\mathbb{E}[l_2(Y)].\end{aligned} \tag{2.3}$$

The last term in (2.3) is induced by the common-shock component, because a common shock simultaneously generates losses in both lines and the mutual independence of X and Y.

Thus, following the classical diffusion approximation for compound Poisson risk processes, the retained aggregate loss process is approximated by the following:

$$\mathrm{d}L^{l,p}(t) = (\beta_1^{(p)}(l_1) + \beta_2^{(p)}(l_2))\mathrm{d}t - \sigma_L(p, l_1, l_2)\,\mathrm{d}B_1(t),$$

where we let

$$\beta_1^{(p)}(l_1) := (\lambda_0(p) + \lambda_1(p))\mathbb{E}[l_1(X)], \quad \beta_2^{(p)}(l_2) := (\lambda_0(p) + \lambda_2(p))\mathbb{E}[l_2(Y)]. \tag{2.4}$$

Substituting this approximation into the surplus dynamics gives the following:

$$\mathrm{d}\hat{\mathcal{W}}^{l,p}(t) = \Big(c - q^{l,p} - C(p) - \beta_1^{(p)}(l_1) - \beta_2^{(p)}(l_2)\Big)\mathrm{d}t + \sigma_L(p, l_1, l_2)\,\mathrm{d}B_1(t).$$

In addition to the reinsurance business, we assume that the insurer can invest the surplus in a financial market consisting of a risk-free asset with a constant interest rate $r > 0$ and a risky asset with price dynamics

$$\mathrm{d}S(t) = \mu(t)\,S(t)\,\mathrm{d}t + \sigma_R\,S(t)\,\mathrm{d}B_2(t),$$

where $\mu(\cdot)$ is a deterministic continuous function satisfying $\mu(t) > r$ for all $t \in [0, T]$, $\sigma_R > 0$, and B_2 is a standard Brownian motion independent of B_1. Let $\pi = \{\pi(t)\}_{t\geqslant 0}$ be the amount invested in the risky asset at time t, and the remaining wealth is invested in the risk-free asset. Then, for a chosen combination of controls $\nu = \{\nu(t)\}_{t\geqslant 0} = (\pi, l_1, l_2, p)$, the wealth process $\hat{\mathcal{W}}^\nu(t)$ satisfies the following stochastic differential equation (SDE):

$$\begin{cases} \mathrm{d}\hat{\mathcal{W}}^\nu(t) = \Big(r\hat{W}^\nu(t) + c - q^{l,p} - C(p) - \beta_1^{(p)}(l_1) - \beta_2^{(p)}(l_2) + (\mu(t) - r)\pi\Big)\mathrm{d}t \\ \qquad\qquad + \sigma_L(l_1, l_2, p)\,\mathrm{d}B_1(t) + \pi\sigma_R\,\mathrm{d}B_2(t) \\ \qquad\quad =: \Big(r\hat{W}^\nu(t) + A(l_1, l_2, p) + (\mu(t) - r)\pi\Big)\mathrm{d}t + \sigma_L(l_1, l_2, p)\,\mathrm{d}B_1(t) + \pi\sigma_R\,\mathrm{d}B_2(t), \\ \hat{\mathcal{W}}^\nu(0) = u_0, \end{cases} \tag{2.5}$$

where $A(l_1, l_2, p) := c - q^{l,p} - C(p) - \beta_1^{(p)}(l_1) - \beta_2^{(p)}(l_2)$.

Let $\gamma > 0$ denote the risk-aversion coefficient in the exponential utility function introduced below. For notational simplicity, we write $\mathbb{E}_{t,w}[\cdot] := \mathbb{E}\Big[\cdot\,\Big|\,\hat{\mathcal{W}}^\nu(t) = w\Big]$.

Definition 2.4 (Admissible control)**.** A control process $\nu = (\pi, l_1, l_2, p)$ is said to be admissible if the following conditions hold:

1) it is adapted to the filtration $\mathbb{F}$;
2) $\pi(\cdot)$ is square integrable, (i.e., $\mathbb{E}\int_0^T \pi^2(s)\,\mathrm{d}s < \infty$);
3) for each $i = 1, 2$, $l_i(t, z)$ is a measurable function of the possible claim size z at time t, and satisfies $0 \leqslant l_i(t, z) \leqslant z$ for all $t \in [0, T]$ and $z \geqslant 0$;
4) $p(t) \geqslant 0$ for all $t \in [0, T]$ a.s., and $\mathbb{E}\int_0^T C(p(s))\,\mathrm{d}s < \infty$;
5) the wealth SDE (2.5) under ν admits a unique strong solution;
6) for every $(t, w) \in [0, T) \times \mathbb{R}$, the following exponential integrability conditions hold:

$$\mathbb{E}_{t,w}\left[\sup_{t\leqslant s\leqslant T} \exp\left\{-2\gamma e^{r(T-s)}\hat{\mathcal{W}}^\nu(s)\right\}\right] < \infty, \tag{2.6}$$

and

$$\mathbb{E}_{t,w}\int_t^T \exp\left\{-2\gamma e^{r(T-s)}\hat{\mathcal{W}}^\nu(s)\right\}\left[\sigma_L^2(p(s), l_1(s), l_2(s)) + \pi^2(s)\sigma_R^2\right]\mathrm{d}s < \infty. \tag{2.7}$$

The set of all admissible controls is denoted by $\mathcal{A}[t, T]$.

3. The insurer's optimization results

3.1. HJB equation and verification theorem

Suppose that the insurer is interested in maximizing its expected utility function of terminal wealth, say at time T. Let $\hat{\mathcal{W}}^{\nu}(\cdot)$ be the wealth process (2.5), and the utility function is $U(x) = -\frac{1}{\gamma}e^{-\gamma x}$, $\gamma > 0$. For an admissible control $\nu := (\pi, l_1, l_2, p) \in \mathcal{A}[t, T]$, we define the utility attained by the insurer from state w at time t as follows:

$$J(t, w; \nu) := \mathbb{E}_{t,w}\left[U(\hat{\mathcal{W}}^{\nu}(T))\right].$$

Our objective is to find the optimal value function

$$V(t, w) := \sup_{\nu\in\mathcal{A}[t,T]} J(t, w; \nu),$$

and the optimal admissible control ν^* such that $V(t, w) = J(t, w; \nu^*)$.

To simplify the subsequent analysis, we define the following infinitesimal generator for $\phi \in C^{1,2}([0, T] \times \mathbb{R})$ and admissible control $\nu = (\pi, l_1, l_2, p)$:

$$\mathcal{L}^{\nu}\phi(t, w) = \phi_t(t, w) + \left[rw + A(l_1, l_2, p) + (\mu(t) - r)\pi\right]\phi_w(t, w) + \frac{1}{2}\left[\sigma_L^2(p, l_1, l_2) + \pi^2\sigma_R^2\right]\phi_{ww}(t, w).$$

Assuming that the value function is sufficiently smooth, the dynamic programming principle formally yields the following HJB equation:

$$\begin{cases} 0 = \sup\limits_{\substack{\pi\in\mathbb{R},\ p\geqslant 0\\ 0\leqslant l_1(x)\leqslant x,\ 0\leqslant l_2(y)\leqslant y}} \mathcal{L}^{\nu}W(t, w), \quad (t, w) \in [0, T) \times \mathbb{R}, \\ W(T, w) = U(w). \end{cases} \tag{3.1}$$

The verification theorem below states that, under certain conditions, the solution to the HJB equation is the optimal value function, and the optimal control can be obtained by solving the maximizer in the HJB equation.

Theorem 3.1 (Verification theorem)**.** *Suppose that the HJB equation* (3.1) *admits a solution* $W \in C^{1,2}([0, T] \times \mathbb{R})$ *such that, for some constant* $K > 0$,

$$|W(t, w)|^2 + |W_w(t, w)|^2 \leqslant K \exp\left\{-2\gamma e^{r(T-t)}w\right\}, \qquad (t, w) \in [0, T] \times \mathbb{R}. \tag{3.2}$$

If there exists a control $\nu^* = (\pi^*, l_1^*, l_2^*, p^*) \in \mathcal{A}[t, T]$ *satisfying*

$$\mathcal{L}^{\nu^*}W(t, w) = \sup_{\substack{\pi\in\mathbb{R},\ p\geqslant 0\\ 0\leqslant l_1(x)\leqslant x,\ 0\leqslant l_2(y)\leqslant y}} \mathcal{L}^{\nu}W(t, w) = 0 \tag{3.3}$$

for all (t, w)*, then* ν^* *is an optimal admissible control and* $V(t, w) = W(t, w) = J(t, w; \nu^*)$.

