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Research article

Semilocal convergence of a modified Newton method for nonlinear systems with singular Jacobians

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Abstract: In this work, we were interested in approximating the solutions of nonlinear systems with singular Jacobian matrices. It is well known that the Newton iterative method loses its quadratic convergence when the Jacobian matrix of the nonlinear operator is singular in a neighborhood of the solution. We introduced a semilocal convergence analysis for a modification of this iterative method that overcomes this drawback. We performed several dynamical experiments to illustrate the effects of singular Jacobian matrices and solved different nonlinear systems, thereby corroborating the theoretical results obtained.

Keywords: nonlinear systems; singularities; modified Newton method; semilocal convergence; dynamical systems

1. Introduction

We consider the problem of finding the solutions of the nonlinear system $H(w) = 0$, where the operator H is defined as

$$\begin{array}{rccl} H: & D \subset \mathbb{R}^m & \longrightarrow & \mathbb{R}^m \\ & w & \longrightarrow & H(w) = (h_1(w), h_2(w), \ldots, h_m(w))^T, \end{array}$$

where D is a nonempty open convex subset of $\mathbb{R}^m$ and h_i, with $1 \leq i \leq m$, is a function

$$\begin{array}{rccc} h_i: & D \subset \mathbb{R}^m & \longrightarrow & \mathbb{R} \\ & w & \longrightarrow & h_i(w). \end{array}$$

Many problems in science or engineering are modeled through a system of nonlinear equations, whose solution does not have an explicit expression. For this reason, the approximation of the solutions of the system $H(w) = 0$ is one of the most relevant fields of research in numerical analysis. The best known and most popular iterative method used in this field is Newton's method, defined for systems by the expression

$$\begin{cases} w^{(0)} \in D, \\ w^{(n+1)} = w^{(n)} - H'(w^{(n)})^{-1}H(w^{(n)}), \quad n \geq 0. \end{cases} \tag{1.1}$$

Newton's method is a well-established iterative scheme that, under appropriate assumptions, exhibits quadratic convergence, computational simplicity, and favorable dynamical behavior. Despite these attractive features, its successful application depends on two fundamental conditions: the availability of an initial approximation $w^{(0)}$ sufficiently close to the solution w^*, and the nonsingularity of the Jacobian matrix $H'(w^{(n)})$ in a neighborhood of w^*.

The case of multiple roots for a nonlinear equation in the unidimensional case has been the object of analysis for a long time, see [1, 2]. In recent years, this case has been widely studied. Among, others representative contributions include [3, 4]. In all of these works, a modification in the iterative expression has been introduced for preserving the convergence order when the solution is a root with multiplicity greater than one. This modification of the iterative functions that describes the method always involves m, the multiplicity of the root. However, the multidimensional case for approximating the solution of nonlinear systems, where the Jacobian is singular or bad conditioned has rarely be studied in the literature. Our aim is to perform a study for this case.

In [5, 6] the authors proposed variants of Newton's method that converge quadratically to the solution despite the fact that the Jacobian matrix is singular in the solution. In [7, 8], authors solved the problem where the matrix becomes singular after a few iterations for certain initial points. In recent years, other authors proposed and studied new iterative schemes for ill-conditioned Jacobians [9, 10].

The work in [5] presented a weighted variant of Newton's method designed for a particular class of nonlinear systems with multiple roots. Building on this idea, [11] proposed the following modified Newton scheme for nonlinear systems:

$$\begin{cases} w^{(0)} \in D, \\ w^{(n+1)} = w^{(n)} - [M(w^{(n)}) + H'(w^{(n)})]^{-1}H(w^{(n)}), \quad n \geq 0, \end{cases} \tag{1.2}$$

where the components of the matrix $M(w^{(n)})$ are given by

$$(M(w^{(n)}))_{ij} = \begin{cases} \lambda_i^{(n)} h_i(w^{(n)}) & \text{if} \quad i = j; \\ 0 & \text{if} \quad i \neq j. \end{cases}$$

Therefore, in matrix form, we can write

$$M(w^{(n)}) = \begin{pmatrix} \lambda_1^{(n)} h_1(w^{(n)}) & 0 & \cdots & 0 \\ 0 & \lambda_2^{(n)} h_2(w^{(n)}) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_m^{(n)} h_m(w^{(n)}) \end{pmatrix},$$

where $\lambda_i^{(n)} \in \mathbb{R}$, $0 < |\lambda_i^{(n)}| < +\infty$ with $i = 1, \ldots, m$ and $n = 0, 1, 2, \ldots$, is the i-th component of the variable vector $\lambda^{(n)}$, chosen so that the matrix $M(w^{(n)}) + H'(w^{(n)})$ is nonsingular in all iterates.

First, our aim in this study is the derivation of a semilocal convergence theorem for this modified Newton method with the establishment of computable existence and uniqueness domains for the solution and the derivation of convergence radii under explicit assumptions.

Moreover, we present a dynamical analysis illustrating the behavior of the method in the presence of singular Jacobians. Finally we present numerical experiments supporting the theoretical results.

The rest of the paper is structured as follows. In Section 2, we will motivate the numerical study of method (1.2). To do so, we study the poor accessibility of Newton method when applied to systems with singular Jacobians and the improvement of the accessibility with the modified Newton's method. In Section 3, we study the semilocal convergence of the method (1.2), a study that has not been carried out in previous articles concerning this method. Finally, in Section 4, we calculate the radii of convergence and uniqueness of the modified Newton's method for a particular singular system and, numerically, we compare both methods for larger systems.

2. Motivation

In the introduction, we have commented on the limitations of Newton's method when $det(H'(w^*)) = 0$, where w^* is one of the solutions of the system $H(w) = 0$. The fact that the Jacobian matrix of the system has a singularity in the solution (or close to it) can generate problems in the convergence of the method, since the number of starting points that converge to this solution is reduced and, in addition, we cannot ensure the convergence speed of Newton's method.

The proposed family of methods is a modification of Newton's method, which is applicable to systems with singularities, since we can adapt the vector $\lambda^{(n)}$ according to the system to be solved. The variable components of this vector are very useful to avoid the situation where the determinant of the matrix $H'(w^*) + M(w^*)$ cancels out, in order that there exists an inverse of the resulting matrix and the proposed method is well-defined near the solution of the system.

The aim of this section is to study the dynamical problems that Newton's method presents for systems with singularities and to see how the proposed method improves their dynamical behavior. To do so, we illustrate by dynamical planes the limited accessibility of Newton's method due to the presence of singularities in the problems. To check how the method (1.2) improves the accessibility of Newton's method, we apply it to the study of the same problems and compare the dynamic planes obtained by each method.

In all the dynamical examples, we work with 2×2 systems due to the dimensional limitations when representing the dynamical planes. Moreover, precisely because of these limitations and the fact that the initial points are of the form $w^{(0)} = (w_1^{(0)}, w_2^{(0)})^T$, we study the dynamics by working with real elements and not complex ones as is usual in the one-dimensional case. Therefore, we have systems of two nonlinear equations with two variables that take real values and whose Jacobian matrix of the system has a singularity in some of the solutions of the problem.