Proof. For any $(t, w) \in [0, T) \times \mathbb{R}$ and any admissible control $\nu = (\pi, l_1, l_2, p) \in \mathcal{A}[t, T]$, define

$$\tau_n^{\nu} := \inf\left\{s \in [t, T] : \left|\hat{\mathcal{W}}^{\nu}(s)\right| \geqslant n\right\} \wedge T.$$

Applying Itô's formula to $W(s, \hat{\mathcal{W}}^{\nu}(s))$ on $[t, \tau_n^{\nu}]$ gives the following:

$$
\begin{aligned}
W(\tau_n^{\nu}, \hat{\mathcal{W}}^{\nu}(\tau_n^{\nu})) = W(t, w) &+ \int_t^{\tau_n^{\nu}} \mathcal{L}^{\nu(s)} W(s, \hat{\mathcal{W}}^{\nu}(s))\, \mathrm{d}s \\
&+ \int_t^{\tau_n^{\nu}} W_w(s, \hat{\mathcal{W}}^{\nu}(s)) \sigma_L(p(s), l_1(s), l_2(s))\, \mathrm{d}B_1(s) \\
&+ \int_t^{\tau_n^{\nu}} W_w(s, \hat{\mathcal{W}}^{\nu}(s)) \pi(s) \sigma_R\, \mathrm{d}B_2(s).
\end{aligned}
$$

By the admissibility of ν and the localization induced by τ_n^{ν}, the finite-variation term is integrable. Moreover, (3.2) and (2.7) imply that

$$
\begin{aligned}
&\mathbb{E}_{t,w} \int_t^T \left| W_w(s, \hat{\mathcal{W}}^{\nu}(s)) \right|^2 \left[\sigma_L^2(p(s), l_1(s), l_2(s)) + \pi^2(s) \sigma_R^2 \right] \mathrm{d}s \\
&\leqslant K \mathbb{E}_{t,w} \int_t^T \exp\left\{ -2\gamma e^{r(T-s)} \hat{\mathcal{W}}^{\nu}(s) \right\} \left[\sigma_L^2(p(s), l_1(s), l_2(s)) + \pi^2(s) \sigma_R^2 \right] \mathrm{d}s < \infty.
\end{aligned}
$$

Hence, the two stochastic integrals are square-integrable martingales and have zero expectation. Thus, taking conditional expectations in the Itô identity yields the following:

$$
\mathbb{E}_{t,w} \left[W(\tau_n^{\nu}, \hat{\mathcal{W}}^{\nu}(\tau_n^{\nu})) \right] = W(t, w) + \mathbb{E}_{t,w} \int_t^{\tau_n^{\nu}} \mathcal{L}^{\nu(s)} W(s, \hat{\mathcal{W}}^{\nu}(s))\, \mathrm{d}s \leqslant W(t, w), \tag{3.4}
$$

where the last inequality follows from the assumption that W satisfies the HJB equation.

Next, (3.2) and (2.6) give the following:

$$
\sup_{n \geqslant 1} \mathbb{E}_{t,w} \left[\left| W(\tau_n^{\nu}, \hat{\mathcal{W}}^{\nu}(\tau_n^{\nu})) \right|^2 \right] \leqslant K \mathbb{E}_{t,w} \left[\sup_{t \leqslant s \leqslant T} \exp\left\{ -2\gamma e^{r(T-s)} \hat{\mathcal{W}}^{\nu}(s) \right\} \right] < \infty.
$$

Therefore, the family $\left\{ W(\tau_n^{\nu}, \hat{\mathcal{W}}^{\nu}(\tau_n^{\nu})) \right\}_{n \geqslant 1}$ is uniformly integrable.

Since $\hat{\mathcal{W}}^{\nu}$ has continuous sample paths, it is bounded on $[t, T]$ almost surely. Hence, $\tau_n^{\nu} \uparrow T$ almost surely, and we further have $\hat{\mathcal{W}}^{\nu}(\tau_n^{\nu}) \longrightarrow \hat{\mathcal{W}}^{\nu}(T)$. Consequently, by the preceding uniform integrability, the continuity of W, and the terminal condition in (3.1), we obtain the following:

$$
\lim_{n \to \infty} \mathbb{E}_{t,w} \left[W(\tau_n^{\nu}, \hat{\mathcal{W}}^{\nu}(\tau_n^{\nu})) \right] = \mathbb{E}_{t,w} \left[\lim_{n \to \infty} W(\tau_n^{\nu}, \hat{\mathcal{W}}^{\nu}(\tau_n^{\nu})) \right] = \mathbb{E}_{t,w} \left[U(\hat{\mathcal{W}}^{\nu}(T)) \right] = J(t, w; \nu).
$$

Thus, for any $\nu \in \mathcal{A}[t, T]$, letting $n \to \infty$ in (3.4) gives the following:

$$
W(t, w) \geqslant \mathbb{E}_{t,w} \left[U(\hat{\mathcal{W}}^{\nu}(T)) \right] = J(t, w; \nu).
$$

Therefore,

$$
W(t, w) \geqslant \sup_{\nu \in \mathcal{A}[t,T]} J(t, w; \nu) = V(t, w). \tag{3.5}
$$

Suppose that ν^* is the admissible control satisfying (3.3). Then, by repeating the previous argument with $\nu = \nu^*$ and using $\mathcal{L}^{\nu^*(s)} W(s, \hat{\mathcal{W}}^{\nu^*}(s)) = 0$, we obtain the following:

$$
W(t, w) = J(t, w; \nu^*).
$$

Combining this with (3.5), we conclude that $V(t, w) = W(t, w) = J(t, w; \nu^*)$, and thus ν^* is optimal. □

3.2. Characterization of the optimal strategies

Now, we try to construct a solution of the HJB equation (3.1). Let W satisfy $W_w > 0$, $W_{ww} < 0$. Then, the HJB equation can be written as follows:

$$
\begin{aligned}
0 = W_t + rwW_w + \sup_{l_1,l_2,p}\left\{A(l_1,l_2,p)\,W_w + \frac{1}{2}\sigma_L^2(p,l_1,l_2)\,W_{ww}\right\} \\
+ \sup_{\pi}\left\{(\mu(t)-r)\pi\,W_w + \frac{1}{2}\pi^2\sigma_R^2\,W_{ww}\right\}.
\end{aligned}
\tag{3.6}
$$

For the π-part supremum, by the first-order condition,

$$
\pi^*(t,w) = -\frac{(\mu(t)-r)W_w(t,w)}{\sigma_R^2 W_{ww}(t,w)}. \tag{3.7}
$$

Therefore, $\sup_\pi\left\{(\mu(t)-r)\pi\,W_w + \frac{1}{2}\pi^2\sigma_R^2 W_{ww}\right\} = -\frac{(\mu(t)-r)^2W_w^2}{2\sigma_R^2 W_{ww}}$, and

$$
0 = W_t + rwW_w - \frac{(\mu(t)-r)^2W_w^2}{2\sigma_R^2 W_{ww}} + \sup_{l_1,l_2,p}\left\{A(l_1,l_2,p)\,W_w + \frac{1}{2}\sigma_L^2(p,l_1,l_2)\,W_{ww}\right\}. \tag{3.8}
$$

To solve the above partial equation, we try to seek a solution of the following form:

$$
W(t,w) = -\frac{1}{\gamma}\exp(-\gamma(g(t)w + f(t))), \tag{3.9}
$$

where $\gamma > 0$ is the coefficient of absolute risk aversion and $g(\cdot)$ is a suitable function such that (3.9) is a solution of (3.8). Clearly, under this form solution, we have the following:

$$
W_t = -\gamma(g'w + f')W, \quad W_w = -\gamma g W, \quad W_{ww} = \gamma^2 g^2 W, \quad W_w^2/W_{ww} = W.
$$

By substituting these quantities into (3.8), we can obtain the following:

$$
0 = -\gamma W\left[(g'(t) + rg(t))w + f'(t) + \frac{(\mu(t)-r)^2}{2\gamma\sigma_R^2} + \sup_{l_1,l_2,p}\left\{g(t)A(l_1,l_2,p) - \frac{\gamma}{2}g(t)^2\sigma_L^2(l_1,l_2,p)\right\}\right].
$$

Hence,

$$
\begin{cases}
g'(t) + rg(t) = 0, & g(T) = 1,\\
f'(t) + \Gamma(t) = 0, & f(T) = 0,
\end{cases}
$$

where

$$
\Gamma(t) := \sup_{l_1,l_2,p}\left\{g(t)A(l_1,l_2,p) - \frac{\gamma}{2}g(t)^2\sigma_L^2(l_1,l_2,p)\right\} + \frac{(\mu(t)-r)^2}{2\gamma\sigma_R^2}.
$$

Therefore,

$$
g(t) = e^{r(T-t)}, \qquad f(t) = \int_t^T \Gamma(s)\,ds. \tag{3.10}
$$

Moreover, from (3.7) and (3.10),

$$
\pi^*(t) = \frac{\mu(t)-r}{\gamma g(t)\sigma_R^2} = \frac{\mu(t)-r}{\gamma\sigma_R^2}e^{-r(T-t)}. \tag{3.11}
$$

Then, (l_1^*, l_2^*, p^*) is characterized by the following:

$$(l_1^*, l_2^*, p^*)(t) \in \arg\max_{(l_1,l_2,p)} \left\{ A(l_1, l_2, p)e^{r(T-t)} - \frac{\gamma}{2}\sigma_L^2(l_1, l_2, p)e^{2r(T-t)} \right\}. \tag{3.12}$$

Next, we characterize the forms of the retention functions l_1 and l_2. To further simplify the analysis, define the truncated moment functions

$$\begin{aligned} I_X(d) &= \mathbb{E}[X \wedge d], \quad I_Y(d) = \mathbb{E}[Y \wedge d], \\ m_X(d) &= \mathbb{E}[(X \wedge d)^2], \quad m_Y(d) = \mathbb{E}[(Y \wedge d)^2], \qquad d \geqslant 0, \end{aligned} \tag{3.13}$$

and for each fixed p, define the auxiliary functions

$$\ell_X(d) = \eta_2 d - \frac{\lambda_0(p)\eta_1}{\lambda_0(p) + \lambda_2(p)} I_X(d), \qquad \ell_Y(d) = \eta_1 d - \frac{\lambda_0(p)\eta_2}{\lambda_0(p) + \lambda_1(p)} I_Y(d).$$

It is clear that $\ell_X(0) = \ell_Y(0) = 0$. Moreover, for $d \geqslant 0$, $I_Y(d) = \mathbb{E}[Y \wedge d] = \int_0^d \mathbb{P}(Y > u)\,\mathrm{d}u$. Hence, I_Y is absolutely continuous with $I_Y'(d) = \mathbb{P}(Y > d) \in [0, 1]$ for a.e. $d \geqslant 0$. Therefore,

$$\ell_Y'(d) = \eta_1 - \frac{\lambda_0(p)\eta_2}{\lambda_0(p) + \lambda_1(p)} I_Y'(d) \geqslant \eta_1 - \frac{\lambda_0(p)\eta_2}{\lambda_0(p) + \lambda_1(p)}.$$

Since $\eta_1 \geqslant \eta_2$ and $\lambda_1(p) > 0$, we have $\frac{\lambda_0(p)}{\lambda_0(p)+\lambda_1(p)} < 1$, and thus $\frac{\lambda_0(p)\eta_2}{\lambda_0(p)+\lambda_1(p)} < \eta_2 \leqslant \eta_1$, which implies $\ell_Y'(d) > 0$ a.e. Hence, $\ell_Y(\cdot)$ is strictly increasing on $[0, \infty)$, and therefore its inverse ℓ_Y^{-1} exists*. Now, we characterize the optimal retention functions l_1^* and l_2^*, as well as the optimal prevention control.