2.1. Example 1

The first example we are going to study dynamically is one of the examples studied in [5]. The equations of the system are

$$\begin{cases} h_1(w) = (w_1 - 1)^2(w_1 - w_2), \\ h_2(w) = (w_2 - 2)^5 cos\left(2\dfrac{w_1}{w_2}\right), \end{cases} \tag{2.1}$$

and the nonlinear system is $H(w) = (h_1(w), h_2(w))^T = 0$, whose solutions are $(1,2)^T$ and $(2,2)^T$. It is easy to see that, in these solutions, we have two singularities, that is, both solutions cancel the determinant of the Jacobian matrix of the system, which is a big problem for Newton's method.

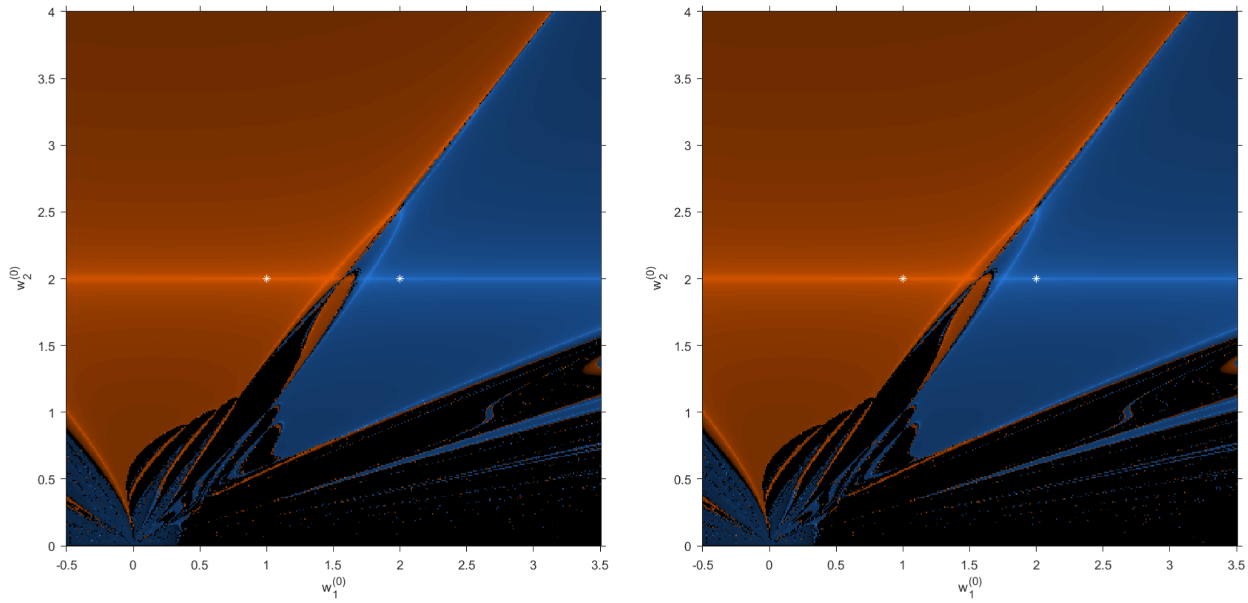
However, since method (1.2) does not use the inverse of $H'(w^{(n)})$, but the inverse of the matrix $M(w^{(n)}) + H'(w^{(n)})$, we can prevent the determinant of this matrix from canceling out in the solutions of the problem by choosing appropriately the vector $\lambda^{(n)} = (\lambda_1^{(n)}, \lambda_2^{(n)})$, which defines the modified Newton method. For this particular example, we have taken $\lambda_i \in \mathbb{R}$, $0 < |\lambda_i| < +\infty$, $i = 1,2$, such that the value of the term $\lambda_i^{(n)} h_i(w^{(n)})$ of the matrix $M(w^{(n)})$ and the value of the term $\dfrac{\partial f_i}{\partial w_i}(w^{(n)})$ have the same sign for each initial point $w^{(0)}$. To achieve this, the components of the vector are modified in each iteration n taking the values $\lambda_i^{(n)} \in \{-1, 1\}$ where it is convenient to adjust the signs of the previously mentioned terms.

To generate the dynamical planes, we choose a mesh of 400×400 points, where each point of the plane represents the two components $w_1^{(0)}$ and $w_2^{(0)}$ that compose each initial vector $w^{(0)}$. Then, we apply Newton and modified Newton methods to each of these initial estimates to represent to which of the solutions they converge in the case that they do. The maximum number of iterations that each point has to converge is 60 and we consider that the starting points converge to one of the solutions if the distance to a solution is less than 10^{-3}. We represent in orange the basin of attraction of the solution vector $(1,2)^T$, in blue the basin of $(2,2)^T$, and in black all those initial estimates that, after 60 iterations, have not converged to any of the two solutions.

For this example, although both solutions are singularities of the system, in Figure 1 we do not observe large differences between the dynamical planes of the methods even though Newton's method has problems in defining the inverse of the Jacobian matrix in these vectors. However, although the dynamical planes are similar, this does not mean that the accessibility of both is good since there is a large area of nonconvergence (the black area) that extends outside the limits we visualize.

It is important to note that the colors of the dynamic planes do not always have the same intensity. This is because there are initial points that converge more quickly to the solution than others. Therefore, the light colors represent initial points that need few iterations to converge to one of the solutions and, in contrast, the dark colors are the guesses that need more iterations to converge.

The previous example is an introduction to the dynamical problems that Newton method presents in the presence of singularities. In the following systems, we do not only highlight the poor accessibility of Newtonme thod when the solutions are singularities, but we also observe a clear improvement when using the modified Newton's method.



(a) Newton's method. **(b)** Modified Newton's method with $\lambda_i^{(n)} \in \{-1, 1\}$.

Figure 1. Basins of attraction for system (2.1).