Proposition 3.2 (Optimal retention functions)**.** *The optimal retentions are of the excess-of-loss type:*

$$l_1^*(x; t, p) = x \wedge d_1^*(t, p), \qquad l_2^*(y; t, p) = y \wedge d_2^*(t, p) \tag{3.14}$$

where

$$\begin{aligned} d_1^*(t, p) &= \left(\frac{\eta_1}{\gamma e^{r(T-t)}} - \frac{\lambda_0(p)}{\lambda_0(p) + \lambda_1(p)} I_Y(d_2^*(t, p)) \right)_+, \\ d_2^*(t, p) &= \left(\frac{\eta_2}{\gamma e^{r(T-t)}} - \frac{\lambda_0(p)}{\lambda_0(p) + \lambda_2(p)} I_X(d_1^*(t, p)) \right)_+. \end{aligned}$$

Moreover, when $\eta_1 \geqslant \eta_2$, the unique optimal pair of retention functions $\big(l_1^*(x; t, p),\ l_2^*(y; t, p)\big)$ admits the following representation:

$$\begin{cases} \Big(x \wedge d_1^*(t, p),\ y \wedge \ell_Y^{-1}\big(\ell_X(d_1^*(t, p))\big)\Big), & d_1^*(t, p) > 0,\ d_2^*(t, p) > 0, \\ \Big(0,\ y \wedge \dfrac{\eta_2}{\gamma e^{r(T-t)}}\Big), & \dfrac{\lambda_0(p)}{\lambda_0(p) + \lambda_1(p)} I_Y\Big(\dfrac{\eta_2}{\gamma e^{r(T-t)}}\Big) \geqslant \dfrac{\eta_1}{\gamma e^{r(T-t)}}, \\ \Big(x \wedge \dfrac{\eta_1}{\gamma e^{r(T-t)}},\ 0\Big), & \dfrac{\lambda_0(p)}{\lambda_0(p) + \lambda_2(p)} I_X\Big(\dfrac{\eta_1}{\gamma e^{r(T-t)}}\Big) \geqslant \dfrac{\eta_2}{\gamma e^{r(T-t)}}. \end{cases} \tag{3.15}$$

*If $\eta_1 < \eta_2$, then one may instead use the monotonicity of ℓ_X. Indeed, since $\lambda_2(p) > 0$,

$$\ell_X'(d) \geqslant \eta_2 - \frac{\lambda_0(p)\eta_1}{\lambda_0(p) + \lambda_2(p)} > 0 \quad \text{a.e.},$$

so that ℓ_X is strictly increasing and ℓ_X^{-1} exists. In this case, when $d_1^*(t, p) > 0$ and $d_2^*(t, p) > 0$, the representation in (3.15) is replaced by

$$\big(l_1^*(x; t, p),\ l_2^*(y; t, p)\big) = \Big(x \wedge \ell_X^{-1}\big(\ell_Y(d_2^*(t, p))\big),\ y \wedge d_2^*(t, p)\Big).$$

Proof. We optimize over measurable retentions l_1, l_2 subject to $0 \leqslant l_1(x) \leqslant x$, $0 \leqslant l_2(y) \leqslant y$. Under these constraints, (3.12) is equivalent to maximizing the functional

$$\begin{aligned}\mathcal{J}_{t,p}(l_1, l_2) :=\ & \eta_1(\lambda_0(p) + \lambda_1(p))\,\mathbb{E}[l_1(X)]e^{r(T-t)} + \eta_2(\lambda_0(p) + \lambda_2(p))\,\mathbb{E}[l_2(Y)]e^{r(T-t)} \\ & - \frac{\gamma}{2}\Big((\lambda_0(p) + \lambda_1(p))\,\mathbb{E}[l_1(X)^2] + (\lambda_0(p) + \lambda_2(p))\,\mathbb{E}[l_2(Y)^2]\Big)e^{2r(T-t)} \\ & - \gamma\lambda_0(p)\,\mathbb{E}[l_1(X)]\mathbb{E}[l_2(Y)]e^{2r(T-t)}.\end{aligned}$$

Let $(l_1^*(\cdot; t, p), l_2^*(\cdot; t, p))$ denote the optimal pair. A direction h_1 is feasible at $l_1^*(\cdot; t, p)$ if, for all $x > 0$ and sufficiently small $\varepsilon > 0$, $0 \le l_1^*(x; t, p) + \varepsilon h_1(x) \le x$. Similarly, a direction h_2 is feasible at $l_2^*(\cdot; t, p)$ if, for all $y > 0$ and sufficiently small $\varepsilon > 0$, $0 \le l_2^*(y; t, p) + \varepsilon h_2(y) \le y$. For such feasible directions, we define the following one-sided directional derivatives by the following:

$$D_1^+\mathcal{J}_{t,p}(l_1^*, l_2^*; h_1) := \lim_{\varepsilon\downarrow 0} \frac{\mathcal{J}_{t,p}(l_1^* + \varepsilon h_1, l_2^*) - \mathcal{J}_{t,p}(l_1^*, l_2^*)}{\varepsilon},$$

$$D_2^+\mathcal{J}_{t,p}(l_1^*, l_2^*; h_2) := \lim_{\varepsilon\downarrow 0} \frac{\mathcal{J}_{t,p}(l_1^*, l_2^* + \varepsilon h_2) - \mathcal{J}_{t,p}(l_1^*, l_2^*)}{\varepsilon}.$$

Since (l_1^*, l_2^*) is optimal, the following inequalities hold for all feasible directions h_1 and h_2:

$$D_1^+\mathcal{J}_{t,p}(l_1^*, l_2^*; h_1) \le 0, \quad D_2^+\mathcal{J}_{t,p}(l_1^*, l_2^*; h_2) \le 0. \tag{3.16}$$

A direct computation yields the following:

$$\begin{aligned}D_1^+\mathcal{J}_{t,p}(l_1^*, l_2^*; h_1) &= \mathbb{E}[\Psi_1(X)\, h_1(X)]\,, \\ D_2^+\mathcal{J}_{t,p}(l_1^*, l_2^*; h_2) &= \mathbb{E}[\Psi_2(Y)\, h_2(Y)]\,,\end{aligned} \tag{3.17}$$

where

$$\Psi_1(x) := \eta_1(\lambda_0(p) + \lambda_1(p))e^{r(T-t)} - \gamma\Big((\lambda_0(p) + \lambda_1(p))\, l_1^*(x; t, p) + \lambda_0(p)\mathbb{E}[l_2^*(Y; t, p)]\Big)e^{2r(T-t)},$$

$$\Psi_2(y) := \eta_2(\lambda_0(p) + \lambda_2(p))e^{r(T-t)} - \gamma\Big((\lambda_0(p) + \lambda_2(p))\, l_2^*(y; t, p) + \lambda_0(p)\mathbb{E}[l_1^*(X; t, p)]\Big)e^{2r(T-t)}.$$

Define the interior regions

$$\mathcal{I}_1 := \{x > 0 :\ 0 < l_1^*(x; t, p) < x\}, \qquad \mathcal{I}_2 := \{y > 0 :\ 0 < l_2^*(y; t, p) < y\}.$$

On $\mathcal{I}_1$ (resp. $\mathcal{I}_2$), perturbations of both signs supported on arbitrarily small neighborhoods are feasible. Therefore, (3.16) and (3.17) imply that $\Psi_1(X) = 0$ a.s. on $\{X \in \mathcal{I}_1\}$ and $\Psi_2(Y) = 0$ a.s. on $\{Y \in \mathcal{I}_2\}$.