2.2. Example 2

The system we are going to study is $H(w) = (h_1(w), h_2(w))^T = 0$, where

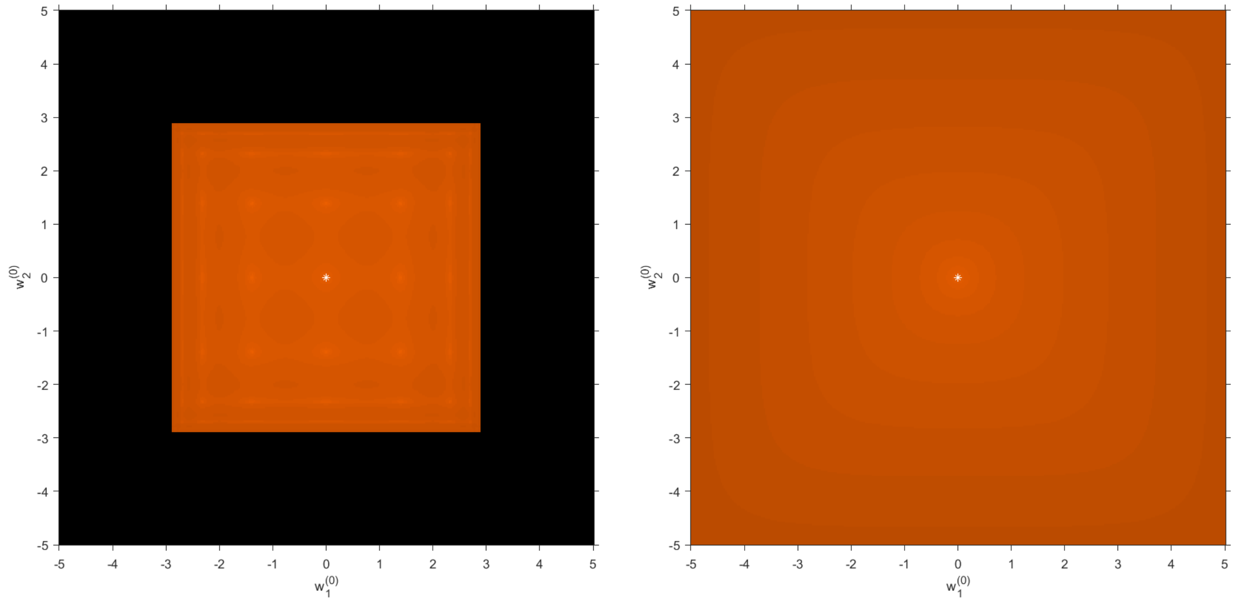
$$\begin{cases} h_1(w) = \arctan^2(w_1), \\ h_2(w) = \arctan^2(w_2), \end{cases} \tag{2.2}$$

whose solution is the vector $(0, 0)^T$. It is easy to see, that in this vector, we have a singularity, that is, the solution cancels the determinant of the Jacobian matrix of the system. However, in the modified Newton case, by taking the vector $\lambda^{(0)}$ as in the previous example, we manage to avoid the determinant of $M(w^{(n)}) + H'(w^{(n)})$ at points close to the solution from canceling.

For the dynamical planes used in this example, we use a mesh similar to the previous one. In this case, since in the dynamical study of the operators, we have only obtained one fixed attractor point, which is the solution vector $(0, 0)^T$, we only have an orange basin of attraction composed of the points that converge to the solution while the black color of the plane represents the initial points that do not converge.

In this example, we see in Figure 2 the increase in accessibility when using the modified Newton method instead of Newton method. Although Newton method presents problems with the starting point $(0, 0)^T$, we observe that the basin of attraction of this solution (Figure 2(a)) is concentrated near this solution because this point is an attractor fixed point of the dynamical Newton operator. That is, although the problems are generated by the solution itself, the points close to it do converge in less than 60 iterations. However, as we move further away, we see that there is a large area of the plane that does not converge.

In contrast, in the dynamical plane presented by the modified Newton method (Figure 2(b)), we observe that initial estimates far from the solution continue to converge to the solution. This is because, by studying the real dynamics of the operator of this method applied to the system (2.2), we obtain that $(0,0)^T$ is a super attractor fixed point of the method.



(a) Newton's method. **(b)** Newton's modified method with $\lambda_i^{(n)} \in \{-1, 1\}$.

Figure 2. Basins of attraction for the system (2.2).

2.3. *Example 3*

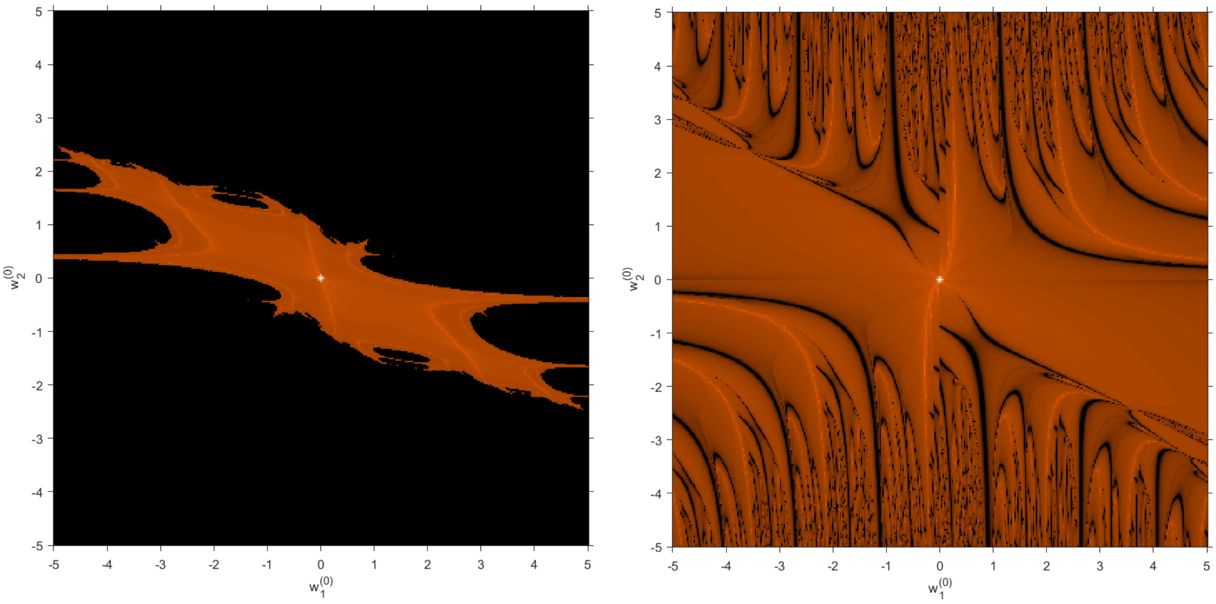
The last system with which we have worked dynamically to motivate the numerical and semilocal convergence study of the method (1.2) is given by the following components,

$$\begin{cases} h_1(w) = \dfrac{w_1 + 2w_2}{1 - w_1 w_2}, \\ h_2(w) = \arctan(w_1) + 2\arctan(w_2), \end{cases} \tag{2.3}$$

where the operator is defined by $H(w) = (h_1(w), h_2(w)) = 0T$ and the solution is the vector $(0,0)^T$. We see, as in the previous example, that in this vector, we have a singularity.

For the dynamical planes, we have again chosen a mesh of 400×400 points, with a maximum number of 60 iterations and the same convergence criterion as in the first two examples. When applying Newton and modified Newton methods to each of the initial vectors of the dynamical plane, we have represented in orange all those starting points that end up converging to the solution $(0,0)^T$ and in black all those vectors that we consider to have not converged under the specified criterion.

Again we see large differences between the two planes in Figure 3 since the convergence zone of Newton method is very small, while the black zone, made up of the points that do not converge, is unlimited. However, by applying method (1.2), with the components of the vector $\lambda^{(0)}$ varying as appropriate, we have greatly reduced the zone of nonconvergence presented by Newton's method.



(a) Newton's method. **(b)** Newton's modified method with $\lambda_i^{(n)} \in \{-1, 1\}$.

Figure 3. Basins of attraction for system (2.3).

3. Convergence study

In this section, first we give the local error equation for the proposed method and then we obtain the convergence domain of existence and uniqueness for the solution, where neither of these things were obtained in [11].

3.1. Local convergence order

In [11], the authors proved the quadratic convergence for the iterative method described by (1.2), see Theorem 1, but we want now give the error equation for this scheme.

Theorem 1. *Let $H : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ be a sufficiently Fréchet differentiable function in a convex neighborhood of D, containing w^*, which is a solution of the nonlinear equation $H(w) = 0$, whose Jacobian matrix is continuous and nonsingular in D. Then, for any $\lambda^{(n)} \in R^m$, with $M(w^{(n)}) = diag(\lambda^{(n)})H(w^{(n)})$,if $M(w^{(n)}) + H'(w^{(n)})$ is also nonsingular in D, for an initial approximation sufficiently close to w^*, the family of methods defined by (1.2) has the following error equation:*

$$e_{n+1} = (\lambda^{(n)} I + c_1^{-1} c_2)\, e_n^2 + O\left(e_n^3\right),$$

where $e_n = w^{(n)} - w^*$ *and* $c_k = \dfrac{H^{(k)}(w^*)}{k!} \in \mathcal{L}_k(\mathbb{R}^m, \mathbb{R}^m)$, $k \geq 1$.

As we can see from the theorem, the error equation of the method is independent of the values of λ_i.

3.2. Semilocal convergence

In this section, we are going to study the semilocal convergence of the iterative method (1.2). For this purpose, we will assume that certain conditions are satisfied in order to prove that, given an initial point $w^{(0)}$, the iterative method (1.2) converges to a solution of the equation $H(w) = 0$. Furthermore, in this section, we define the domains of existence and uniqueness of solution.

For this study, we assume the following necessary conditions:

i. We consider the initial point $w^{(0)} \in D$ and the vector $\lambda^{(0)} = (\lambda_1^{(0)}, \ldots, \lambda_m^{(0)}) \in \mathbb{R}^m$ such that there exists $[M(w^{(0)}) + H'(w^{(0)})]^{-1}$ with $\|H(w^{(0)})\| \leq a$ and $\|[M(w^{(0)}) + H'(w^{(0)})]^{-1}\| \leq b$. To simplify the notation, we denote $c = ab$.

ii. There exists $L \geq 0$ such that

$$\|H(u) - H(w)\| \leq L\|u - w\|, \quad \text{for} \quad u, w \in D.$$

iii. There exists $L' \geq 0$ such that

$$\|H'(u) - H'(w)\| \leq L'\|u - w\|, \quad \text{for} \quad u, w \in D.$$

iv. We consider $\lambda^{(j)} \in \mathbb{R}^m$, $0 \leq j \leq n$, for all $n \geq 0$ such that there exists ε, $\mathcal{N}$ verifying

$$\max_{0\leq i,j\leq n} \|\lambda^{(i)} - \lambda^{(j)}\| \leq \varepsilon, \qquad \|\lambda^{(j)}\| \leq \mathcal{N}, \;\; 0 \leq j \leq n.$$

Throughout the paper, we employ, the infinity norm, in vectors, and its associated induced matrix norm. Under the general conditions (i)–(iv), we present the following convergence lemma.

Lemma 1. *Let $w^{(n)} \in B(w^{(0)}, r) \in D$, with $r > 0$ such that $\gamma(r) = [\varepsilon + b(\mathcal{N}L + L')]r \leq 1$, $\delta(r) = [\mathcal{N} + \frac{1}{2}L'b]r \leq 1$ and $c < r$. Then, there exists $[M(w^{(n)}) + H'(w^{(n)})]^{-1}$ satisfying*

$$\|[M(w^{(n)}) + H'(w^{(n)})]^{-1}\| \leq bg(\gamma(r)), \tag{3.1}$$

where $g(t) = \dfrac{1}{1-t}$.

Proof. Under conditions (i)–(iv), we have

$$\begin{aligned}
&\|I - [M(w^{(0)}) + H'(w^{(0)})]^{-1}[M(w^{(n)}) + H'(w^{(n)})]\| \\
&\qquad \leq \|[M(w^{(0)}) + H'(w^{(0)})]\|\|\|M(w^{(0)}) - M(w^{(n)}) + H'(w^{(0)}) - H'(w^{(n)})\| \\
&\qquad \leq b\left(\max_{1\leq i\leq m}\left\{|\lambda_i^{(0)}h(w^{(0)}) - \lambda_i^{(n)}h(w^{(n)})|\right\} + \|H'(w^{(0)}) - H'(w^{(n)})\|\right) \\
&\qquad \leq b\left(\max_{1\leq i\leq m}\left\{|(\lambda_i^{(0)} - \lambda_i^{(n)})h(w^{(0)}) - \lambda_i^{(n)}(h(w^{(n)}) - h(w^{(0)}))|\right\} + L'\|w^{(n)} - w^{(0)}\|\right) \\
&\qquad \leq b\left(\varepsilon\|H(w^{(0)})\| + \mathcal{N}\|H(w^{(n)}) - H(w^{(0)})\| + L'\|w^{(n)} - w^{(0)}\|\right) \\
&\qquad \leq b\left(\varepsilon a + \mathcal{N}Lr + L'r\right) < \gamma(r) \leq 1.
\end{aligned}$$

Then, by Banach's lemma, there exists $[M(w^{(n)}) + H'(w^{(n)})]^{-1}$ and, moreover, the Eq (3.1) is verified. □

From the above lemma, we deduce that

$$\|w^{(1)} - w^{(0)}\| \leq \|[M(w^{(0)}) + H'(w^{(0)})]^{-1}\|\|H(w^{(0)})\| \leq ab = c < r.$$

Therefore, we obtain that $w^{(1)} \in B(w^{(0)}, r)$ and, since $B(w^{(0)}, r) \subset D$, from the conditions of the lemma, we deduce that there exists $[M(w^{(1)}) + H'(w^{(1)})]^{-1}$. Therefore, $w^{(2)}$ is well-defined.