Recalling that $g(t) = e^{r(T-t)}$ by (3.10), these conditions yield, for $x \in \mathcal{I}_1$ and $y \in \mathcal{I}_2$,

$$\begin{aligned}(\lambda_0(p) + \lambda_1(p))\, l_1^*(x; t, p) + \lambda_0(p)\mathbb{E}[l_2^*(Y; t, p)] &= \frac{\eta_1(\lambda_0(p) + \lambda_1(p))}{\gamma e^{r(T-t)}}, \\ (\lambda_0(p) + \lambda_2(p))\, l_2^*(y; t, p) + \lambda_0(p)\mathbb{E}[l_1^*(X; t, p)] &= \frac{\eta_2(\lambda_0(p) + \lambda_2(p))}{\gamma e^{r(T-t)}}.\end{aligned} \tag{3.18}$$

The unconstrained optimizer on the interior is constant in x (resp. y). Moreover, on the boundary parts where $l_1^*(x; t, p) = 0$ or $l_1^*(x; t, p) = x$, only one-sided perturbations are feasible, and the variational inequality (3.16), together with (3.17), implies the complementary sign conditions $\Psi_1(x) \leqslant 0$ on $\{l_1^*(x;t,p) = 0\}$ and $\Psi_1(x) \geqslant 0$ on $\{l_1^*(x;t,p) = x\}$ (and analogously for Ψ_2). Hence, $l_1^*(\cdot;t,p)$ (resp. $l_2^*(\cdot;t,p)$) is obtained by projecting the interior constant onto the constraints $0 \leqslant l_1(x) \leqslant x$ (resp. $0 \leqslant l_2(y) \leqslant y$), which yields the excess-of-loss forms (3.14) with

$$d_1^*(t,p) = \Big(\frac{\eta_1}{\gamma e^{r(T-t)}} - \frac{\lambda_0(p)}{\lambda_0(p)+\lambda_1(p)}\,\mathbb{E}[l_2^*(Y;t,p)]\Big)_+,$$
$$d_2^*(t,p) = \Big(\frac{\eta_2}{\gamma e^{r(T-t)}} - \frac{\lambda_0(p)}{\lambda_0(p)+\lambda_2(p)}\,\mathbb{E}[l_1^*(X;t,p)]\Big)_+.$$

Furthermore, if $d_1^*(t,p) > 0$ and $d_2^*(t,p) > 0$, then the coupled fixed-point system can be reduced to a one-dimensional representation. By (3.14) and (3.13), we have the following:

$$\mathbb{E}[l_1^*(X;t,p)] = \mathbb{E}[X \wedge d_1^*(t,p)] = I_X(d_1^*(t,p)),$$

and

$$\mathbb{E}[l_2^*(Y;t,p)] = \mathbb{E}[Y \wedge d_2^*(t,p)] = I_Y(d_2^*(t,p)).$$

Together with (3.18), we obtain the following:

$$\begin{aligned}\gamma e^{r(T-t)} &= \frac{\eta_1(\lambda_0(p)+\lambda_1(p))}{(\lambda_0(p)+\lambda_1(p))d_1^*(t,p)+\lambda_0(p)I_Y(d_2^*(t,p))}\\ &= \frac{\eta_2(\lambda_0(p)+\lambda_2(p))}{(\lambda_0(p)+\lambda_2(p))d_2^*(t,p)+\lambda_0(p)I_X(d_1^*(t,p))}.\end{aligned}$$

By equating the two fractions above, we obtain $\ell_X(d_1^*(t,p)) = \ell_Y(d_2^*(t,p))$. Since the inverse function ℓ_Y^{-1} exists, we obtain $d_2^*(t,p) = \ell_Y^{-1}(\ell_X(d_1^*(t,p)))$. Substituting this relation into $l_2^*(y;t,p) = y \wedge d_2^*(t,p)$ yields (3.15). □

Remark 3.3 (Existence of an optimal retention pair). For fixed (t,p), let l_1 and l_2 range over the admissible retention functions specified in Definition 2.4. Define the following:

$$\mathscr{A}_X := \{l_1(X) : 0 \leqslant l_1(x) \leqslant x \text{ for all } x \geqslant 0\},$$
$$\mathscr{A}_Y := \{l_2(Y) : 0 \leqslant l_2(y) \leqslant y \text{ for all } y \geqslant 0\}.$$

Since $0 \leqslant l_1(X) \leqslant X$, $0 \leqslant l_2(Y) \leqslant Y$, and $\mathbb{E}[X^2] < \infty$, $\mathbb{E}[Y^2] < \infty$, the sets $\mathscr{A}_X$ and $\mathscr{A}_Y$ are nonempty, closed, convex, and bounded in their respective L^2 spaces. Since these spaces are reflexive, it follows that $\mathscr{A}_X$ and $\mathscr{A}_Y$ are weakly compact. Moreover, $\mathcal{J}_{t,p}$ is weakly upper semicontinuous on $\mathscr{A}_X \times \mathscr{A}_Y$, since its linear and cross terms are weakly continuous, while its negative quadratic terms are weakly upper semicontinuous. Since $\mathscr{A}_X \times \mathscr{A}_Y$ is weakly compact and $\mathcal{J}_{t,p}$ is weakly upper semicontinuous, $\mathcal{J}_{t,p}$ attains its maximum over $\mathscr{A}_X \times \mathscr{A}_Y$. Hence, an optimal retention pair (l_1^*, l_2^*) exists.

Remark 3.4 (Existence and uniqueness of the optimal retention levels). Fix (t,p) and let $(d_1,d_2) \in \mathbb{R}_+^2$ be a candidate pair of deductibles for the first and second lines, respectively. Define the mapping $\mathcal{T}_{t,p} : \mathbb{R}_+^2 \to \mathbb{R}_+^2$ by the following:

$$\mathcal{T}_{t,p}(d_1,d_2) := \left(\left(\frac{\eta_1}{\gamma e^{r(T-t)}} - \frac{\lambda_0(p)}{\lambda_0(p)+\lambda_1(p)}I_Y(d_2)\right)_+, \left(\frac{\eta_2}{\gamma e^{r(T-t)}} - \frac{\lambda_0(p)}{\lambda_0(p)+\lambda_2(p)}I_X(d_1)\right)_+\right).$$

For any $d, \widetilde{d} \geqslant 0$, we have the following:

$$|I_X(d) - I_X(\widetilde{d})| = \left|\mathbb{E}[X \wedge d] - \mathbb{E}[X \wedge \widetilde{d}]\right| \leqslant \mathbb{E}\left[\left|(X \wedge d) - (X \wedge \widetilde{d})\right|\right] \leqslant |d - \widetilde{d}|,$$
$$|I_Y(d) - I_Y(\widetilde{d})| \leqslant |d - \widetilde{d}|.$$

Since the positive-part mapping is Lipschitz continuous with a constant of one, it follows that

$$\left\|\mathcal{T}_{t,p}(d_1, d_2) - \mathcal{T}_{t,p}(\widetilde{d}_1, \widetilde{d}_2)\right\|_\infty \leqslant \max\left\{\frac{\lambda_0(p)}{\lambda_0(p) + \lambda_1(p)}, \frac{\lambda_0(p)}{\lambda_0(p) + \lambda_2(p)}\right\} \left\|(d_1, d_2) - (\widetilde{d}_1, \widetilde{d}_2)\right\|_\infty,$$

where $\|(a, b)\|_\infty := \max\{|a|, |b|\}$. Noting that $\lambda_0(p) \geq 0$ and $\lambda_i(p) > 0$ for $i = 1, 2$, it follows that

$$\max\left\{\frac{\lambda_0(p)}{\lambda_0(p) + \lambda_1(p)}, \frac{\lambda_0(p)}{\lambda_0(p) + \lambda_2(p)}\right\} < 1.$$

Therefore, $\mathcal{T}_{t,p}$ is a contraction on the complete metric space $(\mathbb{R}^2_+, \|\cdot\|_\infty)$. By the Banach fixed-point theorem, $\mathcal{T}_{t,p}$ admits a unique fixed point $(d_1^*(t, p), d_2^*(t, p)) \in \mathbb{R}^2_+$. Hence, the coupled deductible system in Proposition 3.2 admits a unique solution, and the corresponding excess-of-loss retention functions

$$l_1^*(x; t, p) = x \wedge d_1^*(t, p), \qquad l_2^*(y; t, p) = y \wedge d_2^*(t, p)$$

are uniquely determined.

Recall that the optimization problem in (3.12) is to maximize

$$F(t, p, l_1, l_2) := \left(c - q^{l,p} - C(p) - \beta_1^{(p)}(l_1) - \beta_2^{(p)}(l_2)\right)e^{r(T-t)} - \frac{\gamma}{2}\sigma_L^2(p, l_1, l_2)e^{2r(T-t)}$$

over (l_1, l_2, p). By Proposition 3.2, the optimal retention functions are of the excess-of-loss form. Therefore, substituting $l_1(x) = x \wedge d_1$ and $l_2(y) = y \wedge d_2$ into the objective function, we obtain the reduced objective

$$\begin{aligned}\widetilde{F}(t, p, d_1, d_2) &:= F(t, p, x \wedge d_1, y \wedge d_2)\\ &= \left(c - q^{d,p} - C(p) - \beta_1^{(p)}(d_1) - \beta_2^{(p)}(d_2)\right)e^{r(T-t)} - \frac{\gamma}{2}\sigma_L^2(p, d_1, d_2)e^{2r(T-t)},\end{aligned} \tag{3.19}$$

where

$$q^{d,p} := (1 + \eta_1)(\lambda_0(p) + \lambda_1(p))\mathbb{E}[X - X \wedge d_1] + (1 + \eta_2)(\lambda_0(p) + \lambda_2(p))\mathbb{E}[Y - Y \wedge d_2], \tag{3.20}$$

$$\beta_1^{(p)}(d_1) := (\lambda_0(p) + \lambda_1(p))\mathbb{E}[X \wedge d_1], \quad \beta_2^{(p)}(d_2) := (\lambda_0(p) + \lambda_2(p))\mathbb{E}[Y \wedge d_2], \tag{3.21}$$

$$\begin{aligned}\sigma_L^2(p, d_1, d_2) := &(\lambda_0(p) + \lambda_1(p))\mathbb{E}[(X \wedge d_1)^2] + (\lambda_0(p) + \lambda_2(p))\mathbb{E}[(Y \wedge d_2)^2]\\ &+ 2\lambda_0(p)\mathbb{E}[X \wedge d_1]\mathbb{E}[Y \wedge d_2].\end{aligned} \tag{3.22}$$