On the other hand, applying the definition of the method, we know that

$$[M(w^{(0)}) + H'(w^{(0)})](w^{(1)} - w^{(0)}) = -H(w^{(0)}),$$

and then

$$H(w^{(0)}) + H'(w^{(0)})(w^{(1)} - w^{(0)}) = -M(w^{(0)})(w^{(1)} - w^{(0)}).$$

Thus, we obtain

$$\begin{aligned} H(w^{(1)}) &= H(w^{(0)}) + H'(w^{(0)})(w^{(1)} - w^{(0)}) + \int_{w^{(0)}}^{w^{(1)}} [H'(z) - H'(w^{(0)})]dz \\ &= -M(w^{(0)})(w^{(1)} - w^{(0)}) + \int_0^1 [H'(w^{(0)} + \tau(w^{(1)} - w^{(0)})) - H'(w^{(0)})]d\tau(w^{(1)} - w^{(0)}). \end{aligned}$$

Applying norms, we obtain

$$\begin{aligned} \|H(w^{(1)})\| &\leq \|M(w^{(0)})\|\|w^{(1)} - w^{(0)}\| + \frac{1}{2}L'\|w^{(1)} - w^{(0)}\|^2 \\ &\leq \max_{1\leq i\leq m} \left\{|\lambda_i^{(0)} h_i(w^{(0)})|\right\} \|w^{(1)} - w^{(0)}\| + \frac{1}{2}L'\|w^{(1)} - w^{(0)}\|^2 \\ &\leq [\mathcal{N}\|H(w^{(0)})\| + \tfrac{1}{2}L'\|w^{(1)} - w^{(0)}\|]\|w^{(1)} - w^{(0)}\| \\ &\leq [\mathcal{N}a + \tfrac{1}{2}L'r]\|w^{(1)} - w^{(0)}\|. \end{aligned} \tag{3.2}$$

Now, on the one hand, we get

$$\begin{aligned} \|H(w^{(1)})\| &\leq [\mathcal{N}a + \tfrac{1}{2}L'r]b\|H(w^{(0)})| \\ &\leq [\mathcal{N} + \tfrac{1}{2}L'b]r\|H(w^{(0)})\|. \end{aligned}$$

Then, if $\delta(r) = [\mathcal{N} + \frac{1}{2}L'b]r \leq 1$, it follows that

$$\|H(w^{(1)})\| < \delta(r)\|H(w^{(0)})\| \leq \|H(w^{(0)})\|.$$

On the other hand, by applying (3.2) and Lemma 1, we obtain

$$\begin{aligned} \|w^{(2)} - w^{(1)}\| &\leq \|[M(w^{(1)}) + H'(w^{(1)})]^{-1}\|\|H(w^{(1)})\| \\ &\leq bg(\gamma(r))[\mathcal{N}a + \tfrac{1}{2}L'r]\|w^{(1)} - w^{(0)}\| \\ &< g(\gamma(r))\delta(r)\|w^{(1)} - w^{(0)}\| < \|w^{(1)} - w^{(0)}\|. \end{aligned}$$

In addition,

$$\|w^{(2)} - w^{(0)}\| \leq \|w^{(2)} - w^{(1)}\| + \|w^{(1)} - w^{(0)}\| < [1 + g(\gamma(r))\delta(r)]\|w^{(1)} - w^{(0)}\|. \tag{3.3}$$

Therefore, for $w^{(2)} \in B(w^{(0)}, r)$, we require that $g(\gamma(r)\delta(r)) < 1$, so we consider

$$[1 + g(\gamma(r))\delta(r)]c < \frac{1}{1 - g(\gamma(r))\delta(r)}c = r,$$

and thus, we will have a procedure to obtain r.

Lemma 2. *Under conditions of Lemma 1, we assume that the real equation*

$$\frac{c}{1 - g(\gamma(t))\delta(t)} = t \tag{3.4}$$

has at least one positive real solution and denote by r the smallest of them.

If $B(w^{(0)}, r) \subset D$ and $g(\gamma(r))\delta(r) < 1$, then the following items are verified for $n \geq 1$:

i. There exists $[M(w^{(n)}) + H'(w^{(n)})]^{-1}$, where $\|[M(w^{(n)}) + H'(w^{(n)})]^{-1}\| \leq bg(\gamma(r))$.

ii. $\|H(w^{(n)})\| \leq g(\gamma(r))\delta(r) \leq \|H(w^{(n-1)})\| < \|H(w^{(0)})\|$.

iii. $\|w^{(n+1)} - w^{(n)}\| \leq g(\gamma(r))\delta(r)\|w^{(n)} - w^{(n-1)}\| \leq \|w^{(n)} - w^{(0)}\|$.

iv. $w^{(n+1)} \in B(w^{(0)}, r) \subset D$.

Proof. Taking into account (3.4), it is evident that $c = ab < r$ and, moreover, $\delta(r) < 1$ since $g(\gamma(r)) \geq 1$ and $g(\gamma(r))\delta(r) < 1$.

On the other hand, from (3.3) and (3.4), it follows that

$$\begin{aligned}\|w^{(2)} - w^{(0)}\| &\leq [1 + g(\gamma(r))\delta(r)]\|w^{(1)} - w^{(0)}\| \\ &< \sum_{n\geq 0}[g(\gamma(r))\delta(r)]^n\|w^{(1)} - w^{(0)}\| \\ &= \frac{1}{1 - g(\gamma(r))\delta(r)}c = r.\end{aligned}$$

Therefore, $w^{(2)} \in B(w^{(0)}, r) \in D$. This proves the step $n = 1$.

Next, we prove items (i)–(iv) by applying a mathematical induction procedure. We suppose that they are verified for $n = k - 1$ and we will prove them for k.

We have that, by (iv) for $n = k - 1$, $w^{(k)} \in B(w^{(0)}, r) \in D$. Since we have already indicated that $c = ab < r$ and applying Lemma 1, we obtain that there exists $[M(w^{(k)}) + H'(w^{(k)})]^{-1}$ with $[M(w^{(k)}) + H'(w^{(k)})]^{-1}| \leq bg(\gamma(r))$, which proves item (i) for $n = k$.

To prove (ii) for $n = k$, we take into account that by the expression of the method, it is verified that

$$H(w^{(k-1)}) + H'(w^{(k-1)})(w^{(k)} - w^{(k-1)}) = -M(w^{(k)})(w^{(k)} - w^{(k-1)}),$$

and then

$$H(w^{(k)}) = H(w^{(k-1)}) + H'(w^{(k-1)})(w^{(k)} - w^{(k-1)}) + \int_{w^{(k-1)}}^{w^{(k)}} [H'(z) - H'(w^{(k-1)})]dz$$

$$= -M(w^{(k)})(w^{(k)} - w^{(k-1)}) + \int_0^1 [H'(w^{(k-1)} + \tau(w^{(k)} - w^{(k-1)})) - H'(w^{(k-1)})]d\tau(w^{(k)} - w^{(k-1)}).$$

Therefore, applying (ii) and (iii), for $n = k - 1$:

$$\begin{aligned}
\|H(w^{(k)})\| &\le \left[\mathcal{N}\|H(w^{(k-1)})\| + \tfrac{1}{2}L'\|w^{(k)} - w^{(k-1)}\|\right] \|w^{(k)} - w^{(k-1)}\| \\
&\le \left[\mathcal{N}\|H(w^{(0)})\| + \tfrac{1}{2}L'\|w^{(1)} - w^{(0)}\|\right] \|\|\|w^{(k)} - w^{(k-1)}\| \\
&\le \left[\mathcal{N}a + \tfrac{1}{2}L'r\right] \|\|\|[M(w^{(k-1)}) + H'(w^{(k-1)})]^{-1}\|\|\|H(w^{(k-1)})\| \\
&\le \left[\mathcal{N}a + \tfrac{1}{2}L'r\right] bg(\gamma(r))\|H(w^{(k-1)})\| \\
&\le \left[\mathcal{N} + \tfrac{1}{2}L'b\right] rg(\gamma(r))\|H(w^{(k-1)})\| \\
&\le \delta(r)g(\gamma(r))\|H(w^{(k-1)})\| \le \|H(w^{(k-1)})\|,
\end{aligned} \tag{3.5}$$

which proves (ii) for $n = k$.

On the other hand, taking into account (3.5), it follows that

$$\begin{aligned}
\|w^{(k+1)} - w^{(k)}\| &\le \|[M(w^{(k)}) + H'(w^{(k)})]^{-1}\|\|\|H(w^{(k)})\| \\
&\le bg(\gamma(r))\left[\mathcal{N}\|H(w^{(0)})\| + \tfrac{1}{2}L'\|w^{(1)} - w^{(0)}\|\right] \|w^{(k)} - w^{(k-1)}\| \\
&\le g(\gamma(r))\delta(r)\|w^{(k)} - w^{(k-1)}\| \le \|w^{(k)} - w^{(k-1)}\|,
\end{aligned}$$

which proves (iii) for $n = k$.

Finally, we have to

$$\begin{aligned}
\|w^{(k+1)} - w^{(0)}\| &= \|w^{(k+1)} - w^{(k)}\| + \|w^{(k)} - w^{(k-1)}\| + \ldots + \|w^{(1)} - w^{(0)}\| \\
&\left([g(\gamma(r))\delta(r)]^k + [g(\gamma(r))\delta(r)]^{k-1} + \ldots + [g(\gamma(r))\delta(r)] + 1\right) \|w^{(1)} - w^{(0)}\| \\
&< \frac{1}{1 - g(\gamma(r))\delta(r)} c = r,
\end{aligned}$$

so that (iv) is proved for $n = k$. □

Considering the above conditions and results, we formulate the semilocal convergence theorem.

Theorem 2. *Suppose that the conditions of Lemma 2 and Theorem 1 are verified. Then, for $w^{(0)} \in D$, the sequence $\{w^{(n)}\}$ given in (1.2) is well-defined, $\{w^{(n)}\} \subset B(w^{(0)}, r)$ and it converges to $w^* \in \overline{B(w^{(0)}, r)}$, a solution of the equation $H(w) = 0$.*

Proof. It follows directly from Lemma 2 that the sequence $\{w^{(n)}\}$ given in (1.2) is well-defined and that $\{w^{(n)}\} \subset B(w^{(0)}, r)$. Next, we will see that $w^{(n)}$ is a Cauchy sequence.

For $n, m \ge 0$, we have

$$\begin{aligned}
\|w^{(n+m)} - w^{(n)}\| &\le \|w^{(n+m)} - w^{(n+m-1)}\| + \|w^{(n+m-1)} - w^{(n+m-2)}\| + \ldots + \|w^{(n+1)} - w^{(n)}\| \\
&\le \left([g(\gamma(r))\delta(r)]^{n+m-1} + [g(\gamma(r))\delta(r)]^{n+m-2} + \ldots + [g(\gamma(r))\delta(r)]^n\right) \|w^{(1)} - w^{(0)}\| \\
&\le \frac{[g(\gamma(r))\delta(r)]^n - [g(\gamma(r))\delta(r)]^{n+m}}{1 - g(\gamma(r))\delta(r)} \|w^{(1)} - w^{(0)}\|
\end{aligned}$$

$$\leq \frac{1-[g(\gamma(r))\delta]^m}{1-g(\gamma(r))\delta(r)}[g(\gamma(r))\delta(r)]^n\|w^{(1)}-w^{(0)}\|.$$

Then, $\{w^{(k)}\}$ is a Cauchy sequence in a Banach space, so there exists $w^* = \lim_{n\to\infty} w^{(n)} \in \overline{B(w^{(0)}, r)}$. Furthermore, from (ii) of Lemma 2, we have that

$$\|H(w^{(n)})\| \leq [g(\gamma(r))\delta(r)]^n\|H(w^{(0)})\|,$$

and, by the continuity of the operator H, when $n \to \infty$, we obtain that $H(w^*) = 0$. □

To finish with the study of convergence, we end with the uniqueness theorem, which ensures the uniqueness of the solution in a certain environment under certain conditions.

Theorem 3. *Under the same conditions as in Theorem 2, the solution w^* of the equation $H(w) = 0$ is unique in $B(w^{(0)}, R) \cap D$, where*

$$R = \frac{2(1-\delta(r))}{L'b}.$$

Proof. Let $u^* \in B(w^{(0)}, R) \cap D$ be another solution of the equation $H(w) = 0$.

Then,

$$0 = H(u^*) - H(w^*) = \int_0^1 H'(w^* + \tau(u^* - w^*))d\tau(u^* - w^*),$$

so that if the operator $P = \int_0^1 H'(w^* + \tau(u^* - w^*))d\tau$ has an inverse, we obtain that $u^* = w^*$.

To prove that P^{-1} exists, we will apply Banach's lemma. Thus, we consider

$$\begin{aligned}
\|I - [M(w^{(0)}) + H'(w^{(0)})]^{-1}P\| &\leq \|[M(w^{(0)}) + H'(w^{(0)})]^{-1}\|\|M(w^{(0)}) + H'(w^{(0)}) - \int_0^1 H'(w^* + \tau(u^* - w^*))d\tau\| \\
&\leq b\left(\|M(w^{(0)})\| + \int_0^1 \|H'(w^* + \tau(u^* - w^*)) - H'(w^{(0)})\|d\tau\right) \\
&\leq b\left(\mathcal{N}\|H(w^{(0)})\| + \int_0^1 L'\|(1-\tau)(w^* - w^{(0)}) + \tau(w^* - w^{(0)})\|d\tau\right) \\
&< \mathcal{N}r + L'b\int_0^1[(1-\tau)r + \tau R]d\tau \\
&= \mathcal{N}r + \tfrac{1}{2}L'b(r+R) = 1.
\end{aligned}$$

Therefore, by Banach's lemma, P^{-1} exists and uniqueness is proved.

□

4. Numerical experiments

In this section, we apply the theoretical results obtained above to study the domain of convergence and uniqueness for some nonlinear systems. Moreover, we compare the behavior of the studied method with the classical Newton method.

4.1. Domains of convergence and uniqueness

First of all, we construct the auxiliary functions and the bounds needed to obtain the semilocal convergence ball for some problems that were solved in [12], but in this case, the domain of convergence and uniqueness was not obtained. We note in the introduction that this part involves the establishment of computable existence and uniqueness domains for the solution and the derivation of convergence radii under explicit assumptions.

4.1.1. Example 1

Let us consider the system $H(w) = (h_1(w), h_2(w))^T = 0$, where

$$\begin{cases} h_1(w) = w_1^2 - w_2 + 1, \\ h_2(w) = w_1 - \cos(\frac{\pi}{2} w_2), \end{cases}$$

which has a solution in $w^* = (0, 1)^T$. Then we work in the domain $B(w^*, 1/2)$.