In order to characterize the optimal prevention control, we define the following:

$$\begin{aligned}\mathcal{R}(t, p) := &\eta_1(\lambda_0'(p) + \lambda_1'(p))I_X(d_1^*(t, p)) + \eta_2(\lambda_0'(p) + \lambda_2'(p))I_Y(d_2^*(t, p))\\ &- (1 + \eta_1)(\lambda_0'(p) + \lambda_1'(p))\mathbb{E}[X] - (1 + \eta_2)(\lambda_0'(p) + \lambda_2'(p))\mathbb{E}[Y]\\ &- \frac{\gamma}{2}\Big((\lambda_0'(p) + \lambda_1'(p))m_X(d_1^*(t, p)) + (\lambda_0'(p) + \lambda_2'(p))m_Y(d_2^*(t, p))\Big)e^{r(T-t)}\\ &- \gamma\lambda_0'(p)\, I_X(d_1^*(t, p))I_Y(d_2^*(t, p))e^{r(T-t)}.\end{aligned}$$

The quantity $\mathcal{R}(t,p)$ represents the marginal benefit of prevention after the retention functions have been optimally selected. Accordingly, define the following:

$$G(t,p) := C'(p) - \mathcal{R}(t,p),$$

which measures the net marginal cost of prevention. Economically, the strict increase of $G(t,p)$ means that as the prevention level increases, the marginal prevention cost rises relative to its marginal benefit, thus, reflecting diminishing marginal net returns from prevention. To ensure that the optimal prevention level is uniquely determined, we impose the following condition.

Assumption 3.5. *The function $G : [0,T] \times [0,\infty) \to \mathbb{R}$ is continuous, and for each $t \in [0,T]$, $G(t,\cdot)$ is strictly increasing.*

Proposition 3.6 (Optimal prevention)**.** *Under Assumptions 2.1, 2.2, and 3.5, the optimal prevention level is characterized by the following:*

$$p^*(t) = \begin{cases} 0, & G(t,0) \geqslant 0, \\ (C')^{-1}(\mathcal{R}(t,p^*(t))), & G(t,0) < 0. \end{cases} \tag{3.23}$$

Proof. We apply the envelope theorem[†] to the reduced maximization problem (3.19) with respect to the deductible pair (d_1,d_2). For a fixed t, Proposition 3.2 shows that the maximization with respect to (d_1,d_2) can be restricted to the compact set $\left[0,\frac{\eta_1}{\gamma e^{r(T-t)}}\right] \times \left[0,\frac{\eta_2}{\gamma e^{r(T-t)}}\right]$. The reduced objective is continuous in (d_1,d_2), and its partial derivative with respect to p is jointly continuous in (p,d_1,d_2). Since the optimal deductible pair is unique, the envelope theorem gives the following:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}p}\widetilde{F}(t,p,d_1^*(t,p),d_2^*(t,p)) = \Big(&-\partial_p q^{d,p}\big|_{d=d^*(t,p)} - C'(p) - \partial_p\beta_1^{(p)}(d_1)\big|_{d_1=d_1^*(t,p)} - \partial_p\beta_2^{(p)}(d_2)\big|_{d_2=d_2^*(t,p)}\Big)e^{r(T-t)} \\ &- \frac{\gamma}{2}\,\partial_p\sigma_L^2(p,d_1,d_2)\big|_{d=d^*(t,p)}\,e^{2r(T-t)}.\end{aligned}$$

The same formula holds for the right derivative at $p=0$. Using (3.20), (3.22), (3.21), (3.13), and (3.14), we obtain

$$\begin{aligned}\partial_p q^{d,p}\big|_{d=d^*(t,p)} &= (1+\eta_1)(\lambda_0'(p)+\lambda_1'(p))\Big(\mathbb{E}[X] - I_X(d_1^*(t,p))\Big) \\ &\quad + (1+\eta_2)(\lambda_0'(p)+\lambda_2'(p))\Big(\mathbb{E}[Y] - I_Y(d_2^*(t,p))\Big), \\ \partial_p\beta_1^{(p)}(d_1)\big|_{d_1=d_1^*(t,p)} &= (\lambda_0'(p)+\lambda_1'(p))I_X(d_1^*(t,p)), \\ \partial_p\beta_2^{(p)}(d_2)\big|_{d_2=d_2^*(t,p)} &= (\lambda_0'(p)+\lambda_2'(p))I_Y(d_2^*(t,p)),\end{aligned}$$

and

$$\begin{aligned}\partial_p\sigma_L^2(p,d_1,d_2)\big|_{d=d^*(t,p)} &= (\lambda_0'(p)+\lambda_1'(p))m_X(d_1^*(t,p)) + (\lambda_0'(p)+\lambda_2'(p))m_Y(d_2^*(t,p)) \\ &\quad + 2\lambda_0'(p)\,I_X(d_1^*(t,p))I_Y(d_2^*(t,p)).\end{aligned}$$

[†]See Corollary 4 of Milgrom and Segal [34]. For a maximization problem over a nonempty compact choice set, if the objective function is upper semicontinuous in the choice variable and its partial derivative with respect to the parameter is jointly continuous, then the optimized value function has well-defined one-sided derivatives. When the maximizer is unique, the two one-sided derivatives coincide. Importantly, differentiability of the maximizer with respect to the parameter is not required.

Substituting these identities into $\partial_p F(t,p)$ gives the following:

$$\frac{\mathrm{d}}{\mathrm{d}p}\widetilde{F}(t,p,d_1^*(t,p),d_2^*(t,p)) = e^{r(T-t)}(\mathcal{R}(t,p) - C'(p)) = -e^{r(T-t)}G(t,p). \tag{3.24}$$

By Assumption 3.5, $\partial_p F(t,\cdot)$ is strictly decreasing. Moreover, Assumption 2.2 ensures the boundedness of λ_k', while Assumption 2.1 gives $\mathbb{E}[X^2] < \infty$ and $\mathbb{E}[Y^2] < \infty$. Hence, $\mathcal{R}(t,\cdot)$ is bounded. Since $\lim_{p\to\infty} C'(p) = \infty$ by Assumption 2.2, we have the following:

$$\lim_{p\to\infty} G(t,p) = \infty. \tag{3.25}$$

If $G(t,0) \geqslant 0$, then the strict increase of $G(t,\cdot)$ implies $G(t,p) > 0$ for every $p > 0$. It follows from (3.24) that $\partial_p F(t,p) < 0$. Thus, $F(t,\cdot)$ is strictly decreasing on $(0,\infty)$, and its unique maximizer is as follows:

$$p^*(t) = 0.$$

If $G(t,0) < 0$, then, by the strict increase of $G(t,\cdot)$ and (3.25), there exists a unique $p^*(t) > 0$ such that

$$G(t,p^*(t)) = 0.$$

Specifically, $F(t,\cdot)$ is strictly increasing before $p^*(t)$ and strictly decreasing after $p^*(t)$. Therefore, $p^*(t)$ is the unique maximizer. Furthermore,

$$p^*(t) = (C')^{-1}(\mathcal{R}(t,p^*(t)))\,.$$

This completes the proof. □

Theorem 3.7. *Suppose that Assumptions 2.1, 2.2, and 3.5 hold. Let W be defined by* (3.9) *with $g(t) = e^{r(T-t)}$, $f(t) = \int_t^T \Gamma(s)\,\mathrm{d}s$, and $\Gamma(t) := \sup_{l_1,l_2,p}\left\{g(t)A(l_1,l_2,p) - \frac{\gamma}{2}g^2(t)\sigma_L^2(p,l_1,l_2)\right\} + \frac{(\mu(t)-r)^2}{2\gamma\sigma_R^2}$. Then, W is a solution of the HJB equation* (3.1)*, and*

$$V(t,w) = W(t,w) = J(t,w;\nu^*) = -\frac{1}{\gamma}\exp\{-\gamma(g(t)w + f(t))\}, \qquad (t,w) \in [0,T]\times\mathbb{R}.$$

Moreover, the optimal admissible control is $\nu^* = (\pi^*, l_1^*, l_2^*, p^*)$, where π^*, (l_1^*, l_2^*), and p^* are given by (3.11)*,* (3.14)*, and* (3.23)*, respectively.*

Proof. The identities in (3.10) imply

$$g'(t) + rg(t) = 0, \qquad g(T) = 1,$$

$$f'(t) + \Gamma(t) = 0, \qquad f(T) = 0.$$

Substituting the derivatives of W into (3.8), and using the definition of Γ, shows that W satisfies the HJB equation (3.1). The maximization with respect to π yields (3.11). Propositions 3.2 and 3.6 give the maximizing retention functions and prevention level, respectively. Hence, ν^* attains the supremum in the HJB equation. Next, we verify that the candidate control ν^* satisfies all the conditions of Definition 2.4. The boundedness of π^*, the measurability and integrability of p^*, and the measurability of l_1^* and l_2^* are verified as follows.