4.1.2. Example 2

Next, we have the system $H(w) = 0$, where the components are defined by

$$\begin{cases} h_1(w) = 4(w_1 + w_2), \\ h_2(w) = 4(w_1 + w_2) + (w_1 - w_2)((w_1 - 2)^2 + w_2^2 - 1), \end{cases}$$

which has a solution in $w^* = (0, 0)^T$. We work in the domain $B(w^*, 1/4)$.

4.1.3. Example 3

The last system is composed by

$$\begin{cases} h_1(w) = e^{w_1} - w_2 - 1, \\ h_2(w) = w_1 - w_2, \end{cases}$$

where $H(w) = (h_1(w), h_2(w))^T = 0$ has a solution in $w^* = (0, 0)^T$. Then, we work in the domain $B(w^*, 1/2)$.

In Table 1, we can see the starting point considered for each example, the parameters $\lambda_i^{(0)},\ i = 1, 2$, some bounds implicated in conditions (i)–(iv) for setting the semilocal convergence study, and the radius of existence and uniqueness for the solution, denoted by r and R, respectively. The results show that, in spite of working with ill-conditioned problems, we manage to find the convergence balls.

Table 1. Radii of semilocal convergence balls for different values of $\lambda_i^{(n)}$.

Example	$w^{(0)}$	$\lambda_1^{(n)}$	$\lambda_2^{(n)}$	c	L	L'	r	R
1	$(1/10, 19/20)^T$	1	1	0.05258	$1 + \frac{\pi}{2}$	2	0.15254	0.87477
1	$(1/10, 19/20)^T$	1/2	1/2	0.06301	$1 + \frac{\pi}{2}$	2	0.18077	0.73955
2	$(1/100, 3/100)^T$	1/6	1/6	0.08776	$\frac{43}{4}$	5	0.16922	0.68546
2	$(1/100, 1/100)^T$	1/2	1/2	0.05377	$\frac{43}{4}$	5	0.12731	0.66063
3	$(1/10, 1/10)^T$	1/2	1/6	0.03268	2	$e^{-1/2}$	0.06521	0.43943
3	$(1/10, 1/10)^T$	1	1/5	0.01888	2	$e^{-1/2}$	0.07622	0.75799

4.2. *The applicability of the method*

As an additional numerical assessment, we consider the test problem reported in [13], consisting of a nonlinear system with 50 variables. The purpose of this example is to investigate the performance of the modified Newton method for large-scale problems and to compare its efficiency and convergence characteristics with those of the standard Newton method.

4.2.1. Example 4

The components of our operator $H(w)$ in this example are defined by the following expressions

$$\begin{cases} h_i(w) = w_i \sin(w_{i+1}) - 1, & 1 \le i \le m-1, \\ h_m(w) = w_m \sin(w_1) - 1. \end{cases} \tag{4.1}$$

For the numerical computations, variable-precision arithmetic with 100 significant digits is employed. Several initial approximations are selected, and the iterative process is stopped when the Euclidean distance between two consecutive iterates falls below the tolerance 10^{-25}. We solve Example 4 for different values of m, comparing the performance of the classical Newton method with the modified Newton scheme (1.2). Throughout these experiments, $\lambda^{(n)}$ is taken as a constant vector of dimension m.

We begin with the case $m = 2$. The corresponding nonlinear system possesses infinitely many solutions, as illustrated in Figure 4. Focusing on the solutions located in a neighborhood of the origin, we identify

$$s_1 = (1.1142\ldots, 1.1142\ldots)^T,$$

together with its symmetric counterpart

$$s_2 = -s_1.$$

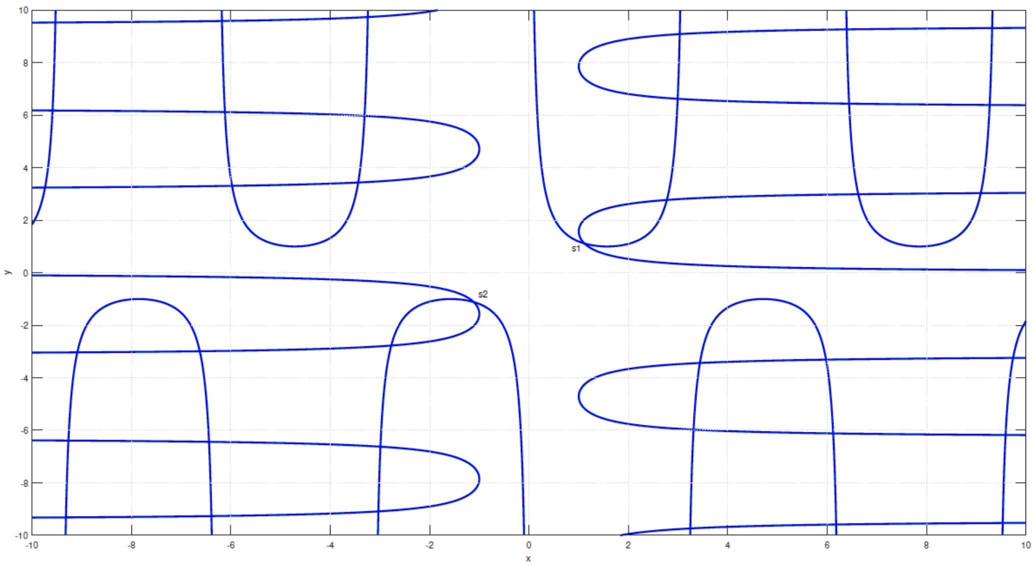


Figure 4. Graphical representation of the nonlinear system for $m = 2$.

Table 2 summarizes the numerical experiments obtained from several initial approximations. It can be observed that the classical Newton method fails to converge for some starting points due to the singularity of the Jacobian matrix. In contrast, by selecting suitable values for the parameters $\lambda_i^{(n)}$, the modified Newton method successfully converges in those cases.

Table 2. Convergence results for Example 4 with $m = 2$.

Method	$w^{(0)}$	$\lambda^{(n)}$	iter	$\|\|w_m^{(k+1)} - w_m^{(k)}\|\|$	$\|\|H(w^{(k+1)})\|\|$	ACOC	Sol
Newton (1.1)	$(0,0)^T$	–	–	–	–	–	–
Mod. Newton (1.2)	$(0,0)^T$	-1	6	1.4542×10^{-31}	2.0195×10^{-31}	2	s_1
Mod. Newton (1.2)	$(0,0)^T$	1	6	1.4542×10^{-31}	2.0195×10^{-31}	2	s_2
Newton (1.1)	$(0.9, 0.2)^T$	–	12	2.4496×10^{-26}	1.1518×10^{-26}	1.9781	s_2
Mod. Newton (1.2)	$(0.9, 0.2)^T$	-0.2	6	2.4496×10^{-26}	7.2143×10^{-28}	1.9992	s_1
Newton (1.1)	$(1,0)^T$	–	–	–	–	–	–
Mod. Newton (1.2)	$(1,0)^T$	-0.2	7	3.2984×10^{-32}	3.0995×10^{-32}	2	s_1

The columns of Table 2 report, respectively, the initial approximation, the constant vector $\lambda^{(n)}$ employed in each experiment, the number of iterations required to satisfy the prescribed tolerance, the distance between the last two iterates, the norm of the nonlinear operator H evaluated at the final approximation, the approximated computational order of convergence (ACOC) introduced in [14], and the computed solution.