Since $\mu(\cdot)$ is continuous on $[0,T]$, the candidate investment strategy $\pi^*(s) = \frac{\mu(s)-r}{\gamma\sigma_R^2}e^{-r(T-s)}$ is bounded and hence satisfies $\mathbb{E}\int_t^T (\pi^*(s))^2\,\mathrm{d}s < \infty$. The characterization of p^* and the strict monotonicity of $G(t,\cdot)$ imply that, for every $a \geqslant 0$, the optimal prevention level $p^*(t)$ exceeds a if and only if $G(t,a)$ is negative. Since $t \mapsto G(t,a)$ is continuous for every fixed a, it follows that $p^*(\cdot)$ is measurable. Moreover, $\mathcal{R}$ is bounded because λ_k' are bounded and X and Y have finite second moments. Since $C'(p) \to \infty$ as $p \to \infty$, the characterization of p^* shows that p^* is bounded on $[0,T]$. Therefore, $\mathbb{E}\int_t^T C(p^*(s))\,\mathrm{d}s < \infty$. In addition, since I_X, I_Y, λ_k, and the positive-part mapping are continuous, the mapping $(t,p,d_1,d_2) \longmapsto \mathcal{T}_{t,p}(d_1,d_2)$ is continuous. Starting from $(d_1^{(0)}, d_2^{(0)}) = (0,0)$, recursively define the following:

$$(d_1^{(n+1)}(t,p), d_2^{(n+1)}(t,p)) = \mathcal{T}_{t,p}(d_1^{(n)}(t,p), d_2^{(n)}(t,p)).$$

Each iterate is measurable, and the contraction property implies that the iterative sequence converges pointwise to the unique fixed point. Hence, (d_1^*, d_2^*) is measurable. Consequently,

$$l_1^*(s,x) = x \wedge d_1^*(s,p^*(s)), \qquad l_2^*(s,y) = y \wedge d_2^*(s,p^*(s))$$

are measurable and satisfy $0 \leqslant l_1^*(s,x) \leqslant x$, $0 \leqslant l_2^*(s,y) \leqslant y$. Under the candidate control ν^*, the drift coefficient of the wealth equation in (2.5) can be written as follows:

$$b(s,w) = rw + b_0^*(s),$$

where $b_0^*(s) = A(l_1^*(s), l_2^*(s), p^*(s)) + (\mu(s)-r)\pi^*(s)$. The two diffusion coefficients, $\sigma_L(p^*(s), l_1^*(s), l_2^*(s))$ and $\pi^*(s)\sigma_R$, do not depend on the wealth variable. Hence, the drift and diffusion coefficients are globally Lipschitz in w. Since p^*, π^*, d_1^*, and d_2^* are bounded, while X and Y have finite second moments, these coefficients also satisfy the required linear-growth and integrability conditions. Therefore, the wealth equation under ν^* admits a unique strong solution.

Next, we verify the two exponential integrability conditions in Definition 2.4. Under ν^*, the wealth equation admits the representation

$$\begin{aligned}\hat{\mathcal{W}}^{\nu^*}(s) = e^{r(s-t)}w &+ \int_t^s e^{r(s-u)}\left[A(l_1^*(u), l_2^*(u), p^*(u)) + (\mu(u)-r)\pi^*(u)\right]\mathrm{d}u \\ &+ \int_t^s e^{r(s-u)}\sigma_L(p^*(u), l_1^*(u), l_2^*(u))\,\mathrm{d}B_1(u) + \int_t^s e^{r(s-u)}\pi^*(u)\sigma_R\,\mathrm{d}B_2(u), \qquad s \in [t,T].\end{aligned}$$

Since $g(s) = e^{r(T-s)}$, it follows that

$$\begin{aligned}g(s)\hat{\mathcal{W}}^{\nu^*}(s) = e^{r(T-t)}w &+ \int_t^s e^{r(T-u)}\left[A(l_1^*(u), l_2^*(u), p^*(u)) + (\mu(u)-r)\pi^*(u)\right]\mathrm{d}u \\ &+ \int_t^s e^{r(T-u)}\sigma_L(p^*(u), l_1^*(u), l_2^*(u))\,\mathrm{d}B_1(u) + \int_t^s e^{r(T-u)}\pi^*(u)\sigma_R\,\mathrm{d}B_2(u).\end{aligned} \tag{3.26}$$

Since B_1 and B_2 are independent and the diffusion coefficients have been shown to be bounded, the quadratic variation of the continuous local martingale in (3.26) satisfies the following:

$$\int_t^T e^{2r(T-u)}\left[\sigma_L^2(p^*(u), l_1^*(u), l_2^*(u)) + (\pi^*(u))^2\sigma_R^2\right]\mathrm{d}u \leqslant K_1, \tag{3.27}$$

for some constant $K_1 > 0$. $A(l_1^*(u), l_2^*(u), p^*(u)) + (\mu(u) - r)\pi^*(u)$ is bounded on $[t, T]$. Since Γ is continuous on $[0, T]$, the definition $f(s) = \int_s^T \Gamma(u)\,\mathrm{d}u$ implies that f is bounded on $[0, T]$. Hence, (3.26) implies that there exists a constant $K_2 > 0$ such that

$$\begin{aligned}
&\sup_{t\leqslant s\leqslant T} \exp\left\{-2\gamma\left(g(s)\widehat{\mathcal{W}}^{\nu^*}(s) + f(s)\right)\right\}\\
&\qquad\leqslant K_2 \sup_{t\leqslant s\leqslant T} \exp\left\{-2\gamma \int_t^s e^{r(T-u)}\sigma_L(p^*(u), l_1^*(u), l_2^*(u))\,\mathrm{d}B_1(u)\right.\\
&\qquad\qquad\left. - 2\gamma \int_t^s e^{r(T-u)}\pi^*(u)\sigma_R\,\mathrm{d}B_2(u)\right\}.
\end{aligned} \tag{3.28}$$

By (3.27), Novikov's condition is satisfied. Consequently, the process

$$\begin{aligned}
\exp\Bigg\{ &- 2\gamma \int_t^s e^{r(T-u)}\sigma_L(p^*(u), l_1^*(u), l_2^*(u))\,\mathrm{d}B_1(u) - 2\gamma \int_t^s e^{r(T-u)}\pi^*(u)\sigma_R\,\mathrm{d}B_2(u)\\
&- 2\gamma^2 \int_t^s e^{2r(T-u)}\left[\sigma_L^2(p^*(u), l_1^*(u), l_2^*(u)) + (\pi^*(u))^2\sigma_R^2\right]\mathrm{d}u\Bigg\}, \qquad s \in [t, T],
\end{aligned} \tag{3.29}$$

is a martingale. Moreover, the terminal value of the exponential martingale (3.29) has a finite second moment. Indeed, applying the exponential-martingale property to the doubled stochastic integrands and using (3.27), we obtain the following:

$$\begin{aligned}
\mathbb{E}_{t,w} \exp\Bigg\{ &- 4\gamma \int_t^T e^{r(T-u)}\sigma_L(p^*(u), l_1^*(u), l_2^*(u))\,\mathrm{d}B_1(u) - 4\gamma \int_t^T e^{r(T-u)}\pi^*(u)\sigma_R\,\mathrm{d}B_2(u)\\
&- 4\gamma^2 \int_t^T e^{2r(T-u)}\left[\sigma_L^2(p^*(u), l_1^*(u), l_2^*(u)) + (\pi^*(u))^2\sigma_R^2\right]\mathrm{d}u\Bigg\} \leqslant e^{4\gamma^2 K_1}.
\end{aligned}$$

Therefore, Doob's L^2 maximal inequality, together with (3.28) and (3.27), yields the following:

$$\mathbb{E}_{t,w}\left[\sup_{t\leqslant s\leqslant T} \exp\left\{-2\gamma\left(g(s)\hat{\mathcal{W}}^{\nu^*}(s) + f(s)\right)\right\}\right] < \infty. \tag{3.30}$$

Since f is bounded on $[0, T]$, there exists a constant $K_3 > 0$ such that

$$\exp\left\{-2\gamma g(s)\hat{\mathcal{W}}^{\nu^*}(s)\right\} \leqslant K_3 \exp\left\{-2\gamma\left(g(s)\hat{\mathcal{W}}^{\nu^*}(s) + f(s)\right)\right\}, \qquad s \in [t, T].$$

Recalling that $g(s) = e^{r(T-s)}$, it follows from (3.30) that

$$\mathbb{E}_{t,w}\left[\sup_{t\leqslant s\leqslant T} \exp\left\{-2\gamma e^{r(T-s)}\hat{\mathcal{W}}^{\nu^*}(s)\right\}\right] < \infty.$$

Hence, Condition (2.6) is satisfied.

Moreover, the boundedness of π^* and $\sigma_L(p^*, l_1^*, l_2^*)$ implies that, for some constant $K_4 > 0$,

$$\mathbb{E}_{t,w} \int_t^T \exp\left\{-2\gamma e^{r(T-s)}\hat{\mathcal{W}}^{\nu^*}(s)\right\}\left[\sigma_L^2(p^*(s), l_1^*(s), l_2^*(s)) + (\pi^*(s))^2\sigma_R^2\right]\mathrm{d}s$$

$$\leqslant K_4(T-t)\mathbb{E}_{t,w}\left[\sup_{t\leqslant s\leqslant T}\exp\left\{-2\gamma e^{r(T-s)}\hat{W}^{\nu^*}(s)\right\}\right]<\infty.$$

Thus, Condition (2.7) also holds. Consequently, all the conditions of Definition 2.4 are satisfied, and $\nu^* \in \mathcal{A}[t,T]$.

It remains to verify the growth condition (3.2). Since

$$W(t,w) = -\frac{1}{\gamma}\exp\{-\gamma(g(t)w + f(t))\}$$

and

$$W_w(t,w) = g(t)\exp\{-\gamma(g(t)w + f(t))\},$$

we have

$$\begin{aligned}|W(t,w)|^2 + |W_w(t,w)|^2 &= \left(\frac{1}{\gamma^2} + g^2(t)\right)e^{-2\gamma f(t)}\exp\{-2\gamma g(t)w\}\\ &= \left(\frac{1}{\gamma^2} + g^2(t)\right)e^{-2\gamma f(t)}\exp\left\{-2\gamma e^{r(T-t)}w\right\}.\end{aligned}$$

Since f and g are bounded on $[0,T]$, there exists a constant $K_5 > 0$ such that

$$|W(t,w)|^2 + |W_w(t,w)|^2 \leqslant K_5\exp\left\{-2\gamma e^{r(T-t)}w\right\}, \qquad (t,w)\in[0,T]\times\mathbb{R}.$$

Thus, Condition (3.2) is satisfied.

We have shown that W solves the HJB equation, that ν^* attains the supremum in the HJB equation, and that ν^* is admissible. Moreover, the growth condition (3.2) has been verified. Therefore, all the conditions of Theorem 3.1 are satisfied, and hence

$$V(t,w) = W(t,w) = J(t,w;\nu^*).$$

Hence, ν^* is an optimal admissible control. □

3.3. Examples

Example 3.8 (No common shock)**.** Assume that $\lambda_0(p)\equiv 0$ for all $p \geqslant 0$. In this case, claim arrivals are independent across the two lines. The reinsurance premium rate (2.1) reduces to

$$q^{l,p} = (1+\eta_1)\lambda_1(p)\mathbb{E}[X - l_1(X)] + (1+\eta_2)\lambda_2(p)\mathbb{E}[Y - l_2(Y)],$$

and the diffusion variance (2.3) becomes

$$\sigma_L^2(p,l_1,l_2) = \lambda_1(p)\mathbb{E}[l_1(X)^2] + \lambda_2(p)\mathbb{E}[l_2(Y)^2].$$

Moreover, the coupled system in (3.14) is decoupled. More precisely,

$$d_1^*(t,p) = \left(\frac{\eta_1}{\gamma e^{r(T-t)}}\right)_+, \qquad d_2^*(t,p) = \left(\frac{\eta_2}{\gamma e^{r(T-t)}}\right)_+.$$

Since $\gamma > 0$ and $\eta_1,\eta_2 > 0$, then the optimal retentions are as follows:

$$l_1^*(x;t) = x \wedge \frac{\eta_1}{\gamma e^{r(T-t)}}, \qquad l_2^*(y;t) = y \wedge \frac{\eta_2}{\gamma e^{r(T-t)}}.$$

Remark 3.9. Example 3.8 describes the special case without common shocks. When $\lambda_0(p) \equiv 0$, the common-shock component disappears, and the two claim arrival processes become independent. In this case, both the reinsurance premium rate in (2.1) and the diffusion variance in (2.3) decompose into two terms—one solely depending on l_1 and the other solely depending on l_2. As a result, the optimization problem for the two retentions decouples.

In contrast, when $\lambda_0(p) \not\equiv 0$, a common shock may generate claims in both lines at the same time. Hence, even though the claim sizes X and Y are independent, the aggregate retained losses of the two lines are coupled through their common arrival component. This coupling appears in the optimality system (3.14) through the follows terms:

$$\frac{\lambda_0(p)}{\lambda_0(p)+\lambda_1(p)} I_Y(d_2^*(t,p)) \quad \text{and} \quad \frac{\lambda_0(p)}{\lambda_0(p)+\lambda_2(p)} I_X(d_1^*(t,p)).$$

Thus, a larger retained exposure in one line increases the risk contribution of common shocks and reduces the optimal retention of the other line. This shows that the common shock component creates a strategic interaction between the two reinsurance retention decisions, whereas such an interaction disappears in the independent two-line model.

Example 3.10 (Closed form prevention level)**.** Assume that $\lambda_k(p) = \bar{\lambda}_k e^{-ap}$ for $k = 0, 1, 2$, with $a > 0$, and that $C(p) = \kappa p^2$ with $\kappa > 0$. Then,

$$\lambda_k'(p) = -a\bar{\lambda}_k e^{-ap}, \qquad \frac{\lambda_0(p)}{\lambda_0(p)+\lambda_i(p)} = \frac{\bar{\lambda}_0}{\bar{\lambda}_0+\bar{\lambda}_i}, \quad i = 1, 2.$$

Thus, the ratios appearing in (3.14) are independent of p. It follows that the optimal retention levels do not depend on p, and we can write

$$d_i^*(t,p) \equiv d_i^*(t), \qquad l_i^*(\cdot;t,p) \equiv l_i^*(\cdot;t), \quad i = 1, 2.$$

Under this specification, the marginal benefit term in Proposition 3.6 can be written as follows:

$$\mathcal{R}(t,p) = ae^{-ap}\mathcal{K}(t),$$

where

$$\begin{aligned}\mathcal{K}(t) = {} & (\bar{\lambda}_0+\bar{\lambda}_1)\Big[(1+\eta_1)\mathbb{E}[X] - \eta_1 I_X(d_1^*(t))\Big] \\ & + \frac{\gamma}{2}e^{r(T-t)}\Big[(\bar{\lambda}_0+\bar{\lambda}_1)m_X(d_1^*(t)) + (\bar{\lambda}_0+\bar{\lambda}_2)m_Y(d_2^*(t)) \\ & + 2\bar{\lambda}_0 I_X(d_1^*(t))I_Y(d_2^*(t))\Big] + (\bar{\lambda}_0+\bar{\lambda}_2)\Big[(1+\eta_2)\mathbb{E}[Y] - \eta_2 I_Y(d_2^*(t))\Big].\end{aligned}$$

Then, the first-order condition $C'(p^*(t)) = \mathcal{R}(t, p^*(t))$ is reduced to the following:

$$2\kappa p^*(t) = ae^{-ap^*(t)}\mathcal{K}(t).$$

If $\mathcal{K}(t) > 0$, then

$$p^*(t) = \frac{1}{a}\mathbb{W}\left(\frac{a^2}{2\kappa}\mathcal{K}(t)\right),$$

where $\mathbb{W}$ denotes the Lambert W function. The associated optimal retention functions are as follows:

$$l_1^*(x;t) = x \wedge d_1^*(t), \qquad l_2^*(y;t) = y \wedge d_2^*(t),$$

where $d_1^*(t)$ and $d_2^*(t)$ are independent of p and satisfy

$$d_1^*(t) = \left(\frac{\eta_1}{\gamma e^{r(T-t)}} - \frac{\bar{\lambda}_0}{\bar{\lambda}_0 + \bar{\lambda}_1} I_Y(d_2^*(t)) \right)_+ ,$$
$$d_2^*(t) = \left(\frac{\eta_2}{\gamma e^{r(T-t)}} - \frac{\bar{\lambda}_0}{\bar{\lambda}_0 + \bar{\lambda}_2} I_X(d_1^*(t)) \right)_+ .$$

Remark 3.11. This example provides a closed-form representation of the optimal prevention level $p^*(t)$ through the Lambert W function. However, the retention levels $d_1^*(t)$ and $d_2^*(t)$ do not decouple in the presence of the common shock component. As shown in Example 3.8, the decoupling occurs when the common shock intensity $\lambda_0(p) \equiv 0$. In the present example, the specification $\lambda_k(p) = \bar{\lambda}_k e^{-ap}$ makes the ratios in (3.14) independent of p, and hence $d_1^*(t, p)$ and $d_2^*(t, p)$ are independent of p. Nevertheless, the common shock terms remain in (3.14), so the two retention levels are still linked through the truncated moments $I_Y(d_2^*(t))$ and $I_X(d_1^*(t))$. Therefore, the above example shows that the prevention control has an explicit formula, whereas the retention levels remain implicitly determined.

4. Numerical illustrations

In the numerical analysis, we adopt a heterogeneous decay specification for the prevention dependent intensities. Specifically, the idiosyncratic intensities are given by $\lambda_i(p) = e^{-a_i p} (i = 1, 2)$, while the common shock intensity is $\lambda_0(p) = e^{-a_0 p}$. This specification is more general than the one used in Example 3.10, where all three intensities share the same decay parameter a in order to obtain a closed form expression for the optimal prevention level. The prevention cost is $C(p) = \kappa p^2$ and the loss severities are $X \sim \mathrm{Exp}(\alpha_1)$ and $Y \sim \mathrm{Exp}(\alpha_2)$. The baseline parameter values are reported in Table 1.

Table 1. The basic parameters for numerical illustrations.

t	T	r	μ	σ_R	η_1	η_2	α_1	α_2	κ	γ	a_0	a_1	a_2
0	1	0.03	0.08	0.20	0.26	0.25	1	2	2	2	1	2	3

First, we examine the value of prevention. We fix $t = 0$ and numerically evaluate the reduced objective $F(0, p)$ over $p \in [0, 1]$. For each candidate value of p, the associated optimal retention pair is computed and substituted into the reduced objective as follows:

$$F(0, p) = -e^{r(T-t)} \Big(q^{l^*(p), p} + C(p) + \beta^{(p)}(l^*(p)) \Big) - \frac{\gamma}{2} e^{2r(T-t)} \sigma_L^2(p, l^*(p)).$$

Figure 1 shows that $F(p)$ is hump-shaped: starting from $p = 0$, moderate prevention improves the objective, while excessive prevention becomes suboptimal due to the increasing cost $C(p)$. The dotted vertical line indicates the threshold of $F(p) > F(0)$, which is close to zero. This indicates that when costs are not particularly high, prevention can yield substantial gains for the insurer. The dashed line marks the maximizer p^* under the parameters in Table 1, which is approximately 0.56. The theoretical characterization guarantees that this is the unique maximizer over the full admissible domain $p \in [0, \infty)$.