We next consider the nonlinear system (4.1) with $m = 3$. Figure 5 depicts the three surfaces defining the system, whose intersections generate infinitely many solutions. The convergence properties of both iterative schemes toward the solutions

$$s_1 = (1.1142\ldots,\ 1.1142\ldots,\ 1.1142\ldots)^T$$

and its symmetric counterpart

$$s_2 = -s_1$$

are essentially the same as those observed for the case $m = 2$. Therefore, we focus on a different solution. In particular, using the initial approximation $w^{(0)} = (-1, 1, 0.5)^T$, the classical Newton method fails to converge, whereas the modified Newton method successfully converges to

$$s_3 = (9.7165\ldots,\ 3.0385\ldots,\ -3.4770\ldots)^T.$$

The corresponding numerical results are presented in Table 3.

The results reported in Table 3, corresponding to different initial approximations, confirm the same behavior observed previously. The singularity of the Jacobian prevents the classical Newton method from converging in several cases, while the proposed modified Newton method remains effective and reaches the corresponding solutions.

Finally, we consider Example 4 with $m = 50$ to evaluate the performance of both iterative schemes when applied to a high-dimensional nonlinear system. As in the previous cases, the system admits infinitely many solutions, including the solutions s_1 and s_2, whose 50 components are the natural extensions of those obtained for smaller values of m.

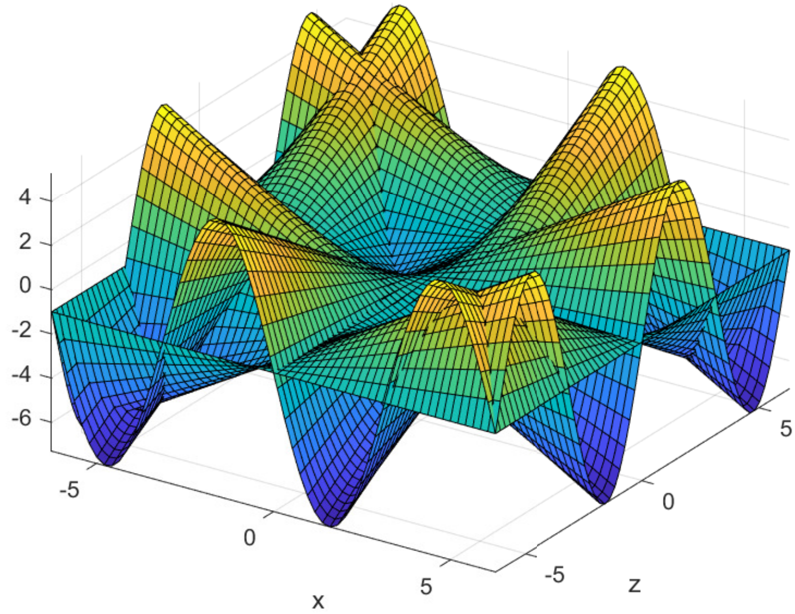


Figure 5. Graphical representation of the nonlinear system for = 3.

Table 3. Convergence results for Example 4 with $m = 3$.

Method	$w^{(0)}$	$\lambda^{(n)}$	iter	$\|\|w_m^{(k+1)} - w_m^{(k)}\|\|$	$\|\|H(w^{(k+1)})\|\|$	ACOC	Sol
Newton (1.1)	$(-1, 1, 0.5)^T$	–	100	NC	NC	–	–
Mod. Newton (1.2)	$(-1, 1, 0.5)^T$	-1	9	1.1912×10^{-26}	3.4423×10^{-26}	2.0908	s_3
Mod. Newton (1.2)	$(-1, 1, 0.5)^T$	-1.2	11	5.8898×10^{-49}	5.9379×10^{-49}	1.9999	s_1
Newton (1.1)	$(9, 3, -3)^T$	–	7	4.0701×10^{-48}	1.3556×10^{-47}	1.9999	s_3
Mod. Newton (1.2)	$(9, 3, -3)^T$	0.1	6	4.2756×10^{-26}	1.5069×10^{-25}	2.0014	s_3
Mod. Newton (1.2)	$(9, 3, -3)^T$	0.01	7	1.7024×10^{-48}	5.6709×10^{-48}	1.9998	s_3
Mod. Newton (1.2)	$(9, 3, -3)^T$	1	7	5.0351×10^{-46}	3.2056×10^{-45}	1.9411	s_3

The initial approximation is chosen as

$$w^{(0)} = [w_1^{(0)}, w_2^{(0)}]^T,$$

where $w_1^{(0)}$ and $w_2^{(0)}$ are subvectors of dimensions 20 and 30, respectively, given by

$$w_1^{(0)} = \text{ones}(1, 20), \qquad w_2^{(0)} = \frac{1}{4}\,\text{ones}(1, 30).$$

Table 4 summarizes the numerical results obtained for this large-scale problem. In addition to the quantities reported in the previous experiments, the last column provides the CPU time (in seconds) required by each method to compute the corresponding approximate solution.

Table 4. Convergence results for Example 4 with $m = 50$.

Method	$w^{(0)}$	$\lambda^{(n)}$	iter	$\|\|w_m^{(k+1)} - w_m^{(k)}\|\|$	$\|\|H(w^{(k+1)})\|\|$	ACOC	Sol	Time
Newton (1.1)	$w^{(0)}$		100	NC	NC	–	–	54.9641
Mod. Newton (1.2)	$w^{(0)}$	-1	8	2.3557×10^{-32}	2.4208×10^{-32}	2.0	s_1	4.3789
Mod. Newton (1.2)	$w^{(0)}$	-1.2	8	3.4272×10^{-26}	3.3661×10^{-26}	2	s_1	4.3982
Mod. Newton (1.2)	$w^{(0)}$	-1.5	9	2.2719×10^{-38}	3.1553×10^{-38}	2.0	s_1	5.1588
Mod. Newton (1.2)	$w^{(0)}$	-2	10	6.6908×10^{-48}	9.2922×10^{-48}	1.9999	s_1	5.8910

5. Conclusions

In this work, we performed the semilocal convergence study for a modification of Newton's iterative method for approximating the solution of nonlinear systems when the Jacobian matrix associated is singular in the starting point. The numerical results of Example 4 showed that in cases where the classical Newton method does not converge, the modified version can reach the solution in a few iterations. Also, in the last case of Table 4, that Newton method works correctly, and we have checked that for values of λ decreasing to zero, the behavior of modified method is similar to that of Newton method. In future research we will try to develop an adaptive or automatic selection strategy for choosing the parameter λ, and will also extend present framework to stochastic settings.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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