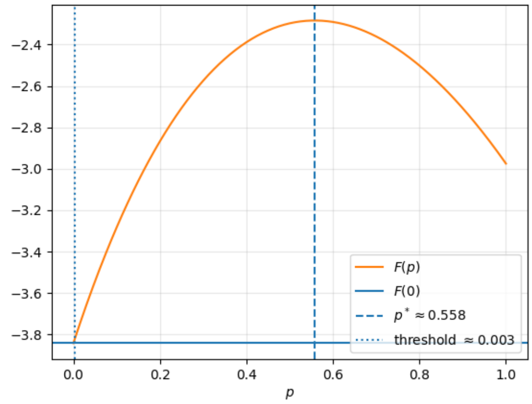

Figure 1. The impact of prevention effort p on the reduced objective.

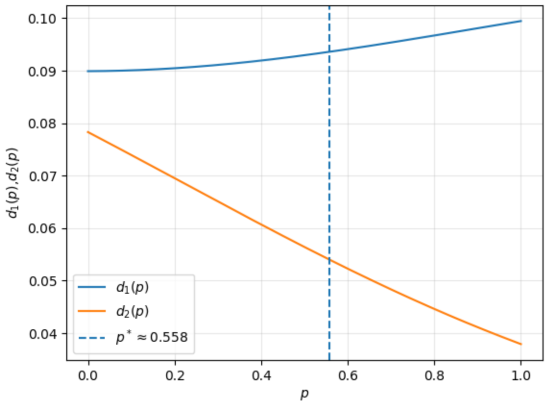

Figure 2. The impact of prevention effort p on the deductible levels d_1 and d_2.

Figure 2 illustrates how $d_1^*(t,p)$ and $d_2^*(t,p)$ vary with the prevention level p, with their values computed from the coupled fixed-point system (3.14). The variation of $d_1(t,p)$ and $d_2(t,p)$ reflects the interaction between prevention and common shock dependence. First, prevention changes the relative weights $\lambda_0(p)/(\lambda_0(p)+\lambda_1(p))$ and $\lambda_0(p)/(\lambda_0(p)+\lambda_2(p))$. Under the baseline choice, the common shock intensity decreases more slowly than the idiosyncratic intensities, and hence its relative contribution becomes larger as p increases. Second, $(d_1(t,p), d_2(t,p))$ are coupled through the truncated expectations $\mathbb{E}[X \wedge d_1^*]$ and $\mathbb{E}[Y \wedge d_2^*]$. Therefore, the effect of prevention on $(d_1(t,p), d_2(t,p))$ is jointly determined by changes in the relative arrival intensities and the feedback between the two dependent lines. Under the baseline parameters, $d_1^*(t,p)$ increases with p, thus indicating that the insurer retains more risk from line X as prevention becomes stronger. In contrast, d_2 decreases with p, thus indicating a stronger reliance on reinsurance for line Y. The vertical dashed line indicates the optimal prevention level $p^*(t)$. Its intersections with the two curves determine the optimal deductible levels: $d_1^*(t) = d_1^*(t, p^*(t))$ and $d_2^*(t) = d_2^*(t, p^*(t))$. These deductible levels further determine the optimal retention functions.

We further examine the coupling induced by the common shock. Figure 3 reports two measures: the absolute coupling term $2\lambda_0(p)\mathbb{E}[l_1(X)]\mathbb{E}[l_2(Y)]$ and the relative coupling share

$2\lambda_0(p)\mathbb{E}[l_1(X)]\mathbb{E}[l_2(Y)]/\sigma_L^2(l_1, l_2, p)$. The latter measures the proportion of the total variance attributable to the common shock. Under the baseline parameters, the absolute coupling term increases with the common shock intensity, while the relative coupling share decreases. This indicates that, although the common shock strengthens the absolute dependence between the two claim lines, the idiosyncratic variance components grow more strongly and dominate the total variance. As a result, the relative contribution of the common shock component becomes smaller.

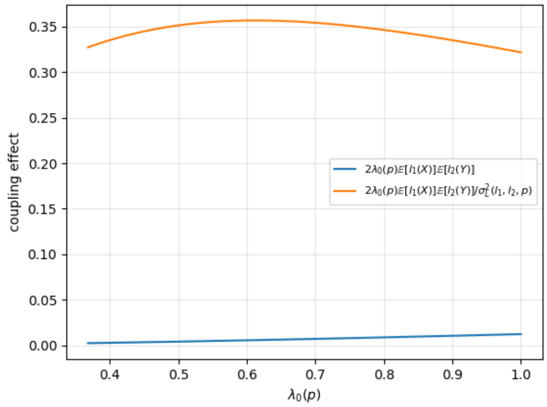

Figure 3. The impact of common shock $\lambda_0(p)$ on the coupling effect.

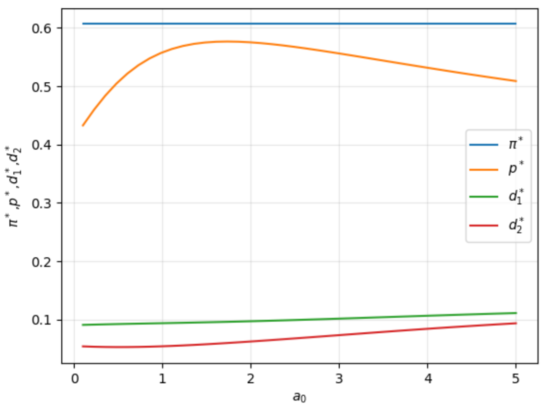

Figure 4. The sensitivity of the control variables to a_0.

Figures 4–6 present the sensitivity analyses of the four optimal controls $(\pi^*, d_1^*, d_2^*, p^*)$ with respect to a_0, κ, and γ. Figure 4 examines the effect of the prevention effectiveness parameter a_0. Since $\lambda_0(p) = e^{-a_0 p}$, a larger a_0 means that prevention is more effective in reducing the common shock intensity. Under the baseline parameters, p^* first increases with a_0, because the marginal benefit of prevention becomes larger. However, when a_0 is sufficiently large, the additional cost of further prevention becomes less attractive. As a result, p^* starts to decrease. At the same time, (d_1^*, d_2^*) also mildly increase, thus indicating that the insurer retains slightly more risk when the common shock is easier to control.

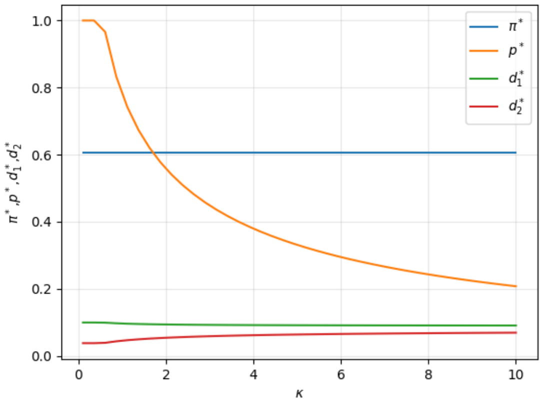

Figure 5. The sensitivity of the control variables to κ.

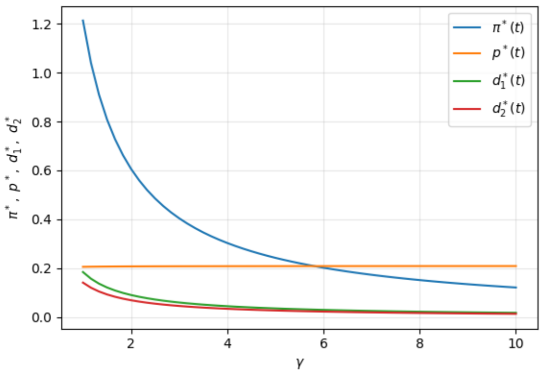

Figure 6. The sensitivity of the control variables to γ.

Figure 5 shows the impact of the prevention cost coefficient κ. Since $C(p) = \kappa p^2$, a larger κ makes the cost function steeper and raises the marginal cost $C'(p) = 2\kappa p$. As a result, the optimal prevention level p^* clearly decreases as κ increases. In contrast, the changes in d_1^* and d_2^* are relatively small under the baseline parameters. This suggests that κ mainly affects the prevention decision through the cost channel, while its indirect effect on the optimal reinsurance structure is limited. Figure 6 illustrates the effect of the risk aversion coefficient γ. A larger γ makes the insurer more conservative. Accordingly, the optimal investment π^* decreases, and d_1^* and d_2^* also decline, meaning that the insurer transfers more losses to the reinsurer. Contrastingly, p^* remains almost unchanged in this numerical setting. This indicates that, under the baseline parameters, the optimal prevention level is mainly driven by claim intensity reduction and prevention cost, rather than by the insurer's risk aversion.

5. Conclusions

This paper studied how an insurer simultaneously chooses investment, reinsurance, and prevention efforts. The claim arrivals are dependent due to a common shock, and reinsurance is modeled using a general framework that is not restricted to any specific type. Prevention reduces the claim frequency at a cost. Under exponential utility, we derive the optimal controls. The key findings are as follows: first, the optimal risky investment admits an explicit formula; second, starting from unrestricted retention functions, we show that the optimal retentions take the excess-of-loss form, with the levels d_1 and d_2 determined by coupled equations induced by the common shock; and third, the optimal prevention strategy is expressed through the inverse of the marginal cost function, which provides a tractable characterization of the optimal prevention level.

The numerical results confirm the value of prevention for the insurer, especially when the prevention cost is not too high. Additionally, they also show that prevention affects the optimal retention structure through changes in relative arrival intensities and the feedback induced by common shock dependence. The common shock increases the absolute coupling between the two lines, but its relative share in the total variance may decrease when idiosyncratic variance components dominate. The sensitivity analysis indicates that higher prevention effectiveness leads to a nonmonotone response of the optimal prevention level, while the optimal levels d_1^* and d_2^* only mildly change. A larger prevention cost coefficient leads to a lower optimal prevention level, while greater risk aversion makes both investment and retention decisions more conservative, with only a limited effect on prevention.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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